Euclidean hypersurfaces with genuine deformations in codimension two

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1 Introduction

Isometrically deformable Euclidean hypersurfaces were classified at the beginning of last century by Sbrana [**Sb**] and Cartan [**Ca**], after earlier work by Schur [**Sc**] and Bianchi [**Bi**]. Apart from flat hypersurfaces, they have rank two, that is, exactly two nonzero principal curvatures everywhere, and are divided into four distinct classes. The two less interesting ones consist of ruled and surface-like hypersurfaces, that is, products with Euclidean factors of surfaces in $\mathbb{R}^3$ and cones over surfaces in the sphere $\mathbb{S}^3$. The main part of Sbrana's classification is a description of the remaining classes in terms of what is now called the Gauss parametrization (see [**DG**]). The latter allows to recover a hypersurface of constant rank by means of its Gauss image and its support function.

The Sbrana-Cartan theory was extended to hypersurfaces of the sphere and hyperbolic space in [**DFT**]. Moreover, some questions left over in the works of Sbrana and Cartan, as the possibility of smoothly attaching different types of hypersurfaces in the Sbrana-Cartan classification and the existence of hypersurfaces that admit a unique deformation, were also settled in [**DFT**].

The proper setting for attempting to extend the Sbrana-Cartan theory to submanifolds of higher codimension was developed in [$\mathbf{DF}_1$] by means of the concept of *genuine* isometric deformations of a submanifold. An isometric immersion $g\colon M^n \to \mathbb{R}^{n+q}$ of an n-dimensional Riemannian manifold M^n into Euclidean space with codimension q is said to be a *genuine isometric deformation* of a given isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ if f and g are nowhere (i.e., on no open subset of M^n) compositions, $f = F\circ j$ and $g = G\circ j$, of an isometric embedding $j\colon M^n \hookrightarrow N^{n+r}$ into a Riemannian manifold N^{n+r} with $r > 0$ and isometric immersions $F\colon N^{n+r} \to \mathbb{R}^{n+p}$ and $G\colon N^{n+r} \to \mathbb{R}^{n+q}$:

$$\begin{tikzcd}
 & & \mathbb{R}^{n+p} \\
M^n \arrow[urr, "f"] \arrow[drr, "g"'] \arrow[r, hook, "j"] & N^{n+r} \arrow[ur, "F"'] \arrow[dr, "G"] & \\
 & & \mathbb{R}^{n+q}
\end{tikzcd}$$

An isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ with the property that no open subset $U \subset M^n$ admits a genuine isometric deformation in $\mathbb{R}^{n+q}$ for a fixed $q > 0$ is said to be *genuinely rigid* in $\mathbb{R}^{n+q}$.

In particular, that an isometric immersion $g\colon M^n \to \mathbb{R}^{n+q}$ is a genuine isometric deformation of a hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$ means that g is nowhere a composition $g = H \circ f$, where $H\colon V \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+q}$ is an isometric immersion of an open subset V containing $f(M)$. In terms of this concept, the main result of [$\mathbf{DT}_1$] states that a necessary condition for a hypersurface f in $\mathbb{R}^{n+1}$ to admit a genuine isometric deformation in $\mathbb{R}^{n+q}$, $2 \le q \le n-2$, is that rank $f \le q+1$. It follows that f must have rank at most 3 in order to admit a genuine isometric deformation in $\mathbb{R}^{n+2}$.

In this paper, we give a complete local description, in terms of the Gauss parametrization, of rank two Euclidean hypersurfaces of dimension $n \ge 3$ that admit a genuine isometric deformation in $\mathbb{R}^{n+2}$. They are divided into three distinct classes. Again, ruled and surface-like hypersurfaces form the less interesting ones. On the other hand, the characterization of the Gauss images of hypersurfaces in the remaining class turns out to be significantly more involved than that in the Sbrana-Cartan theory.

Our work was in part motivated by the results in [$\mathbf{DF}_2$] on genuine rigidity of isometric immersions $g\colon M^n \to \mathbb{R}^{n+2}$ with rank two, which can be divided into three distinct classes, called *elliptic*, *parabolic* and *hyperbolic*. Any elliptic submanifold $g\colon M^n \to \mathbb{R}^{n+2}$ of rank two was shown in [$\mathbf{DF}_2$] to be *genuinely rigid* in $\mathbb{R}^{n+2}$, unless M^n admits an isometric immersion into $\mathbb{R}^{n+1}$. However, it was not clear whether there exists any elliptic submanifold of rank two in $\mathbb{R}^{n+2}$ that can be isometrically immersed into Euclidean space as a hypersurface. In fact, it was shown that this is never the case if g is minimal. Our result gives strong indication that such submanifolds do exist, although this remains an open problem.

2 The result

In order to state our result, we first recall the Gauss parametrization of Euclidean hypersurfaces with constant rank.

Let $f\colon M^n \to \mathbb{R}^{n+1}$ be an oriented hypersurface such that its Gauss map $N\colon M^n \to \mathbb{S}^n$ has constant rank k. Let $\pi\colon M^n \to L^k$ denote the projection onto the quotient space $L^k := M^n/\Delta$ of totally geodesic leaves of the relative nullity distribution of f defined by

$$\Delta(x) = \ker A(x).$$

Here $A = A_N$ stands for the shape operator of f, given by $AX = -\nabla_X N$ for any $X \in TM$. The quotient L^k is a differentiable manifold, except possibly for the Hausdorff condition, but this is always satisfied if we work locally.

Since the Gauss map is constant along the leaves of Δ, it induces an immersion $h\colon L^k \to \mathbb{S}^n$ given by $h \circ \pi = N$, which we also call the Gauss map of f. Accordingly, the support function $\tilde{\gamma} = \langle f, N\rangle$ of f gives rise to $\gamma \in C^\infty(L)$ defined by $\gamma \circ \pi = \tilde{\gamma}$.

The *Gauss parametrization* given in [**DG**] allows to locally recover f in terms of the pair (h, γ). Namely, there exists locally a diffeomorphism $\Phi\colon V \subset T_h^\perp L \to M^n$ of an open neighborhood of the zero section of the normal bundle $T_h^\perp L$ of h such that $\pi \circ \Phi = \hat{\pi}$, where $\hat{\pi}\colon T_h^\perp L \to L$ is the canonical projection, and

$$f \circ \Phi(y, w) = \gamma(y)h(y) + h_* \nabla\gamma(y) + w.$$

Next, we introduce the class of spherical surfaces that appear as Gauss images of hypersurfaces of rank two admitting genuine isometric deformations in codimension two.

Let $h\colon L^2 \to \mathbb{S}^n$ be a surface and let α_h denote its second fundamental form with values in its normal bundle. Assume $n \geq 4$ and that the first normal spaces N_h^1 of h, that is, the subspaces of the normal spaces spanned by the image of α_h, have dimension two everywhere. Then, given $x \in L$ and a basis $\{X, Y\}$ of T_xL, there exist $a, b, c \in \mathbb{R}$ such that

$$a\alpha_h(X, X) + 2c\alpha_h(X, Y) + b\alpha_h(Y, Y) = 0.$$

We say that h is *elliptic* (resp., *hyperbolic* or *parabolic*) if $ab - c^2 > 0$ (resp., < 0 or $= 0$) everywhere, a condition that is independent of the given basis.

This is easily seen to be equivalent to the existence of a tensor J on L^2 satisfying $J^2 = \epsilon I$, with $\epsilon = -1$, 1 or 0 respectively, and

$$\alpha_h(JX, Y) = \alpha_h(X, JY). \tag{1}$$

Moreover, in the first two cases the tensor J is unique up to sign. Note that (1) is equivalent to requiring any height function $h^v = \langle h, v\rangle$ of h to satisfy

$$(\mathrm{Hess}_{h^v} + h^v I)J = J^t(\mathrm{Hess}_{h^v} + h^v I),$$

where Hess denotes the Hessian with respect to metric induced by h, as an endomorphism of TL. We also use the same notation for the corresponding symmetric bilinear form.

When the first normal space of h has dimension less than two, we still call h elliptic, hyperbolic or parabolic with respect to such a tensor J if (1) is satisfied.

Now assume that h: $L^2 \to \mathbb{S}^n$ is hyperbolic. Let (u, v) be coordinates around $(0, 0)$ whose coordinate vector fields $\{\partial_u, \partial_v\}$ are eigenvectors of J. Then, condition (1) says that (u, v) are conjugate coordinates for h, that is,

$$\alpha_h(\partial_u, \partial_v) = 0.$$

Write

$$\nabla_{\partial_u}\partial_v = \Gamma^u\partial_u + \Gamma^v\partial_v,$$

and set $F = \langle\partial_u, \partial_v\rangle$. Then, any height function h_v of h belongs to the kernel of the differential operator

$$Q(\theta) := \mathrm{Hess}_\theta(\partial_u, \partial_v) + F\theta = \theta_{uv} - \Gamma^u\theta_u - \Gamma^v\theta_v + F\theta.$$

For each pair of smooth functions $U = U(u)$ and $V = V(v)$ define

$$\varphi^U = U(u)e^{2\int_0^v \Gamma^u(u,s)ds} \quad\text{and}\quad \psi^V = V(v)e^{2\int_0^u \Gamma^v(s,v)ds}.$$

In other words, the functions φ^U and ψ^V satisfy

$$\varphi^U_v = 2\Gamma^u\varphi^U \quad\text{and}\quad \psi^V_u = 2\Gamma^v\psi^V \tag{2}$$

as well as the initial conditions $\varphi^U(u, 0) = U(u)$ and $\psi^V(0, v) = V(v)$.

Assume, in addition, that one of the following conditions hold:

$$U, V > 0 \quad\text{or}\quad 0 < 2\varphi^U < -(2\psi^V + 1) \quad\text{or}\quad 0 < 2\psi^V < -(2\varphi^U + 1). \tag{3}$$

Define
$$\rho^{UV}=\sqrt{|2(\varphi^U+\psi^V)+1|}$$
and
$$\mathcal{C}_h=\{(U,V):(3)\text{ holds and }Q(\rho^{UV})=0\}.$$

Now assume that h: $L^2\to\mathbb{S}^n$ is elliptic. If (u,v) are coordinates on L^2 such that
$$\partial_z=\frac{1}{2}(\partial_u-\partial_v)\ \text{ and }\ \partial_{\bar z}=\frac{1}{2}(\partial_u+\partial_v)$$
are eigenvectors of the complex linear extension of J to $TL\otimes\mathbb{C}$, where $\{\partial_u,\partial_v\}$ is the frame of coordinate vector fields, then
$$\alpha_h(\partial_z,\partial_{\bar z})=0.$$

Define a complex–valued Christoffel symbol Γ by
$$\nabla_{\partial_z}\partial_{\bar z}=\Gamma\partial_z+\bar\Gamma\partial_{\bar z}$$
and set $F=\langle\partial_z,\partial_{\bar z}\rangle$, where $\langle\ ,\ \rangle$ also stands for the complex bilinear extension of the metric induced by h. In this case, the height functions of h belong to the kernel of the differential operator
$$Q(\theta):=\mathrm{Hess}\,_\theta(\partial_z,\partial_{\bar z})+F\theta=\theta_{z\bar z}-\Gamma\theta_z-\bar\Gamma\theta_{\bar z}+F\theta.$$

For each holomorphic function ζ, let $\varphi^\zeta(z,\bar z)$ be the unique complex valued function such that
$$\varphi^\zeta_{\bar z}=2\Gamma\varphi^\zeta\ \text{ and }\ \varphi^\zeta(z,0)=\zeta(z).$$

Assume further that
$$\varphi^\zeta\neq-1/2\ \text{ and }\ 4\mathrm{Re}(\varphi^\zeta)+1<0.\tag{4}$$

In this case, define
$$\rho^\zeta=\sqrt{-(4\mathrm{Re}(\varphi^\zeta)+1)}$$
and
$$\mathcal{C}_h=\{\zeta\text{ holomorphic}:(4)\text{ holds and }Q(\rho^\zeta)=0\}.$$

To state our main result, we first recall a few definitions. That an isometric immersion g: $M^n\to\mathbb{R}^{n+2}$ is *locally substantial* means that there exists no open subset $U\subset M^n$ such that $g(U)$ is contained in an affine hyperplane of $\mathbb{R}^{n+2}$. A hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$ is *ruled* if M^n admits a foliation by leaves of codimension one that are mapped by f into affine subspaces of $\mathbb{R}^{n+1}$. It is *surface like* if $f(M) \subset L^2 \times \mathbb{R}^{n-2}$ where L^2 is a surface in $\mathbb{R}^3$, or $f(M) \subset CL^2 \times \mathbb{R}^{n-3}$ where $CL^2 \subset \mathbb{R}^4$ is the cone over a surface $L^2 \subset \mathbb{S}^3$.

Theorem 1. *Let $f\colon M^n \to \mathbb{R}^{n+1}$ be a nowhere surface–like or ruled rank two hypersurface that admits a locally substantial genuine isometric deformation $g\colon M^n \to \mathbb{R}^{n+2}$. Then, on each connected component of an open dense subset, the Gauss map h of f is either an elliptic or hyperbolic surface, $\mathcal{C}_h$ is nonempty, and the support function γ of f satisfies $Q(\gamma) = 0$.*

Conversely, any simply connected hypersurface f whose Gauss map h and support function γ satisfy the above conditions admits genuine isometric deformations in $\mathbb{R}^{n+2}$, which are parametrized by the set $\mathcal{C}_h$.

Remarks 2. (i) Excluding surface–like and ruled hypersurfaces has the only purpose of emphasizing the really interesting class of hypersurfaces that admit substantial genuine isometric deformations in $\mathbb{R}^{n+2}$. In fact, it was shown in [$\mathbf{DF}_2$] that ruled hypersurfaces in $\mathbb{R}^{n+1}$ admit locally as many genuine isometric deformations in $\mathbb{R}^{n+2}$ as triples of smooth functions in one variable, all of them being also ruled with the same rulings. On the other hand, if f is surface–like then genuine deformations of f in $\mathbb{R}^{n+2}$ are given by genuine deformations of L^2 in either $\mathbb{R}^4$ or $\mathbb{S}^4$ (see Remark 17).

(ii) For $n \geq 4$ and h as in Theorem 1, it follows from Theorem 1 in [$\mathbf{DT}_2$] that if N_1^h is one-dimensional everywhere and $h(L)$ is not contained in a totally geodesic $\mathbb{S}^3 \subset \mathbb{S}^n$, then h carries a relative nullity distribution of rank one. Moreover, it is easily seen that in this case h is necessarily hyperbolic.

It follows from Theorem 1 that hypersurfaces as in the statement are rather special. In fact, if $n \geq 4$ then their Gauss maps h must have first normal spaces with dimension less than or equal to two everywhere, which is already a strong restriction for $n \geq 5$. Moreover, h can not be parabolic, and even if it is elliptic or hyperbolic the condition that the set $\mathcal{C}_h$ be nonempty occurs only in very special cases. Furthermore, one has the condition $Q(\gamma) = 0$ on the support function. Still, we show next that our result can be used to easily construct a family of hypersurfaces admitting a large set of genuine deformations in codimension two.

Example 3. Let us analyze the case where h has flat normal bundle, that is, (u, v) are real orthogonal conjugate coordinates. Setting $E = \langle \partial_u, \partial_u \rangle$ and $G = \langle \partial_v, \partial_v \rangle$, it follows from $F = \langle \partial_u, \partial_v \rangle = 0$ that

$$2\Gamma^u = E_v/E \quad \text{and} \quad 2\Gamma^v = G_u/G.$$

Hence $\varphi^U = UE$ and $\varphi^V = VG$, after replacing the smooth functions $U(u)$ and $V(v)$ if necessary. Moreover,

$$Q(\theta) = \theta_{uv} - \frac{E_v}{2E}\theta_u - \frac{G_u}{2G}\theta_v.$$

Assume further that (u, v) are also isothermic coordinates, say, $E = e^{2\lambda} = G$, so that h is an isothermic surface. Then $Q(\theta) = \theta_{uv} - \lambda_v \theta_u - \lambda_u \theta_v$. Hence, we have $\mathcal{C}_h \neq \emptyset$ if and only if there exist smooth functions $U(u)$ and $V(v)$ such that $\rho := \sqrt{(U+V)e^{2\lambda} + 1}$ satisfies $Q(\rho) = 0$, that is, $\rho_{uv} = \lambda_v \rho_u + \lambda_u \rho_v$. The latter equation is equivalent to

$$4(U+V)(1-e^{2\lambda}(U+V))\lambda_{uv} - 4e^{2\lambda}(U+V)^2\lambda_u\lambda_v + 2U'\lambda_v + 2V'\lambda_u - e^{2\lambda}U'V' = 0.$$

In addition, now suppose that $\lambda = \lambda(u)$. Then, these isothermic surfaces are precisely warped products of curves in the sense of [**No**], parametrized in isothermic coordinates (see the main theorem in [**No**]). Then, the preceding equation reduces to

$$V'(2\lambda_u - e^{2\lambda}U') = 0.$$

It follows that $\mathcal{C}_h$ is the set of pairs of smooth functions (U, V) satisfying

$$V = d \in \mathbb{R} \quad or \quad U = c - e^{-2\lambda} \text{ for } c \in \mathbb{R},$$

with the restrictions arising from (3). By Theorem 1, any simply connected hypersurface given in terms of the Gauss parametrization by a pair (h, γ), where h is a surface as above and $Q(\gamma) = \gamma_{uv} - \lambda_u \gamma_v = 0$, i.e., $\gamma = \nu e^{\lambda}$ for some smooth function $\nu(v)$, admits genuine isometric deformations in $\mathbb{R}^{n+2}$ which are parametrized by $\mathcal{C}_h$.

3 Outline of the proof

The starting point of the proof of Theorem 1 is the following consequence of Proposition 7 in [$\mathbf{DF}_2$].

Proposition 4. *Let $f\colon M^n \to \mathbb{R}^{n+1}$ be a rank two hypersurface. Then, any genuine isometric deformation of f in $\mathbb{R}^{n+2}$ has the same relative nullity distribution as f.*

The remaining of the proof is divided into four main steps. In the first one, we approach the problem in the light of the fundamental theorem of submanifolds. We show that the existence of a locally substantial isometric immersion $g\colon M^n \to \mathbb{R}^{n+2}$ with the same relative nullity distribution as f implies (and is implied by, if M^n is simply connected) the existence of a pair of tensors and a one-form satisfying several conditions that arise from the Gauss, Codazzi and Ricci equations for an isometric immersion into $\mathbb{R}^{n+2}$.

A key role in the proof is played by the splitting tensor of the relative nullity distribution Δ of f. Recall that the *splitting tensor* C of a totally geodesic distribution Δ on a Riemannian manifold M^n assigns to each $T \in \Delta$ the endomorphism C_T of $\Delta^\perp$ given by

$$C_T X = -(\nabla_X T)_{\Delta^\perp}.$$

When $\Delta^\perp$ has rank two, we say that Δ is *elliptic*, *hyperbolic* or *parabolic* if there exists a tensor $J\colon \Delta^\perp \to \Delta^\perp$ satisfying $J^2 = \epsilon I$, with $\epsilon = -1$, 1 or 0, respectively, such that the image $C(\Delta)$ of C at each point of M^n is not spanned by the identity tensor I but is contained in $\operatorname{span}\{I, J\}$. Accordingly, we call an Euclidean submanifold of rank two *elliptic*, *hyperbolic* or *parabolic* if its relative nullity distribution is elliptic, hyperbolic or parabolic, respectively.

Remark 5. Let $g\colon M^n \to \mathbb{R}^{n+p}$ be an isometric immersion of rank two of a Riemannian manifold without flat points. Assume that the first normal spaces of g have dimension two everywhere. As in the case of surfaces with first normal bundle of rank two, there exists a tensor $J\colon \Delta^\perp \to \Delta^\perp$, where Δ is the relative nullity distribution, satisfying $J^2 = \epsilon I$, with $\epsilon = -1$, 1 or 0, and $\alpha(JX, Y) = \alpha(X, JY)$. In [$\mathbf{DF}_1$] the immersion g was called accordingly *elliptic*, *hyperbolic* or *parabolic*, respectively, and it was shown that this is equivalent to the preceding definition.

Proposition 6. *Let $f\colon M^n \to \mathbb{R}^{n+1}$ be a rank two hypersurface that is nowhere surface-like or ruled. Assume that there exists a locally substantial isometric immersion $g\colon M^n \to \mathbb{R}^{n+2}$ that also has Δ as its relative nullity distribution. Then, on an open dense subset of M^n, the hypersurface f is* *either elliptic or hyperbolic and there exist a unique (up to signs and permutation) pair (D_1, D_2) of tensors in $\Delta^\perp$ contained in span$\{I, J\}$ and a unique one-form ϕ on M^n satisfying the following conditions:*

(i) $\Delta \subset \ker \phi$,

(ii) $AD_i = D_i^t A$,

(iii) $\det D_i = 1/2$,

(iv) $\nabla_T D_i = 0 = [D_i, C_T]$ *for all* $T \in \Delta$,

(v) $(\nabla_X AD_i)Y - (\nabla_Y AD_i)X = (-1)^j A(\phi(X)D_j Y - \phi(Y)D_j X),\ i \neq j$,

(vi) $d\phi(Z, T) = 0$ *for all* $Z \in TM$ *and* $T \in \Delta$,

(vii) $d\phi(X, Y) = \langle [AD_1, AD_2]X, Y\rangle$,

(viii) $D_2^2 \neq \pm D_1^2$.

Conversely, if M^n is simply connected and f is an elliptic or hyperbolic hypersurface that carries such a triple (D_1, D_2, ϕ), then there exists a locally substantial isometric immersion $g\colon M^n \to \mathbb{R}^{n+2}$ with Δ as its relative nullity distribution. Moreover, distinct triples (up to signs and permutation) yield noncongruent isometric immersions, and conversely.

The second step is to show that the problem of finding a triple (D_1, D_2, ϕ) satisfying all conditions in Proposition 6 can be reduced to a similar but easier problem for the Gauss map h of f.

Proposition 7. *Let $f\colon M^n \to \mathbb{R}^{n+1}$ be a hypersurface of rank two given in terms of the Gauss parametrization by a pair (h, γ). Let $\pi\colon M^n \to L^2$ be the projection and ∇' the Levi-Civita connection on L^2 for the metric $\langle\ ,\ \rangle'$ induced by h. If f is elliptic (resp., hyperbolic) and (D_1, D_2, ϕ) is a triple on M^n satisfying all conditions in Proposition 6, then there exist a unique tensor $\bar{J}$ on L^2 with $\bar{J}^2 = -I$ (resp., $\bar{J}^2 = I$), a unique pair of tensors $(\bar{D}_1, \bar{D}_2)$ in span$\{I, \bar{J}\}$, and a unique one-form $\bar{\phi}$ on L^2 such that h is elliptic (resp., hyperbolic) with respect to $\bar{J}$ and*

$$\bar{J} \circ \pi_* = \pi_* \circ J, \quad \bar{D}_i \circ \pi_* = \pi_* \circ D_i \quad \text{and} \quad \bar{\phi} \circ \pi_* = \phi.$$

Moreover,

$$(\mathrm{Hess}_\gamma + \gamma I)\bar{J} = \bar{J}^t\,(\mathrm{Hess}_\gamma + \gamma I) \tag{5}$$

and the triple $(\bar{D}_1, \bar{D}_2, \bar{\phi})$ satisfies:

(a) $\det \bar{D}_i = 1/2$,

(b) $(\nabla'_X \bar{D}_i)Y - (\nabla'_Y \bar{D}_i)X = (-1)^j(\bar{\phi}(X)\bar{D}_j Y - \bar{\phi}(Y)\bar{D}_j X),\ i \neq j$,

(c) $d\bar{\phi}(X,Y) = \langle [\bar{D}_1, \bar{D}_2]X, Y\rangle'$,

(d) $\bar{D}_2^2 \neq \pm \bar{D}_1^2$.

Conversely, if h is elliptic (resp., hyperbolic) then f is elliptic (resp., hyperbolic) and any such triple $(\bar{D}_1, \bar{D}_2, \bar{\phi})$ on L^2 gives rise to a unique triple (D_1, D_2, ϕ) on M^n satisfying all conditions in Proposition 6.

In the third step, we determine under which additional conditions the isometric deformation g of f in Proposition 6 is genuine.

Proposition 8. *Under the assumptions of Proposition 6, we have that g is a genuine isometric deformation of f if and only if rank $(D_1^2 + D_2^2 - I) = 2$. This is always the case if f is elliptic.*

The last and crucial step in the proof of Theorem 1 is to characterize the pairs (h, γ), where $h\colon L^2 \to \mathbb{S}^n$ is an elliptic or hyperbolic surface and $\gamma \in C^\infty(L)$, such that L^2 carries a triple $(\bar{D}_1, \bar{D}_2, \bar{\phi})$ satisfying all conditions in Proposition 7 and rank $(\bar{D}_1^2 + \bar{D}_2^2 - I) = 2$.

Proposition 9. *Let $h\colon L^2 \to \mathbb{S}^n$ be an elliptic or hyperbolic surface and let $\gamma \in C^\infty(L)$. Then there exists a triple $(\bar{D}_1, \bar{D}_2, \bar{\phi})$ satisfying all conditions in Proposition 7 and rank $(\bar{D}_1^2 + \bar{D}_2^2 - I) = 2$ if and only if $\mathcal{C}_h$ is nonempty and γ satisfies $Q(\gamma) = 0$. Distinct triples (up to signs and permutation) give rise to distinct elements of $\mathcal{C}_h$, and conversely.*

The proofs of Propositions 6 to 9 will be provided in the following sections. Theorem 1 follows easily by putting them together with Proposition 4.

Proof of Theorem 1: Under the assumptions of Theorem 1, it follows from Propositions 4, 6 and 8 that, on an open dense subset of M^n, the hypersurface f is elliptic or hyperbolic and there exists a unique (up to signs and permutation) triple (D_1, D_2, ϕ) satisfying all conditions in Proposition 6 as well as that in Proposition 8. Let f be locally given in terms of the Gauss parametrization by a pair (h, γ). By Proposition 7, the surface h is elliptic or hyperbolic, respectively, and the triple (D_1, D_2, ϕ) projects to a (unique) triple $(\bar{D}_1, \bar{D}_2, \bar{\phi})$ on L^2 satisfying all conditions in Proposition 7. Moreover, rank $(\bar{D}_1^2+\bar{D}_2^2-I) = \text{rank } (D_1^2+D_2^2-I) = 2$. We conclude from Proposition 9 that $(\bar{D}_1, \bar{D}_2, \bar{\phi})$ gives rise to a unique element of $\mathcal{C}_h$, and that $Q(\gamma) = 0$.

Conversely, assume that $f\colon M^n \to \mathbb{R}^{n+1}$ is a simply connected hypersurface given in terms of the Gauss parametrization by a pair (h, γ), where h is an elliptic or hyperbolic surface such that $\mathcal{C}_h$ is nonempty and $Q(\gamma) = 0$. By Proposition 9, each element of $\mathcal{C}_h$ gives rise to a unique triple $(\bar{D}_1, \bar{D}_2, \bar{\phi})$ on L^2 satisfying all conditions in Proposition 7 and rank $(\bar{D}_1^2 + \bar{D}_2^2 - I) = 2$. Then, it follows from Proposition 7 that f is elliptic or hyperbolic, respectively, and that $(\bar{D}_1, \bar{D}_2, \bar{\phi})$ can be lifted to a unique triple (D_1, D_2, ϕ) on M^n satisfying all conditions in Proposition 6. Moreover, rank $(D_1^2 + D_2^2 - I) = 2$. By Proposition 6, such triple yields a unique (up to rigid motions of $\mathbb{R}^{n+2}$) isometric immersion $g\colon M^n \to \mathbb{R}^{n+2}$ sharing with f the same relative nullity distribution. Proposition 8 implies that g is a genuine isometric deformation of f in $\mathbb{R}^{n+2}$.

Finally, we also have from Propositions 4 to 9 that (congruence classes of) genuine isometric deformations of f are in one-to-one correspondence with triples $(\bar{D}_1, \bar{D}_2, \bar{\phi})$ on L^2 satisfying all conditions in Proposition 7 and rank $(\bar{D}_1^2 + \bar{D}_2^2 - I) = 2$ (up to signs and permutation), and these are in turn in one-to-one correspondence with elements of $\mathcal{C}_h$. ∎

4 Projectable tensors and one-forms

In this section, we establish some facts that will be needed in the proof of Proposition 7 and also have interest on their own.

Given a submersion $\pi\colon M \to L$, a vector field X on M is said to be *projectable* if it is π-related to a vector field on L, that is, if there exists a vector field $\bar{X}$ on L such that $\pi_* X = \bar{X} \circ \pi$.

Proposition 10. *Let Δ be an integrable distribution on a differentiable manifold M, let $L = M/\Delta$ be the (local) quotient space of leaves of Δ and let $\pi\colon M \to L$ be the projection. Then, a vector field X on M is projectable if and only if $[X, T] \in \Delta$ for any $T \in \Delta$.*

Proof: If $\pi_* X = Z \circ \pi$ and $T \in \Delta$, then $\pi_*[X, T] = [\pi_* X, \pi_* T] = [Z, 0] \circ \pi = 0$, hence $[X, T] \in \Delta$. For the converse, in order to prove that X is projectable we must show that, for each leaf F of Δ, the map $\psi\colon F \to T_qL$, $q = \pi(F)$, given by $\psi(p) = \pi_*(p)X_p$, is constant. Given $p \in F$ and $v \in T_pF$, choose $T \in \Delta$ with $T(p) = v$ and let g_t be the flow of T. By the assumption and since $\pi \circ g_t = \pi$, we have

$$\begin{aligned} 0 = \pi_*[X,T](p) &= \lim_{t\mapsto 0} \frac{1}{t}(\pi_* X(g_t(p)) - \pi_* g_{t*} X(p)) \\ &= \lim_{t\mapsto 0} \frac{1}{t}(\pi_* X(g_t(p)) - \pi_* X(p)) = \psi_*(p)v. \ \blacksquare \end{aligned}$$

The conclusion of Proposition 10 when applied to a totally geodesic distribution Δ on a Riemannian manifold can be expressed in terms of its splitting tensor C.

Corollary 11. *Let Δ be a totally geodesic distribution on a Riemannian manifold M and let $L = M/\Delta$ be the (local) quotient space of leaves of Δ. Then, a vector field $X \in \Delta^\perp$ on M is projectable if and only if*

$$\nabla_T X + C_T X = 0 \quad \textit{for any } T \in \Delta.$$

Proof: For any $T \in \Delta$ we have

$$[X,T] = (\nabla_X T)_\Delta - C_T X - \nabla_T X.$$

Since Δ is totally geodesic, then $\nabla_T X \in \Delta^\perp$. Hence $[X,T] \in \Delta$ if and only if $\nabla_T X + C_T X = 0$, and the statement follows from Proposition 10. $\blacksquare$

If $\pi \colon M \to L$ is a submersion, we say that

(i) A one-form ω on M is *projectable* if there exist a one-form $\bar{\omega}$ on L such that $\bar{\omega} \circ \pi_* = \omega$.

(ii) A tensor D on M is *projectable* if there exists a tensor $\bar{D}$ on L such that $\bar{D} \circ \pi_* = \pi_* \circ D$.

Clearly, a one-form ω on M is projectable if and only if $\omega(X)$ is constant along the fibers of π for any projectable vector field X on M. Similarly, a tensor D on M is projectable if and only if DX is projectable for any projectable vector field X.

Corollary 12. *Let Δ be an integrable distribution Δ on a differentiable manifold M, let $L = M/\Delta$ be the (local) quotient space of leaves of Δ and let $\pi \colon M \to L$ be the quotient map. Then, a one-form ω on M is projectable if and only if $\omega(T) = 0$ and $d\omega(T,X) = 0$ for any $T \in \Delta$ and $X \in \Delta^\perp$.*

Proof: If $\omega = \bar{\omega} \circ \pi_*$, then $\omega(T) = \bar{\omega}(\pi_* T) = 0$. In order to prove that $d\omega(T, X) = 0$ we can assume that X is projectable. Then $T\omega(X) = 0$, hence

$$d\omega(T, X) = T\omega(X) - X\omega(T) - \omega([X, T]) = 0$$

where the vanishing of the last term follows from Proposition 10.

Conversely, if $X \in \Delta^\perp$ is projectable then $[X, T] \in \Delta$ by Proposition 10, hence the assumptions give

$$T\omega(X) = d\omega(T, X) + X\omega(T) + \omega([X, T]) = 0. \quad \blacksquare$$

Corollary 13. *Let Δ be a totally geodesic distribution on a Riemannian manifold M and let $L = M/\Delta$ be the (local) quotient space of leaves of Δ. Then a tensor field D on M is projectable if and only if*

$$\nabla_T D = [D, C_T] \ \textit{ for any } \ T \in \Delta.$$

Proof: We have

$$\nabla_T DX + C_T DX = (\nabla_T D - [D, C_T])\, X + D(\nabla_T X + C_T X). \tag{6}$$

If D and X are projectable then DX is also projectable. Thus, the preceding equality and Corollary 11 show that $\nabla_T D - [D, C_T]$ vanishes on projectable vector fields, hence it vanishes for this is a tensorial property.

Conversely, if $\nabla_T D = [D, C_T]$, then (6) and Corollary 11 imply that DX is projectable whenever X is projectable. $\blacksquare$

5 Proof of Proposition 6

Let A^g_ξ denote the shape operator of g with respect to $\xi \in T^\perp_g M$ given by

$$\langle A^g_\xi X, Y \rangle = \langle \alpha_g(X, Y), \xi \rangle.$$

Denote by $D_\xi \colon \Delta^\perp \to \Delta^\perp$ the endomorphism defined by

$$D_\xi = A^{-1} A^g_\xi,$$

where A and A^g_ξ are regarded as endomorphisms of $\Delta^\perp$.

Lemma 14. *The subspace of endomorphisms $W = span\{D_\xi : \xi \in T^\perp_g M\}$ has dimension two on an open dense subset of M^n.*

Proof: If W has dimension one at a point, then the first normal space N_1^g of g is also one-dimensional. If this happens on an open subset $V \subset M^n$ and N_1^g is not parallel in the normal connection along V, then V is flat by Theorem 1 in [**DT**$_2$]. But this contradicts the fact that f has rank two. In case N_1^g is parallel along V, then $g(V)$ is contained in a hyperplane of $\mathbb{R}^{n+2}$, again a contradiction with our assumption that g is locally substantial. ∎

Lemma 15. *The following holds:*

(i) $[D_\xi, C_T] = 0$ *for all* $T \in \Delta$*,*

(ii) $\nabla_T D_\xi = 0$ *for all* $T \in \Delta$ *and* $\xi \in T_g^\perp M$ *parallel along* Δ*.*

Proof: We obtain from the Codazzi equation that

$$\nabla_T A = AC_T. \tag{7}$$

Moreover,

$$\nabla_T AD_\xi = \nabla_T A_\xi^g = A_\xi^g C_T = AD_\xi C_T$$

if $\xi \in T_g^\perp M$ is parallel along Δ. An easy computation yields

$$\nabla_T AD_\xi - AD_\xi C_T = (\nabla_T A - AC_T)D_\xi + A(\nabla_T D_\xi - [D_\xi, C_T]), \tag{8}$$

hence

$$\nabla_T D_\xi = [D_\xi, C_T]. \tag{9}$$

On the other hand, we obtain from (7) that AC_T is symmetric, i.e.,

$$AC_T = C_T^t A.$$

A similar equation holds for $A_\xi^g = AD_\xi$, thus

$$AD_\xi C_T = A_\xi^g C_T = C_T^t A_\xi^g = C_T^t AD_\xi = AC_T D_\xi.$$

This gives (i), and then (ii) in view of (9). ∎

We now determine the structure of the splitting tensor C. We make use of the following well-known fact (cf. [**DFT**]).

Proposition 16. *Let* $f\colon M^n \to \mathbb{R}^{n+1}$ *be a hypersurface of rank two. If the image of the splitting tensor* C *is either trivial or spanned by the identity tensor* I*, then* f *is surface-like.*

Remark 17. Proposition 16 holds for submanifolds of any codimension. Therefore, since the splitting tensor C is intrinsic, when f is surface-like its genuine deformations in $\mathbb{R}^{n+2}$ are surface-like submanifolds over genuine deformations of L^2 in $\mathbb{R}^4$ or $\mathbb{S}^4$, respectively.

Lemma 18. *There exists a tensor J on $\Delta^\perp$ such that $J^2 = \epsilon I$, $\epsilon \in \{-1, 1, 0\}$ and span$\{I\} \subset C(\Delta) \subset$ span$\{I, J\} = W$.*

Proof: We have from Proposition 16 and our assumption that f is not surface-like that $C(\Delta)$ is not spanned by I on an open dense subset. By part (i) of Lemma 15, it is contained in the subspace S of linear operators on $\Delta^\perp$ that commute with all elements of the subspace W. Since W is two-dimensional by Lemma 14, it must contain I. Otherwise, it is easily seen that S would have to be the subspace spanned by I, in contradiction with the fact that S contains $C(\Delta)$. Therefore, $W = \mathrm{span}\{I, J\}$, where J is a tensor on $\Delta^\perp$ satisfying $J^2 = \epsilon I$, $\epsilon \in \{-1, 1, 0\}$. In particular, $W \subset S$ and, on the other hand, the fact that any element of S commutes with J implies that the dimension of S is at most two. Hence $W = S$ and $C(\Delta) \subset S = \mathrm{span}\,\{I, J\}$. ∎

Lemma 19. *There exists a unique (up to signs and permutation) orthonormal frame ξ_1, ξ_2 of $T_g^\perp M$ such that $D_i := D_{\xi_i}$, $1 \le i \le 2$, satisfy*

$$\det D_1 = 1/2 = \det D_2.$$

Moreover, $D_2^2 \neq -D_1^2$ and ξ_i, $1 \le i \le 2$, is parallel along Δ.

Proof: For any orthonormal frame $\{\xi_1, \xi_2\}$ of $T_g^\perp M$, the Gauss equations of f and g give

$$\det D_1 + \det D_2 = 1.$$

Since W has dimension two, we have $D_1 \neq \pm D_2$.

We now show that $D_2^2 \neq -D_1^2$. It is easily seen that $D_2^2 = -D_1^2$ could only happen if $W = \mathrm{span}\{I, J\}$ with $J^2 = -I$. Assume this to be the case. Let $\hat{D}_i$ denote the complex linear extension of D_i to $\Delta^\perp \otimes \mathbb{C}$. Then, there exists $\theta \in \mathbb{S}^1$ such that

$$\sqrt{2}\,\hat{D}_1 = \begin{bmatrix} \theta & 0 \\ 0 & \bar{\theta} \end{bmatrix} \quad \text{and} \quad \sqrt{2}\,\hat{D}_2 = \begin{bmatrix} i\theta & 0 \\ 0 & -i\bar{\theta} \end{bmatrix},$$

with respect to the frame of $\Delta^\perp \otimes \mathbb{C}$ of eigenvectors of $\hat{D}_i$, $1 \leq i \leq 2$. We obtain writing $\theta = e^{i\beta}$ that

$$\cos\beta D_1 - \sin\beta D_2 = I \quad \text{and} \quad \sin\beta D_1 + \cos\beta D_2 = J.$$

Hence, the orthonormal frame $\{\xi, \eta\}$ of $T_g^\perp M$ given by

$$\xi = \cos\beta\, \xi_1 - \sin\beta\, \xi_2 \quad \text{and} \quad \eta = \sin\beta\, \xi_1 + \cos\beta\, \xi_2$$

satisfies

$$\sqrt{2} A_\xi^g = A \quad \text{and} \quad \sqrt{2} A_\eta^g = AJ.$$

Using the preceding equations, and comparing the Codazzi equation of f with the Codazzi equation

$$(\nabla_X A_\xi^g)Y - (\nabla_Y A_\xi^g)X = \psi(X) A_\eta^g Y - \psi(Y) A_\eta^g X,$$

where $\psi(Z) = \langle \nabla_Z^\perp \xi, \eta \rangle$, we obtain that ψ vanishes identically. By the Ricci equation, this implies that $[A_\xi^g, A_\eta^g] = 0$, or equivalently, that $[A, AJ] = 0$. The latter equation can only be satisfied if $A = \lambda I$ for some $\lambda \in C^\infty(M)$. From $\nabla_T A = AC_T$ for any $T \in \Delta$, it follows that $C_T = \langle \text{grad}\, \lambda, T \rangle I$. In view of Proposition 16, this is in contradiction with the assumption that f is nowhere surface-like. Hence $D_2^2 \neq -D_1^2$.

Using the preceding condition, it is easily seen that there exists a unique (up to signs and permutation) pointwise choice of unit orthogonal vectors ξ_1 and ξ_2 such that $\det D_1 = 1/2 = \det D_2$, thus defining a smooth orthonormal normal frame with this property. We now show that ξ_1 and ξ_2 are parallel along Δ. Given $x \in M^n$, $T \in \Delta$ and an integral curve γ of T starting at x, let $\hat{\xi}_i(t)$ denote the parallel transport of $\xi_i(x)$ along γ at $\gamma(t)$. By Lemma 15-(ii), we have that $\nabla_{\gamma'(t)} D_{\hat{\xi}_i(t)} = 0$, hence $\det D_{\hat{\xi}_i(t)} = 1/2$. Since ξ_1 and ξ_2 are unique (up to signs and permutation) with this property, by continuity we must have $\hat{\xi}_i(t) = \xi_i(\gamma(t))$ for any t. It follows that $\nabla_T^\perp \xi_i = 0$ for any $T \in \Delta$, $1 \leq i \leq 2$. ∎

Lemma 20. *There is no open subset $U \subset M^n$ where $W = span\{I, J\}$ with $J^2 = 0$. Therefore, the hypersurface f is either elliptic or hyperbolic.*

Proof: It suffices to show that $f|_U$ is ruled for U as in the statement, and the proof follows from our assumption. Let $\{X, Y\}$ be an orthonormal frame of

$\Delta^\perp$ such that Y spans the image of J, that is, $JY=0$ and $JX=\lambda Y$ for some $\lambda\neq 0$. Since $C(\Delta)\subset W$, we have

$$\langle C_T Y, X\rangle = 0 \ \text{ for all } T\in\Delta. \tag{10}$$

It follows easily from the fact that AC_T is symmetric that

$$\langle AY, Y\rangle = 0. \tag{11}$$

We claim that the distribution $x\mapsto \mathrm{span}\{Y(x)\}\oplus\Delta(x)$ is totally geodesic. From (10) we have

$$\langle \nabla_Y T, X\rangle = -\langle C_T Y, X\rangle = 0, \ \text{ for all } T\in\Delta. \tag{12}$$

Now, let $\{\xi_1,\xi_2\}$ be the orthonormal normal frame given by Lemma 19. From $D_i = D_{\xi_i}\in \mathrm{span}\{I,J\}$ and $\det D_i = 1/2$ we obtain $\sqrt{2}D_iY = Y$, $1\leq i\leq 2$, after replacing ξ_i by $-\xi_i$ if necessary. Moreover, since $\sqrt{2}D_1$ and $\sqrt{2}D_2$ cannot be both multiples of I, we can assume that $\sqrt{2}D_1X\neq X$. On the other hand, it follows from Lemma 15-(ii) that $\sqrt{2}D_1\nabla_TY = \nabla_TY$ for all $T\in\Delta$, hence

$$\nabla_T Y = 0 \ \text{ for all } T\in\Delta. \tag{13}$$

Now write

$$A = \begin{bmatrix} \lambda & \mu \\ \mu & 0 \end{bmatrix} \quad \text{and} \quad \sqrt{2}A^g_{\xi_1} = \begin{bmatrix} \lambda+\theta & \mu \\ \mu & 0 \end{bmatrix}$$

with respect to the frame $\{X,Y\}$ for smooth functions λ,μ and $\theta\neq 0$ on M^n. The Codazzi equations for A and $A^g_{\xi_1}$ yield

$$Y(\mu) - \lambda\langle\nabla_YY,X\rangle - 2\mu\langle\nabla_XX,Y\rangle = 0$$

and

$$Y(\mu) - (\lambda+\theta)\langle\nabla_YY,X\rangle - 2\mu\langle\nabla_XX,Y\rangle = 0,$$

respectively. It follows that

$$\langle\nabla_YY,X\rangle = 0. \tag{14}$$

The claim follows from (12), (13), (14) and the fact that Δ is totally geodesic. In view of (11), this implies that $f|_U$ is ruled. ∎

To complete the proof of the direct statement of Proposition 6, for the orthonormal frame $\{\xi_1,\xi_2\}$ given by Lemma 19 set $\phi(Z)=\langle\nabla^\perp_Z\xi_1,\xi_2\rangle$. Then, the last assertion in Lemma 19 shows that condition (i) holds, and hence (iv) in view of Lemma 15. Condition (v) follows from the Codazzi equation for g, whereas (vi) and (vii) from the Ricci equation.

The proof of the converse statement is a straightforward application of the fundamental theorem of submanifolds. Choose an orthonormal frame $\{\xi_1, \xi_2\}$ of the trivial bundle $E = M \times \mathbb{R}^2$ and define a connection $\hat{\nabla}$ on E by $\langle \hat{\nabla}_X \xi_1, \xi_2 \rangle = \phi(X)$ for $X \in TM$. It follows from (i) that ξ_1 and ξ_2 are parallel along Δ. Let $\alpha\colon TM \times TM \to E$ be defined by setting $\ker \alpha = \Delta$ and

$$\alpha(X,Y) = \sum_{i=1}^{2} \langle AD_i X, Y \rangle \xi_i \ \text{ for all } \ X, Y \in \Delta^\perp.$$

Condition (ii) implies that α is symmetric, and from (iii) and the Gauss equation for f it satisfies the Gauss equation for an isometric immersion into $\mathbb{R}^{n+2}$. The Codazzi equation follows from (iv), (v) and (8), whereas the Ricci equation is a consequence of (vi) and (vii). By the fundamental theorem of submanifolds, there exists an isometric immersion $g\colon M^n \to \mathbb{R}^{n+2}$ having α as second fundamental form and $\hat{\nabla}$ as normal connection. Since $D_1 \neq \pm D_2$, it follows that the first normal spaces of g have dimension 2 everywhere, hence g is locally substantial.

Finally, the last assertion is a consequence of the uniqueness of the frame $\{\xi_1, \xi_2\}$ such that $\det D_1 = 1/2 = \det D_2$, together with the uniqueness part of the fundamental theorem of submanifolds.

6 Proof of Proposition 7

We start with the following lemma.

Lemma 21. *The tensors D_1, D_2 and the one-form ϕ are projectable with respect to the quotient map $\pi\colon M^n \to L^2 = M/\Delta$.*

Proof: That D_1 and D_2 are projectable follows from (iv) and Corollary 13. On the other hand, ϕ is projectable by (vi) and Corollary 12. ∎

Thus, there exist tensors $\bar{D}_i$, $1 \leq i \leq 2$, and a one-form $\bar{\phi}$ on L^2 such that $\bar{D}_i \circ \pi_* = \pi_* \circ D_i$ and $\bar{\phi} \circ \pi_* = \phi$. In particular, $[\bar{D}_1, \bar{D}_2] \circ \pi_* = \pi_* \circ [D_1, D_2] = 0$, hence there exists a unique tensor $\bar{J}$ on L^2 such that $\bar{J}^2 = \epsilon I$, $\epsilon \in \{1, -1\}$, and $\bar{D}_i \in \{I, \bar{J}\}$. Write $\bar{J} = a\bar{D}_1 + b\bar{D}_2$, $a, b \in \mathbb{R}$, and define $\hat{J} = aD_1 + bD_2$.

From $\bar{D}_i \circ \pi_* = \pi_* \circ D_i$ we obtain that $\hat{J} \circ \pi_* = \pi_* \circ \bar{J}$, hence $\hat{J}^2 = \epsilon I$, $\epsilon \in \{1, -1\}$. Since J is (up to sign) the unique tensor in $\operatorname{span}\{D_1, D_2\}$ with this property, it follows that $\hat{J} = J$, after a change of sign if necessary. Summarizing, we have proved the existence of a unique tensor $\bar{J}$ on L^2 such that $\hat{J} \circ \pi_* = \pi_* \circ \bar{J}$ and $\bar{D}_i \in \operatorname{span}\{I, \bar{J}\}$.

Conditions (a) and (d) are clear, for these properties are inherited from D_1 and D_2. In order to verify the remaining conditions we first make a few computations.

Let X and Y be projectable vector fields on M^n. Then, we have

$$f_* AX = -N_* X = -h_* \pi_* X, \tag{15}$$

$$f_* AD_i X = -h_* \pi_* D_i X = -h_* \bar{D}_i \pi_* X$$

and

$$\begin{aligned} f_* \nabla_X AD_i Y &= \nabla_X f_* AD_i Y - \langle AX, AD_i Y \rangle N \\ &= -\nabla_{\pi_* X} h_* \bar{D}_i \pi_* Y - \langle h_* \pi_* X, h_* \bar{D}_i \pi_* Y \rangle h \circ \pi \\ &= -h_* \nabla'_{\pi_* X} \bar{D}_i \pi_* Y - \alpha_h(\pi_* X, \bar{D}_i \pi_* Y). \end{aligned} \tag{16}$$

In view of (15) and (16), condition (v) implies (b) and

$$\alpha_h(\bar{D}_i X, Y) = \alpha_h(X, \bar{D}_i Y), \;\; 1 \leq i \leq 2, \tag{17}$$

which is equivalent to h being elliptic or hyperbolic with respect to $\bar{J}$. On the other hand, condition (vii) gives (c).

To prove (5), let $\hat{\pi}\colon T_h^\perp L \to L$ be the canonical projection. By the Gauss parametrization, there exists a local diffeomorphism $\Phi\colon U \subset T_h^\perp L \to M$ of an open neighborhood of the zero section such that $\pi \circ \Phi = \hat{\pi}$ and

$$\psi(x, w) := f \circ \Phi(x, w) = \gamma h + h_* \nabla \gamma + w.$$

For any horizontal vector $X \in T_{(x,w)}(T_h^\perp L)$ we have

$$\psi_* X = h_* P \hat{\pi}_* X + \alpha_h(\hat{\pi}_* X, \nabla \gamma),$$

where P is the endomorphism of TL given by

$$P = \operatorname{Hess}_\gamma + \gamma I - B_w. \tag{18}$$

Here B_w stands for the shape operator of h in direction w. Thus,

$$f_*\Phi_*X = h_*P\hat{\pi}_*X + \alpha_h(\hat{\pi}_*X, \nabla\gamma) = h_*P\pi_*\Phi_*X + \alpha_h(\pi_*\Phi_*X, \nabla\gamma),$$

and hence

$$-\langle AD_i\Phi_*X, \Phi_*Y\rangle = \langle h_*\bar{D}_i\pi_*\Phi_*X, h_*P\pi_*\Phi_*Y\rangle = \langle \bar{D}_i\hat{\pi}_*X, P\hat{\pi}_*Y\rangle' \quad (19)$$

for all horizontal vectors $X, Y \in T(T_h^\perp L)$. Therefore, condition (ii) implies that $P\bar{D}_i = \bar{D}_i^tP$. Since $B_w\bar{D}_i = \bar{D}_i^tB_w$ by (17), this gives (5).

We now prove the converse. Set $\omega = \bar{\omega}\circ\pi_*$ and let D_i be the horizontal lift of $\bar{D}_i$ to M^n, $1 \leq i \leq 2$, that is, $\Delta \subset \ker D_i$ and, for any $x \in M^n$ and $X \in \Delta^\perp(x)$, D_iX is the unique vector in $\Delta^\perp(x)$ such that $\pi_*D_iX = \bar{D}_i\pi_*X$. Define in a similar way a tensor J on $\Delta^\perp$ such that $\pi_*\circ J = \bar{J}\circ\pi_*$, so that $\mathrm{span}\{D_1, D_2\} = \mathrm{span}\{I, J\}$.

Conditions (i), (iii) and $(viii)$ are clear. Since $B_w\bar{J} = \bar{J}^tB_w$ for any $w \in T_h^\perp L$, for h is elliptic or hyperbolic with respect to $\bar{J}$, it follows from (5) that $P\bar{J} = \bar{J}^tP$, where P is given by (18). This implies that $P\bar{D}_i = \bar{D}_i^tP$ for $1 \leq i \leq 2$, hence $AD_i = D_i^tA$ by (19). This proves (ii).

It follows from Corollary 13 that $\nabla_TD_i = [C_T, D_i]$, hence (8) gives $\nabla_TAD_i = AD_iC_T$. This implies that AD_iC_T is symmetric, which is equivalent to $A[D_i, C_T] = 0$, bearing in mind (ii) and the fact that AC_T is symmetric. This proves (iv). Moreover, it implies that $C(\Delta) \subset \mathrm{span}\{I, J\}$, hence f is elliptic or hyperbolic, according as $J^2 = -I$ or I, that is, according as h is elliptic or hyperbolic, respectively.

Since (17) is satisfied, for h is elliptic or hyperbolic with respect to $\bar{J}$, using (15) and (16) and condition (b) we obtain (v). Finally, condition (vi) follows from Corollary 12 and (vii) is a consequence of (c), by using (15). ∎

7 Proof of Proposition 8

We make use of the following special case of Theorem 5 in [**DT**$_1$].

Proposition 22. *Let $f\colon M^n \to \mathbb{R}^{n+1}$ and $g\colon M^n \to \mathbb{R}^{n+2}$ be isometric immersions. If g is the composition $g = H\circ f$ of f with an isometric immersion $H\colon W \to \mathbb{R}^{n+2}$ of an open subset $W \supset f(M)$ of $\mathbb{R}^{n+1}$, then there exists an orthonormal frame $\{\xi, \eta\}$ of $T_g^\perp M$ such that $A_\xi^g = A^f$ and rank $A_\eta^g \leq 1$. The converse also holds if rank $A_\eta^g = 1$ everywhere and η is parallel along* $\ker A_\eta$.

For f and g as in Proposition 6, assume that there exist an open subset $U \subset M^n$ and an isometric immersion $H\colon W \to \mathbb{R}^{n+2}$ of an open subset $W \supset f(U)$ of $\mathbb{R}^{n+1}$ such that $g|_U = H \circ f|_U$. Then, by Proposition 22 there exists $\theta \in (0, 2\pi)$ such that

$$D_\theta := \cos\theta D_1 + \sin\theta D_2 = I \text{ and rank } (D_{\theta+\pi/2} := -\sin\theta D_1 + \cos\theta D_2) < 2.$$

Since this can never happen if $D_1, D_2 \subset \{I, J\}$ with $J^2 = -I$, the last assertion is proved.

From now on assume that $D_1, D_2 \subset \{I, J\}$ with $J^2 = I$. Let $\{X, Y\}$ be a frame of $\Delta^\perp$ of eigenvectors of J, say,

$$\sqrt{2}\, D_1 = \begin{bmatrix} \theta_1 & 0 \\ 0 & 1/\theta_1 \end{bmatrix} \quad \text{and} \quad \sqrt{2}\, D_2 = \begin{bmatrix} \theta_2 & 0 \\ 0 & 1/\theta_2 \end{bmatrix}, \quad \theta^2 \neq \pm\theta^1.$$

Then, it is easily checked that $a_1 D_1 + a_2 D_2 = I$ if and only if

$$a_i = \frac{\sqrt{2}\theta_i(1-\theta_j^2)}{\theta_i^2 - \theta_j^2}, \ 1 \leq i \neq j \leq 2.$$

Moreover, for these values of a_1 and a_2, the rank of $-a_2 D_1 + a D_2$ is less than two if and only if either $\theta_1^2 + \theta_2^2 = 2$ or $\theta_1^{-2} + \theta_2^{-2} = 2$, that is, if and only if rank $D_1^2 + D_2^2 - I < 2$. Note that $\sqrt{2}a_i = \theta_i$ if $\theta_1^2 + \theta_2^2 = 2$, whereas $\sqrt{2}a_i = 1/\theta_i$ if $\theta_1^{-2} + \theta_2^{-2} = 2$. In either case we have $a_1^2 + a_2^2 = 1$.

Summarizing, there exists an orthonormal frame $\{\xi, \eta\}$ of $T_g^\perp M$ such that $A_\xi^g = A^f$ and rank $A_\eta^g \leq 1$ if and only if rank $D_1^2 + D_2^2 - I < 2$. This already shows the "if" part of the statement.

Let us prove the converse. Assume, say, that $\theta_1^2 + \theta_2^2 = 2$ on an open subset $U \subset M^n$. Then $\sqrt{2}(\theta_1 D_1 + \theta_2 D_2) = I$ and X belongs to the kernel of $-\theta_2 D_1 + \theta_1 D_2$. Therefore, the orthonormal frame $\{\xi, \eta\}$ of $T_g^\perp M$ given by

$$\sqrt{2}\, \xi = \theta_1 \xi_1 + \theta_2 \xi_2 \ \text{ and } \ \sqrt{2}\, \eta = -\theta_2 \xi_1 + \theta_1 \xi_2$$

satisfies $A_\xi^g = A$ and rank $A_\eta^g = 1$. By Proposition 22, in order to conclude that $g|_U$ is a composition $g|_U = H \circ f|_U$, where $H\colon W \to \mathbb{R}^{n+2}$ is an isometric immersion of an open subset $W \supset f(U)$, we must still show that η is parallel along ker A_η^g, that is, that $\nabla_X^\perp \eta = 0$. The latter condition is equivalent to

$$X(\theta_1) = \theta_2 \phi(X) \tag{20}$$

bearing in mind that $\theta_1^2+\theta_2^2=2$.

In order to prove (20) it is easier to work on the quotient space L^2. First note that Proposition 6-(iv) implies that θ_1 and θ_2 are constant along the leaves of Δ, hence give rise to functions on L^2 that we also denote by θ_1 and θ_2. Moreover, these are also the eigenvalues of $\bar{D}_1$ and $\bar{D}_2$. Let (u,v) be coordinates in L^2 whose coordinate vector fields are eigenvectors of $\bar{D}_i$, i.e.,

$$\sqrt{2}\,\bar{D}_1=\begin{bmatrix}\theta_1 & 0\\ 0 & 1/\theta_1\end{bmatrix}\quad\text{and}\quad\sqrt{2}\,\bar{D}_2=\begin{bmatrix}\theta_2 & 0\\ 0 & 1/\theta_2\end{bmatrix},\tag{21}$$

with respect to the frame $\{\partial_u,\partial_v\}$ of coordinate vector fields. Then, equation (20) is equivalent to

$$(\theta_1)_u=\phi^u\theta_2.\tag{22}$$

The equation in Proposition 7-(b) can be written as

$$\nabla_{\partial_u}\bar{D}_i\partial_v-\nabla_{\partial_v}\bar{D}_i\partial_u=(-1)^j(\bar{\phi}^u\bar{D}_j\partial_v-\bar{\phi}^v\bar{D}_j\partial_u),\ \ 1\le i\ne j\le 2,$$

where $\bar{\phi}^u=\bar{\phi}(\partial_u)$ and $\bar{\phi}^v=\bar{\phi}(\partial_v)$. This is equivalent to

$$\frac{\theta^i_u}{(\theta^i)^2}+\left(\theta^i-\frac{1}{\theta^i}\right)\Gamma^v=(-1)^i\frac{\phi^u}{\theta^j}$$

and

$$\theta^i_v+\left(\theta^i-\frac{1}{\theta^i}\right)\Gamma^u=(-1)^j\theta^j\phi^v,\ \ 1\le i\ne j\le 2,$$

where we write $\Gamma^u=\Gamma^u_{uv}$ and $\Gamma^v=\Gamma^v_{uv}$ for simplicity. In terms of $\tau^i:=(\theta^i)^2$, the preceding equations become

$$\left(\frac{1}{\tau^i}\right)_u+2\left(\frac{1}{\tau^i}-1\right)\Gamma^v=2(-1)^j\frac{\phi^u}{\theta_1\theta_2}\tag{23}$$

and

$$\tau^i_v+2(\tau^i-1)\Gamma^u=2(-1)^j\phi^v\theta_1\theta_2,\ \ 1\le i\ne j\le 2.\tag{24}$$

Equation (22) takes the form

$$(\tau_1)_u=2\phi^u\theta_1\theta_2,$$

and we must show that it is satisfied if $\tau_1+\tau_2=2$. We obtain from (23) for $i=1$ that

$$2\phi^u\theta_1\theta_2=-\frac{\tau_2}{\tau_1}(\tau_1)_u+2(2-\tau_1)(1-\tau_1)\Gamma^v.\tag{25}$$

On the other hand, equation (23) for $i = 2$ gives

$$2\phi^u \theta_1 \theta_2 = -\frac{\tau_1}{\tau_2}(\tau_1)_u + 2\tau_1(1-\tau_1)\Gamma^v. \tag{26}$$

It follows from (25) and (26) that

$$\frac{(\tau_1)_u}{\tau_2} = \tau_1(1-\tau_2)\Gamma^v.$$

Replacing into (26) yields

$$2\phi^u \theta_1 \theta_2 = -\frac{\tau_1}{\tau_2}(\tau_1)_u + 2\frac{(\tau_1)_u}{\tau_2} = \frac{2-\tau_2}{\tau_1}(\tau_1)_u = (\tau_1)_u. \quad \blacksquare$$

8 Proof of Proposition 9

In order to prove the direct statement, we consider separately the hyperbolic and elliptic cases.

8.1 The hyperbolic case

As in the proof of Proposition 8, let (u, v) be conjugate coordinates on L^2 such that $\bar{D}_1$ and $\bar{D}_2$ are given by (21) with respect to the frame $\{\partial_u, \partial_v\}$ of coordinate vector fields. As pointed out in the proof of Proposition 8, condition (b) can be written as (23) and (24) in terms of $\tau^i := (\theta^i)^2$, $1 \le i \le 2$. On the other hand, (c) takes the form

$$2(\phi^v_u - \phi^u_v) = \frac{\tau^1 - \tau^2}{\theta^1\theta^2} F. \tag{27}$$

It follows from (23) and (24) that

$$\left(\frac{1}{\tau^1} + \frac{1}{\tau^2}\right)_u + 2\left(\frac{1}{\tau^1} + \frac{1}{\tau^2} - 2\right)\Gamma^v = 0$$

and

$$(\tau^1 + \tau^2)_v + 2(\tau^1 + \tau^2 - 2)\Gamma^u = 0.$$

In terms of

$$\alpha = \tau_1 + \tau_2 \ \text{ and } \ \beta = 1/\tau_1 + 1/\tau_2$$

the preceding equations can be written as

$$\beta_u + 2(\beta - 2)\Gamma^v = 0 \text{ and } \alpha_v + 2(\alpha - 2)\Gamma^u = 0. \tag{28}$$

Notice that $\alpha, \beta > 0$. Moreover, since τ_1 and τ_2 are distinct real roots of $\tau^2 - \alpha\tau + (\alpha/\beta) = 0$, it follows that $\alpha\beta > 4$ and that τ_1 and τ_2 can be recovered from α and β by

$$2\tau_i = \alpha - (-1)^i \sqrt{(\alpha/\beta)(\alpha\beta - 4)}, \quad 1 \le i \le 2. \tag{29}$$

Since $\bar{D}_1$ and $\bar{D}_2$ satisfy the condition in Proposition 8, we have that $\alpha \neq 2$ and $\beta \neq 2$. Then, we can define

$$\varphi = 1/(\alpha - 2) \text{ and } \psi = 1/(\beta - 2).$$

From $\alpha > 0$, $\beta > 0$ and $\alpha\beta - 4 > 0$, it follows that (φ, ψ) satisfies (3). Moreover, we obtain from (28) that

$$\varphi_v/\varphi = 2\Gamma^u \text{ and } \psi_u/\psi = 2\Gamma^v.$$

Set

$$\rho := \sqrt{|2(\varphi + \psi) + 1|} = \sqrt{\alpha\beta - 4}/\sqrt{|(\alpha - 2)(\beta - 2)|}.$$

Writing ϕ^u, ϕ^v and the τ^i in terms of α and β by means of (23), (24) and (29), and replacing into (27), a rather long but straightforward computation shows that it reduces to $Q(\rho) = 0$. Thus, the set $\mathcal{C}_h$ is nonempty. Moreover, (5) reduces to

$$\text{Hess}_\gamma(\partial_u, \partial_v) + F\gamma = 0,$$

that is, to $Q(\gamma) = 0$. Finally, distinct triples $(\bar{D}_1, \bar{D}_2, \bar{\phi})$ (up to signs and permutation of $\bar{D}_1$ and $\bar{D}_2$) give rise to distinct 4-tuples $(\tau^1, \tau^2, \phi^u, \phi^v)$, and hence to distinct pairs (φ, ψ).

8.2 The elliptic case

Assume that (u, v) are coordinates on L^2 such that the frame $\{\partial_z, \partial_{\bar{z}}\}$ defined by

$$\partial_z = \frac{1}{2}(\partial_u - i\partial_v) \text{ and } \partial_{\bar{z}} = \frac{1}{2}(\partial_u + i\partial_v),$$

in terms of the frame $\{\partial_u, \partial_v\}$ of coordinate vector fields, are eigenvectors of the complex linear extension of $\bar{J}$ to $TL \otimes \mathbb{C}$. Write the complex linear extensions of $\bar{D}_i$ as

$$\sqrt{2}\,\bar{D}_1 = \begin{bmatrix} \theta^1 & 0 \\ 0 & \bar{\theta}^1 \end{bmatrix}, \quad \sqrt{2}\,\bar{D}_2 = \begin{bmatrix} \theta^2 & 0 \\ 0 & \bar{\theta}^2 \end{bmatrix}, \quad |\theta^1| = 1 = |\theta^2|,\ \theta^2 \neq \pm\theta^1, \tag{30}$$

with respect to the frame $\{\partial_z, \partial_{\bar{z}}\}$. Define a complex valued Christoffel symbol Γ by

$$\nabla_{\partial_z}\partial_{\bar{z}} = \Gamma\partial_z + \bar{\Gamma}\partial_{\bar{z}}$$

and set $\phi^z = \phi(\partial_z)$. As in the hyperbolic case, define $\tau^i = (\theta^i)^2$, $1 \leq i \leq 2$. Then, the complex versions of (23) and (24) are

$$\tau^i_{\bar{z}} + 2(\tau^i - 1)\Gamma = 2(-1)^j \bar{\phi}^z\theta^1\theta^2, \tag{31}$$

whereas (27) becomes

$$4\mathrm{Im}\,(\phi^z)_{\bar{z}} = i(\tau^1 - \tau^2)\bar{\theta}^1\bar{\theta}^2 F. \tag{32}$$

Notice that, since $\bar{\theta}^i = 1/\theta^i$ these are the same equations as in the hyperbolic case with (u, v), (ϕ^u, ϕ^v) and (Γ^u, Γ^v) replaced by $(z, \bar{z})$, $(\phi^z, \bar{\phi}^z)$ and $(\Gamma, \bar{\Gamma})$, respectively. Set $\alpha = \tau^1 + \tau^2$ as before. We obtain from (31) that

$$\alpha_{\bar{z}} + 2(\alpha - 2)\Gamma = 0.$$

Note that $|\alpha| < 2$, since $|\tau^1| = 1 = |\tau^2|$ and $\tau^1 \neq \tau^2$. Thus $\varphi = 1/(\alpha - 2)$ is well-defined and satisfies

$$\frac{\varphi_{\bar{z}}}{\varphi} = 2\Gamma.$$

From $|\alpha| < 2$ and

$$4\mathrm{Re}(\varphi) + 1 = \frac{|\alpha|^2 - 4}{|\alpha - 2|^2}$$

it follows that $4\mathrm{Re}(\varphi) + 1 < 0$. On the other hand, condition (d) implies that $\tau^2 \neq -\tau^1$, hence $\alpha \neq 0$. This gives $\varphi \neq -1/2$, thus conditions (4) are satisfied. Observe that, since $\alpha \neq 0$, we can recover τ^1 and τ^2 from α by

$$\tau^i = \frac{\alpha}{2}\left(1 \pm i\frac{\sqrt{4 - |\alpha|^2}}{|\alpha|}\right). \tag{33}$$

Now set

$$\rho = \sqrt{-(4\mathrm{Re}(\varphi) + 1)} = \frac{\sqrt{4 - |\alpha|^2}}{|\alpha - 2|}.$$

As before, writing ϕ^z and the τ^i in terms of α by means of (31) and (33), and replacing into (32), we arrive at the equation $Q(\rho) = 0$. This shows that the set $\mathcal{C}_h$ is nonempty. Finally, (5) reduces in this case to

$$\mathrm{Hess}_\gamma(\partial_z, \partial_{\bar{z}}) + F\gamma = 0,$$

that is, to $Q(\gamma) = 0$. Again, distinct triples $(\bar{D}_1, \bar{D}_2, \bar{\phi})$ (up to signs and permutation of $\bar{D}_1$ and $\bar{D}_2$) yield distinct triples (τ^1, τ^2, ϕ^z), and hence distinct φ's.

8.3 Proof of the converse statement

We argue first in the hyperbolic case. Let (φ, ψ) be a pair of smooth functions on L^2 satisfying (2) and (3). Assume also that $\rho = \sqrt{|2(\varphi + \psi) + 1|}$ satisfies $Q(\rho) = 0$. Set

$$\alpha = 2 + 1/\varphi \ \text{ and } \ \beta = 2 + 1/\psi.$$

Since (φ, ψ) satisfies (3), it follows that $\alpha > 0$, $\beta > 0$ and $\alpha\beta - 4 > 0$. Then, we can define τ^i by (29). Let ϕ^u and ϕ^v be given by (23) and (24), respectively. It follows from $Q(\rho) = 0$ that (27) is satisfied. Write $\tau^i = (\theta^i)^2$, let $\bar{D}_1$ and $\bar{D}_2$ be defined by (21) with respect to the frame $\{\partial_u, \partial_v\}$ of coordinate vector fields, and set $\bar{\phi} = \phi^u\, du + \phi^v\, dv$. Then condition (a) is clear, whereas (b) follows from (23) and (24). Condition (c) is a consequence of (27), (5) follows from $Q(\gamma) = 0$ and (d) is automatic in this case. It is also clear that distinct pairs (φ, ψ) give rise to distinct 4-tuples $(\tau^1, \tau^2, \phi^u, \phi^v)$, and hence to distinct triples $(\bar{D}_1, \bar{D}_2, \bar{\phi})$.

Assume now that h elliptic with complex conjugate coordinates $(z, \bar{z})$. Suppose that φ satisfies (4) and $\rho := \sqrt{-(4\mathrm{Re}(\varphi) + 1)}$ satisfies $Q(\rho) = 0$. Set

$$\alpha = 2 + 1/\varphi.$$

It follows from (4) that $\alpha \neq 0$ and $|\alpha| < 2$. Then (33) gives τ^i, $1 \leq i \leq 2$, with $\tau^2 \neq \pm\tau^1$, $|\tau^1| = 1 = |\tau^2|$ and $\tau^1 + \tau^2 = \alpha$. Write $\tau^i = (\theta^i)^2$ and define a triple $(\bar{D}_1, \bar{D}_2, \bar{\phi})$ by requiring that $\bar{D}_1$ and $\bar{D}_2$ be given by (30) with respect to the frame $\{\partial_z, \partial_{\bar{z}}\}$, and $\bar{\phi} = \phi^z\, dz + \bar{\phi}^z\, d\bar{z}$ where ϕ^z is given by (31). Then, condition (a) is immediate and (b) follows from (31). From $Q(\rho) = 0$ we obtain (32), and hence condition (c). Finally, (5) follows from $Q(\gamma) = 0$, and (d) holds because $\tau^2 \neq -\tau^1$. Again, distinct φ's yield distinct triples $(\bar{D}_1, \bar{D}_2, \bar{\phi})$. ∎

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