Submanifold Theory: class guide

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Prerequisites: Basics about manifolds, tensors, vector bundles, at least up to page 16 here. Basics of Riemannian geometry, fundamental groups and covering maps.

Bibliography: [**DT**], [**dC**], [**ON**], [**Pe**], [**Sp**], [**KN**]....

DO ALL THE EXERCISES IN [**DT**] !!!!

Clickable index

§1. Notations

Top. manifolds: Hausdorff + countable basis. Partitions of unity.
n-dimensional differentiable manifolds: M^n. Everything is C^∞.
$\mathcal{F}(M) := C^\infty(M, \mathbb{R})$; $\mathcal{F}(M, N) := C^\infty(M, N)$.
(x, U) chart $\Rightarrow$ coordinate vector fields $= \partial_i := \partial/\partial x_i \in \mathfrak{X}(U)$.
Tangent bundle TM, vector fields $\mathfrak{X}(M) := \Gamma(TM) \cong \mathcal{D}(M)$.
Submersions, immersions, embeddings, local diffeomorphisms.
Vector bundles, trivializing charts, transition functions, sections.
Tensor fields $\mathfrak{X}^{r,s}(M)$, k-forms $\Omega^k(M)$, orientation, integration.
Pull-back of a vector bundle $\pi : E \to N$ over N: $f^*(E)$.
Vector fields along a map $f : M \to N \;\Rightarrow \mathfrak{X}_f \cong \Gamma(f^*(TN))$.
f-related vector fields.

Distributions: Definition. Integrable and involutive distributions.

Theorem 1 (Frobenius)**.** *A distribution $D \subset TM$ is integrable if and only if it is involutive, namely,* $[X, Y] \in \Gamma(D)$ $\forall X, Y \in \Gamma(D)$. *We will write* $[D, D] \subset D$.

§2. Riemannian metrics

Gauss, 1827: $M^2 \subset \mathbb{R}^3 \Rightarrow \langle\, ,\rangle|_{M^2}$, $K_M = K_M(\langle\, ,\rangle)$, distances, areas, volumes... Non-Euclidean geometries.
Riemann, 1854: $\langle\, ,\rangle \Rightarrow K_M$ (relations proved decades later).
Slow development. General Relativity pushed up!
Riemannian metric, Riemannian manifold: $(M^n, \langle\, ,\rangle) = M^n$.
$g_{ij} := \langle \partial_i, \partial_j \rangle \in \mathcal{F}(U) \Rightarrow (g_{ij}) \in C^\infty(U, S(n, \mathbb{R}) \cap Gl(n, \mathbb{R}))$.
Isometries, local isometries, isometric immersions.
Product metric. $T_p\mathbb{V} \cong \mathbb{V}$, $T\mathbb{V} \cong \mathbb{V} \times \mathbb{V}$.

Examples: $(\mathbb{R}^n, \langle\, ,\rangle_{can})$, Euclidean submanifolds. Nash.

Example: (bi-)invariant metrics on Lie groups.

Proposition 2. *Every differentiable manifold admits a Riemannian metric.*

Angles between vectors at a point. Norm.
Riemannian vector bundles: $(E, \langle\,,\rangle)$.
The natural induced metric on $f^*(E)$ is $\langle\,,\rangle^f := \langle\,,\rangle \circ f$.
It always exists <u>local</u> orthonormal frames: $\{e_1, \ldots, e_n\}$.
Length of a piecewise differentiable curve $\Rightarrow$ Riem. distance d.
The topology of d coincides with the original one on M.

§3. Linear connections

If $M^n = \mathbb{R}^n$, or even if $M^n \subset \mathbb{R}^N$, there is a natural way to differentiate vector fields. And this depends only on $\langle\,,\rangle$.

Def.: An *affine connection* or a *linear connection* or a *covariant derivative* on M is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

with $\nabla_X Y$ being $\mathbb{R}$-bilinear, tensorial in X and a derivation in Y.

Tensoriality in $X \Rightarrow (\nabla_X Y)(p) = \nabla_{X(p)} Y$ makes sense.
Local oper.: $Y|_U = 0 \Rightarrow (\nabla_X Y)|_U = 0 \Rightarrow (\nabla_X Z)|_U = \nabla^U_{X|_U}(Z|_U)$
$\Rightarrow$ The *Christoffel symbols* Γ^k_{ij} of ∇ in a coordinate system $\Rightarrow$ Christoffel symbols completely determine the connection: all that is needed is to have <u>local basis of sections</u> $\Rightarrow$
Affine vector bundle $= (E, \nabla)$: formally exactly the same.
The local property above is a particular case of the following:

Proposition 3. *(or "Everything I know about connections!")* *Let ∇ be a linear connection on a vector bundle $\pi : E \to M$. Then, for every smooth map $f : N \to M$, there exists a unique linear connection ∇^f on $f^*(E)$ such that*

$$\nabla^f_Y(\xi \circ f) = \nabla_{f_* Y}\xi, \quad \forall\, Y \in \mathfrak{X}(N), \xi \in \Gamma(E).$$

Exercise. Give meaning and prove that $g^*(f^*(E)) = (f \circ g)^*(E)$ and $\left(\nabla^f\right)^g = \nabla^{f\circ g}$.

We will omit the superindex f in ∇^f.
In particular, Proposition 3 holds for any smooth curve $\alpha(t) = \alpha : I \subset \mathbb{R} \to M$, and if $V \in \mathfrak{X}_\alpha$ we denote $V' := \nabla_{\partial_t} V \in \mathfrak{X}_\alpha$.
So, if $\alpha'(0) = v$, $\nabla_v Y = (Y \circ \alpha)'(0)$. But beware of "$\nabla_{\alpha'}\alpha'$"!!

Def.: $V \in \mathfrak{X}_\alpha$ is *parallel* if $V' = 0$. We denote by $\mathfrak{X}''_\alpha$ the set of parallel vector fields along α.

Proposition 4. *Let $\alpha : I \subset \mathbb{R} \to M$ be a piecewise smooth curve, and $t_0 \in I$. Then, for each $v \in T_{\alpha(t_0)}M$, there exists a unique parallel vector field $V_v \in \mathfrak{X}_\alpha$ such that $V_v(t_0) = v$.*

The map $v \mapsto V_v$ is an isomorphism between $T_{\alpha(t_0)}M$ and $\mathfrak{X}''_\alpha$, and the map $(v, t) \mapsto V_v(t)$ is smooth when α is smooth $\Rightarrow$

Def.: The *parallel transport of $v \in T_{\alpha(t)}M$ along α between t and s* is the map $P^\alpha_{ts} : T_{\alpha(t)}M \to T_{\alpha(s)}M$ given by $P^\alpha_{ts}(v) = V_v(s)$.

Notice that $\mathcal{F}(M) = \mathfrak{X}^0(M) = \mathfrak{X}^{0,0}(M)$ and $\mathfrak{X}(M) = \mathfrak{X}^{0,1}(M)$.
Covariant differentiation of 1-forms and tensors: $\forall r, s \geq 0$,

$$\nabla \Rightarrow \begin{cases} \nabla : \mathfrak{X}^r(M) \to \mathfrak{X}^{r+1}(M); \\ \nabla : \mathfrak{X}^{r,s}(M) \to \mathfrak{X}^{r+1,s}(M); \\ \nabla : \mathfrak{X}^{r,s}(E, \hat{\nabla}) \to \mathfrak{X}^{r+1,s}(E, \hat{\nabla}); \end{cases}$$

for any affine vector bundle $(E, \hat{\nabla})$ (in partic., for $E = (TM, \nabla)$).

3.1 The Levi-Civita connection

Def.: A linear connection ∇ on a Riemannian manifold $(M, \langle\, ,\rangle)$ is said to be *compatible* with $\langle\, ,\rangle$ if, for all $X, Y, Z \in \mathfrak{X}(M)$,

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

Exercise. ∇ is compatible with $\langle\, ,\rangle \iff \forall\, V, W \in \mathfrak{X}_\alpha$, $\langle V, W\rangle' = \langle V', W\rangle + \langle V, W'\rangle \iff \forall\, V, W \in \mathfrak{X}''_\alpha$, $\langle V, W\rangle$ is constant $\iff P^\alpha_{ts}$ is an isometry, $\forall\, \alpha, t, s \iff \nabla\langle\, ,\rangle = 0$.

Def.: The tensor $T_\nabla(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ is called the *torsion* of ∇. We say that ∇ is *symmetric* if $T_\nabla = 0$.

Miracle: *Every Riemannian manifold $(M, \langle\, ,\rangle)$ has a unique linear connection that is symmetric and compatible with $\langle\, ,\rangle$, called the Levi-Civita connection of $(M, \langle\, ,\rangle)$.*

This is a consequence of the *Koszul formula*: $\forall\, X, Y, Z \in \mathfrak{X}(M)$,

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle - \langle Y, [X, Z]\rangle + \langle Z, [X, Y]\rangle.$$

Exercise. Verify that this formula defines a linear connection with the desired properties.

This is the only connection that we will work with. In coordinates, if $(g^{ij}) := (g_{ij})^{-1}$,

$$\Gamma^k_{ij} = \frac{1}{2}\sum_r \left(\frac{\partial g_{ir}}{\partial x_j} + \frac{\partial g_{jr}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_r}\right) g^{rk}\ .$$

Exercise. Show that, for $(\mathbb{R}^n, \langle\, ,\rangle_{can})$, $\Gamma^k_{ij} = 0$ and ∇ is the usual vector field derivative.

Exercise. Use Koszul formula to show that the Levi-Civita connection of a bi-invariant metric of a Lie Group satisfies, and is characterized, by the property that $\nabla_X X = 0\ \forall\, X \in \mathfrak{g}$.

Lemma 5. *(Symmetry and Compatibility Lemma) Let N be any manifold, and $f : N \to M$ a smooth map into a Riemannian manifold M. Then:*

- ∇^f *is symmetric, that is,* $\nabla^f_X f_* Y - \nabla^f_Y f_* X = f_*[X, Y]$, $\forall\ X, Y \in \mathfrak{X}(N)$;
- ∇^f *is compatible with the natural metric* $\langle\, , \rangle^f$ *on* $f^*(TM)$.

Example: For every isometric immersion $f\colon N \to M$ we have the natural decomposition of N-bundles

$$f^*(TM) = f_*(TN) \oplus^{\perp} T_f^{\perp} N$$

Accordingly, $\forall\ Z \in \mathfrak{X}_f$, we write $Z = Z^{\top} + Z^{\perp}$ $\Rightarrow$ the relation between the Levi-Civita connections is $f_* \nabla^N_X Y = (\nabla^f_X f_* Y)^{\top}$.

§4. Geodesics

When do we have minimizing curves? What are those curves? Critical points of the arc-length funct. $L : \Omega_{p,q} \to \mathbb{R}$: *geodesics*:

$$\gamma'' := \nabla_{\frac{d}{dt}} \gamma' = 0.$$

Geodesics are second order nonlinear nice ODE that depends only on ∇ (and not the metric) $\Rightarrow$

Proposition 6. $\forall\ v \in TM$, $\exists\ \epsilon > 0$ *and a unique geodesic* $\gamma_v : (-\epsilon, \epsilon) \to M$ *such that* $\gamma_v'(0) = v$ ($\Rightarrow \gamma_v(0) = \pi(v)$).

γ a geodesic $\Rightarrow \|\gamma'\| =$ constant.
γ and $\gamma \circ r$ nonconstant geodesics $\Rightarrow r(t) = at + b$, $a, b \in \mathbb{R} \Rightarrow$ $\gamma_v(at) = \gamma_{av}(t)$; $\gamma_v(t+s) = \gamma_{\gamma_v'(s)}(t) \Rightarrow$ *geodesic field* G of M:

Proposition 7. *There is a unique vector field* $G \in \mathfrak{X}(TM)$ *such that its trajectories are* γ'*, where* γ *are geodesics of* M.

The local flux of G is called the *geodesic flow* of M. In particular:

Corollary 8. *For each $p \in M$, there is a neighborhood $U_p \subset M$ of p and positive real numbers $\delta, \epsilon > 0$ such that the map*

$$\gamma : T_\epsilon U_p \times (-\delta, \delta) \to M, \quad \gamma(v,t) = \gamma_v(t),$$

is differentiable, where $T_\epsilon U_p := \{v \in TU_p \colon \|v\| < \epsilon\}$.

Since $\gamma_v(at) = \gamma_{av}(t)$, changing ϵ by $\epsilon\delta/2$ we can assume $\delta = 2 \Rightarrow$ We have the *exponential map* of M (terminology from $O(n)$):

$$\exp : T_\epsilon U_p \to M, \quad \exp(v) = \gamma_v(1).$$

$\Rightarrow \exp(tv) = \gamma_v(t) \Rightarrow \exp_p = \exp|_{T_pM} \colon B_\epsilon(0_p) \subset T_pM \to M \Rightarrow$

Proposition 9. *For every $p \in M$ there is $\epsilon > 0$ such that $B_\epsilon(p) := \exp_p(B_\epsilon(0_p)) \subset M$ is open and $\exp_p \colon B_\epsilon(0_p) \to B_\epsilon(p)$ is a diffeomorphism.*

An open set $p \in V \subset M$ onto which $\exp_p$ is a diffeomorphism as above is called a *normal neighborhood* of p, and when $V = B_\epsilon(p)$ it is called a *normal* or *geodesic ball* centered at p.

Exercise. Show that if ∇ is any affine connection on M, then there is another torsion free connection on M which has the same geodesics as ∇ (up to reparametrizations).

Exercise. ∇ and $\bar{\nabla}$ are two torsion free connections on M with the same geodesics $\iff$ there exists a 1-form ω such that $\nabla_X Y - \bar{\nabla}_X Y = \omega(X)Y + \omega(Y)X$ for all X, Y.

Proposition 9 $\Rightarrow \left(\exp_p|_{B_\epsilon(0_p)}\right)^{-1}$ is a chart of M in $B_\epsilon(p) \Rightarrow$ We always have (local!) *polar coordinates* for any $(M, \langle\,,\rangle)$:

$$\varphi : (0, \epsilon) \times \mathbb{S}^{n-1} \to B_\epsilon(p) \setminus \{p\}, \qquad \varphi(s, v) = \gamma_v(s), \qquad (1)$$

where $\mathbb{S}^{n-1} = \{v \in T_pM : \|v\| = 1\}$ is the unit sphere in T_pM.

Examples: $(\mathbb{R}^n, can)$; $(\mathbb{S}^n, can)$.

Exercise. Show that for a bi-invariant metric on a Lie Group, it holds that $exp_e = exp^G$.

4.1 Geodesics are (local) arc-length minimizers

Lemma 10. *(Gauss' Lemma) Let $p \in M$ and $v \in T_pM$ such that $\gamma_v(s)$ is defined up to time $s = 1$. Then,*

$$\langle (\exp_p)_{*v}(v), (\exp_p)_{*v}(w) \rangle = \langle v, w \rangle, \quad \forall\ w \in T_pM.$$

Proof. If $f(s,t) := \gamma_{v+tw}(s) = \exp_p(s(v + tw))$ then, for $t = 0$, $f_s = (\exp_p)_{*sv}(v)$, $f_t = (\exp_p)_{*sv}(sw)$ and $\langle f_s, f_t \rangle_s = \langle v, w \rangle$. ∎

Gauss' Lemma $\Rightarrow \mathbb{S}_\epsilon(p) := \partial B_\epsilon(p) \subset M$ is a regular hypersurface of M orthogonal to the geodesics emanating from p, called the *geodesic sphere* of radius ϵ centered at p.
Now, $B_\epsilon(p) := \exp_p(B_\epsilon(0_p)) \subset M$ as in Proposition 9 agrees with the metric ball of (M, d) !!!!! More precisely:

Proposition 11. *Let $B_\epsilon(p) \subset U$ a normal ball centered at $p \in M$. Let $\gamma : [0, a] \to B_\epsilon(p)$ be the geodesic segment with $\gamma(0) = p$, $\gamma(a) = q$. If $c : [0, b] \to M$ is another piecewise differentiable curve joining p and q, then $l(\gamma) \leq l(c)$. Moreover, if equality holds, then c is a monotone reparametrization of γ.*

Proof. In polar coordinates, $c(t) = \exp_p(s(t)v(t))$ in $B_\epsilon(p)\backslash\{p\}$, and if $f(s,t) := \exp_p(sv(t)) = \gamma_{v(t)}(s)$, we have that $c' = s' f_s + f_t$. Now, use that $f_s \perp f_t$, by Gauss' Lemma. ∎

Corollary 12. *d is a distance on M, $d_p := d(p, \cdot)$ is differentiable in $B_\epsilon(p)\backslash\{p\}$, and d_p^2 is differentiable in $B_\epsilon(p)$.*

Exercise. Compute $\|\nabla d_p\|$ and the integral curves of ∇d_p inside $B_\epsilon(p)\backslash\{p\}$.

Remark 13. Proposition 11 is LOCAL ONLY, and $\epsilon = \epsilon(p)$: $\mathbb{R}^n$; $\mathbb{S}^n$; $\mathbb{R}^n \setminus \{0\}$.

§5. Curvature

Gauss: $K(M^2 \subset \mathbb{R}^3) = K(\langle\,,\rangle)$. Riemann: $K(\sigma) = K_p(\exp_p(\sigma))$.

Def.: The *curvature tensor* or *Riemann tensor* of M is (sign!)

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

We also call R the (4,0) tensor given by

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle.$$

Curvature tensor $R_{\hat{\nabla}}$ of a vector bundle E with a connection $\hat{\nabla}$: exactly the same.

Proposition 14. *For all $X, Y, Z, W \in \mathfrak{X}(M)$, it holds that:*

- *R is a tensor;*
- *$R(X,Y,Z,W)$ is skew-symmetric in X,Y and in Z,W;*
- *$R(X,Y,Z,W) = R(Z,W,X,Y)$;*
- *$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ (first Bianchi id.);*
- *$R^s_{ijk} = \sum_l \Gamma^l_{ik}\Gamma^s_{jl} - \sum_l \Gamma^l_{jk}\Gamma^s_{il} + \partial_j \Gamma^s_{ik} - \partial_i \Gamma^s_{jk}$ ($\Rightarrow R \cong \partial^2 \langle\,,\rangle$).*

Proof. Exercise. ∎

$\langle\,,\rangle \Rightarrow \mathfrak{X}(M) \cong \Omega^1(M)$ and $\langle\,,\rangle$ extends to the tensor algebra $\Rightarrow$ the *curvature operator* $R : \Omega^2(M) \to \Omega^2(M)$ is self-adjoint.

Def.: If $\sigma \subset T_pM$ is a plane, then the *sectional curvature* of M at σ is given by

$$K(\sigma) := \frac{R(u,v,v,u)}{\|u\|^2\|v\|^2 - \langle u,v\rangle^2}, \qquad \sigma = \mathrm{span}\{u,v\}.$$

Proposition 15. *If R and R' are tensors with the symmetries of the curvature tensor and Bianchi such that $R(u,v,v,u) = R'(u,v,v,u)$ for all u, v, then $R = R'$ (i.e., K determines R).*

Corollary 16. *If M has constant sectional curvature $c \in \mathbb{R}$, then $R(X,Y,Z,W) = c(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle)$.*

Def.: The *Ricci tensor* is the symmetric (2,0) tensor given by

$$Ric(X,Y) := \frac{1}{n-1}\mathrm{trace}\, R(X,\cdot,\cdot,Y),$$

and the *Ricci curvature* is $Ric(X) = Ric(X,X)$ for $\|X\| = 1$.

Example: $\mathbb{CP}^n$ as $\mathbb{S}^{2n+1}/\mathbb{S}^1$ has $K(X,Y) = 1 + 3\langle JX,Y\rangle^2$ and $Ric \equiv (n+2)/(n-1)$.

Def.: The *scalar curvature* of M is $scal = \frac{1}{n}\mathrm{trace}\, Ric$.

Lemma 17. *(Compare with Lemma 5) Let $f : U \subset \mathbb{R}^2 \to M$ be a map into a Riemannian manifold and $V \in \mathfrak{X}_f$. Then,*

$$\nabla_{\partial_t}\nabla_{\partial_s}V - \nabla_{\partial_s}\nabla_{\partial_t}V = R(f_*\partial_t, f_*\partial_s)V.$$

More generally, $R_{\hat\nabla^f} = f^*(R_{\hat\nabla})$ for any affine vector bundle $(E,\hat\nabla) \to M$ and every smooth map $f : N \to M$.

Proof. Since $R_{\hat\nabla^f}$ is a tensor, it is enough to check for $\xi \circ f$ with $\xi \in \Gamma(E)$: Take $X, Y \in \mathfrak{X}(N)$ and $\bar X, \bar Y \in \mathfrak{X}(M)$ f-related to them near a point $p \in N$. Now compute $\hat\nabla^f_X\hat\nabla^f_Y(\xi \circ f)$. But this prove fails if f is not an immersion! Instead write in a chart of M, $f_*Z = \sum_i Z(f_i)\partial_i \circ f$ and perform the same computation. And don't use charts of N! ∎

Exercise. Let $\pi : E \to M$ be a vector bundle of rank k with a linear connection ∇. Then ∇ is flat if and only if each $\xi \in E$ has a (unique!) local parallel extension. If M is simply connected, such an extension exists globally and therefore $E \cong M \times \mathbb{R}^k$ is trivial.

§6. Isometric immersions (finally!)

As we have seen in the Example in page 6, if $f : M \to N$ is an isometric immersion $\Rightarrow f^*(TN) = f_*(TM) \oplus^\perp T_f^\perp M$, and $\nabla_X^M Y = (\nabla_X^f f_* Y)^\top$, $\forall X, Y \in TM$. Moreover, we have that

$$\alpha(X, Y) := \left(\nabla_X^f f_* Y\right)^\perp$$

is a symmetric tensor, called the *second fundamental form* of f. In addition, the map $\nabla^\perp : TM \times \Gamma(T_f^\perp M) \to \Gamma(T_f^\perp M)$ given by

$$\nabla_X^\perp \eta := \left(\nabla_X^f \eta\right)^\perp$$

is a connection in $T_f^\perp M$, called the *normal connection* of f. Identifications.

Exercise. Show that $\nabla^\perp$ is a compatible connection with the induced metric on $T_f^\perp M$.

$\alpha(p)$ is the quadratic approximation of $f(M) \subset N$ at $p \in M$.
$\alpha(v, v) = \gamma_v'(0)$: Picture!
$\eta \in T_{f(p)}^\perp M \Rightarrow$ (self-adjoint!) *shape operator* $A_\eta : T_pM \to T_pM$.
The Fundamental Equations. Particular case: K = constant $\Rightarrow$ the *Fundamental Theorem of Submanifolds*.
Gauss equation $\Leftrightarrow K(\sigma) = \overline{K}(\sigma) + \langle \alpha(u, u), \alpha(v, v)\rangle - \|\alpha(u, v)\|^2$

$\Rightarrow$ Riemann notion of sectional curvature agrees with ours.

Example: $\mathbb{S}^{n-1}(r) \subset \mathbb{R}^n \Rightarrow K \equiv 1/r^2$ (it *had* to be constant!).

Model of the hyperbolic space $\mathbb{H}^n$ as a submanifold of $\mathbb{L}^{n+1}$:
For $\epsilon = \pm 1$ and $r > 0$, let $i = inc\colon \mathbb{Q}^m_{\epsilon r^2} \to \mathbb{R}^{m+1}_\epsilon =: \mathbb{E}^{m+1}$, where

$$\mathbb{Q}^m_c = \{x \in \mathbb{E}^{m+1} : \langle x, x \rangle = 1/c\}, \quad c = \epsilon r^2.$$

Exercise. Show that $\alpha_i = -c \langle \, , \rangle \, i$, $A_i = -Id$, $K \equiv c$.

The second fundamental form of a composition of is.immersions.

Example: Second fund. form of a graph of a real function.

Example: The catenoid without a meridian and a periodic piece of the helicoid are isometric, but not congruent. Yet, the helicoid second fundamental form is periodic, hence it serves as a 'candidate' second fundamental form of the full catenoid. Yet, it is not realized by an isometric immersion of the catenoid! Reason: the catenoid is not simply connected.

Lemma 18. *Given $f : M^n \to \mathbb{R}^m$ and $v \in \mathbb{R}^m$, the set of critical points of the height function $h^v := \langle f, v \rangle$ is the set $\{x \in M : v \in T_x^\perp M\}$. Moreover,* $\mathrm{Hess}_{h^v} = A_{v^\perp}$.

§7. Hypersurfaces

Principal curvatures and directions; mean curvature;
Gauss-Kronecker curvature; Gauss map.
The fundamental equations for hypersurfaces.
Convex, locally convex and strictly locally convex hypersurfaces.

Lemma 19. *Given a compact $M^n \subset \mathbb{R}^{n+p}$, for every $0 \neq v \in \mathbb{R}^{n+p}$ there exists $x \in M^n$ such that v is normal to M^n at x and $A_v \geq 0$. Moreover, $\exists$ such a v with $A_v > 0$.*

Theorem 20. *For a compact Euclidean hypersurface M^n: The Gauss-Kronecker curvature never vanishes $\iff K > 0$ $\iff$ M is orientable and the Gauss map is a diffeomorphism $\iff$ The second fundamental form is definite everywhere $\Rightarrow$ M is a convex hypersurface ($M = \partial B$ for a convex body B).*

§8. Totally geodesic and umbilical submanifolds

$f : M \to N$ totally geodesic $\Leftrightarrow f_*(TM)$ is parallel in $f^*(TN)$. Umbilical distributions, submanifolds and extrinsic spheres.

Lemma 21. *A distribution (or submanifold) D is umbilical $\Leftrightarrow \nabla_X Y \in D$ for every $X, Y \in D$ with $X \perp Y$.*

Umbilic $\mathbb{Q}^m_{\tilde{c}} \subset \mathbb{Q}^{m+p}_c$ for $\tilde{c} \geq c$. Same if $\tilde{c} \leq c$ for the Lorentzian $\mathbb{Q}^{m+p}_c$.

Lemma 22. *For a curvature-like tensor R on $\mathbb{V}^n$ with $n \geq 3$ the following assertions are equivalent:*

1. *There exists $2 \leq r \leq n-1$ such that R preserves every r-dimensional subspace, i.e., $R(V,V)V \subset V$;*
2. *$\langle R(X,Y)Z, X\rangle = 0$ for every o.n. X, Y, Z;*
3. *All sectional curvatures of R are constant;*
4. *R preserves every subspace of $\mathbb{V}^n$.*

Axiom of r-planes. Axiom of r-spheres.

§9. Nullity distributions

The (relative) nullity distribution (Δ) Γ_c and the index of (relative) nullity ($\nu = \dim \Delta$) $\mu_c = \dim \Gamma_c$. $\Gamma := \operatorname{Ker}(R - f^*(\tilde{R})) = \{X : R(X, ...) = \tilde{R}(X, ...)\}$ and $\mu := \dim \Gamma$ are extrinsic.

Remark 23. Γ_c is always an intrinsic totally geodesic distribution where μ_c is constant (why?). Moreover, $\Delta \subset \Gamma$.

Proposition 24. *For an isometric immersion $f : M \to \tilde{M}$, the following assertions hold:*

i) ν, μ and μ_c are upper semicontinuous. Hence, the subsets where ν, μ and μ_c attain their minimum values are open, and there is an open and dense subset of M^n where ν, μ and μ_c are locally constant;

ii) Δ (resp. Γ and Γ_c) is smooth on any open subset of M^n where ν (resp. μ and μ_c) is constant;

iii) If $\tilde{M}$ has constant sectional curvature, then Δ is a totally geodesic (hence integrable) distribution on any open subset where ν is constant, and the restriction of f to each leaf of Δ is totally geodesic.

Exercise. Every umbilical distribution in a Riemannian manifold is integrable, and its leaves are umbilical submanifolds.

§10. Principal Normals and flat normal bundle

Principal and Dupin principal normals. Eigendistributions.

Proposition 25. *If η is a principal normal of $f : M^n \to \mathbb{Q}_c^m$:*

1. $E_\eta := \operatorname{Ker}(\alpha_f - \langle\, , \,\rangle \eta)$ *is smooth and umbilical;*

2. *If* $\operatorname{rank} E_\eta \geq 2$*, then* η *is Dupin;*

3. η *is Dupin* $\Leftrightarrow$ E_η *is spherical, and the leaves of* f *are mapped to extrinsic spheres in* $\mathbb{Q}_c^m$*;*

4. $\eta \neq 0$ *Dupin and* $c = 0$ $\Rightarrow$ $f + \frac{\eta}{\|\eta\|^2}$ *is constant along* E_η*.*

Proof. The only tricky part is to show that E_η is spherical in (3). If $\operatorname{rank} E_\eta \geq 2$ take $X, Y \in E_\eta$ with $X \perp Y$ and $\|X\| = 1$. Then, $0 = \overline{R}(Y,X)X)_{E_\eta^\perp} = (\overline{\nabla}_Y(\overline{\nabla}_X X)_{E_\eta^\perp})_{E_\eta^\perp}$ and so E_η is spherical, even in $\mathbb{Q}_c^m$. We leave the case $\operatorname{rank} E_\eta = 1$ as an exercise. ∎

Theorem 26. *If* M^n *is compact and* $f : M^n \to \mathbb{R}^m$ *has a principal curvature of (constant or not) multiplicity* $\geq k \geq \frac{n}{2}$*, then* M^n *has the homotopy type of a CW-complex with no cells of dimension* $n-k<r<k$*. In particular,* $H_r(M,G) = 0$*, for all* $n-k<r<k$ *and any coefficient group* G*.*

Proof. Let $v \in \mathbb{R}^m$. By Lemma 18, $\operatorname{Hess}_{h_v}|_{E_\eta} = \lambda I_{E_\eta}$, with $\lambda(p) = \langle \eta(p), v\rangle$. In particular, if we choose v in such a way that h_v is a Morse function, at a critical point x we have $\lambda(x) \neq 0$, and the index of x is at least k if $\lambda(x) < 0$ and at most $n-k$ if $\lambda(x) > 0$. Morse Theory implies the result. ∎

• Submanifolds with flat normal bundle.

§11. Reduction of codimension

First normal spaces $N_1(x) := \operatorname{span}\{\alpha(X,Y) : X, Y \in T_xM\}$.

Proposition 27. *Let* $f : M^n \to \mathbb{Q}_c^{n+p}$ *be an isometric immersion. Suppose that there exists a parallel normal subbundle* $L^q \subset T^\perp M$ *of rank* $q < p$ *such that* $N_1(x) \subset L^q(x)$ *for all* $x \in M^n$*. Then the codimension of* f *reduces to* q*.*

s-nullities: ν_s and ν_s^*.
1-regular and substantial isometric immersions.

Example: Draw a globally substantial curve in $\mathbb{R}^3$ that is nowhere locally substantial (better than Example 2.3 in [**DT**]).

Proposition 28. *Let $f : M^n \to \mathbb{Q}_c^m$ be a 1-regular (necessary!) isometric immersion such that rank $N_1 = q \leq n-1$. If $\nu_s^*(x) < n - s$ for all $1 \leq s \leq q$ at any point $x \in M^n$, then N_1 is parallel and thus f reduces codimension to q.*

§12. Minimal submanifolds

Let $f_t : M^n \to \bar{M}$ be an isotopy of $f = f_0$. Write $f_0' = f_* Z + \eta \in \mathfrak{X}_f$, with $Z \in TM$ and $\eta \in T_f^\perp M$ (i.e., $\mathfrak{X}_f = T_f \mathcal{F}(M, \overline{M})$). We will denote by $H = \text{trace}\, \alpha/n \in \Gamma(T_f^\perp M)$ the *mean curvature vector* of f. Then,

$$(dvol_t)'(0) = (-n\langle H, \eta\rangle + div\ Z)\, dvol.$$

Proposition 29. *M^n compact orientable with boundary and $Z|_{\partial M} = 0 \;\Rightarrow Vol(f_t(M))'(0) = -n\int_M \langle H, \eta\rangle dvol$. In particular, minimal submanifolds are precisely the critical points of the volume functional for compactly supported variations.*

$f : M^n \to \mathbb{R}^m \;\Rightarrow \Delta f = nH$. Hence, minimal $\Rightarrow$ harmonic $\Rightarrow$ *There are no compact minimal Euclidean submanifolds.* Also:

Proposition 30. *A compact minimal Euclidean submanifold with boundary is contained in the convex hull of its boundary. When substantial, it is contained in the interior of this hull.*

If $f : M^n \to \mathbb{Q}_c^{n+p}$ is minimal, then $Ric_M \le c$ since

$$Ric_M(X) = c + \frac{n}{n-1}\langle A_H X, X\rangle - \frac{1}{n-1}\sum_{i=1}^{p}\langle A_{\xi_i}^2 X, X\rangle. \quad (2)$$

In particular, $scal_M = c + \frac{n}{n-1}\|H\|^2 - \frac{1}{n(n-1)}\|\alpha\|^2$.

Lemma 31. *Given $F : M^n \to \mathbb{R}^{m+1}$, there exists a minimal $f : M^n \to \mathbb{S}_c^m$ such that $F = inc \circ f \iff \Delta F = -ncF$.*

§13. Veronese embeddings

Let $\mathcal{H}(m, d)$ be the real vector space of homogeneous harmonic polynomials of degree d in $(m + 1)$ real (similarly for $\mathbb{C}, \mathbb{H}$) variables. Then, $\dim \mathcal{H}(m, d) = n + 1$, where $n = n(m, d) = \frac{(2d+m-1)(d+m-2)!}{d!(m-1)!} - 1$. Then, $W = W(m, d) = \{f|_{\mathbb{S}^m} : f \in \mathcal{H}(m, d)\}$ is contained in (actually, it is equally to) the eigenspace of $\Delta_{\mathbb{S}^m}$ with eigenvalue $\lambda = \lambda(m, d) = -d(m + d - 1) < 0$. Fix $\langle\,,\rangle$ the L^2-inner product on W, and $\{f_0, \ldots, f_n\}$ an orthonormal basis of W. Set $G := O(m + 1)$,

$$F := (f_0, \ldots, f_n) : \mathbb{S}^m \to \mathbb{R}^{n+1}.$$

Since $\langle\,,\rangle$ is invariant under the G-action $A \cdot f = f \circ A$, the basis $\{A \cdot f_0, \ldots, A \cdot f_n\}$ is also orthonormal. So, identifying W with $\mathbb{R}^{n+1}$ via $f_i \mapsto e_i$, we conclude that there is $\tilde{A} \in O(W) \cong O(n + 1)$ such that $F \circ A = \tilde{A} \circ F$, and the map $A \mapsto \tilde{A}$ is a group homomorphism (such an F is said to be *G-equivariant*). In particular, G acts isometrically and transitively with the metric induced by F, and the isotropy groups $O(m)$ act irreducibly on on each tangent space and transitively on its Grassmannians. Thus, there exists $\tilde{c} > 0$ such that $F^*\langle\, ,\rangle = \tilde{c}\langle\, ,\rangle$, and hence F induces an isometric immersion of $\mathbb{S}^m_{1/\tilde{c}}$ into $\mathbb{R}^{n+1}$ with $\Delta F = (1/\tilde{c})\lambda F$. We conclude by Lemma 31 that there is a minimal equivariant isometric immersion $g : \mathbb{S}^m_{1/\tilde{c}} \to \mathbb{S}^n_c$, $c = -\lambda/m\tilde{c}$, $F = inc \circ g$. We just constructed the (essentially *unique!*) minimal, equivariant, and substantial *Veronese embeddings*,

$$g : \mathbb{S}^m_k \to \mathbb{S}^n, \quad k = k(m,d) := \frac{m}{d(m+d-1)}$$

(g is an embedding if d is odd, embedding of the projective space if d is even, and always induces an embedding $\mathbb{RP}^m_k \to \mathbb{RP}^n$). For $d = 1$ we get the identity, while for $d = 2$ we have, setting $\Pi = \{x_4 + x_5 + x_6 = 0\} \subset \mathbb{R}^6$, that $g = g(x,y,z)$ can be given by

$$\left(\frac{xy}{\sqrt{3}}, \frac{yz}{\sqrt{3}}, \frac{zx}{\sqrt{3}}, \frac{x^2-y^2}{3\sqrt{2}}, \frac{y^2-z^2}{3\sqrt{2}}, \frac{z^2-x^2}{3\sqrt{2}}\right) \subset \mathbb{S}^5 \cap \Pi = \mathbb{S}^4. \quad (3)$$

§14. Minimal rigidity of hypersurfaces

Kahler structure of orientable Riemannian surfaces.
The associated family of a minimal $M^2 \subset \mathbb{Q}^3_c$.

Exercise. Any minimal submanifold $M^n \subset \mathbb{Q}^m_c$ with $\mu_c \equiv n-2$ has an associated family.

Deformability and rigidity.

Theorem 32. *Let $f : M^n \to \mathbb{Q}^{n+1}_c$ be a minimal immersion of a Riemannian manifold with $\mu_c \not\geq n-2$. Then, f is rigid among minimal immersions $g : M^n \to \mathbb{Q}^{n+p}_c$, i.e., $g = inc \circ f$.*

Proof. By Gauss equation, $\lambda_i^2 = \sum_k \|\alpha_{ik}\|^2$, where λ_i are the principal curvatures of f and $\alpha_{ij} := \alpha_g(e_i, e_j)$, $1 \leq i, j \leq n$. So,

$$(\langle \alpha_{ii}, \alpha_{jj} \rangle - \|\alpha_{ij}\|^2)^2 = \lambda_i^2 \lambda_j^2 = \sum_k \|\alpha_{ik}\|^2 \sum_k \|\alpha_{jk}\|^2$$
$$\geq (\|\alpha_{ii}\|^2 + \|\alpha_{ij}\|^2)(\|\alpha_{jj}\|^2 + \|\alpha_{ij}\|^2) \geq (\langle \alpha_{ii}, \alpha_{jj} \rangle + \|\alpha_{ij}\|^2)^2.$$

Hence, $\alpha_{ij} \neq 0 \Rightarrow \langle \alpha_{ii}, \alpha_{jj} \rangle \leq 0 \Rightarrow \lambda_i \lambda_j \leq -\|\alpha_{ij}\|^2 < 0$. Thus, at the open subset U with minimum $\nu = \mu_c \leq n-3$, there should be a pair with $\alpha_{ij} = 0$. The above equation implies that α_{ii} and α_{jj} are linearly dependent, and $\alpha_{is} = 0$ for $i \neq s \neq j$. Changing the roles of s and j we get $\alpha_{ij} = 0$. We conclude by the first equation that $(\alpha_g)_{N_g^1} = \pm \alpha_f$. Done, since N_g^1 is parallel in U by Proposition 28 and g is analytic. ∎

§15. Local rigidity and flat bilinear forms

In local coordinates, an isometric immersion is a solution of a nonlinear PDE, so if the codimension is small it should be overdetermined. Hence rigidity should hold under *generic conditions*.
Analyze the proof of Theorem 32: It's just Gauss equation!
But: f rigid $\Rightarrow$ Find $\tau : T_f^\perp M \to T_g^\perp M$ satisfying

$$\tau \circ \alpha_f = \alpha_g.$$

Such τ is unique if f is full (or unique in N_1^f), and its parallelism is not hard to check (see Proposition 39 below). Now, a necessary condition for the existence of such a bundle isometry τ is that

$$\|\alpha_f(X, Y)\| = \|\alpha_g(X, Y)\|, \quad \forall X, Y \in TM. \tag{4}$$

which is equivalent by polarization to that, $\forall\, X, Y, X', Y' \in TM$,

$$\langle \alpha_f(X, Y), \alpha_f(X', Y') \rangle = \langle \alpha_g(X, Y), \alpha_g(X', Y') \rangle.$$

But this is also sufficient(!!): just define τ as $\tau \circ \alpha_f = \alpha_g$ and extend linearly. In other words, we need to understand when the flat bilinear form (FBF) $\beta = (\alpha_f, \alpha_g)$ is null, where

$$\beta = (\alpha_f, \alpha_g) : TM \times TM \to (T_f^\perp M \times T_g^\perp M, \langle\, , \rangle_f - \langle\, , \rangle_g).$$

15.1 Flat bilinear forms

Let $\beta : \mathbb{V} \times \mathbb{V}' \to \mathbb{W}^{p,q}$ a FBF.

Def.: $RE(\beta)$. $S(\beta)$. β_X for $X \in \mathbb{V}$. Isotropic (null) subspaces. $\nu_\beta := \dim N(\beta)$. For $X \in RE(\beta)$ set $\mathcal{U}(X) := \operatorname{Im} \beta_X \cap \operatorname{Im} \beta_X^\perp$.

Proposition 33. *The subset $RE(\beta)$ is open and dense in V.*

Observe that, by flatness: *if $\beta_X(\mathbb{V}') \subset \mathbb{W}$ is isotropic for all X in a dense subset, then β is null.*

Proposition 34. *For any bilinear form β and $X \in RE(\beta)$, $\beta(\mathbb{V}, \operatorname{Ker} \beta_X) \subset \operatorname{Im} \beta_X$. If β is also flat, then $\beta|_{\mathbb{V}\times \operatorname{Ker}\beta_X}$ is null* (since $\beta(\mathbb{V}, \operatorname{Ker} \beta_X) \subset \mathcal{U}(X)$).

Proof. For any $Y \in \mathbb{V}$ and t small, $L_t = \operatorname{Im} \beta_{X+tY} \subset \mathbb{W}$ is a *continuous* family of subspaces that contain $\beta_{X+tY}(\operatorname{Ker} \beta_X) = \beta_Y(\operatorname{Ker} \beta_X)$, which does not depend on t. ∎

Corollary 35. $\beta\colon \mathbb{V} \times \mathbb{V}' \to \mathbb{W}^{p,0}$ *FBF with $S(\beta)$ definite $\Rightarrow$ $\nu_\beta \geq \dim \mathbb{V}' - \dim S(\beta) \geq \dim \mathbb{V}' - \dim \mathbb{W}$.*

Theorem 36 (Chern-Kuiper)**.** *$M^n \subset \tilde{M}^{n+p} \Rightarrow \nu \leq \mu \leq \nu + p$.*

Corollary 37. *$M^n \subset \mathbb{R}^{n+p}$ compact $\Rightarrow \mu (= \mu_0) \not> p$.*

Corollary 38. *$M^n \subset \mathbb{R}^{n+p}$ compact and flat $\Rightarrow p \geq n$.*

15.2 Uniqueness of the normal connection

Proposition 39. *Let $f, f' : M^n \to \mathbb{Q}_c^{n+p}$ be isometric immersions and let $\tau : T_f^\perp M \to T_{f'}^\perp M$ be a vector bundle isometry that preserves the second fundamental forms. Then it also preserves the normal connections on the first normal bundles. In particular, it is parallel if either immersion is full.*

Def.: *Type number τ of f.*
Obs.: *$\tau(x) \geq 1 \ \Rightarrow f$ is full at x, $\nu_s(x) \leq n - s\tau(x)$, and $p \geq [n/\tau(x)]$.*

Remark 40. Allendoerfer: if $\tau \geq 4$, then G $\Rightarrow$ C+R.

§16. Local algebraic rigidity

Lemma 41 (Lorenzian Corollary 35)**.** *If β is a FBF with $S(\beta)$ Lorentzian $\Rightarrow \nu_\beta \geq \dim \mathbb{V}' - \dim S(\beta) \geq \dim \mathbb{V}' - \dim \mathbb{W}$.*

<u>*Exercise*</u>. *Replace the Lorentzian hypothesis in the above by $\dim \mathcal{U}(X) = 1$ for $X \in RE(\beta)$. Actually, by $\dim \mathcal{U}(X) \leq 3$, so it holds even if $\mathbb{W}$ has index ≤ 3 (see Lemma 45).*

Theorem 42 (Beez-Killing)**.** *$M \subset \mathbb{Q}_c^{n+1}$ with $\tau \geq 3$ is rigid.*

Corollary 43. *Let $f, f' : M^n \to \mathbb{Q}_c^{n+1}$ be nowhere congruent isometric immersions of a Riemannian manifold with no points of constant sectional curvature c. Then f and f' carry a common relative nullity distribution of rank $n-2$.*

Theorem 44 (Allendoerfer)**.** *Any $f : M^n \to \mathbb{Q}_c^{n+p}$ with $\tau \geq 3$ everywhere is rigid.*

Proof. By Proposition 39, we only need to show that $\beta = \alpha \oplus \alpha'$ is null, since $\tau \geq 1$ implies that f is full.

Let $k := \min\{\dim \mathcal{U}(X) \colon X \in RE(\beta)\}$. Similarly to $RE(\beta)$, $RE^o(\beta) := \{X \in \mathbb{V} \colon \dim \mathcal{U}(X) = k\}$ is also open and dense in $\mathbb{V}$. So, we only need to show that $k = p$.
$\tau \geq 3 \;\Rightarrow \exists\, L^{n-3p} := (\text{span}\{A_{\xi_j} X_i \colon 1 \leq j \leq p, 1 \leq i \leq 3\})^{\perp} = \cap_{i=1}^{3} \text{Ker}\, \alpha_{X_i}$. But $\dim \text{Ker}\, \beta_{X_1} = n - \text{rank}\, \beta_{X_1} \geq n - 2p + k$. Proposition 34 $\Rightarrow \dim \text{Ker}\, \beta_{X_1} \cap \text{Ker}\, \beta_{X_2} \geq \dim \text{Ker}\, \beta_{X_1} - \dim \mathcal{U}(X_1) \geq n - 2p$ and similarly $\dim \cap_{i=1}^{3} \text{Ker}\, \beta_{X_i} \geq n - 2p - k$. Done, since $\cap_{i=1}^{3} \text{Ker}\, \beta_{X_i} \subset L^{n-3p}$. ∎

§17. The Main Lemma

Def.: Nondegenerate FBFs.

Given a FBF $\beta : \mathbb{V}^n \times \mathbb{V}^n \to \mathbb{W}^{p,q}$, set $U := S(\beta) \cap S(\beta)^{\perp}$, $W = U \oplus \hat{U} \oplus^{\perp} L$ nondegenerate with $\hat{U}$ null, $S(\beta) = U \oplus L$, and $\beta = \beta_U + \beta_L$, with β_U null and β_L nondegenerate.

Lemma 45 (The Main Lemma)**.** *β symmetric nondegenerate with $\min\{p, q\} \leq 5 \;\Rightarrow \nu_\beta \geq n - p - q$.*

Remark 46. The above is false for $\min\{p, q\} \geq 6$. In fact, there are not even linear estimates: $\forall r \in \mathbb{N}, \exists$ a SFBF $\beta : \mathbb{V}^n \times \mathbb{V}^n \to \mathbb{W}^{p,p}$, $p = r(r+1)/2$, with $S(\beta) = \mathbb{W}$ and $\nu_\beta = n - 2p - \binom{r}{3}$.

Theorem 47 (do Carmo-Dajczer)**.** *$f \colon M^n \to \mathbb{Q}_c^{n+p}$ with $p \leq 5$. If $\nu_s \leq n - 2s - 1$ for all $1 \leq s \leq p \;\Rightarrow f$ is rigid.*

Proof. Just show that U above has dimension p, and notice that f is full since $\nu_1 \leq n - 3$. ∎

§18. Submanifolds with constant curvature

FBF were introduced by Cartan to study $f \colon M_c^n \to \mathbb{Q}_c^{n+p}$. Moore.

Examples: Product immersion $T^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \to \mathbb{S}^{2n-1}_{1/n}$, and local immersions $U^n \subset \mathbb{H}^n \to \mathbb{R}^{2n-1}$. Hilbert: $\nexists \mathbb{H}^n \to \mathbb{R}^{2n-1}$??

Lemma 48. $f\colon M^n_c \to \mathbb{Q}^{n+p}_{\tilde c}$. *$c < \tilde c \ \Rightarrow p \geq n-1$. $c > \tilde c$ and $p \leq n-2 \ \Rightarrow \alpha_f = \gamma + \sqrt{c-\tilde c}\,\langle\ ,\ \rangle\xi$ with γ flat and ξ unit.*

Proof. Define

$$\beta = \alpha \oplus \sqrt{|c-\tilde c|}\langle\,,\,\rangle\colon TM \times TM \to \mathbb{W} := T^\perp_f M \oplus \mathbb{R} \quad (5)$$

with the natural Lorentzian (resp. Riemannian) inner product in $\mathbb{W}$ if $c > \tilde c$ (resp. $c < \tilde c$) and apply the Main Lemma 45. ∎

A point $x \in M$ where $\alpha_f(x) = \gamma(x) + \sqrt{c-\tilde c}\,\langle\ ,\ \rangle\xi(x)$ is called a *weak umbilic* of f. Weak umbilic everywhere $\Rightarrow$ composition??? What happens if $c > \tilde c$, f free of weak umbilics, and $p = n-1$?

Proposition 49 (Moore)**.** *$\beta : \mathbb{V}^n \times \mathbb{V}^n \to \mathbb{W}^{p,q}$ symmetric FBF, with $q = 0, 1$ and $\nu_\beta = n-p-q$. If $q = 1$, assume further that β is nondegenerate and that there exist a vector $e \in \mathbb{W}$ such that $\langle \beta, e\rangle > 0$. Then, β decomposes uniquely as the direct sum of $p+q$ rank one flat forms.*

Proof. Let's do the case $q=0$ only. We may assume $p=n$, $\nu_\beta=0$. Fix $X_0 \in RE(\beta) \ \Rightarrow \beta_{X_0}$ is an isomorphism $\ \Rightarrow C(Y) := \beta_Y \circ \beta^{-1}_{X_0} \in End(\mathbb{W})$ are all self-adjoint and commuting by flatness $\Rightarrow$ $\exists$ an O.N.B. of $\mathbb{W}$ such that $C(Y)\xi_i = \mu_i(Y)\xi_i$. Set $\beta_i = \langle \beta, \xi_i\rangle$, $\beta_{X_0}X_i = \xi_i \Rightarrow \beta(Y, X_i) = \mu_i(Y)\xi_i \ \Rightarrow \beta(X_i, X_j) = 0$ if $i \neq j$ $\Rightarrow \beta = \sum_i a_i \rho_i \otimes \rho_i\, \xi_i$, where $\{\rho_i\} = \{X_i\}^*$ and $a_i = \mu_i(X_i)$. Uniqueness follows easily from this. ∎

Corollary 50. *Let* $f\colon M_c^n \to \mathbb{Q}_c^{2n}$ *with* $\nu = 0 \Rightarrow \exists$ *unique basis* $\{X_i\}$ *of unit vectors and o.n.b.* $\{\eta_i\}$ *such that* $\alpha(X_i, X_j) = \delta_{ij}\eta_j$. *The basis* $\{X_i\}$ *is orthogonal if and only if* $R^\perp = 0$, *in which case the* $\{\eta_i\}$ *are the principal normals of* f.

Corollary 51. *Let* $f\colon M_c^n \to \mathbb{Q}_{\tilde{c}}^{2n-1}$ *with* $c \neq \tilde{c}$. *If* $c > \tilde{c}$ *assume that* f *has no weak umbilics. Then* $R^\perp = 0$.

Proof. β in (5) is nondegenerate and has $\nu_\beta = 0$. By the Main Lemma $S(\beta) = \mathbb{W}$. The proof follows from Proposition 49. ∎

Exercise. If $\mu = \nu + p$ in Theorem 36 $\Rightarrow R = f^*(\tilde{R})$ and so $\mu = n$. (Sug: use Proposition 49).

§19. Nonpositive extrinsic curvature

Asymptotic vectors of α: $A(\alpha) := \{X \in \mathbb{V} : \alpha(X, X) = 0\}$.
As we saw in the proof of Lemma 48, we have:

Lemma 52 (Otsuki). *Let* $\alpha : \mathbb{V}^n \times \mathbb{V}^n \to \mathbb{W}^{p,0}$ *symmetric and* $\lambda > 0$ *such that* $K_\alpha \leq \lambda$ *and* $\alpha(X, X) > \sqrt{\lambda}\|X\|^2 \Rightarrow p \geq n$.

Corollary 53 (Otsuki). *Let* $\alpha : \mathbb{V}^n \times \mathbb{V}^n \to \mathbb{W}^{p,0}$ *symmetric such that* $K_\alpha \leq 0$ *and* $A(\alpha) = 0 \Rightarrow p \geq n$.

Corollary 54. $f\colon M^n \to \mathbb{Q}_c^{n+p}$ *with* $K_M(x_0) < c \Rightarrow p \geq n-1$.

Corollary 55. $f\colon M^n \to \mathbb{Q}_c^{n+p}$ *such that* $\exists x \in M$ *and* $V_x^m \subset T_xM$ *with* $K(\sigma) < c\ \forall \sigma \subset V_x^m \Rightarrow p \geq m - 1$.

Corollary 56. $f\colon M^n \to \mathbb{R}^{n+p}$ *compact such that,* $\forall x \in M$ *there is* $V_x^m \subset T_xM$ *with* $K(\sigma) \leq 0\ \forall \sigma \subset V_x^m \Rightarrow p \geq m$.

19.1 The relative nullity in nonpositive extrinsic curvature

The following is a deep generalization of Otsuki Corollary 53, whose first step is a deeper understanding of the structure of the set of asymptotic vectors $A(\alpha)$. Picture with $R^\perp \equiv 0$:

Theorem 57. $\alpha : \mathbb{V}^n \times \mathbb{V}^n \to \mathbb{W}^{p,0}$ *symmetric with* $K_\alpha \leq 0$. *Then,* $\nu_\alpha \leq n - 2p$ *(and this estimate is sharp!).*

This follows from a sequence of three propositions:

- $X_0 \in A(\alpha)$, $\hat{V} = \operatorname{Ker} \alpha_{X_0} \;\Rightarrow S(\alpha|_{\hat{V}\times\hat{V}}) \subset (\operatorname{Im} \alpha_{X_0})^\perp$.
  Proof. $K_\alpha(X_0 + tY, Z) \leq 0$ for $Z \in \hat{V}$. ∎
- $\exists\, T^m \subset A(\alpha)$ subspace with $m \geq n-p$ ($\Rightarrow$ Otsuki Lemma 52!).
  Proof. Induction in p using $X \in A(\alpha)$ with max rank α_X. ∎
- $\nu_\alpha \geq \dim T - p$.
  Proof. Let $T \oplus T' = \mathbb{V}$, $\beta := \alpha|_{T'\times T}$, $Y_0 \in RE(\beta)$. Use that $K_\alpha(Y_0 + tZ, Y + sZ') \leq 0$ for $Z' \in \operatorname{Ker} \beta_{Y_0} \subset T$, $Z \in T$, $Y \in T'$ is affine in $s \;\Rightarrow \beta(Y, \operatorname{Ker} \beta_{Y_0}) \subseteq \beta_{Y_0}(T')^\perp \;\Rightarrow \operatorname{Ker} \beta_{Y_0} \subset \Delta_\alpha$. ∎

Many corollaries (no proofs):

Corollary 58. M^n *complete and finite volume,* $K \leq c < 0$. $f\colon M^n \to N_c^{n+p}$ *for* $2p < n \;\Rightarrow f$ *totally geodesic.*

Corollary 59. $f\colon M^{2n} \to \mathbb{Q}_c^{2n+p}$ *Kahler. If* $\exists\; x_0 \in M$ *with* $K(x_0) \leq c \neq 0 \;\Rightarrow p \geq n$.

Corollary 60. $M^n \subset \mathbb{R}^{n+p}$, $p \leq n/2$, $K \leq 0$ *and* $Ric < 0 \Rightarrow 2p = n$*, local and global product of* p *surfaces* $K < 0$ *in* $\mathbb{R}^3$.

Special cases of Thm. 57: $R^\perp = 0$, $\nu_f = n-2p$ and $\nu_f = n-2p+1$.

Remark 61. $\nu_f = n - 2p$ for $M^n \subset \mathbb{Q}_c^{n+p}$ when $c \neq 0$.

19.2 The Omori-Yau maximum principle

Compactness in Corollary 56 can be relaxed. In order to do this, we first need a slight generalization of Lemma 52:

Lemma 62. *Let* $\alpha : \mathbb{V}^n \times \mathbb{V}^n \to \mathbb{W}^p$ *symmetric. If* $p < n$ *and* $A(\alpha) = 0$*, there are* $X, Y \in \mathbb{V}$ *L.I. such that* $\alpha(X, X) = \alpha(Y, Y)$ *and* $\alpha(X, Y) = 0$*.*

Proof. Complexifying, it is equivalent to p quadratic equations $\alpha(Z, Z) = 0$ in $n > p$ variables, which is well-known to always have a nontrivial solution that cannot be real by assumption. ∎

Def.: The *Omori-Yau maximum principle for the Hessian* (OYMP for short) is said to hold on a given Riemannian manifold M if for any function $g \in C^2(M)$ with $g^* := \sup g < +\infty$ there exists a sequence of points $\{x_k\}$ in M satisfying the following: $g(x_k) > g^* - 1/k$, $\|\nabla g(x_k)\| < 1/k$, $\mathrm{Hess}_g(x_k) < 1/k$.

The following result by Pigola-Rigoli-Setti gives conditions for the OYMP to hold in a complete manifold (no proof):

Theorem 63. *Let* M *be a complete noncompact R.M,* $\rho(x) := d(x, x_0)$*. If* $K_M \geq -\phi^2 \circ \rho$*, where* $\phi \in C^1([0, +\infty))$ *satisfies* $\phi(0) > 0$*,* $\phi' > 0$*,* $1/\phi \notin L^1$*, then* M *satisfies the OYMP.*

Theorem 64. *Let* $f : M^n \to P^m \times \mathbb{R}^\ell$*,* $2 \leq m \leq 2(n - \ell) - 1$*, i.i. between complete R.M. such that* $f(M) \subset B_R(o) \times \mathbb{R}^\ell$ *with* $K_P|_{B_R(o)} \leq b$ *and* $R < \min\{inj_P(o), \pi/2b\}$ $(\pi/2b = +\infty$ *if* $b \leq 0)$*. If* $scal_M \geq -C\rho^2(\Pi_{j=1}^N log^{(j)} \circ \rho)^2$ *outside of a compact set for certain* $C > 0$ *and* $N \in \mathbb{N}$*, then* $\sup K_f \geq c_b^2(R)(= cot...)$*.*

Proof. May assume $\sup K < +\infty$. Then, $K \geq -C'\rho^2(...)$ also $\Rightarrow$ OYMP by Theorem 63. Find a contradiction like Otsuki. ∎

Corollary 65. *$f : M^n \to N^{n+p}$, $p \leq n-1$, M complete, N Hadamard. Assume that the scalar curvature of M is bounded from below. If $K_f \leq 0$, then $f(M)$ is unbounded.*

§20. The relative nullity

Splitting tensor $C : D \times D^\perp \to D^\perp$ of a distribution D.
C_T is self-adjoint $\forall T \iff D^\perp$ is integrable.
C_T is a multiple of the identity $\forall T \iff D^\perp$ is umbilic.
$C \equiv 0 \iff D^\perp$ is totally geodesic $\iff$ locally a product.

Proposition 66. *Let $f\colon M^n \to \mathbb{Q}_c^{n+p}$ and $D \subset \Delta$ a totally geodesic distribution. Then, $\forall \xi \in T^\perp M, S, T \in D, X, Y \in D^\perp$, and $\gamma' \subset D$ geodesic with parallel transport P_γ, we have:*

1. $\nabla_T C_S = C_S C_T + C_{\nabla_T S} + c\langle T, S\rangle I$;
2. $C'_{\gamma'} = C^2_{\gamma'} + cI$ *(Riccati!!)*
3. $P_\gamma^{-1} \circ C_{\gamma'} \circ P_\gamma = (\sin(t)I + \cos(t)B)(\cos(t)I - \sin(t)B)^{-1}$ *for e.g.* $c = 1$ *and* $\|\gamma'\| = 1$, *where* $B = C_{\gamma'}(\gamma(0))$;
4. $(\nabla_X C_T)Y - (\nabla_Y C_T)X = C_{(\nabla_X T)_D} Y - C_{(\nabla_Y T)_D} X$;
5. $\nabla_T A_\xi = A_\xi C_T + A_{\nabla^\perp_T \xi}$ *($A_\bullet$ restricted to $D^\perp$)*;
6. $A'_\xi = A_\xi C_{\gamma'}$, *if ξ is parallel along geodesic $\gamma \subset D$;*
7. $A_\xi C_T = C_T^t A_\xi$;
8. *Both* $\operatorname{Ker} A_\xi$ *and* $\operatorname{Im} A_\xi$ *are parallel along γ if ξ also is.*

Remark 67. Notice that, by item 2, the tensor J defined by $J' + C_{\gamma'} J = 0$ with $J(0) = Id$ is Jacobi: $J'' + cJ = 0$. Hence, item 7 can be equivalently written as $(A_\xi J)' = 0)$.

Corollary 68. *By Proposition 66.3, B and $C_{\gamma'}$ cannot have a real eigenvalue if the length of γ is $\geq \pi$.*

Definitions 69. 1. Given $g : M \to \mathbb{R}^m$, the *k-cylinder over g* is the product immersion $f = g \times Id : N \times \mathbb{R}^k \to \mathbb{R}^{m+k}$.
2. Given $g : M \to \mathbb{Q}_{\tilde{c}}^m$ and $i : \mathbb{Q}_{\tilde{c}}^m \to \mathbb{Q}_c^{m+k}$ umbilic, the *generalized cone over g* is (the local regular image of) the map $f : g^*(T_i^{\perp}\mathbb{Q}_{\tilde{c}}^m) \to \mathbb{Q}_c^{m+k}$ defined by $f(x, \xi) = \exp_{i\circ g(x)}(\xi)$, where where exp is the exponential map of $\mathbb{Q}_c^{m+k}$ (e.g., for $\tilde{c} = 1, c = 0$, it is just $f(x, s, t) = (sg(x), t)$, with $s \in \mathbb{R}, t \in \mathbb{R}^r$.)

Proposition 70. *a) If $D^{k\perp}$ as in Proposition 66 is totally geodesic, then $c = 0$ and f is (locally) a k-cylinder. b) If $D^{k\perp}$ is umbilic and $k \leq n-2$, then f is (locally) a generalized cone.*

Proof. Do (a) and leave (b) as an exercise. ∎

§21. Completeness of the relative nullity

Proposition 71. *Let $f\colon M^n \to \mathbb{Q}_c^{n+p}$, and $U \subset M$ an open subset where $\nu_f = s > 0$. If $\gamma\colon [0, b] \to M$ is a geodesic such that $\gamma([0, b))$ is contained in a leaf of Δ in U, then $\Delta(\gamma(b)) = P_\gamma(\Delta(\gamma(0))$ and $\nu_f(\gamma(b)) = s$. Moreover, $C_{\gamma'}$ extends smoothly to $\gamma(b)$ and Proposition 66 items 2, 6, 7 and 8 hold on $[0, b]$.*

Proof. Let $V_t = \Delta(\gamma(t))^{\perp} = P_{0,t}^{\gamma}(V_0)$, $C := C_{\gamma'}$, and consider $J(t) \in End(V_t)$ its Jacobi tensor (Remark 67). Since $J'' + cJ = 0$, J smoothly extends to b in $End(P_{0,b}^{\gamma}(V_0))$. If $Z, Y \in \mathfrak{X}_{\gamma}''$ with $Y \in \Delta^{\perp} \Rightarrow \alpha(JY, Z)' = 0 \Rightarrow J$ invertible in $[0, b]$, $P_{0,b}^{\gamma}(V_0) = V_b$ and C smoothly extends to b since $C = -J'J^{-1}$. ∎

Corollary 72 (**!!!!**)**.** *The minimum relative nullity distribution is complete if M is complete.*

Remark 73. Propositions 66,71 and Corollary 72 hold for the intersection of the relative nullities of a finite number of immersions (since $(\alpha_1, \alpha_2, ...)$ is Codazzi).

Let $\kappa(m) = (\#$ L.I. vector fields in $\mathbb{S}^{m-1}$+1) be the Radon-Hurwitz number, which is given by $\kappa((odd)2^{4d+b}) = 8d + 2^b$, with $d \in \mathbb{N} \cup \{0\}$ and $b = 0, 1, 2, 3$ (F. Adams, 1962). Set

$$\rho_n := \max\{k : \kappa(n-k) \geq k+1\}.$$

Some values of ρ_n are: $\rho_n = n-$(highest power of $2 \leq n$) for $n \leq 24$, $\rho_n \leq 8d-1$ for $n < 16d$, and $\rho_{2^d} = 0$.

Corollary 74. *M^n complete and $f : M^n \to \mathbb{S}^{n+p}$ with $\nu > \rho_n$ $\Rightarrow$ f totally geodesic.*

Proof. Pick x in the open set with min relative nullity $r < n$, $\{X_1, \ldots, X_r\}$ a basis of $\Delta(x) \cong \mathbb{R}^r$ and $0 \neq Z \in \Delta(x)^\perp \cong \mathbb{R}^{n-r}$. Then, $\{Z, C_{X_1}Z, \ldots, C_{X_r}Z\}$ are L.I. by Corollary 68. So $\exists$ a cross-section between the Stiefel manifolds $V_{r+1}(\mathbb{R}^{n-r}) \to V_1(\mathbb{R}^{n-r})$ ($\iff \exists\, r$ L.I. vector fields in $\mathbb{S}^{n-r-1}$) $\Rightarrow r \leq \rho_n$. ∎

Corollary 75. *M^n complete with $K \leq 1$ and $f : M^n \to \mathbb{S}^{n+p}$ with $2p < n - \rho_n$ $\Rightarrow$ f is totally geodesic (by Theorem 57).*

Remark 76. At least for some dimensions, the hypothesis on p in the above cannot be improved to $2p < n$. For example, the simplest of Cartan's isoparametric hypersurfaces, i.e., the unit normal bundle of the Veronese surface (3) $M^3 \subset \mathbb{S}^4$, is a compact non-totally geodesic homogeneous hypersurface of $\mathbb{S}^4$ with $K \leq 1$ and $\nu \equiv 1$. No problem, since the sphere in $\Delta(x)^\perp \cong \mathbb{R}^2$ is $\mathbb{S}^1$ which has a nonvanishing vector field.

Exercise. Generalizing the last Remark 76: for $M^3 \subset \mathbb{S}^4$ as in Remark 76, let N^{4r-1} be the product of r factors M^3 with $\mathbb{S}^{r-1}$, and use metric induced by the warp product representation of $\mathbb{S}^{5r-1}$ to get $N^{4r-1} \subset \mathbb{S}^4 \times \cdots \times \mathbb{S}^4 \times \mathbb{S}^{r-1} \to \mathbb{S}^{5r-1}$. Then, N^{4r-1} is compact and, as we have seen, with the induced metric this has $K \leq 1$ and $\nu \equiv n - 2p = 2r - 1$. Hence, we have $2r - 1$ linearly independent vector fields in the unit sphere of $\Delta^\perp(x_0) \cong \mathbb{R}^{2r}$. Therefore, every odd-dimensional sphere is... parallelizeable?! Where is the mistake??

21.1 Zero extrinsic curvature: The spherical case

Two leaves of minimum relative nullity $\nu_0 < n$ of a non-totally geodesic $f\colon \mathbb{S}^n \to \mathbb{S}^{n+p}$ have dimension $\nu_0 \geq n - p$ by Chern-Kuiper, are complete, and cannot intersect in $\mathbb{S}^n$. Then $\nu_0 + 1 \leq 2(n+1)$, and hence $p > n/2$. In fact, Corollary 74 and Chern-Kuiper imply that $p \geq n - \rho_n$. But we can do better:

Theorem 77. *[DG] M^n complete, $f\colon M^n \to \mathbb{S}^{n+p}$ with $\nu > 0$. Then, at any point in $U = \{\nu = \nu_0 > 0\}$ where is ν is minimal, and for any normal direction at that point, the numbers of positive and negative principal curvatures are equal.*

Proof. Let $\gamma\colon \mathbb{R} \to U$ a geodesic in a leaf of Δ, ξ normal parallel along γ. By Proposition 66.6, $\operatorname{Ker} A_\xi$ is parallel along γ $\Rightarrow$ $\operatorname{rank} A_\xi(\gamma(t))$ is constant $\Rightarrow$ so are the number of positive and negative eigenvalues. But the antipodal map I of $\mathbb{S}^{n+p}$ leaves U invariant $\Rightarrow \exists\, \tau \in \operatorname{Iso}(U)$ such that $f \circ \tau = I \circ f|_U$. But $inc_*\xi$ is constant in $\mathbb{R}^{n+p+1}$ along γ, so $\xi \circ \tau = -I_*\xi$ and $A_{\xi\circ\tau} \circ \tau_* = -\tau_* \circ A_\xi$. Hence $\mathbf{spec}(A_\xi(\gamma(\pi))) = -\mathbf{spec}(A_\xi(\gamma(0)))$. ∎

Corollary 78. *$Ric \geq 1$ at some $x \in U \Rightarrow f$ is totally geodesic.*

Proof. Use Theorem 77 and (2). ∎

Corollary 79. *If $p \le n-1$, the only $f : M_1^n = \mathbb{S}^n/\Gamma \to \mathbb{S}^{n+p}$ is the totally geodesic inclusion ($\Rightarrow \Gamma = \{Id\}$). In particular, it is rigid.*

Example 80. The product isometric immersion $F\colon \mathbb{R}^{n+1} \to \mathbb{R}^{2n+2}$ given by $F(t) = \frac{1}{\sqrt{n+1}} e^{i\sqrt{n+1}t}$ induces $f = F|_{\mathbb{S}^n}\colon \mathbb{S}^n \to \mathbb{S}^{2n+1}$ which is not totally geodesic.

What about $p = n$?? Recall Corollary 50...

21.2 Zero extrinsic curvature: The Euclidean case

Theorem 81 (Hartman's extrinsic splitting thm)**.** *M^n complete with $Ric_M \ge 0$ and $f\colon M^n \to \mathbb{R}^{n+p}$ containing r independent lines $\Rightarrow$ f is an r-cylinder. In particular, f is an ν_0-cylinder.*

Proof. (by Johel Beltran) Let $g : I \times \mathbb{R} \to \mathbb{R}^m$ i.i. containing a straight line L. Write $g(x, y) = (u(x, y), v(x, y))$ with $u(0, y) = 0 \in \mathbb{R}^{m-1}$, $v(0, y) = y$. But $v(x, y) = y$: indeed, if $p = (x, y)$ and $c = v(p) - y$, let $q = (0, y - \lambda c) \in L \subset \mathbb{R}^2$ with $\lambda >> 1$ so that $d(p, q) - d((0, y), q) \le |c|/2$. So, $(\lambda+1)|c| \le d(g(p), g(q)) \le d(p, q) \le (\lambda+1/2)|c|$, and thus $c = 0$. Since $1 = \|g_y\|^2 = \|u_y\|^2 + 1$, we get $g(x, y) = (u(x), y)$ and $g = u \times Id_{\mathbb{R}}$.
Now, by the splitting theorem, $M = N \times \mathbb{R}^r$. Take $\gamma\colon I \to N$ a unit curve and set $f_\gamma : I \times \mathbb{R}^r \to \mathbb{R}^m$, $f_\gamma(s, v) = f(\gamma(s), v)$. By the above $f_\gamma(x, v) = (g(\gamma(x), v), v)$. But $\|f_*(0, v)\| = \|v\| \Rightarrow g_*(0, v) = 0$. ∎

Corollary 82. *If $f : \mathbb{R}^n \to \mathbb{R}^{n+p}$ is an i.i. with $p < n$, then f is a $(n-p)-$cylinder.*

§22. The Gauss Parametrization

Motivation. Let $g : I \times \mathbb{R} \to \mathbb{R}^m$ a non-cylindrical ruled surface, with rulings $R = \text{span}\{Z\} \Rightarrow g(t,s) = \beta(t) + sZ(t)$, where $Z : I \to \mathbb{S}^{m-1}$, $\|Z'\| = 1$, W.L.G. $\langle \beta', Z' \rangle = 0$ (β is called the *striction curve*). Let $J = \{t : \beta'(t) || Z(t)\}$. Then, $K \leq 0$, and $K^{-1}(0) = \{\nu > 0\} = \{\nu = 1\} = \{\Delta = R\} = J \times \mathbb{R}$, and $g(\text{sing}(g)) = \beta(J)$. In particular $K \equiv 0 \iff g(\text{sing}(g)) = \beta$. But sing$(g)$ is not just the singular set of the map g, but of the submanifold, since $C_Z = -s^{-1}Id$. We just gave another proof of Hartman's Theorem 81, certainly much less elegant and elementary, but the parametrization idea is much more powerful!

In fact, we classified all ruled flat surfaces in $\mathbb{R}^n$, and hence all flat surfaces in $\mathbb{R}^3$: *each connected component of an open dense subset* is either a cylinder over a curve, or a cone over a spherical curve (with β=constant as the vertex), or the surface of tangents of the regular curve β: $g = \beta + s\beta'$. Moreover, all these connected components are glued together along rulings. Observe that looking at the problem with the singularities actually helped!

$\sim \cdot \sim$

Let $f : M^n \to \mathbb{R}^{n+1}$ be an (orientable) Euclidean hypersurface with constant index of relative nullity $\nu_f \equiv n-k$ and Gauss map $\eta : M^n \to \mathbb{S}^n$. Let $\gamma := \langle f, \eta \rangle$ be the *support function* of f. Then, locally, we have a projection

$$\pi : M^n \mapsto V^k := M^n/\Delta$$

onto the leaf space, which is a k-dimensional manifold. Since both η and γ are constant along Δ, both η and γ descend to the quotient, namely, $\eta = h \circ \pi$ and $\gamma = r \circ \pi$, for certain *Gauss data* of f,

$$h\colon V^k \to \mathbb{S}^n \text{ and } r\colon V^k \to \mathbb{R}.$$

Therefore, at regular points, $f(M)$ is locally parametrized by

$$\hat{f}: T_h^{\perp}V \to \mathbb{R}^{n+1}, \quad \hat{f} \circ \xi = rh + \nabla r + \xi, \qquad \xi \in \Gamma(T^{\perp}V),$$

and the map π becomes the normal projection $\pi\colon T^{\perp}V \to V$. For $x \in V$ and $w = \xi(x) \in T_x^{\perp}V$, set

$$P_w := rI + \mathrm{Hess}_r - A_w : T_xV \to T_xV,$$

which gives rise to a bundle map $P\colon \pi^*(TV) \to \pi^*(TV)$. Clearly, the (open!) maximal set $\hat{M}^n := \{w \in T^{\perp}V : P_w \text{ is invertible}\}$ where P is a bundle isomorphism is also the regular set of $\hat{f}$. Endowing $\hat{M}^n$ with the metric induced by $\hat{f}$ we thus *saturate* the original isometric immersion f to obtain $\hat{f}|_{\hat{M}^n}\colon \hat{M}^n \to \mathbb{R}^{n+1}$. Now, since the subspaces $h_{*x}(T_xV)$ and $\hat{f}_{*w}(\Delta^{\perp}(w))$ are parallel, we identify T_xV with $\Delta^{\perp}(w)$ via the isometry $j_w\colon T_xV \to \Delta^{\perp}(w)$ defined by $\hat{f}_{*w} \circ j_w = h_{*x}$ (equivalently, $j \circ \pi_* = -A|_{\Delta^{\perp}}$). In fact:

Proposition 83. *The map $j\colon \pi^*(TV) \to \Delta^{\perp}$ defined by the relation $\hat{f}_* \circ j = h_{*\pi}$ is a parallel bundle isometry over $T^{\perp}V$.*

Proof. Just compute: $\hat{f}_*(\nabla_{\xi_*X}j)(Y\circ\pi) = (\overline{\nabla}_{\xi_*X}\hat{f}_*j(Y\circ\pi))_{TV} - \hat{f}_*j(\nabla^{\pi}_{\xi_*X}Y\circ\pi) = (\overline{\nabla}_{\xi_*X}(h_*Y)\circ\pi)_{TV} - h_{*\pi}(\nabla_XY) = 0.$ ∎

Since $j_w \circ P_w = (\xi_{*x})_{\Delta^{\perp}(w)}$, or equivalently $\pi_{*w} \circ j_w = P_w^{-1}$, we conclude that

$$\pi_* \circ j = P^{-1}.$$

This also shows that the singular set of $\hat{f}$ is singular for $\hat{f}(T^{\perp}V)$!

Remark 84. Not needed, but notice that the total space of the bundle $\pi^*(TV)$ is just the Whitney sum $TV \oplus T^\perp V = h^*(T\mathbb{S}^n)$, but with projection to the first factor $T^\perp V$ instead of V.

We now compute the geometry of $\hat{f}$ in terms of the one for h and r. In order to do this, we introduce some natural notations:

- For each $X \in \mathfrak{X}(V)$ we set $\hat{X} := j(X \circ \pi) \in \mathfrak{X}(\Delta^\perp)$.
- Define $PX \in \Gamma(\pi^*(TV))$ by $PX(w) := P_w(X(x))$, i.e., PX is just a notation for $P(X \circ \pi) = P\pi^* X$. Same for $P^{-1}X$.
- Given $\xi \in \Gamma(T^\perp V)$ we define $\hat{\xi} \in \mathfrak{X}(\Delta)$ with the expression: $\hat{\xi}(w) := \frac{d}{dt}|_{t=0}(w + t\xi(x)) \in T_x^\perp V \subset T_w(T^\perp V)$. In particular,

$$\hat{f}_* \hat{X} = (h_* X) \circ \pi \qquad \text{and} \qquad \hat{f}_* \hat{\xi} = \xi \circ \pi.$$

Proposition 85. *Up to conjugation with j we have:*

a) $\Delta(w) = T_x^\perp V$ and $\Delta^\perp(w) = T_x V$ by construction;

b) $w \in T_x^\perp V$ is a regular point of $\hat{f} \iff P_w$ is invertible;

c) $\forall X \neq 0, (\xi_* X)_{\Delta^\perp} \neq 0$ and $\|(\xi_* X)_{\Delta^\perp}\|_{\hat{f}} = \|PX\|_h$;

d) The shape op. $\mathcal{A}$ of f in $\Delta^\perp(w)$ is $-P_w^{-1}$, i.e., $\mathcal{A} = -P^{-1}$;

e) The singular set of $\hat{f}$ is the singular set of $\hat{f}(T^\perp V)$;

f) The connection $\hat{\nabla}$ of $\hat{f}$ on $\Delta^\perp$ and the tangent connection ∇ of h are related by $(\hat{\nabla}_{\hat{X}} \hat{Y})_{\Delta^\perp} = \nabla_{P^{-1}X} Y$;

g) The connection $\hat{\nabla}$ of $\hat{f}$ on Δ and the normal connection $\nabla^\perp$ of h are related by $(\hat{\nabla}_{\hat{X}} \hat{\xi})_\Delta = \nabla^\perp_{P^{-1}X} \xi$;

h) The splitting tensor of Δ for $\hat{\xi}$ is $C_{\hat{\xi}} = A_\xi P^{-1}$.

Proof. (*f*): Formally, we get $(\hat{\nabla}_{\hat{X}}\hat{Y})_{\Delta^\perp} = (\nabla_{jX\circ\pi}\, jY\circ\pi)_{\Delta^\perp} = j(\nabla_{\pi_* jX\circ\pi}Y)_{\Delta^\perp} = j(\nabla_{P^{-1}X\circ\pi}Y)$.
$(g)+(h)$: $\hat{f}_*(\hat{\nabla}_{\hat{X}}\hat{\xi}) = \overline{\nabla}^h_{\pi_*\hat{X}}\xi = -\hat{f}_* j A_\xi P^{-1}X + \nabla^\perp_{P^{-1}X}\xi$. ∎

Corollary 86. *Every spherical submanifold $h : V^k \to \mathbb{S}^n$ is the Gauss image of some Euclidean hypersurface. The set of hypersurfaces with Gauss image h is parametrized by $\mathcal{F}(V^k)$.*

The Gauss parametrization can be used to get several relations between the geometries of a hypersurface and its Gauss map:

Corollary 87. *f is a cylinder $\iff$ h reduces codimension.*

Corollary 88. *$\Delta^\perp$ is integrable $\iff$ h has flat normal bundle and $[\mathrm{Hess}_r, A_w] = 0$ $\forall w \in T^\perp V$.*

There is also a Gauss parametrization in space forms:

Corollary 89. *In $\mathbb{Q}_c^{n+1}$ we also have Gauss parametrization.*

Corollary 90. *<u>Local</u> classification of $f : U \subset \mathbb{Q}_c^n \to \mathbb{Q}_c^{n+1}$.*

The Gauss parametrization is local, but it is a very powerful tool to prove <u>global</u> results:

Corollary 91. *Hartman's Theorem 81 for hypersurfaces.*

Corollary 92. *$f\colon M^n \to \mathbb{R}^{n+1}$ with $\mu \equiv n-2$ and complete leaves of Δ along which the mean curvature of f does not change sign. Then, $h : V^2 \to \mathbb{S}^n$ is minimal and f is a cylinder over $g : N^{2+\epsilon} \to \mathbb{R}^{3+\epsilon}$ with $\nu_g = \epsilon$, where $\epsilon = 0, 1$.*

The last one and Theorem 32 give:

Corollary 93. *$f\colon M^n \to \mathbb{R}^{n+1}$ complete minimal without euclidean factors, $n \geq 4 \Rightarrow f$ is rigid in $\mathbb{R}^{n+p}$ among minimal.*

Corollary 94. *$f\colon M^n \to \mathbb{R}^{n+1}$ with $\mu \equiv n-2$ and $scal_M$ constant. Then f is locally a cylinder over a surface. If in addition M^n is complete $\Rightarrow f(M) = \mathbb{S}^2_c \times \mathbb{R}^{n-2} \subset \mathbb{R}^3 \times \mathbb{R}^{n-2}$.*

Proof. Need to prove that h is totally geodesic (global part then follows from Hilbert's $\mathbb{H}^2 \not\subset \mathbb{R}^3$ and the rigidity of $\mathbb{S}^2 \subset \mathbb{R}^3$). Otherwise $\Rightarrow \nu_h = 1$ and $K_{V^2} = 1$ in some open subset $U \subset V^2$. Let $\{X, Y\}$ o.n.b. of TU and $Y \in \Delta_h$. So, $\nabla_Y X = \nabla_Y Y = 0$. Moreover $YX(r) = scal_M^{-1} \neq 0$ is constant and $YY(r) + r = 0 \Rightarrow \langle \nabla_Y \nabla_X \nabla r, Y\rangle = 0$ and $0 = X(r) + XYY(r) = 2\langle \nabla_{[X,Y]} \nabla r, Y\rangle = 2\langle \nabla_X Y, X\rangle YX(r) \Rightarrow \nabla Y = 0 \Rightarrow K_{V^2} = 0$, a contradiction. ∎

§23. Homogeneous hypersurfaces

Def.: $f\colon M^n \to \mathbb{Q}^{n+1}_c$ is *isoparametric* if it has constant principal curvatures ($\lambda_1 < \cdots < \lambda_g$ with multiplicity m_i).

Lemma 95. *$f\colon M^n \to \mathbb{Q}^{n+1}_c$ with M^n homogeneous $\Rightarrow$ Either $\tau \leq 1$ or τ is constant. If $\tau \geq 3$, then f is isoparametric.*

Theorem 96 (Cartan fund. formula)**.** *$\forall i$, $\sum_{j\neq i}^{g} m_j \frac{\lambda_i\lambda_j + c}{\lambda_i - \lambda_j} = 0$.*

Proof. $Ae_i = \lambda_i e_i$, $E_i = \mathrm{Ker}\,(A - \lambda_i I)$, $\Gamma^k_{ij} = \langle \nabla_{e_i} e_j, e_k\rangle = -\Gamma^j_{ik}$. Codazzi: $(\lambda_j - \lambda_k)\Gamma^k_{ij} = (\lambda_i - \lambda_k)\Gamma^k_{ji} \Rightarrow E_i$ totally geodesic $\Rightarrow$ WLG $g \geq 3$. Gauss: $c + \lambda_i\lambda_j = \sum_k (\Gamma^k_{ij}\Gamma^k_{ji} + \Gamma^k_{ij}\Gamma^j_{ki} + \Gamma^k_{ji}\Gamma^i_{kj}) = 2\sum_k \Gamma^k_{ij}\Gamma^k_{ji} = \sum_{i\neq k\neq j} \frac{(\lambda_i - \lambda_j)^2}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_k)} (\Gamma^j_{ki})^2$. Now just sum. ∎

Corollary 97. *Suppose that $f\colon M^n \to \mathbb{R}^{n+1}$ is isoparametric. Then, $f(M) \subset \mathbb{S}^k_c \times \mathbb{R}^{n-k}$ for some $0 \leq k \leq n$.*

Proof. We may assume that f is not umbilic, and that $A \not\leq 0$. Let i in Theorem 96 s.t. λ_i is the smallest positive one $\Rightarrow$ all others are 0, $E_{\lambda_i} = \Delta^\perp$ is tot. geod., done by Proposition 70. ∎

Corollary 98. *Suppose that* $f\colon M^n \to \mathbb{H}^{n+1}$ *is isoparametric. Then,* f *is either umbilical, or there is some* $0 < k < n$ *for which* $f(M) \subset \mathbb{S}^k_c \times \mathbb{H}^{n-k}_{-\frac{c}{c+1}} \subset \mathbb{H}^{n+1} \subset \mathbb{R}^{k+1} \times \mathbb{L}^{n-k+1} = \mathbb{L}^{n+2}$*.*

Proof. If $|\lambda_i| \leq 1$ for all i, pick the biggest λ_i to conclude from Theorem 95 that f is flat and umbilic. If not, the same proof as the one for Corollary 97 works by taking i as the smallest index for which $\lambda_i > 1$. We end up with only two different eigenvalues satisfying $\lambda_1\lambda_2 - 1 = 0$. The splittings follow from deRham's Theorem and Theorem 108 below. ∎

Theorem 99. $f\colon M^n \to \mathbb{R}^{n+1}$ *with* M^n *homogeneous* $\Rightarrow f$ *is either a complete cylinder over a plane curve, or* $\mathbb{S}^k_c \times \mathbb{R}^{n-k}$*.*

23.1 Curvature homogeneous hypersurfaces

There are weaker (and local!) notions than the above:

Definition 100. A Riemannian manifold M is said to be *curvature homogeneous* if $\forall x, y \in M$ $\exists$ a linear isometry $\tau_{xy} : T_xM \to T_yM$ such that its curvature tensor satisfies $R_x = J^*_{xy}R_y$.

Definition 101. We say that $f\colon M^n \to \mathbb{Q}^{n+p}_c$ is *weakly isoparametric* if $\forall x, y \in M$ $\exists$ linear isometries $\tau_{xy} : T_xM \to T_yM$ and $\hat{\tau}_{xy} : T^\perp_x M \to T^\perp_y M$ such that $\hat{\tau}_{xy} \circ \alpha_x = \tau^*_{xy}\alpha_y$.

By Gauss eqn, weakly isoparametric $\Rightarrow$ curvature homogeneous. WLOG, we can fix $y = x_0$ and work with just $\tau_x = \tau_{xx_0}, \hat{\tau}_x = \hat{\tau}_{xx_0}$. For $f\colon M^n \to \mathbb{Q}^{n+p}_c$ with M^n curvature homogeneous, define

$$\beta_x = (\alpha_x, \tau^*_x\alpha_{x_0}) : T_xM \times T_xM \to W^{p,p}_x = T_xM \oplus T_{x_0}M.$$

Then, β_x is flat $\forall x$, and f is weakly isoparametric if and only if β_x is null $\forall x$. The Main Lemma 45 then immediately implies:

Proposition 102. *$f\colon M^n \to \mathbb{Q}_c^{n+1}$ is curvature homogeneous if and only if either it is isoparametric, or has constant curvature c, or has rank two and constant scalar curvature $\neq c$.*

Besides the isoparametric case, Corollary 90 and Corollary 94 tell us that what is left are the rank two hypersurfaces with constant scalar curvature for $c \neq 0$. By the Gauss parametrization, this is equivalent to the classification of $V^2 \subset \mathbb{S}_{\pm 1}^{n+1}$ for which all shape operators of unit vectors have constant determinant $\neq 0$. Tsukada proved in 1988 that the only case for $n \geq 4$ was a single complete hypersurface $M^4 \subset \mathbb{H}^5$ related to the unit normal bundle of the Veronese surface. Now, the case $n = 3$ has just been solved:

Theorem 103 ([Bryant-Florit-Ziller])**.** *Let $\mathcal{M}$ be the set of immersed rank two hypersurfaces in $\mathbb{Q}_c^4$, $c = \pm 1$, whose induced metrics have constant scalar curvature. Then, $\mathcal{M}$ contains a one parameter family of hypersurfaces admitting no continuous symmetries, and an isolated rotationally symmetric hypersurface. None of these examples is complete, and any connected hypersurface in $\mathcal{M}$ is congruent to an open subset of one of them.*

23.2 An introduction to isoparametric hypersurfaces

By Corollaries 97 and 98 the only interesting isoparametric hypersurfaces live in $\mathbb{S}^{n+1}$. Münzner showed that $g = 1, 2, 3, 4$ or 6, while Abresh proved that, if $g = 6$, then the multiplicities are all equal to 1 or 2, and so in this case $n = 6$ or 12. Using representations of Clifford algebras, Ferus-Karcher-Münzner gave in [**FKM**] a beautiful construction of a large family with $g = 4$. Lots of things are understood, but the full classification is still(!) an important open problem. Let's see what Münzner did in [**Mu1**].

Given $f\colon M^n \to \mathbb{S}^{n+1}$, let $\rho : M^n \to \mathbb{S}^{n+1}$ be its Gauss map and A its shape operator, i.e.,

$$\langle f, f\rangle = \langle \rho, \rho\rangle = 1, \quad \langle f, \rho\rangle = \langle df, \rho\rangle = 0, \quad d\rho = -df \circ A.$$

For each $\tau \in \mathbb{S}^1$, the gauss map ρ_τ of the *parallel hypersurface*

$$f_\tau := \cos(\tau) f + \sin(\tau)\rho$$

is $\rho_\tau := \cos(\tau)\rho - \sin(\tau) f$, whose shape operator for f_τ is

$$A_\tau = (I + \cot(\tau)A)(\cot(\tau)I - A)^{-1}.$$

Write the different principal curvatures of f as $\cot(\theta_i)$, $0 < \theta_i < \pi$, $1 \leq i \leq g$, and m_i their multiplicities. Then the principal curvatures of f_τ are $\cot(\theta_i - \tau)$ with same multiplicities, and its mean curvature is $h_\tau = \frac{1}{n}\sum_i m_i \cot(\theta_i - \tau)$ which as an analytic function of τ gives the principal curvatures of f as its poles. So:

Proposition 104. *If f is isoparametric then all f_τ also are, and* <u>*all*</u> *f_τ's have CMC if and only if they are isoparametric.*

If f is isoparametric, then the totally geodesic distribution E_j integrates as umbilical round sphere $\mathbb{S}^{m_j}_{r_j}$ in $\mathbb{S}^{n+1}$ contained in the fixed subspace $E_j \oplus \operatorname{span}\{f, \rho\}$, with $r_j = \sin(\theta_j)^{-2}$ hence of radius θ_j centered at $c_j = \cos(\theta_j) f_{\theta_j}$ (since $\overline{\nabla}_{E_j} c_j = 0$). In particular, we have an orientation-reversing involution φ_j of M^n that sends each x to its antipodal point in its leaf $\mathbb{S}^{m_j}_{r_j}$. But this point is precisely $f_{2\theta_j}(x)$, so $f_{2\theta_j} = f \circ \varphi_j$ (picture!). Hence the sets $\{\cot(\theta_k), 1 \leq k \leq g\}$ and $\{-\cot(\theta_k - 2\theta_j), 1 \leq k \leq g\}$

coincide for every $1 \leq j \leq g$, together with multiplicities. Thus,

$$\{\mathbb{R}e^{i\theta_k}, 1 \leq k \leq g\} = \{\mathbb{R}e^{i(2\theta_j - \theta_k)}, 1 \leq k \leq g\}, \;\; \forall\, 1 \leq j \leq g.$$

Since the latter is the reflection of the former along the line $\mathbb{R}e^{i\theta_j}$, we conclude that

$$\theta_i = \theta_1 + (i-1)\pi/g, \quad \text{and} \quad m_i = m_{i+2} \; (i \bmod g). \tag{6}$$

Notice that the focal (singular) element of the family f_{θ_j} is an immersion of the leaf space $M_j^{n-m_j} = M^n/E_j$, whose unit normal bundle at $[x]$ is given by $\rho_{\theta_j}(y)$ for y in the leaf $\mathbb{S}_{r_j}^{m_j}$ containing x. In particular, all its (unit) shape operators have the same eigenvalues $\cot(\theta_i - \theta_j)$, $i \neq j$. Moreover, it is minimal since their trace is precisely Cartan fundamental formula Theorem 96. :-)

Now, let $\mu : \mathbb{S}^{n+1} \to \mathbb{R}$ be a smooth function with the property than $\|\nabla\mu\|$ is constant over each level set $M_r = \mu^{-1}(r)$. Then, the regular levels are parallel hypersurfaces with gauss maps $\frac{\nabla\mu}{\|\nabla\mu\|}$ and shape operators $-\|\nabla\mu\|^{-1}\nabla_{\bullet}\nabla\mu$ and mean curvature $-(n\|\nabla\mu\|)^{-1}\Delta\mu$. Thus M_r form an isoparametric family precisely when $\Delta\mu$ is constant on the levels, that is, when both of "the first two differential parameter" functions are functions of μ itself (whence the name "isoparametric"):

$$\|\nabla\mu\|^2 = a(\mu), \quad \Delta\mu = b(\mu).$$

Conversely, given an isoparametric family and a function μ on the oriented distance from a fixed hypersurface of the family, μ obviously satisfies the above differential equations.
For f isoparametric, following the spirit of Cartan's classification for $g = 3$, Münzner proved the following beautiful result:

Proposition 105. *The map defined in a neighborhood of $f(M)$ in $\mathbb{R}^{n+2}$ by $F(rf_\tau(x)) = r^g\cos(\beta(\tau))$ for $\beta(\tau) = g(\theta_1 - \tau)$ satisfies that $\|\nabla F\| = g\,r^{g-1}$ and $2\Delta F = (m_2 - m_1)g^2r^{g-2}$. Moreover, F is a homogeneous polynomial of degree g.*

Proof. Set $G = F - ar^g$ for $a = \frac{g(m_2-m_1)}{2(g+n-1)}$. Since $\nabla F = gr^{g-1}f_{\beta+\tau}$ we get using (6) and that $g\cot(gs) = \sum_{i=1}^{g}\cot(s + i\pi/g)$ $\forall s$ that $\Delta G = 0$ (exercise). But for every harmonic function H in $\mathbb{R}^N$ one has $\Delta^g\|\nabla H\|^2 = 2^g\|\partial^{g+1}H\|^2$, and $\Delta^g\|\nabla G\|^2 = 0$. ∎

In particular, one can extend every open connected subset of an isoparametric hypersurface to a compact embedded orientable algebraic hypersurface. Moreover, setting $\mu = F|_{\mathbb{S}^{n+1}}$, we have that $\|\nabla\mu\|^2 = g^2(1-\mu^2)$. Hence $\mathrm{Im}\,(\mu) = [-1,1]$ with $M_{\pm 1}$ the only singular levels of μ. This implies that $M_{-1}^{n-m_2} = f_{\theta_1}(M)$ and $M_1^{n-m_1} = f_{\theta_g}(M)$, the two closest focal sets of f, which correspond to the collapsing of the two sphere foliations $\mathbb{S}^{m_1}_{r_1}$ and $\mathbb{S}^{m_2}_{r_g}$. We conclude that $\mathbb{S}^{n+1}$ is written as the union of the two disk bundles $\mu^{-1}[-1,b] \cup \mu^{-1}[b,1]$ over M_{-1} and M_1, glued along the compact $f(M)$, $b = \cos(g\theta_1)$. But Mayer-Vietoris $\Rightarrow$ $H^*(M) = H^*(M_1) \oplus H^*(M_{-1})$ for $* \neq 0, n$. [**Mu2**] is devoted to compute this, that $2g = \dim H^*(M)$, and conclude that $g = 1, 2, 3, 4$ or 6. Hard!

Corollary 106. *There is a unique f_τ that is minimal.*

Proof. By the formula above, the mean curvature h_τ is strictly increasing in τ, positive as $\tau \to \theta_1$, negative as $\tau \to \theta_g$. ∎

Corollary 107. *M_r is connected for all $r \in [-1,1]$.*

Proof. Since codim $M_{\pm 1} \geq 2$, $U = \mathbb{S}^{n+1} \setminus M_{\pm 1}$ is connected, and M_r is a retract of U for all $r \in (-1,1)$. ∎

§24. Immersions of Riemannian products

Orthogonal nets. Ex: Riemannian products. Adapted tensors. Recall: if $X_i \in \mathfrak{X}(M_i)$ we have lifts $\tilde{X}_i \overset{\pi_i}{\sim} X_i$, and for the injections $\tau_j = \tau_j^{x_{j+1}} : M_j \to M_1 \times M_2$ we have $X_i \overset{\tau_i}{\sim} \tilde{X}_i$. We conclude that $T_{(x_1,x_2)}(M_1 \times M_2) \cong T_{x_1}M_1 \oplus T_{x_2}M_2$ canonically.

Theorem 108 (Moore)**.** *If the second fundamental form of $f : M_1 \times M_2 \to \mathbb{R}^m$ is adapted, then f is an extrinsic product.*

Proof. Taking lifts of vector fields in each factors we see that $f_* T_x M_1 \perp f_* T_y M_2 \;\; \forall (x, y) \in M_1 \times M_2$. Thus, $V_1 \perp V_2$, where

$$V_i := \operatorname{span}\{f_{*(x_1,x_2)}(T_{x_i}M_i) : (x_1, x_2) \in M_1 \times M_2\}.$$

Decomposing $\mathbb{R}^m = V_0 \oplus^\perp V_1 \oplus^\perp V_2$ and $f = f_0 + f_1 + f_2$, we conclude that $f_1 = f_1(x_1)$, $f_2 = f_1(x_2)$, and that f_0 is constant. ∎

Remark 109. The decomposition is unique (if f_i is substantial).

Corollary 110. *Same in $\mathbb{S}^m$. Almost the same in $\mathbb{H}^m$.*

24.1 Splitting under a curvature condition

Theorem 111. *Let $f : M^n = \times_{i=0}^{p} M_i^{n_i} \to \mathbb{R}^{n+p}$ such that the set of flat points of M_j has empty interior, $\forall 1 \leq j \leq p$. Then, $M_0^{n_0}$ is flat and $f(M)$ is an open subset of a n_0-cylinder over an extrinsic product of p hypersurfaces.*

Proof. Fix $1 \leq j \neq j' \leq p$, and let $\sigma_j = \operatorname{span}\{e_{2j-1}, e_{2j}\} \subset T_{x_j}M_j$ with $k_j := K(\sigma_j) \neq 0$, $L_j := \operatorname{span} \alpha(e_{2j}, e_{2j}) \neq 0$, and $T_x^\perp M = L_1 \oplus^\perp \cdots \oplus^\perp L_p$. By Gauss equation, if $V^{2p} = \sigma_1 \oplus \cdots \oplus \sigma_p$,

$$\beta := \alpha \oplus B_1 \oplus \cdots \oplus B_p : V^{2p} \times V^{2p} \to T_x^\perp M \oplus \mathbb{R}^p = W^{2p,0}$$

is flat, where $B_j = \sqrt{|k_j|}(e^{2j-1} \otimes e^{2j-1} - sign(k_j) e^{2j} \otimes e^{2j})$. By Proposition 49, there is a basis $\{e'_1, \ldots, e'_{2p}\}$ of V^{2p} such that $\beta(e'_r, e'_s) = 0$, $\forall 1 \leq r \neq s \leq 2p$. In particular, $\operatorname{Ker} B_i$ and $\operatorname{Im} B_i$ are spanned by vectors in this basis. Hence, up to order, $e'_{2j-1}, e'_{2j} \in \sigma_j$, $\alpha(\sigma_j, \sigma_{j'}) = 0$, and $\alpha(\sigma_j, \sigma_j) = L_j$. So, for the conullities $\Gamma_j^\perp \subset T_{x_j} M_j$ we get $\alpha(\Gamma_j^\perp, \Gamma_{j'}^\perp) = 0$ and $\alpha(\Gamma_j^\perp, \Gamma_j^\perp) = L_j$ by dimension reasons. Now it's Gauss equation: $\langle \alpha(\Gamma, \Gamma_j^\perp), \alpha(\Gamma_{j'}^\perp, \Gamma_{j'}^\perp) \rangle = 0 \ \Rightarrow \alpha(\Gamma, \Gamma_j^\perp) \subset L_j$. But if $X \in \Gamma_j^\perp$, there is $Y \in \Gamma_j^\perp$ such that $\alpha(X, Y) = 0$ thus $0 \neq \alpha(Y, Y) \in L_j$. Therefore we have $\langle \alpha(\Gamma, X), \alpha(Y, Y) \rangle = 0 \ \Rightarrow \alpha(\Gamma, \Gamma_j^\perp) = 0 \ \Rightarrow \langle \alpha(\Gamma, \bullet), \alpha(\Gamma_j^\perp, \Gamma_j^\perp) = 0 \Rightarrow \Gamma = \Delta(x)$. Since $TM_0 \subset \Gamma$ we conclude that $\alpha(x)$ is adapted. Therefore α is everywhere adapted to the product structure and the result follows from Proposition 70. ∎

Remark 112. Similar in $\mathbb{Q}_c^{n+p}$. If $p = 2$ we can say a bit more.

24.2 Splitting under an algebraic condition

Lemma 113. *Let $\beta : V^n \times V^n \to W^{p,0}$ symmetric, $V = V_1 \oplus V_2$ and $R_\beta(V_1, V, V, V_2) = 0$. If $\nu_s < n - 2s \ \ \forall 1 \leq s \leq p$, then $\beta(V_1, V_2) = 0$.*

Proof. For $i \neq j$, let $\beta_{ij} := \beta|_{V_i \times V_j}$, $\Delta_{ij} := \operatorname{Ker} \beta_{ij} \subset V_i$, $S^s := S(\beta_{12}) \Rightarrow \langle \beta(\Delta_{12}, V), \beta(V, V_2) \rangle = 0 \Rightarrow \beta(\Delta_{12}, V) \perp S^s$. Similarly, we have $\beta(V, \Delta_{21}) \perp S^s \ \Rightarrow \Delta_{12} \oplus \Delta_{21} \subset \operatorname{Ker} \beta_S \Rightarrow \nu_s \geq \dim(\Delta_{12} \oplus \Delta_{21}) \geq \dim V_2 - s + \dim V_1 - s = n - 2s \Rightarrow s = 0$. ∎

Corollary 114. *If $f : M^n := \times_{i=1}^p M_i^{n_i} \to \mathbb{Q}_c^{n+p}$ satisfies $\nu_s < n - 2s \ \forall\ 1 \leq s \leq p$, then f is an extrinsic p-product.*

24.3 Splitting under a global condition

The following is a generalization of Chern-Kuiper Theorem 36:

Theorem 115. *Given* $f\colon M^n \to \tilde{M}^{n+p}$*, decompose* $\Gamma^\perp(x)$ *as* $\Gamma^\perp(x) = T_1 \oplus^\perp \cdots \oplus^\perp T_s$*, where all* T_i*'s are non-zero and* $(R - f^*\tilde{R})$*-invariant. Then,* $\nu(x) \le \mu(x) \le \nu(x) + p - s$*.*

Proof. If $S = (\Gamma \cap \Delta^\perp) \oplus \operatorname{span}\{Y_1, \ldots, Y_s\}$ for $0 \ne Y_i \in T_i$ $\Rightarrow$ $(R - f^*\tilde{R})(S, S) = 0$. Now, if $Z \in RE(\alpha|_{TM\times S})$ then $\operatorname{Ker}(\alpha_Z|_S) = 0$ since $S \cap \Delta = 0$. Hence, $p \ge \dim S = \mu - \nu + s$. ∎

Corollary 116. *We always have that* $s \le p$ *and, if* $\mu = \nu + p$*, then* $R = f^*\tilde{R}$ *and* α_f *is flat.*

Corollary 117. *Let* $f\colon M^n = \times_{i=1}^p M_i^{n_i} \to \mathbb{R}^{n+p}$*,* $n_i \ge 2$*. If* $\alpha_f(x)$ *is* <u>*not*</u> *adapted, then* $0 < r(x) \le \mu(x) - r(x) \le \nu(x) \le \mu(x)$*, where* $r(x)$ *is the number of factors that are flat at* x*.*

Proof. By Theorem 111 at least one factor M_i is flat at $x_i = \pi_i(x)$, so $r(x) > 0$. Moreover, $\mu(x) \ge 2r(x)$ since $n_i \ge 2$. The third inequality follows from Theorem 115. ∎

Theorem 118. *Let* $M_i^{n_i}$ *be compact with* $n_i \ge 2$*. Then, every* $f\colon M^n = \times_{i=1}^p M_i \to \mathbb{R}^{n+p}$ *splits as a product of* p *hypersurfaces.*

Proof. Let $U \subset M$ be the open subset where α_f is not adapted, and $U_0 \subset U$ where the relative nullity of $f|_U$ is minimum ν_0 $\Rightarrow$ $\nu_0 \ge \mu - r \ge r > 0$ by Corollary 117. Since M^n is compact and U is open, by Proposition 71 a maximal geodesic in a leaf of Δ in U_0 has to leave U. At the end point α is adapted, hence it is adapted inside U by Proposition 66.6; see also the proof of Proposition 71. So $U = \emptyset$ and the result follows from Theorem 108. ∎

Questions: If $f_i\colon M_i^{n_i} \to \mathbb{R}^{n_i + p_i}$ with p_i the minimal codimension, is $q = p_1 + p_2$ the minimal codimension for an is.im.

$f\colon M_1^{n_1}\times M_2^{n_2}\to\mathbb{R}^{n_1+n_2+q}$? If yes, is it necessarily a product?

Remark 119. Similar results to those in this section exist for *warped products*; see [**DT**].

§25. Conformal immersions

General philosophy: $\mathbb{Q}_c^m\cong\mathbb{R}^m$; conformal immersions in $\mathbb{R}^m\cong$ isometric immersions in the *light cone* $\mathbb{V}^{m+1}\subset\mathbb{L}^{m+2}$.
If $w\in\mathbb{V}^{m+1}$ and $v\in\mathbb{E}^m=\mathbb{E}_w:=\{v\in\mathbb{V}^{m+1}\colon\langle v,w\rangle=1\}\cong\mathbb{R}^m$, and $C\colon\mathbb{R}^m\to\operatorname{span}\{v,w\}^\perp\subset\mathbb{L}^{m+2}$ is a linear isometry, then

$$\iota(x)=v+Cx-\frac{1}{2}\|x\|^2w\colon\mathbb{R}^m\to\mathbb{E}^m$$

is an isometric embedding in $\mathbb{L}^{m+2}$. Fix one. In fact, if $z\in\mathbb{L}^{m+1}$ with $\langle z,z\rangle=-1/c$, then $\mathbb{Q}_c^m\cong\{u\in\mathbb{V}^{m+1}:\langle u,z\rangle=-1/c\}$.

Exercise. The set of transformations $(v,w,C)\mapsto(v',w',C')$ is $\mathbb{O}_1(m+2)$.

Exercise. If $R_z(u):=u-2\langle u,z\rangle z\in\mathbb{O}_1(m+2)$ is a reflection w.r.t the space-like vector z with $\langle z,w\rangle\neq 0$ in $\mathbb{E}_w^m$, then $\hat{R}_z$ is the inversion with respect to the hypersphere $\mathbb{E}_w^m\cap\{z\}^\perp$.

Remark 120. Given a hypersurface $f\colon M^n\to\mathbb{V}^{n+1}$ we have that $f\in T_f^\perp M$ is parallel and $A_f^f=-I$. Hence, if η is the normal parallel with $\langle\eta,\eta\rangle=0$, $\langle\eta,f\rangle=1$, Gauss equation gives

$$\alpha_f(X,Y)=-\langle X,Y\rangle\eta-\langle LX,Y\rangle f,$$

where L is the *Schouten tensor* (not Einstein's!) given by

$$L:=\frac{n-1}{n-2}Ric-\frac{n}{2(n-2)}\,scal_M\,\mathrm{Id}.$$

In particular, $T_\iota^\perp\mathbb{R}^m=\operatorname{span}\{\iota,w\}$, $A_w^\iota=0$, and $\alpha_\iota=-\langle\ ,\ \rangle w$.

Corollary 121. *Every hypersurface of* $\mathbb{V}^{m+1}$ *is rigid. In particular, if* $F : U \subset \mathbb{R}^m \to \mathbb{V}^{m+1}$ *is an isometric immersion, then* $F = \iota|_U$ *for some* (v, w, C)*.*

Proof. 2nd part: by Remark 120, $\alpha(X,Y) = -\langle X, Y\rangle\eta$. But $\tilde{\nabla}_X \eta = -LX = 0 \Rightarrow \eta$ is a constant vector and $F(U) \subset \mathbb{E}^m_\eta$. ∎

25.1 The light cone representative

Conformal structure. Pull back.

Proposition 122. *Let* M^n *be a Riemannian manifold, and* $f : M^n \to \mathbb{R}^m \cong \mathbb{E}^m_w$ *a conformal immersion with conformal factor* φ*. Then,* $\hat{f} := \varphi^{-1}\, \iota \circ f : M^n \to \mathbb{V}^{m+1}$ *is an isometric immersion. Conversely, if* $\hat{f} : M^n \to \mathbb{V}^{m+1} \setminus \mathbb{R}w$ *is an isometric immersion, then* $f := \iota^{-1}(\langle \hat{f}, w\rangle^{-1}\hat{f}) : M^n \to \mathbb{R}^m$ *is a conformal immersion with conformal factor* $\varphi = \langle \hat{f}, w\rangle^{-1}$*. We call* $\hat{f}$ *the isometric light cone representative of* f*.*

Corollary 123. M^n *simply connected is conformally flat if and only if it is a hypersurface of the light cone.*

Remark 124. The space of curvature tensors can be decomposed in three $O(n)-$invariant subspaces: the one generated by the inner product (manifolds of constant curvature), the one spanned by the Ricci flat tensor (conformally flat manifolds), and the complement of these. So, we define the *Weil tensor* W by

$$\begin{aligned}\langle W(X,Y)Z, V\rangle &= \langle R(X,Y)Z, V\rangle - \langle LX, V\rangle\langle Y, Z\rangle - \langle LY, Z\rangle\langle X, V\rangle \\ &\quad + \langle LX, Z\rangle\langle Y, V\rangle + \langle LY, V\rangle\langle X, Z\rangle.\end{aligned}$$

A well-known theorem by Schouten states that, for $n \geq 4$, M^n is conformally flat if and only if $W = 0$. In fact, this can be easily seen using Corollary 123: $W = 0$ is precisely the Gauss equation of an hypersurface in the light-cone ($W = 0$ implies that L is Codazzi for $n \geq 4$, while $W = 0$ if $n = 3$); see Remark 120.

Corollary 125. *For any conformal map $\mathcal{T} : U \subset \mathbb{R}^m \to \mathbb{R}^m$, there is $T \in \mathbb{O}_1(m+2)$ such that $\hat{\mathcal{T}} = T \circ \iota|_U$.*

Conformal congruence can be regarded as a special case of isometric congruence (so we can use the isometric methods!):

Proposition 126. *Two is.im. $f', f \colon M^n \to \mathbb{R}^{n+p}$ are conformally congruent $\iff$ $\hat{f}'$ and $\hat{f}$ are isometrically congruent.*

Proof. Observe that the conformal factor of a composition $i \circ j$ satisfies $\varphi_{i\circ j} = \varphi_j \varphi_i \circ j$. If $f' = \mathcal{T} \circ f$ for a conformal diffeo $\mathcal{T}$ of $\mathbb{R}^{n+p}$ $\Rightarrow \hat{\mathcal{T}} = T \circ \iota$ for $T \in \mathbb{O}_1(n+p+2)$, and $\hat{f}' = \varphi^{-1}_{\mathcal{T}\circ f}\, \iota \circ \mathcal{T} \circ f = \varphi_f^{-1}(\varphi_{\mathcal{T}}^{-1} \iota \circ \mathcal{T}) \circ f = \varphi_f^{-1} T \circ \iota \circ f = T \circ \hat{f}$. ∎

Remark 127. See [**DT**] for equations relating the second fundamental forms, normal connections, etc, between a conformal immersion and its light-cone representative, and the Fundamental Theorem in Moebius geometry. Not surprisingly, by Proposition 126 many isometric results have natural conformal counterparts, that usually can be proved adapting isometric methods.

25.2 The conformal Gauss parametrization

Motivation. Classify conformally flat Euclidean hypersurfaces in terms of curves, as in the flat case (Corollary 90). The first step:

Proposition 128. *Let $f \colon M^n \to \mathbb{R}^{n+1}$ with $n \geq 4$. Then, M^n is conformally flat and if and only if f has a principal curvature of multiplicity at least $n-1$.*

Proof. We can assume f is not umbilic. By Proposition 122, there is a local i.i. $g\colon M^n \to \mathbb{V}^{n+1}$. By Remark 120 $\beta = (\alpha_f, \alpha_g) = (A^f, I, L)$ is flat with $\nu_\beta = 0$. By the Main Lemma 45 β is degenerate, so $L \in \operatorname{span}\{A^f, Id\}$ and $\dim \operatorname{Ker}(A^f - \lambda I) \geq n-1$. The converse is left as an exercise (show that $W = 0$). ∎

Remark 129. Cartan's examples: Proposition 128 false for $n=3$.

Proposition 130. *If $f\colon M^n \to \mathbb{R}^{n+p}$ is conformally flat and $p \leq n-3$, then f has a Dupin principal normal of multiplicity at least $n-p \geq 3$.*

Proof. Adapt the proof of Proposition 128 (exercise). ∎

So let's classify hypersurfaces with umbilic distributions:

$\sim \cdot \sim$

Let $f\colon M^n \to \mathbb{R}^{n+1}$ be orientable with Gauss map η and a Dupin principal curvature $\lambda \neq 0$ of multiplicity $n-k$. Since the corresponding eigendistribution E_λ is umbilical (hence integrable), we have the leaf space $V^k := M^n/E_\lambda$ and a submersion $\pi : M^n \to V^k$. The map

$$h = f + \lambda\eta$$

is constant along the leaves of E_λ, hence it descends to the quotient and we have an immersion $g : V^k \to \mathbb{R}^{n+1}$ and a function $r \in \mathcal{F}(V)$ given by

$$g \circ \pi = h, \quad r \circ \pi = \lambda^{-1}.$$

We endow V^k with the metric induced by g. In particular, $f = g \circ \pi - (r \circ \pi)\,\eta$. Since η is normal to f, $\eta^\top = (\nabla r) \circ \pi$ and therefore, by dimension reasons, we can parametrize f over the unit normal bundle $T_1^\perp V$ of g by

$$f \circ \xi = g - r\left(\nabla r + \sqrt{1 - \|\nabla r\|^2}\,\xi\right), \quad \xi \in \Gamma(T_1^\perp V).$$

§26. Deformable hypersurfaces

Let Δ an integrable distribution on M, and $L = M/\Delta$ the (local) space of leaves with projection $\pi : M \to L$. A vector field $X \in \mathfrak{X}(M)$ is called *projectable* if there is $\bar{X} \in \mathfrak{X}(L)$ π-related to X. Equivalently, the *horizontal lift* $\bar{X}^h$ of $\bar{X}$ agrees with $X_{\Delta^\perp}$.

Lemma 131. *$X \in \mathfrak{X}(M)$ is projectable $\iff$ $[X, \Delta] \subset \Delta$. $D \in End(\Delta^\perp)$ is projectable $\iff$ $\nabla_\Delta D = [D, C_\Delta]$.*

Proof. Use the usual flux formula for the Lie bracket: $[X, T] = \lim_{t\to 0} \frac{1}{t}(X \circ \varphi_t - \varphi_{t*}X)$, where $\varphi_t{}' = T \circ \varphi_t$ ($\Rightarrow \pi \circ \varphi_t = \pi$). ∎

Lemma 132. *Let $\mathbb{V} = End(\mathbb{R}^2)$, $D \in \mathbb{V} \setminus \mathbb{R}I$. Then $\dim \mathcal{C}^D = 2$, where $\mathcal{C}^D := \{C \in \mathbb{V} \colon [C, D] = 0\}$. In addition, up to sign, $\exists!\ J \in End(\mathbb{R}^2)$ such that $J^2 = \epsilon I$, $\epsilon = 1, 0, -1$, $\|J\| = 1$ if $\epsilon = 0$, satisfying $span\{I, J\} = span\{I, D\} = \mathcal{C}^D$.*

Proof. If there is $C \in \mathcal{C}^D$ symmetric such that $C \neq aI$, then D and all elements in $\mathcal{C}^D$ diagonalize in the same basis. ∎

Definition 133. We say that a nowhere flat Euclidean hypersurface f is a *Sbrana-Cartan hypersurface* if there is another $\hat{f}$ nowhere congruent to f (i.e., f is *locally deformable*).

Example: Associated family of a minimal rank 2 hypersurface.

Nowhere flat surfaces in $\mathbb{R}^3$ are locally deformable, but no classification exists. Pogorelov-Nadirashvili-Yuan's examples.

Proposition 70: Surface-like hypersurfaces $\iff$ $\operatorname{Im} C \subset \operatorname{span}\{I\}$.

From now on in this section, assume that $f\colon M^n \to \mathbb{R}^{n+1}$ is a nowhere surface-like Sbrana-Cartan hypersurface with deformation $\hat{f}$, and A and $\hat{A}$ their shape operators in $\Delta^\perp$. Then:

(a) $\hat{\Delta} = \Gamma = \Delta$ agree and are intrinsic (since $\hat{\nu} = \mu = \nu \equiv n-2$);

(b) Hence, the splitting tensor C of Δ is the same and intrinsic!

(c) Gauss $\iff$ $D := A^{-1}\hat{A} \in End(\Delta^\perp)$ satisfies $\det D = 1$;

(d) Noncongruent $\iff$ $D \notin \operatorname{span}\{I\}$ on an open dense $U \subset M^n$;

(e) $[D, C_T] = 0 \;\; \forall T \in \Delta$ (by Proposition 66.7);

(f) $\nabla_\Delta D = 0$ (by the last and 2 Codazzi's in $\nabla_\Delta \hat{A} = \nabla_\Delta AD$);

(g) $\dim(\operatorname{span}\{I\} + \operatorname{Im} C) = 2$ a.e. U (by (e) and Lemma 132) $\Rightarrow$

(h) Up to sign, $\exists !\, J \in End(\Delta^\perp)$ such that $J^2 = \epsilon I$, $\epsilon = 1, 0, -1$, $\|J\| = 1$ if $\epsilon = 0$, satisfying $\operatorname{span}\{I\} \neq \operatorname{Im} C \subset \operatorname{span}\{I, J\}$;

(i) $AJ = J^t A$ (by (h), Lemma 132 and Proposition 66.7);

(j) $D \in \operatorname{span}\{I, J\}$ (again by Lemma 132) $\Rightarrow$

(k) $\nabla_\Delta J = 0$ (by (f) and $J^2 = \epsilon I$, since also $J \in \operatorname{span}\{I, D\}$).

Def.: A Riemannian manifold M^n with $\mu \equiv n-2$ is called *parabolic* ($\epsilon = 0$), *hyperbolic* ($\epsilon = 1$), *elliptic* ($\epsilon = -1$) if there is $J \in End(\Gamma^\perp)$ satisfying (h)+(k) ($\Rightarrow M^n$ is nowhere surface-like).

Corollary 134. *Both D and J project to $V^2 := M^n/\Delta$, i.e., $\exists\ \bar{D}, \bar{J}$ such that $\bar{D} \circ \pi_* = \pi_* \circ D$ and $\bar{J} \circ \pi_* = \pi_* \circ J$ on $\Delta^\perp$.*

Parabolic, hyperbolic and elliptic surfaces: existence of conjugate coordinates, first normal space of dimension 2.
Set
$$\mathcal{D}_M := \{f : M^n \to \mathbb{R}^{n+1}\}/\text{congruence}.$$

Corollary 135. *$f\colon M^n \to \mathbb{R}^{n+1}$ rank two nowhere surface-like. Then, M^n is parabolic (resp hyperbolic, elliptic) w.r.t. J $\iff$ the Gauss data is parabolic (resp. hyperbolic, elliptic) w.r.t. $\bar{J}$. In particular, every member of $\mathcal{D}_M$ is parabolic (resp. hyperbolic, elliptic).*

Proof. Since $\bar{J} \circ \pi_* = \pi_* \circ J$, and $AJ = J^t A$, by Section 22 $P_w^{-1} = \pi_* = -A$ (omitting j). So, $\bar{J} = -J^t$ and $\bar{J}^t P_w = P_w \bar{J}$. ∎

Corollary 136. *By $(h), (i)$ and Corollary 135, the Gauss data is parabolic, hyperbolic or elliptic with respect to $\bar{J}$: $Q(h) = 0$ and $Q(r) = 0$.*

Proposition 137. *Assume M^n is hyperbolic or elliptic. Then, $\hat{A}$ is Codazzi $\iff$ $\bar{D}$ is Codazzi $\iff$ the Gauss data is of first or second species.*

Proof. Use Section 22: $(\nabla^h_X \bar{D}Y)' = j\nabla^h_{X\circ\pi} \bar{D}Y = j\nabla^h_{X^h} \bar{D}Y \circ \pi = j\nabla^h_{X^h} \pi_*(DY^h) = (\nabla^M_{X^h}(j \circ \pi_*)(DY^h))_{\Delta^\perp} = -(\nabla^M_{X^h} \hat{A}DY^h)_{\Delta^\perp}$. ∎

We first deal with the easiest parabolic case:

Proposition 138. *M^n is parabolic $\iff$ f is ruled. In this case $\mathcal{D}_M = \mathbb{R}$ and every $g \in \mathcal{D}_M$ is ruled with the rulings of f.*

Proof. Let $\{X, Y\}$ o.n.b. of $\Delta^\perp$ such that $JX = Y$, $JY = 0$, and $R = \Delta \oplus^\perp \text{span}\{Y\}$. $J^t A = AJ \Rightarrow \langle AY, Y\rangle = 0$. $\nabla_\Delta J = 0 \Rightarrow \nabla_\Delta Y = 0$. Since $\text{Im}\, C \subset \text{span}\{I, J\}$ we get $\nabla_Y \Delta \subset R$. Write

$D = I + \theta J$. AD Codazzi $\iff$ θAJ Codazzi $\iff$ $\nabla_Y Y \in R$, $\Delta(\theta) = 0$, and $Y(\theta\mu) = \theta\mu\langle\nabla_X X, Y\rangle$ where $\mu := \langle AX, Y\rangle$.

Or: Let (u, v) coord. on V with $\partial_v \in \operatorname{Ker} \overline{J}$, $J\partial_u = \partial_v$. $\overline{D} = I + \overline{\theta J}$ is Codazzi $\iff \theta_v = \theta\Gamma^v_{vv}$ and $\nabla_{\partial_v}\partial_v \in \operatorname{span}\{\partial_v\}$. So, if $Y \overset{\pi_i}{\sim} \theta\partial_v$ then $[Y, \Delta] \subset \Delta$, $JY = 0$, $\langle AY, Y\rangle = 0$ (since $AJ = J^t A$), $\nabla_Y \Delta \subset R$ (since $\operatorname{Im} C \subset \operatorname{span}\{I, J\}$) and $\nabla_Y Y \in R$ (since $(\nabla_Y AY)_{\Delta^\perp} = -\nabla_{\theta\partial_v}\theta\partial_v \circ \pi = 0$). ∎

Def.: Gauss data (h, r) of *first or second species* (with conjugate coordinate system (u, v)): hyperbolic (resp. elliptic) h and r, such that

$$\tau(\Gamma^v_v - 2\Gamma^u\Gamma^v) = (\Gamma^u_u - 2\Gamma^u\Gamma^v)$$

(resp. $\operatorname{Im}(\rho(\Gamma_z - 2\Gamma\bar{\Gamma})) = 0$).

Finally, we can give the complete Sbrana-Cartan classification:

Theorem 139. *Let M^n be any Riemannian manifold. Then, each connected component U of an open dense subset of M^n falls, even locally, exactly into one of these categories:*

i) $\mathcal{D}_U = \emptyset$, i.e., U is not even locally a Eucl. hypersurface;

ii) U is rigid, i.e., $\mathcal{D}_U$ is a point;

iii) U is flat, and $\mathcal{D}_U = \mathcal{F}(\mathbb{R}, \mathbb{S}^n) \times \mathcal{F}(\mathbb{R})$;

iv) U is nonflat surface-like and $\mathcal{D}_U$ is the one of the surface;

v) U is parabolic and ruled, $\mathcal{D}_U = \mathcal{F}(\mathbb{R})$, and every element in $\mathcal{D}_U$ is ruled with same rulings as U;

vi) The Gauss data is of first species, and $\mathcal{D}_U = \mathbb{R}$;

vii) The Gauss data is of second species, and $\mathcal{D}_U = \mathbb{Z}_2$.

Case (vi) is called the *continuous type* (e.g., g minimal), while case (vii) is called the *discrete type* (e.g....?????????).

Remark 140. Recently, Diego Navarro Guajardo extended Sbrana-Cartan Theory to higher codimension ([**Gu1**]).

§27. Intersections

These were the first known Sbrana-Cartan hypersurfaces of the discrete type. They can be obtained intersecting two flat hypersurfaces, or, better, as in [**FF**] as rank two hyperbolic submanifolds in codimension two that extend as flat hypersurfaces in two different ways. We briefly describe this work.

Let $f\colon M^n \to \mathbb{R}^{n+2}$ be a hyperbolic rank two submanifold. It is easy to see that f has a (hyperbolic) *polar surface* that "integrates" its normal bundle, i.e, there is $g : V^2 = M/\Delta \to \mathbb{R}^{n+2}$ such that

$$g_{*[x]}(T_{[x]}V) = T^{\perp}_{f(x)}M \quad \forall\, x \in M^n.$$

In [**FF**] it was shown that f extends as a flat hypersurface in two different ways $\iff$ $\Gamma^u = \Gamma^v = 0$ for g, i.e., if

$$g(u,v) = c_1(u) + c_2(v),$$

with c_1', c_1'', c_2', c_2'' pointwise L.I.. The *shared dimension* $I(c_1, c_2) \in \mathbb{N}_0$ is the smallest integer k for which there is an orthogonal decomposition in affine subspaces, $\mathbb{R}^{n+2} = \mathbb{V}_1 \oplus^{\perp} \mathbb{V}^k \oplus^{\perp} \mathbb{V}_2$, satisfying that $\operatorname{span}(c_i) \subset \mathbb{V}_i \oplus^{\perp} \mathbb{V}^k$, $i = 1, 2$. It turns out that:

- $I(c_1, c_2) = 0 \iff M^n$ is flat;
- $I(c_1, c_2) = 1 \iff M^n$ is of the continuous type;
- $I(c_1, c_2) \geq 2 \iff M^n$ is of the discrete type.

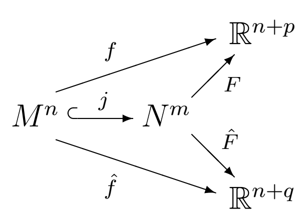
Corollary 141. *Crazy collage of the different types.*

Remark 142. Diego Navarro Guajardo also extended this intersection construction to higher codimension ([**Gu2**]).

§28. Genuine rigidity ([DF2])

Rigidity in higher codimensions: rigidity and compositions are particular cases of *isometric extensions*. In this context, algebraic rigidity results like Theorem 44 and Theorem 47 disregard information about the normal connections. As such, they should be understood as "generic" without much usefulness for classification. Now we search for more geometry.

Def.: We say that a pair $f\colon M^n \to \mathbb{R}^{n+p}$ and $\hat{f}\colon M^n \to \mathbb{R}^{n+q}$ of isometric immersions *extends isometrically* when there are an isometric embedding $j\colon M^n \hookrightarrow N^m$ into a Riemannian manifold N^m with $m > n$ and isometric immersions $F\colon N^m \to \mathbb{R}^{n+p}$ and $\hat{F}\colon N^m \to \mathbb{R}^{n+q}$ such that $f = F \circ j$ and $\hat{f} = \hat{F} \circ j$. In other terms, the following diagram commutes:



(1)

We want to *discard* deformations $\hat{f}$ that arise in this way, since the deformation problem essentially depends on the codimension, and not on the dimension. This gives rise to the following:

Def.: We say that $\hat{f}\colon M^n \to \mathbb{R}^{n+q}$ is a *genuine deformation* of a given $f\colon M^n \to \mathbb{R}^{n+p}$ (or that $\{f_\lambda\colon M^n \to \mathbb{R}^{n+p_\lambda}\}$ is a *genuine set*) if $\nexists U \subset M^n$ s.t. $f|_U$ and $\hat{f}|_U$ extend isometrically.

Def.: We say that $f\colon M^n \to \mathbb{R}^{n+p}$ is *genuinely rigid in* $\mathbb{R}^{n+q}$ if every $\hat{f}\colon M^n \to \mathbb{R}^{n+q}$ is *nowhere* a genuine deformation of f.
Motivation of the following. Structure of the second fundamental forms and normal connections when a pair extends isometrically: the extensions induce a natural parallel bundle isometry between the normal subbundles $F_*(T_j^\perp M) \to \hat{F}_*(T_j^\perp M)$ that preserves second fundamental forms. *When is the converse statement true?*

Def.: D^d-ruled submanifolds, mutually ruled (sets!), ruled extensions.
Let $f\colon M^n \to \mathbb{R}^{n+p}$ and $\hat{f}\colon M^n \to \mathbb{R}^{n+q}$, and τ a parallel vector bundle isometry that preserves second fundamental forms,

$$\tau : L^\ell \subset T_f^\perp M \to \hat{L}^\ell \subset T_{\hat{f}}^\perp M. \tag{7}$$

Equivalently, the induced v.b. isometry $\bar{\tau}$ is parallel, where

$$\bar{\tau} = Id \oplus \tau\colon\ f_*TM \oplus L \to \hat{f}_*TM \oplus \hat{L}.$$

Define $\phi_\tau\colon TM \times (TM \oplus L) \to (L^\perp \times \hat{L}^\perp, \langle\,,\rangle_{L^\perp} - \langle\,,\rangle_{\hat{L}^\perp})$ by

$$\phi_\tau(X,\eta) = \left((\tilde{\nabla}_X \eta)_{L^\perp}, (\tilde{\nabla}_X \bar{\tau}\eta)_{\hat{L}^\perp}\right).$$

Proposition 143. *The bilinear form ϕ_τ is Codazzi and flat.*

Proof. Exercise. ∎

Notice that $\alpha_{L^\perp} \oplus \hat{\alpha}_{\hat{L}^\perp} = \phi_\tau|_{TM \times TM}$. Assume that the subspaces

$$D = D_\tau := \mathcal{N}(\alpha_{L^\perp} \oplus \hat{\alpha}_{\hat{L}^\perp}) \subset TM,$$

$$\Delta = \Delta_\tau := \mathcal{N}_r(\phi_\tau) \subset TM \oplus L$$

have constant dimensions $d_\tau \le \nu_\tau$ respectively (observe that

$\Delta \cap TM = D$). It follows that $\bar\tau\,|_\Delta\colon\ \Delta \to \hat\Delta$ is a parallel vector bundle isometry, and hence, we can identify $\hat\Delta$ with Δ.

Corollary 144. *$\mathcal{N}_l(\phi_\tau) \subset D \subset TM$ is integrable. In particular, if L and $\hat L$ are parallel along D, namely, if $D = \mathcal{N}_l(\phi_\tau)$, then $D \subset \Delta$ is integrable.*

Corollary 145. *If L and $\hat L$ are parallel along D,* $\operatorname{Im}(\phi_\tau)$ *and Δ are smooth and parallel along its leaves. In particular, the leaf through $x \in M^n$ is also given by $\Delta(x) \cap M^n$.*

Let $\pi\colon \Lambda \to M^n$ be the vector bundle $\Lambda := D^\perp \subset \Delta \subset TM \oplus L$, and consider the extensions $F : \Lambda \to \mathbb{R}^{n+p}$ and $\hat F : \Lambda \to \mathbb{R}^{n+q}$,

$$F \circ \xi = f \circ \pi + \xi, \quad \hat F \circ \xi = \hat f \circ \pi + \bar\tau\xi, \quad \forall \xi \in \Gamma(\Lambda), \qquad (8)$$

restricted to a neighborhood N of $M^n \cong 0 \subset \Lambda$ to get immersions. Observe that $L^\perp$, D, Δ, etc, induce natural corresponding bundles over Λ (e.g., $L^\perp(\xi_x) = L^\perp(x)$, namely, $\pi^*(L^\perp)$).

The following is the main result on isometric extensions:

Theorem 146. *If L and $\hat L$ are parallel along D, then F and $\hat F$ are isometric $\pi^*(\Delta)$-ruled extensions of f and $\hat f$. Moreover, there are orthogonal splittings*

$$T_F^\perp N = \mathcal{L} \oplus^\perp \pi^*(L^\perp), \quad T_{\hat F}^\perp N = \hat{\mathcal{L}} \oplus^\perp \pi^*(\hat L^\perp),$$

and a parallel vector bundle isometry $\mathcal{T} : \mathcal{L} \to \hat{\mathcal{L}}$ that preserves second fundamental forms such that $\pi^*(\Delta) = D_{\mathcal{T}}$. In addition, $\mathcal{L}$ and $\hat{\mathcal{L}}$ are parallel along $\pi^*(\Delta)$.

Proof. Let's argue first for F, $\hat{F}$ being similar. Observe first that $\operatorname{Im} F_* \subset \pi^*(T_f M \oplus L)$ which is parallel along $\pi^*(D) \subset T\Lambda$. Thus, $\pi^*(L^\perp) \subset T_F^\perp \Lambda$, and we can write

$$T_F\Lambda \oplus^\perp \mathcal{L} = \pi^*(T_f M \oplus L), \quad \mathcal{L}^\perp = \pi^*(L^\perp). \tag{9}$$

Since D is integrable and $\tilde{\nabla}_D \Delta \subset \Delta$ by Corollary 145, we easily get that F is $\pi^*(\Delta)$-ruled. (Equivalently, we could have worked on $V := M/D$ and defined F over the bundle $\pi^*(\Delta) \to V$ instead!). Using (9) we define $\bar{\mathcal{T}}$ by $\bar{\mathcal{T}} \circ \pi^* = \pi^* \circ \bar{\tau}$, which is clearly parallel, and the extensions are isometric since $\hat{F}_{*w_x} = \bar{\tau}_x \circ F_{*w_x}$.
Now, take $\eta \in \Gamma(L^\perp)$ and $Z \in \Gamma(\Delta)$. Since $\pi^*(\Delta) = \pi_*^{-1}(D)$,

$$(\tilde{\nabla}_{Z\circ\pi} \eta \circ \pi)_{T_F\Lambda \oplus \mathcal{L}} = (\tilde{\nabla}_Z \eta \circ \pi)_{T_f M \oplus L} = (\tilde{\nabla}_{\pi_* Z}\, \eta)_{T_f M \oplus L} = 0.$$

This proves the last assertion and $\pi^*(\Delta) \subset D_{\mathcal{T}}$. For the opposite inclusion, since $\alpha_{L^\perp} \oplus \hat{\alpha}_{\hat{L}^\perp} = \alpha^F_{\mathcal{L}^\perp} \oplus \hat{\alpha}^{\hat{F}}_{\hat{\mathcal{L}}^\perp}|_{T_j M \times T_j M}$, equality holds along M^n, and hence in a neighborhood by semicontinuity. ∎

Corollary 147. *The extensions F and $\hat{F}$ are trivial $\iff$ f and $\hat{f}$ are mutually D-ruled.*

Lemma 148. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ be a D-ruled submanifold. $\Rightarrow$ $L_D := S(\alpha|_{D\times TM})$ is parallel along D (constant dimension).*

Corollary 149. *$\{f, \hat{f}\}$ is genuine $\Rightarrow$ f and $\hat{f}$ are mutually D-ruled and we have:*

$$T_f^{\perp}M = L_D \oplus L_D^{\perp} \qquad\qquad L_D := \text{span}\ \{\alpha(D,TM)\}$$

$\mathcal{T}_D$:
$(\nabla^{\perp})_{L_D} = (\hat{\nabla}^{\perp})_{\hat{L}_D}$
$\alpha_{L_D} = \hat{\alpha}_{\hat{L}_D}$

$$D = \mathcal{N}\left(\alpha_{L_D^{\perp}} \oplus \hat{\alpha}_{\hat{L}_D^{\perp}}\right) \text{ are rulings!!}$$

$$T_{\hat{f}}^{\perp}M = \hat{L}_D \oplus \hat{L}_D^{\perp} \qquad\qquad \hat{L}_D := \text{span}\ \{\hat{\alpha}(D,TM)\}$$

In other words, if $\{f, \hat{f}\}$ is genuine, then they have a *"partial relative nullity"* in common, which, if not relative nullity ($L_D \neq 0$), then it is much bigger than it should be, i.e.: *if we lose relative nullity we gain dimension.*

There are always isometries such as τ as in (7), e.g., $\tau = 0$! In this case $D^d = \Delta_f \cap \Delta_{\hat{f}}$, where the result is obvious. But we have no estimate on d for $\tau = 0$, since ϕ_0 may be degenerate...
Yet, in [**DF2**] we explicitly constructed τ, L, and D^d for which the rulings are big:

Theorem 150. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ and $\hat{f}\colon M^n \to \mathbb{R}^{n+q}$ a genuine pair with $p+q < n$ and $\min\{p,q\} \leq 5$. Then, $\{f, \hat{f}\}$ are mutually D^d-ruled a.e., with $d \geq n - p - q + 3\dim L_D$. Moreover, the isometry τ_D is parallel and preserves second fundamental forms.*

Just the proof of the above (sharp!) estimate on d takes 5 pages and it is quite delicate. But as we will see in the next section, the above generalizes all known result about compositions, rigidity with s-nullities, etc, i.e., the ones we studied so far (exercise).

Not surprisingly, we get several corollaries, like:

Corollary 151. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ and q a positive integer with $p+q<n$. If $\min\{p,q\} \le 5$ and f is not $(n-p-q)$–ruled on any open subset of M^n, then f is genuinely rigid in $\mathbb{R}^{n+q}$.*

Corollary 152. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ and q a positive integer with $p+q<n$. If $\min\{p,q\} \le 5$ and $Ric_M > 0$ then f is genuinely rigid in $\mathbb{R}^{n+q}$.*

Corollary 153. *Any $f\colon U \subset \mathbb{S}^N \to \mathbb{R}^{2n-2}$ is a composition a.e.*

We even get topological criteria for genuine rigidity in line with the rigidity question proposed by M. Gromov in *Partial Differential Relations* p.259 (and answered in [**DG**]):

Corollary 154. *Let M^n be a compact manifold whose first Pontrjagin class satisfies that $[p_1]^2 \neq 0$. If $n > p+q$ and $p+q \le 6$, then any analytic immersion $f\colon M^n \to \mathbb{R}^{n+p}$ is (with the induced metric) genuinely rigid in $\mathbb{R}^{n+q}$.*

§29. Better s-nullities

Since L_D is always parallel along D by Lemma 148, Corollary 147 screams to use the s-nullity of another bilinear form instead of the ones for the second fundamental form. Indeed, given $V^s \subset T^\perp M$ a normal subbundle of rank s of an isometric immersion $f : M^n \to \mathbb{R}^{n+p}$, define as in [**FG**] the tensor

$$\phi_V : TM \times (TM \oplus V^\perp) \to V, \quad \phi_V(X, v) = (\tilde{\nabla}_X v)_V.$$

Notice that $\phi_\tau = (\phi_{L^\perp}, \hat{\phi}_{\hat{L}^\perp})$. As before, since ϕ_V is Codazzi, its left nullity

$$\mathcal{N}_l(\phi_V) = \{X \in \mathcal{N}(\alpha_V) \colon \nabla^\perp_X V \subset V\}$$

is integrable where it has constant dimension. Set

$$\bar{\nu}^f_s := \max_{V^s \subset T^\perp M} \dim \mathcal{N}_l(\phi_V).$$

Thus Lemma 148, Corollary 147 and Theorem 150 imply that $D = \mathcal{N}_l(\phi_\tau) \subset \mathcal{N}_l(\phi_{L_D^\perp}) \ \Rightarrow$

Corollary 155. *Let $f \colon M^n \to \mathbb{R}^{n+p}$, and $q \in \mathbb{N}$ such that $\min\{p, q\} \leq 5$. If $\bar{\nu}^f_s < n + 2p - q - 3s$ almost everywhere for all $1 \leq s \leq p$, then f is genuinely rigid in $\mathbb{R}^{n+q}$.*

Remark 156. This result is stronger than all the ones with s-nullities cited before (and probably all the ones not cited too...):

- We can work with $q \neq p$;
- $\bar{\nu}^f_s \leq \nu^f_s$ since $\mathcal{N}_l(\phi_V) \subset \mathcal{N}(\alpha_V)$;
- The bound on $\bar{\nu}^f_s$ is weaker than the usual one for ν^f_s by $p - s$;
- $\mathcal{N}_l(\phi_V)$ is always integrable, and 'almost' totally geodesic;
- We can require in the definition of $\bar{\nu}^f_s$ to $\mathcal{N}_l(\phi_V)$ be totally geodesic, or asymptotic, since the leaves of D are rulings, which makes $\bar{\nu}^f_s$ even smaller. That is:

We define the (local) *ruling index* $\nu_R(f)$ for $f \colon M^n \to \mathbb{R}^{n+p}$ by

$$\nu_R(f) = \max\{d - 3\ell_D : f|_U \text{ is } D^d\text{-ruled for some } U \subset M^n\}.$$

Corollary 157. *Let* $f\colon M^n \to \mathbb{R}^{n+p}$ *and let* q *be a positive integer such that* $p+q<n$ *and* $\min\{p,q\} \leq 5$. *If* $\nu_R(f) \leq n-p-q-1$, *then* f *is genuinely rigid in* $\mathbb{R}^{n+q}$.

§30. Global rigidity

Global rigidity results in submanifold theory are *way* more scarce than local ones, the most beautiful of which is Sacksteder's:

Theorem 158. *A compact Euclidean hypersurface of dimension at least* 3 *is rigid provided its set of non-totally geodesic points is connected. (And we understand if not!). Same for complete bounded.*

Proof. By Proposition 143, $\beta := \phi_0$ is flat and Codazzi. Hence, Propositions 66, 71 and Corollary 72 hold for $\Delta_0 = \mathcal{N}(\beta)$ (see Remark 73). Now use the spirit of the proof of Theorem 118 to show that $A = \pm\hat{A}$ everywhere. ∎

Remark 159. Same result and proof hold for $f\colon M^n \to \mathbb{H}^{n+1}$. For $f\colon M^n \to \mathbb{S}^{n+1}$ complete and $n \geq 4$ it also holds by the proof of Corollary 74 since no leaf of relative nullity with $\nu \geq n-2$ can be complete: $\{X, \ldots, C_{T_\nu}X\}$ would be $n-1$ L.I. vectors in $\Delta^\perp$.

In [**DG**] the codimension two case was solved by showing that, giving a pair of is.ims., along each connected component of an open dense subset, the immersions are either congruent or extend isometrically to flat hypersurfaces, or to singular Sbrana-Cartan hypersurfaces. That singularities are necessary was proved much later in [**FF**] (also in the flat case, filling a gap in [**DG**]). In other words, compact codimension two Euclidean submanifolds are *singularly genuinely rigid*, and singularities are needed!

§31. Singular genuine rigidity ([FG])

As we saw in Theorem 146, given $\tau : L \to \hat{L}$ we get isometric (possibly trivial) ruled extensions as in (8). In particular, this holds for $\tau = 0$, in which case $\phi_0 = \beta := (\alpha, \hat{\alpha})$. The key distribution here was thus $\Delta_0 = \operatorname{Ker}\beta$. The extensions in (8) are then obviously isometric since $\hat{F}_* = \bar{\tau} \circ F_*$. This is a sufficient condition, but not a necessary one (!!!). Indeed, a tautology:

Proposition 160. *Let $f, \hat{f}$ and $\tau : L \to \hat{L}$ parallel that preserves second fundamental forms. Let $\Lambda \subset TM \oplus L$ be any subbundle. Then, F and $\hat{F}$ in (8) are isometric $\iff$*

$$\phi_\tau(TM, \Lambda) \subset L^\perp \oplus \hat{L}^\perp \;\; \textit{is null.} \tag{10}$$

Of course this holds if $\Lambda \subset \Delta_\tau$ as before, but it has two very important advantages:

- No Main Lemma! In particular, no need for $\min\{p, q\} \leq 5$ in an analogous to Theorem 150.
- Null subspaces are much, Much, <u>MUCH</u> easier to get than nullities due to Proposition 34. Thanks J.D.Moore!

Observe that Proposition 160 holds even if Λ is not transversal to M^n. In this case, F and $\hat{F}$ are not immersions along $M^n \subset \Lambda$. Actually, the only problem to extend is when $\Lambda = D \subset TM$, otherwise we just take a subbundle of Λ transversal to M^n.
If f is D-ruled, then F is not an immersion, it has constant rank equal to n, and $F(D) = f(M)$. But what if not?

Def.: We say that $F = F_{\Lambda,f}$ in (8) is a *singular extension* of f if it is an immersion in some open neighborhood of M^n (the

0-section of Λ), except of course at M^n itself.

Def.: We say that $\hat{f}$ is a *strongly genuine deformation* of f, or that $\{f, \hat{f}\}$ is a *strongly genuine pair*, if there is no open subset U where $f|_U$ and $\hat{f}|_U$ singularly extend isometrically.

Def.: Given $q \in \mathbb{N}$, the isometric immersion f is said to be *singularly genuinely rigid in* $\mathbb{R}^{n+q}$ if, for any isometric immersion $\hat{f}\colon M^n \to \mathbb{R}^{n+q}$, $\{f, \hat{f}\}$ singularly extend isometrically a.e..

We say that $F = F_{\Lambda,f}$ *nowhere induces a singular extension* of f if, for every open subset $U \subset M^n$ and every subbundle $\Lambda' \subset \Lambda|_U$, the restriction of $F|_{\Lambda'}$ is not a singular extension of $f|_U$.

The key point is that this only happens when f is $\bar{\Lambda}$-ruled (observe that now $\Lambda \subset TM$ is not necessarily integrable, so $\bar{\Lambda}$*-ruled* means that $f_{*x}(\Lambda(x) \cap U) \subset f(M)$):

Proposition 161. *Let $\Lambda \subset TM$ any smooth distribution. Then, $F_{\Lambda,f}$ nowhere induces a singular extension of f $\iff$ f is $\bar{\Lambda}$-ruled.*

Proof. We only need to prove the direct statement. SPG, $\Lambda \subset TM$ with $\operatorname{rank} \Lambda = 1$. So we may parametrize $F(x,t) = f(x) + tX(x)$ where $\|X\| = 1$. Then, $\forall p \in M$ there is $(p_m, t_n) \to (p, 0)$ with $t_m \neq 0$ such that $\operatorname{rank} F_*(p_m, t_n) = n$. Define the tensors on M by $K = \nabla_{\bullet} X$ and $H_t = I + tK$. Hence, there is $Y_m \in T_{p_m}M$ such that $F_{*(p_m,t_m)} Y_m = X(p_m)$, i.e., $H_{t_m} Y_m = X(p_m)$ and

$$\alpha(X(p_m), H_{t_m}^{-1} X(p_m)) = 0. \tag{11}$$

Consider a precompact open neighborhood $U \subset M^n$ of p, so $\|\alpha\| < c$ and $\|K\| < c$ for some constant $c > 1$. Hence for

$t \in I = (-\frac{1}{c^2}, \frac{1}{c^2})$ we have that H_t is invertible on U, and

$$H_t^{-1} = \sum_{i \geq 0} (-t)^i K^i,$$

since $H_t \circ \sum_{i=0}^{N} (-t)^i K^i = Id - (-t)^{N+1} K^{N+1}$.
We claim that $\alpha(X, S_X) = 0$ on M^n, where S_X is the K-invariant subspace generated by X, i.e., $S_X = \text{span}\{X, KX, K^2X, \dots\}$. If otherwise, set $j := \min\{k \in \mathbb{N} : \alpha(X(q), K^k(X(q))) \neq 0,\ q \in M^n\}$ and take $p \in M^n$ such that $\alpha(X(p), K^j(X(p))) \neq 0$. By (11),

$$\sum_{i \geq 0} (-t_m)^i \alpha(X(p_m), K^{j+i}(X(p_m))) = 0.$$

Taking $m \to \infty$ we get $\alpha(X(p), K^j(X(p))) = 0$, a contradiction. Now, since $\alpha(X, S_X) = 0$ on M^n, for any $t \in I$ and $p \in U$ we get $F_{*(p,t)}(H_t^{-1}(X)) = X$ since $H_t^{-1}(X) \in S_X$. It follows that $\text{rank}(F_*) = n$ in all $U \times I$, and therefore $F(U \times I) = f(U)$. Hence a segment of the line generated by X is contained in $f(U)$. ∎

We have thus shown:

Theorem 162. *Let $\{f, \hat{f}\}$ be a strongly genuine pair and $\tau\colon L^\ell \subset T_f^\perp M \to \hat{L}^\ell \subset T_{\hat{f}}^\perp M$ a parallel vector bundle isometry that preserves second fundamental forms. Let $D \subset TM \oplus L^\ell$ be a subbundle such that $\phi_\tau(TM, D)$ is a null subset. Then $D \subset TM$ and f and $\hat{f}$ are mutually $\bar{D}$-ruled.*

From Proposition 34 we immediately get the following, where

$$i(\phi_\tau)(x) := \max\{\text{rank}\,(\phi_\tau(X, \cdot))\colon X \in T_xM\}.$$

Corollary 163. *Under the assumptions of Theorem 162, along each connected component of an open dense subset of* M^n, $i(\phi_\tau)$ *is constant and* f *and* $\hat{f}$ *are mutually* $\bar{D}^d_Y$*-ruled for any smooth vector field* $Y \in Re(\phi_\tau)$, *where* $D^d_Y := \mathrm{Ker}\,(\phi^Y_\tau) \subset TM$. *In particular,* f *and* $\hat{f}$ *are mutually* d*-ruled with*

$$d = n + \ell - i(\phi_\tau) \geq n - p - q + 3\ell.$$

By allowing singular extensions we recover all the corollaries in [**DF2**], and even without the technical restrictions on the codimensions required there due to the Main Lemma. For example:

Corollary 164. *Any* $M^n \subset \mathbb{R}^{n+p}$ *with positive Ricci curvature is singularly genuinely rigid in* $\mathbb{R}^{n+q}$, *for every* $q < n-p$.

Now, we did all this to apply to global rigidity. Indeed we have:

Theorem 165. *Let* $f\colon M^n \to \mathbb{R}^{n+p}$ *and* $\hat{f}\colon M^n \to \mathbb{R}^{n+q}$ *be isometric immersions of a* <u>*compact*</u> *Riemannian manifold with* $p+q < n$. *Then, along each connected component of an open dense subset of* M^n, *either* f *and* $\hat{f}$ *singularly extend isometrically, or* f *and* $\hat{f}$ *are mutually* d*-ruled, with* $d \geq n-p-q+3$.

This is an immediate consequence of Corollary 163 and the next, which shows that we have $\ell \geq 1$ a.e.:

Lemma 166. *Under the assumptions of Theorem 165, at each point of* M^n *either* $i(\beta) \leq p+q-3$, *or* $S(\beta)^\perp$ *is not definite. The last possibility holds globally if* $\min\{p,q\} \leq 5$.

Proof. Let W be the complement, i.e., where either $S(\beta)^\perp$ is definite, and $i(\beta) \geq p+q-2$ if $\min\{p,q\} \geq 6$. So, $\nu_0 > 0$ on W since this is the easy part of the Main Lemma where no hypothesis is needed; see the first exercise in Section 16. Then, use Sacksteder's trick: not only ν_0, but also $i(\beta)$ (by the proof of Proposition 71), are constant along a geodesic in Δ_0. ∎

In particular, for $p+q \leq 4$, Theorem 165 easily unifies Sacksteder and Dajczer-Gromoll Theorems above, states that the only way to isometrically immerse a compact Euclidean hypersurface in codimension 3 is through compositions (which in turn were classified in [**DF1**]), and provides a global version of the main result in [**DFT**]:

Corollary 167. *Any* *compact* *(or complete and bounded) isometrically immersed submanifold* M^n *of* $\mathbb{R}^{n+p}$ *is singularly genuinely rigid in* $\mathbb{R}^{n+q}$ *for all* $q < \min\{5, n\} - p$*.*

Proof. The only case left is the $(n-1)$-ruled one, which is not hard, or you can attack it directly; see Section 3.1 in [**FG**]. ∎

From Theorem 165 we also get the following topological criteria for singular genuine rigidity with the same spirit as Corollary 154, yet without any *a priori* assumption on the codimensions:

Corollary 168. M^n *a compact manifold whose* k*-th Pontrjagin class* $[p_k] \neq 0$ *for some* $k > \frac{3}{4}(p+q-3)$*. Then, any analytic immersion* $f\colon M^n \to \mathbb{R}^{n+p}$ *(with the induced metric) is singularly genuinely rigid in* $\mathbb{R}^{n+q}$ *in the* C^∞*-category.*

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