

On conformally flat submanifolds

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§1 Introduction

An immersed submanifold $f: M^n \rightarrow \mathbf{R}^{n+p}$ in standard flat Euclidean space, endowed with the induced metric, is said to be *conformally flat* if each point has a neighborhood conformal to \mathbf{R}^n . Around 1919, nonflat conformally flat hypersurfaces ($p = 1, n \geq 4$) were completely described by E. Cartan ([**Ca**₂]) as being any envelope of a 1-parameter family of spheres. In this case, the geometric parametric description (see [**CY**], [**CDM**], [**AD**₂] or [**Da**]) is an immediate consequence of the existence, at any point, of a principal curvature of multiplicity at least $n - 1$.

For higher but still low codimension, namely, $p \leq n - 3$, the algebraic structure of the second fundamental form of a conformally flat submanifold has also been well understood for some time due to the work of Moore ([**Mo**₁]; see also [**MM**]). His extension of Cartan's result implies that generically, M^n carries a $(n-p)$ -dimensional foliation by extrinsic spheres which immerse in \mathbf{R}^{n+p} as round spheres.

It is well known ([**AD**₁], [**Da**]) that a simply connected Riemannian manifold $M^n, n \geq 3$, is conformally flat if and only if it can be realized as a hypersurface of the light cone \mathbf{V}^{n+1} of the standard flat Lorentzian space \mathbf{L}^{n+2} . Recall that

$$\mathbf{V}^{n+1} = \{X \in \mathbf{L}^{n+2}: \langle X, X \rangle = 0, X \neq 0\}.$$

Hence, in order to obtain examples of conformally flat submanifolds M^n of \mathbf{R}^{n+p} , it suffices to produce a Riemannian manifold N^{n+1} which admits isometric immersions $F: N^{n+1} \rightarrow \mathbf{R}^{n+p}$ and $G: N^{n+1} \rightarrow \mathbf{L}^{n+2}$ and then take M^n as the intersection $G(N^{n+1}) \cap \mathbf{V}^{n+1}$.

Our first main result is that, for $p \leq n - 3$, the above procedure generates all simply connected examples. In particular, Moore’s spherical foliation is nothing else but the intersection with \mathbf{V}^{n+1} of the at least $(n - p + 1)$ -dimensional relative nullity foliation common to F and G .

The remaining part of the paper is devoted to the classification of all local conformally flat submanifolds in codimension $p = 2$. In fact, this goal is achieved by two different means. Our first approach consists in putting together the above result with a description of all Riemannian manifolds N^m which can be realized, simultaneously, as hypersurfaces in \mathbf{R}^{m+1} and \mathbf{L}^{m+1} .

This last result is of independent interest and has other consequences. On one hand, it allows us to construct explicit counterexamples to the claims in [No₁] and [No₂]. More importantly, it reveals a completely unexpected strong relation with the classical Sbrana–Cartan theory ([Sb], [Ca₁]) of isometrically deformable Euclidean hypersurfaces. It turns out that in order to admit an isometric immersion in \mathbf{L}^{m+1} , a nonflat hypersurface N^m of \mathbf{R}^{m+1} is either in one of three (out of five) classes of deformable Sbrana–Cartan hypersurfaces or has a similar structure as the elements of a fourth class. As a consequence, in codimension 2, we show that any ‘generic’ conformally flat submanifold either has as many isometric deformations as a certain surface in \mathbf{R}^3 or in the sphere \mathbf{S}^3 , admits precisely a 1-parameter family of deformations, or is isometrically rigid.

Making use of our previous results, we are then able to describe all nonflat codimension 2 conformally flat submanifolds in a parametric form. This second approach turns out to be much more involved than Cartan’s description because, aside from hypersurfaces, the existence alone of a spherical foliation as above is far from sufficient to conclude conformal flatness.

Our parametrization and an observation due to Cartan then enable us to explicitly construct a large family of what seems to be the first known generic examples. Finally, in order to complete our classification, we extend the parametrization of flat surfaces in \mathbf{R}^4 obtained in [CD] to flat n -dimensional submanifolds of \mathbf{R}^{n+2} .

§2 The general case

We first recall some basic facts and definitions. Given an isometric immersion $F: N^m \rightarrow \mathbf{R}^{m+p}$, the *relative nullity subspace* $\Delta(x) \subset T_x N$ at $x \in N^m$ is defined as

$$\Delta(x) = \{X \in T_x N: \alpha_F(X, Y) = 0, \forall Y \in T_x N\},$$

where $\alpha_F: TN \times TN \rightarrow T_F^\perp N$ stands for the vector valued second fundamental form. It is a standard fact that on any open subset where the *index of relative nullity* $\nu_F(x) := \dim \Delta(x)$ is constant, the relative nullity distribution is integrable and its leaves are totally geodesic in N^m and \mathbf{R}^{m+p} .

Consider a hypersurface $F: N^m \rightarrow \mathbf{R}^{m+1}$ with constant index $\nu_F = \ell$, $0 \leq \ell \leq m - 1$. In this situation, we may locally parametrize F by means of the *Gauss parametrization* which we briefly describe next for later use and refer to [DG₁] for further details.

Let $V^{m-\ell}$ be the quotient space of relative nullity leaves in an open subset $U \subset N^m$ with projection $\pi: U \rightarrow V^{m-\ell}$. The Gauss map $\xi: U \rightarrow \mathbf{S}^m(1)$ induces an isometric (with the induced metric) immersion $h: V^{m-\ell} \rightarrow \mathbf{S}^m(1)$ so that $h \circ \pi = \xi$. Let \mathcal{N} denote the normal bundle of h in $\mathbf{S}^m(1) \subset \mathbf{R}^{m+1}$ and let γ be the “support function” defined by $\gamma \circ \pi = \langle F, \xi \rangle$. The Gauss parametrization $\Psi: \mathcal{N} \rightarrow \mathbf{R}^{m+1}$ is given by

$$\Psi(\vartheta) = \gamma(x)h(x) + \text{grad } \gamma(x) + \vartheta, \quad x = \pi(\vartheta),$$

where we fiberwise identify the affine relative nullity bundle over a cross section with the vector bundle \mathcal{N} by parallel transport in Euclidean space.

All of the above easily extends to isometric immersions $G: N^m \rightarrow \mathbf{L}^{m+p}$. In particular, the Gauss parametrization is now given in terms of a submanifold $k: V^{m-\ell} \rightarrow \mathbf{H}^m(-1) \subset \mathbf{L}^{m+1}$ in hyperbolic space and takes the form

$$\Upsilon(\vartheta) = \rho k - \text{grad } \rho + \vartheta,$$

where $\rho \circ \pi = -\langle G, \xi \rangle$.

When $f: M^n \rightarrow \mathbf{R}^{n+p}$, $p \leq n - 3$, is a conformally flat submanifold, from the work of Moore ([Mo₁]; see also the proof below) we know that at each point there is an *umbilical* subspace $\mathcal{U}(x) \subset T_x M$ with $\dim \mathcal{U}(x) \geq n - p$.

Hence, there is a unit vector $\eta \in T_{f(x)}^\perp M$ and $\mathbf{R} \ni \lambda \geq 0$, so that the second fundamental form satisfies

$$\alpha_f(Z, X) = \lambda \langle Z, X \rangle \eta, \quad \forall Z \in \mathcal{U}(x), \quad \forall X \in \widetilde{T}_x M. \quad (1)$$

It is a well known fact (c.f. [Ki]) that the umbilical distribution \mathcal{U} is integrable on any open subset where the *index of conformal nullity* $\nu_f^c(x) := \dim \mathcal{U}(x)$ is constant, and its leaves are extrinsic spheres in M^n and part of round spheres in \mathbf{R}^{n+p} . Recall that an *extrinsic sphere* Σ of a Riemannian manifold M^n is an umbilical submanifold with parallel mean curvature vector such that the sectional curvature of M^n is constant along planes tangent to Σ .

We say that an isometric immersion $F: N^{n+1} \rightarrow \widetilde{N}^{n+p}$ extends an isometric immersion $f: M^n \rightarrow \widetilde{N}^{n+p}$ when there exists an isometric embedding of M^n into N^{n+1} such that $F|_M = f$.

Theorem 1. *Let $f: M^n \rightarrow \mathbf{R}^{n+p}$, $n \geq 5$, $p \leq n - 3$, be a simply connected conformally flat submanifold without flat points. If f has constant index of conformal nullity $\nu_f^c = \ell$, then there exist an extension $F: N^{n+1} \rightarrow \mathbf{R}^{n+p}$ of f and an isometric immersion $G: N^{n+1} \rightarrow \mathbf{L}^{n+2}$ so that $M^n = G(N^{n+1}) \cap \mathbf{V}^{n+1}$. Moreover, F and G carry a common $(\ell + 1)$ -dimensional relative nullity foliation.*

Proof: We first sketch (see [AD₁] or [Da] for details) how one produces the isometric immersion $g: M^n \rightarrow \mathbf{V}^{n+1} \subset \mathbf{L}^{n+2}$ since G , in the statement, will be an extension of g in \mathbf{L}^{n+2} . Take the trivial Lorentzian vector bundle $M^n \times \mathbf{L}^2$ over M^n endowed with the compatible fiber connection which makes a canonical orthonormal basis $\{\xi, \nu\}$ parallel. Here $\|\xi\| = -1$. Now consider the symmetric bilinear form $\alpha: TM \times TM \rightarrow \mathbf{L}^2$ given by

$$\alpha(X, Y) = - \left(L(X, Y) - \frac{1}{2} \langle X, Y \rangle \right) \xi + \left(L(X, Y) + \frac{1}{2} \langle X, Y \rangle \right) \nu,$$

where L is the tensor defined as

$$L(X, Y) = \frac{1}{n-2} \left(\text{Ric}(X, Y) - \frac{1}{2} n s \langle X, Y \rangle \right)$$

and s stands for the scalar curvature. Then α verifies the fundamental Gauss, Codazzi and Ricci equations. In fact, the Gauss equations are equivalent to

the vanishing of the conformal curvature tensor (Weyl tensor), the Codazzi equations are equivalent to L being a Codazzi tensor and the Ricci equations are trivially verified. We obtain from the Lorentzian version of the fundamental theorem for submanifolds that there exists an isometric immersion of M^n into \mathbf{L}^{n+2} with second fundamental form α . Finally, in order to conclude that the immersion lies in the light cone \mathbf{V}^{n+1} , one has to use that the normal vector field $\xi - \nu$ is umbilic, parallel and light-like, i.e., $\|\xi - \nu\| = 0$.

At each point, let $\beta: TM \times TM \rightarrow \mathbf{L}^2 \oplus T_f^\perp M$ be the symmetric bilinear form defined by

$$\beta(X, Y) = \left(L(X, Y) + \frac{1}{2}\langle X, Y \rangle \right) \xi + \left(L(X, Y) - \frac{1}{2}\langle X, Y \rangle \right) \nu + \alpha_f(X, Y).$$

Since β is flat and $\beta(X, X) \neq 0$ for all $X \neq 0$, by Proposition 2 of [Mo₁] (or Lemma 7.12 of [Da]) there exist a subspace $\mathcal{U} \subset TM$, a real valued bilinear form $\phi: TM \times TM \rightarrow \mathbf{R}$ and a light-like vector $e = \xi + a\nu + b\eta$, where $\eta \in T_f^\perp M$ and $\|\eta\| = 1$, such that:

$$i) \dim \mathcal{U} \geq n - p,$$

$$ii) \beta(Z, X) = \phi(Z, X)e, \quad \forall Z \in \mathcal{U}, X \in TM.$$

We conclude from $\|e\| = 0$ and $ii)$ above by a straightforward computation that there exists $\mu \in T_g^\perp M$, $\|\mu\| = 1$, such that

$$\alpha_f(Z, X) = \lambda\langle Z, X \rangle \eta, \quad \alpha_g(Z, X) = \lambda\langle Z, X \rangle \mu, \quad (2)$$

for all $Z \in \mathcal{U}, X \in TM$.

We claim for the tangent valued second fundamental forms of f and g that

$$A_\eta^f = A_\mu^g. \quad (3)$$

Recall that a Riemannian manifold M^n is conformally flat if and only if its sectional curvature K_M satisfies

$$K_M(X, Y) + K_M(Z, W) = K_M(X, W) + K_M(Z, Y) \quad (4)$$

for any orthonormal set of vectors. Using this and our assumption on M^n , it follows easily that $\lambda \neq 0$ in equations (2). On the other hand, we get from the Gauss equations,

$$\lambda\langle \alpha_f(X, Y), \eta \rangle = \lambda\langle \alpha_g(X, Y), \mu \rangle, \quad \forall X, Y \in \mathcal{U}^\perp,$$

and the claim follows.

From the assumption that \mathcal{U} has constant dimension, we have that all of the above data can be taken to be smooth. Let us denote by ∇ and $\widetilde{\nabla}$ the Riemannian connections in the manifold and the ambient spaces, respectively. For any vector field $Z \in \mathcal{U}$, we have

$$\widetilde{\nabla}_Z Z = \nabla_Z Z + \lambda \langle Z, Z \rangle \eta. \quad (5)$$

It follows from (5) that

$$L^f = \text{span}\{(\widetilde{\nabla}_Z Z)_{\mathcal{U}^\perp \oplus T_f^\perp M}, \forall Z \in \mathcal{U}\}$$

is a smooth line bundle over M^n . Moreover, $L^f \not\subset TM$ because $\lambda \neq 0$.

Since η is parallel along \mathcal{U} in the normal connection, the Codazzi equation for any normal vector field $\delta \perp \eta$ takes the form

$$\nabla_Z A_\delta^f X - A_\delta^f \nabla_Z X + A_{\nabla_X^\perp \delta}^f Z + A_\delta^f \nabla_X Z = 0, \quad \forall Z \in \mathcal{U}.$$

Taking inner product with Z , we easily get

$$\langle \widetilde{\nabla}_Z Z, \widetilde{\nabla}_X \delta \rangle = 0.$$

Since $\langle \widetilde{\nabla}_Z Z, \delta \rangle = 0$, we conclude that

$$\widetilde{\nabla}_X \widetilde{\nabla}_Z Z \in TM \oplus \text{span}\{\eta\}, \quad \forall X \in TM.$$

For any unit vector $Z \in \mathcal{U}$, we have from (5) and the above that

$$\widetilde{\nabla}_X \widetilde{\nabla}_Z Z = \nabla_X \nabla_Z Z - \lambda A_\eta^f X + \langle A_\eta^f \nabla_Z Z + \text{grad } \lambda, X \rangle \eta,$$

from where we easily obtain that

$$\widetilde{\nabla}_X (\widetilde{\nabla}_Z Z)_{\mathcal{U}^\perp} = \nabla_X (\nabla_Z Z)_{\mathcal{U}^\perp} - \lambda A_\eta^f X + \langle A_\eta^f (\nabla_Z Z)_{\mathcal{U}^\perp} + \text{grad } \lambda, X \rangle \eta. \quad (6)$$

Let L^g be the line bundle over M^n similarly defined as L^f and denote by $\tau: L^f \rightarrow L^g$ the obvious bundle isometry. Then, the maps $F: L^f \rightarrow \mathbf{R}^{n+p}$ and $G: L^f \rightarrow \mathbf{L}^{n+2}$ defined as

$$F(\zeta) = f(x) + \zeta, \quad G(\zeta) = g(x) + \tau(\zeta), \quad x = \pi(\zeta),$$

are immersions when restricted to a tubular neighborhood N^{n+1} of the zero section. A similar calculation shows that equation (6) also holds for g when η is replaced by μ . We easily conclude from (3) and (6) that F and G induce the same metric.

Notice that the maps F and G have been produced by “replacing” the extrinsic spheres by the affine subspaces of one dimension higher which contain them. This is so because the fibers of L^f and L^g are spanned by the mean curvature vectors of the umbilical foliation. We now show that these affine subspaces are contained in the relative nullity distributions. We have from (6) that the normal bundles to F and G are, respectively, the orthogonal complements to η and μ in the normal bundles of f and g . Being η and μ parallel in the normal connection along \mathcal{U} , we easily see that their orthogonal complements are constant along \mathcal{U} . Hence, the normal bundles of F and G must be constant along the above affine subspaces. This concludes the proof of the theorem. ■

Remark 2. It may well happen that M^n in Theorem 1 carries a foliation by k -dimensional extrinsic spheres with $k > \ell$. If this is the case, we have for the indices of relative nullity that $\nu_F = \ell + 1$ but, as we show below, $\nu_G = k + 1$. See also Proposition 9 for further understanding of this situation.

From Theorem 1, any conformally flat hypersurface in \mathbf{R}^{n+1} without flat points may always be obtained as the intersection of a flat hypersurface in \mathbf{L}^{n+2} with the light cone. In fact, we have the following general result.

Theorem 3. *Any simply connected conformally flat manifold M^n foliated by extrinsic spheres of dimension $\ell \geq 3$ can be realized as the intersection $G(N^{n+1}) \cap \mathbf{V}^{n+1}$, where $G: N^{n+1} \rightarrow \mathbf{L}^{n+2}$ is an isometric immersion with constant index of relative nullity $\nu_G = \ell + 1$. Conversely, any conformally flat manifold, obtained by an intersection as above, carries a foliation by ℓ -dimensional extrinsic spheres.*

Proof: We first prove the converse. In terms of a Gauss parametrization

$$\Upsilon(\vartheta) = \rho k - \text{grad } \rho + \vartheta$$

for G , the intersection $g: M^n \rightarrow \mathbf{V}^{n+1} \subset \mathbf{L}^{n+2}$ of $G(N^{n+1})$ with \mathbf{V}^{n+1} can be parametrized as

$$g(w) = \rho k - \text{grad } \rho + \sqrt{\rho^2 - \|\text{grad } \rho\|^2} w = \rho(k - \mu) \quad (7)$$

along the unit normal bundle \mathcal{N}_1 of k in $\mathbf{H}^{n+1}(-1) \subset \mathbf{L}^{n+2}$.

All we have to show is that each fiber of \mathcal{N}_1 , already part of an umbilical sphere in \mathbf{L}^{n+1} , is an extrinsic sphere in M^n . Taking derivatives of $\langle g, g \rangle = 0$, we see that $g \in T_g^\perp M$. Thus, $T_g^\perp M = \text{span}\{k, g\}$. Hence, the pair $\{k, \mu\}$ is an orthonormal basis for $T_g^\perp M$ where μ (defined by (7)) is normal to M^n in N^{n+1} . Since k and ρ are constant along the leaves, we have for any Z tangent to a leaf that

$$g_*Z = -\rho\widetilde{\nabla}_Z\mu,$$

which concludes the proof in one direction.

To prove the direct statement, let \mathcal{U} denote the distribution tangent to the extrinsic spheres. Conditions (4) and $\ell \geq 3$ yield

$$K_M(X, Y) = K_M(X, Z), \quad \forall Y, Z \in \mathcal{U}(x), \quad \forall X \in \mathcal{U}^\perp(x).$$

For the tensor L , this implies that

$$L(Y, Z) = \kappa\langle Y, Z \rangle, \quad \forall Y, Z \in \mathcal{U}, \quad \kappa \in C^\infty(M).$$

Moreover, using that L is a Codazzi tensor, it follows easily that κ must be constant along each extrinsic sphere. Therefore, the second fundamental form of $g: M^n \rightarrow \mathbf{V}^{n+1} \subset \mathbf{L}^{n+2}$ satisfies

$$\alpha_g(Y, Z) = \langle Y, Z \rangle \left(\left(\frac{1}{2} - \kappa \right) \xi + \left(\frac{1}{2} + \kappa \right) \eta \right) = \langle Y, Z \rangle \delta.$$

Hence, each extrinsic sphere is umbilic in \mathbf{L}^{n+2} . By the same argument that the one in the proof of Theorem 1, we now conclude that extension G of g as required exists. Finally, observe that the condition $\nu_G = \ell + 1$ follows from the first half of the proof. ■

§3 Hypersurfaces of \mathbf{R}^N and \mathbf{L}^N

Our classification of Riemannian manifolds which can be simultaneously realized as hypersurfaces in both, \mathbf{R}^N and \mathbf{L}^N , makes use of a special class of spherical and hyperbolic surfaces which we describe next.

For a spherical surface $h: V^2 \rightarrow \mathbf{S}^m(1) \subset \mathbf{R}^{m+1}$, $m \geq 2$, together with a global coordinate system (u, v) , we denote by Γ^1, Γ^2 , the two connection functions of the Riemannian connection ∇' of V^2 determined by

$$\nabla'_{\partial_u} \partial_v = \Gamma^1 \partial_u + \Gamma^2 \partial_v,$$

where ∂_u, ∂_v stand for the coordinate vector fields. Notice that h is just a coordinate system when $m = 2$. We say that the coordinates (u, v) are *conjugate* whenever the second fundamental form of h satisfies

$$\alpha_h(\partial_u, \partial_v) = 0. \quad (8)$$

In terms of the coordinate functions (h^1, \dots, h^{m+1}) of h in \mathbf{R}^{m+1} , equation (8) takes the form

$$\text{Hess}_{h^j}(\partial_u, \partial_v) + \langle \partial_u, \partial_v \rangle h^j = 0, \quad 1 \leq j \leq m+1. \quad (9)$$

We call an *associated function* to a given spherical surface with conjugate coordinates $\{h, (u, v)\}$ any negative solution τ of the system of equations

$$\begin{cases} \tau_u = 2\Gamma^2\tau(1-\tau) \\ \tau_v = 2\Gamma^1(1-\tau). \end{cases} \quad (10)$$

The integrability condition for system (10) turns out to be

$$(\Gamma_v^2 - 2\Gamma^1\Gamma^2)\tau - \Gamma_u^1 + 2\Gamma^1\Gamma^2 = 0.$$

We then say that $\{h, (u, v)\}$ is a surface of *first type* when its metric satisfies

$$\Gamma_u^1 = \Gamma_v^2 = 2\Gamma^1\Gamma^2. \quad (11)$$

Surfaces of first type posses a 1-parameter family of associated functions. In fact, as already observed in [Sb] and [Ca₁], equations (11) can be easily integrated. We get

$$\Gamma^1 = \frac{V'}{2(U-V)}, \quad \Gamma^2 = \frac{-U'}{2(U-V)}, \quad (12)$$

where $U = U(u)$ and $V = V(v)$ are smooth functions of one of the coordinates. Then (10) and (12) yield for the associated functions

$$\tau = \frac{c + V(v)}{c + U(u)},$$

where $c \in \mathbf{R}$ has to be chosen so that τ is negative.

When $\{h, (u, v)\}$ is not of first type, we call it of *second type* if

$$\tau = \frac{\Gamma_u^1 - 2\Gamma^1\Gamma^2}{\Gamma_v^2 - 2\Gamma^1\Gamma^2} \quad (13)$$

is a (necessarily unique) associated function.

Remark 4. In their classification of isometrically deformable Euclidean hypersurfaces without flat points, Sbrana ([**Sb**]) and Cartan ([**Ca**₁]) considered two classes of spherical surfaces (called by Sbrana of first and second species) which carry either real or complex conjugate coordinates. While surfaces of first type are nothing else but surfaces of first species for real conjugate coordinates, surfaces of second type are not of second species but of a similar kind. Namely, τ given by (13) is still a solution of system (10) but it is a positive one.

The notion of spherical surfaces of first or second type extends immediately to surfaces $k: V^2 \rightarrow \mathbf{H}^m(-1) \subset \mathbf{L}^{m+1}$ in hyperbolic space. In this case, equation (9) takes the form

$$\text{Hess}_{k^j}(\partial_u, \partial_v) - \langle \partial_u, \partial_v \rangle k^j = 0, \quad 1 \leq j \leq m+1.$$

The following is our main result in this section.

Theorem 5. *Let N^m , $m \geq 2$, be a Riemannian manifold without flat points and let $F: N^m \rightarrow \mathbf{R}^{m+1}$ and $G: N^m \rightarrow \mathbf{L}^{m+1}$ be isometric immersions. Then, F is locally given by the Gauss parametrization $\Psi: \mathcal{N} \rightarrow \mathbf{R}^{m+1}$,*

$$\Psi(\vartheta) = \gamma h + \text{grad} \gamma + \vartheta,$$

in terms of a surface of first or second type $\{h, (u, v)\}$ and a solution γ of the differential equation

$$\text{Hess}_\gamma(\partial_u, \partial_v) + \langle \partial_u, \partial_v \rangle \gamma = 0. \quad (14)$$

Conversely, any parametrized hypersurface in \mathbf{R}^{m+1} as above can be locally isometrically immersed in \mathbf{L}^{m+1} . A similar description holds for G .

Proof: From Corollary 2 of [Mo₂] and our assumption, we have everywhere for the relative nullity distributions and their indices that $\Delta_F = \Delta_G$ and $\nu_F = m - 2 = \nu_G$. Define $D: \Delta^\perp \rightarrow \Delta^\perp$ by

$$D = (A^F)^{-1} \circ A^G, \quad (15)$$

where A^F (respectively, A^G) denotes the second fundamental form of F (respectively, G) restricted to Δ^\perp . From the Gauss equations,

$$\det D = -1.$$

Hence, for a basis $\{X_1, X_2\}$ of eigenvectors D takes the form

$$D = \begin{bmatrix} \theta & 0 \\ 0 & -1/\theta \end{bmatrix}. \quad (16)$$

In particular, from

$$\langle A^G X_1, X_2 \rangle = \langle A^F D X_1, X_2 \rangle = \theta \langle A^F X_1, X_2 \rangle = -\theta^2 \langle A^G X_1, X_2 \rangle,$$

we conclude that X_1, X_2 are conjugate for F and G , i.e.,

$$\langle A^F X_1, X_2 \rangle = 0 = \langle A^G X_1, X_2 \rangle.$$

We argue for F . We have to find necessary and sufficient conditions for the existence of a tensor field D as in (16) so that $A^F \circ D$ satisfies the Codazzi equations for conjugate X_1, X_2 . First recall that the *splitting tensor* C assigns to each $T \in \Delta$ the endomorphism C_T of Δ^\perp given by

$$C_T X = -(\nabla_X T)_{\Delta^\perp}.$$

From the Codazzi equation, we get

$$\nabla_T A^F = A^F \circ C_T, \quad \forall T \in \Delta. \quad (17)$$

Being the term on the left hand side symmetric, we have

$$A^F \circ C_T = C_T^* \circ A^F, \quad (18)$$

where C_T^* denotes the adjoint operator of C_T . Similarly, $A^F \circ D$ also has to satisfy (18). Thus,

$$A^F D C_T = C_T^* A^F D = A^F C_T D.$$

Therefore,

$$[D, C_T] = 0, \quad \forall T \in \Delta. \quad (19)$$

On the other hand, equation (17) for $A^F \circ D$ yields

$$A^F DC_T X = (\nabla_T A^F) DX + A^F (\nabla_T D) X, \quad \forall X \in \Delta^\perp, \forall T \in \Delta.$$

Using (17), we get

$$A^F [D, C_T] X = A^F (\nabla_T D) X, \quad \forall X \in \Delta^\perp, \forall T \in \Delta.$$

We conclude from (19) and the above that

$$\nabla_T D = 0, \quad \forall T \in \Delta. \quad (20)$$

Conversely, equations (19) and (20) imply that the Codazzi equation for $A^F \circ D$ combining vectors in Δ and Δ^\perp is satisfied.

Observe that equations (19) and (20) can be substituted by

$$T(\theta) = 0, \quad \forall T \in \Delta, \quad (21)$$

and

$$\nabla_T X_j \in \text{span}\{X_j\}, \quad \forall T \in \Delta, \quad 1 \leq j \leq 2, \quad (22)$$

where (21) says that θ is a function on V^2 . Recall that V^2 is the quotient space of relative nullity leaves and that $h: V^2 \rightarrow \mathbf{S}^m(1)$ an immersion induced by the Gauss map ξ . Moreover, condition (22) means that there exists a local coordinate system (u, v) on V^2 such that

$$\partial_u \circ \pi = \pi_* X_1, \quad \partial_v \circ \pi = \pi_* X_2,$$

for X_1, X_2 of appropriated length. In particular, this implies that $[X_1, X_2] \in \Delta$. Thus, the Codazzi equation we still have to consider reduces to

$$\nabla_{X_1}(A^F DX_2) = \nabla_{X_2}(A^F DX_1). \quad (23)$$

We have,

$$\begin{aligned} \nabla_{X_2}(A^F DX_1) &= \nabla_{X_2}(A^F \theta X_1) = \widetilde{\nabla}_{X_2}(A^F \theta X_1) - \theta \langle A^F X_1, A^F X_2 \rangle \xi \\ &= -\widetilde{\nabla}_{X_2} \xi_* \theta X_1 - \theta \langle \xi_* X_1, \xi_* X_2 \rangle \xi = -\widetilde{\nabla}_{\partial_v} \theta \partial_u + \theta \langle \partial_u, \partial_v \rangle \xi \\ &= -\theta_v \partial_u - \theta (\nabla_{\partial_u} \partial_v + \alpha_h(\partial_u, \partial_v)). \end{aligned}$$

Analogously,

$$\nabla_{X_1}(A^F DX_2) = -\frac{\theta_u}{\theta^2}\partial_v + \frac{1}{\theta}(\nabla_{\partial_u}\partial_v + \alpha_h(\partial_u, \partial_v)).$$

Setting $\tau = -\theta^2$, we easily see that equation (23) reduces to conditions (8) and (10).

It remains to find the condition for X_1, X_2 to be conjugate. Recall that in terms of the Gauss parametrization,

$$A^F(\vartheta) = -[\text{Hess}_\gamma + \gamma\text{Id} - B_\vartheta]^{-1},$$

where Hess_γ stands for the linear operator associated to the Hessian of γ and B_ϑ denotes the second fundamental form of h in direction ϑ . Hence, in the presence of (8), we have

$$\langle A^F X_1, X_2 \rangle = 0 \iff \text{Hess}_\gamma(\partial_u, \partial_v) + \langle \partial_u, \partial_v \rangle \gamma = 0. \quad (24)$$

The statement for F follows from (8), (10) and (24), and a similar argument takes care of G . The converse is straightforward. ■

Remarks 6. 1) For G , equation (14) becomes

$$\text{Hess}_\rho(\partial_u, \partial_v) - \langle \partial_u, \partial_v \rangle \rho = 0. \quad (25)$$

2) The intersection of $G(N^{m+1})$ with a foliation of \mathbf{L}^{m+2} by light cones provides a local foliation of N^{m+1} by m -dimensional conformally flat submanifolds of \mathbf{R}^{m+2} . See also Theorem 1.10 of [AD₁].

3) The following special examples satisfy the assumptions of the above result:

- i)* $N^m = N^2 \times \mathbf{R}^{m-2}$, where N^2 admits isometric immersions $F': N^2 \rightarrow \mathbf{R}^3$ and $G': N^2 \rightarrow \mathbf{L}^3$, where $F = F' \times \text{Id}$ and $G = G' \times \text{Id}$;
- ii)* $N^m = CN^2 \times \mathbf{R}^{m-3}$, where $CN^2 \cong N^2 \times \mathbf{R}_+$ is a cone over a surface N^2 which admits isometric immersions $F': N^2 \rightarrow \mathbf{S}^3(1) \subset \mathbf{R}^4$ and $G': N^2 \rightarrow \mathbf{S}_1^3(1) \subset \mathbf{L}^4$ into Lorentzian unit sphere, where $F = CF' \times \text{Id}$ and $G = CG' \times \text{Id}$.

We now introduce two new definitions in order to deal with the rigidity question. We call a conformally flat submanifold $f: M^n \rightarrow \mathbf{R}^{n+2}$, $n \geq 5$, *generic* when its umbilical direction $\eta \in T_f^\perp M$ (recall (1)) possesses everywhere a nonzero principal curvature λ of multiplicity $n - 2$. Notice that $\nu_f^c = n - 2$. From (4) and the Gauss equations, we easily get that generic conformally flat submanifolds cannot have flat points. We say that a generic f is *surface-like* if its isometric extension $F: N^{n+1} \rightarrow \mathbf{R}^{n+2}$ is as either one of the examples in Remark 6.3.

Theorem 7. *Any local isometric deformation of a generic conformally flat submanifold $f: M^n \rightarrow \mathbf{R}^{n+2}$, $n \geq 5$, is the restriction to M^n of an isometric deformation of its isometric extension $F: N^{n+1} \rightarrow \mathbf{R}^{n+2}$. Moreover, if nowhere surface-like, f admits, precisely, a 1-parameter family of isometric deformations when F is generated by a surface of first type, and is isometrically rigid otherwise. In the surface-like situation, all deformations of f are determined by isometric deformations of the surface in the first factor of F .*

Proof: The first statement is a consequence of Theorem 5 in [DG₂]. The remaining part follows from the classical Sbrana–Cartan classification of deformable Euclidean hypersurfaces ([Sb], [Ca₁]). In fact, they proved that isometric deformations of hypersurfaces splitting a surface factor or a cone of a spherical surface are given by deformations of the surface. Moreover, they also showed that hypersurfaces given by the Gauss parametrizations in terms of a surface of first species (see Remark 4) and a support function satisfying equation (14), which do not split a factor as above, admit a 1-parameter family of isometric deformations. Finally, unless it splits a factor, F does not belong to any class of deformable hypersurfaces when generated by a surface of second type. ■

Given a conformally flat submanifold $f: M^n \rightarrow \mathbf{R}^{n+2}$ without flat points and constant index $\nu_f^c = n - 2$, it is easy to conclude from the result in [MM] that M^n possesses two foliations by conformally flat hypersurfaces each leaf having constant $\nu^c = n - 2$ in \mathbf{R}^{n+2} . Moreover, the foliation by extrinsic spheres is generate by the intersection of the two foliations. In addition, the foliations are orthogonal if and only if f has flat normal bundle. Next, we present two families of examples showing that the converse to this observation, claimed in [No₁] and [No₂], does not hold.

Examples 8. 1) Consider a surface $g': N^2 \rightarrow \mathbf{L}^3$ which does not admit any local isometric immersion into \mathbf{R}^3 . The intersection $g: N^2 \times \mathbf{S}^{n-2}(1) \rightarrow \mathbf{L}^{n+2}$ of $G = g' \times \text{Id}: N^2 \times \mathbf{R}^{n-1} \rightarrow \mathbf{L}^{n+2}$ with \mathbf{V}^{n+1} , has the form

$$g(x, v) = g'(x) + \phi(x)v$$

where $\phi^2 = -\langle g', g' \rangle$. Orthogonal foliations by conformally flat hypersurfaces as above, are produced by attaching the spheres to the integral curves of $\text{grad } \phi$ and $(\text{grad } \phi)^\perp$. It follows from Theorem 5 that $N^2 \times \mathbf{S}^{n-2}(1)$ cannot be isometrically immersed in \mathbf{R}^{n+2} with the metric induced by g .

2) Take any surface $k: N^2 \rightarrow \mathbf{H}^{n+1}(-1)$ with flat normal bundle and principal coordinates (u, v) which is not of first or second type. Let $G: N^{n+1} \rightarrow \mathbf{L}^{n+2}$ be the hypersurface defined in terms of the Gauss parametrization by k and the support function $\rho = 1$. The intersection $g: N^2 \times \mathbf{S}^{n-2}(1) \rightarrow \mathbf{L}^{n+2}$ of $G(N^{n+1})$ with \mathbf{V}^{n+1} is of the form

$$g(u, v, t_1, \dots, t_{n-2}) = k(u, v) + \sum_{j=1}^{n-1} \psi_j(t_1, \dots, t_{n-2}) \xi_j(u, v),$$

where ξ_1, \dots, ξ_{n-1} is a parallel orthonormal normal frame for k and $\psi = (\psi_1, \dots, \psi_{n-1})$ a parametrization of the unit sphere in \mathbf{R}^{n-1} . A straightforward calculation shows that the foliations obtained fixing values of u or v verify all of the above conditions but $N^2 \times \mathbf{S}^{n-2}(1)$ cannot be isometrically immersed in \mathbf{R}^{n+2} with the metric induced by g .

§4 The parametrization

In this section we provide an explicit parametrization for any nonflat conformally flat submanifolds in codimension 2.

That $f: M^n \rightarrow \mathbf{R}^{n+2}$ is a *composition* we mean that there exist an open subset $U \subset \mathbf{R}^{n+1}$ and isometric immersions $\tilde{f}: M^n \rightarrow U$ and $H: U \rightarrow \mathbf{R}^{n+2}$ such that $f = H \circ \tilde{f}$.

Proposition 9. *Any conformally flat submanifold $f: M^n \rightarrow \mathbf{R}^{n+2}$, $n \geq 5$, without flat points is locally along an open dense subset either generic or a composition.*

Proof: Since we are dealing with a local statement, we may assume that the principal curvature $\lambda \neq 0$ of the umbilical direction η (see (1)) has constant multiplicity ℓ . When $\ell \geq n - 1$, we have from (4) that $\text{rank} A_\xi \leq 1$ for unit $\xi \perp \eta$. On any open subset of M^n where A_ξ has constant rank, we verify by a straightforward computation that A_η satisfies the Gauss and Codazzi equations for a hypersurface $\tilde{f}: M^n \rightarrow \mathbf{R}^{n+1}$. It follows from Theorem 8 of [DT₁] that f is a composition. ■

Notice that compositions as in the above result can easily be described parametrically by using together the Gauss parametrization and Cartan's parametrization of conformally flat hypersurfaces referred to in the introduction. We now consider the generic case.

Theorem 10. *Let $h: V^2 \rightarrow \mathbf{S}^{n+1}(1) \subset \mathbf{R}^{n+2}$, $n \geq 2$, be a surface of first or second type with conjugate coordinates (u, v) . For a given associated function τ , let Θ^* be the adjoint to the tensor $\Theta: TV \rightarrow TV$ defined by*

$$\Theta \partial_u = \frac{1}{\theta} \partial_u, \quad \Theta \partial_v = -\theta \partial_v,$$

where $\theta = \sqrt{-\tau}$. Moreover, let ρ be a solution of the differential equation

$$\rho_{uv} + \theta^2 \Gamma^2 \rho_v + \frac{1}{\theta^2} \Gamma^1 \rho_u + \rho \langle \partial_u, \partial_v \rangle = 0, \quad (26)$$

and let $\beta: V^2 \rightarrow \mathbf{R}^{n+2}$ be a solution, unique up to translations, of the completely integrable system of first order

$$\begin{cases} \beta_u = \theta \rho h_u - \frac{\rho_u}{\theta} h \\ \beta_v = -\frac{\rho}{\theta} h_v + \theta \rho_v h. \end{cases}$$

Then, on the open subset of regular points, the map $\varphi: \mathcal{N}_1 \rightarrow \mathbf{R}^{n+2}$, defined on the unit normal bundle \mathcal{N}_1 of h in the sphere and given by

$$\varphi(w) = \beta - \Theta^* \text{grad} \rho + \sqrt{\rho^2 - \|\Theta^* \text{grad} \rho\|^2} w \quad (27)$$

is a parametrization of a generic n -dimensional conformally flat submanifold of \mathbf{R}^{n+2} . Conversely, for $n \geq 5$, any generic conformally flat submanifold $f: M^n \rightarrow \mathbf{R}^{n+2}$ can be locally parametrized this way.

Proof: We prove the converse. Consider a local isometric immersion of M^n into the light cone $g: M^n \rightarrow \mathbf{V}^{n+1} \subset \mathbf{L}^{n+2}$, and the isometric extensions $F: N^{n+1} \rightarrow \mathbf{R}^{n+2}$ of f and $G: N^{n+1} \rightarrow \mathbf{L}^{n+2}$ of g . From the proofs of Theorem 1 and Proposition 3, there exist orthonormal bases $\{h, \eta\}$ of $T_f^\perp M$ and $\{k, \mu\}$ of $T_g^\perp M$, where h, k are seen simultaneously as Gauss maps for F and G and as certain immersions $h: V^2 \rightarrow \mathbf{S}^{n+1}(1)$ and $k: V^2 \rightarrow \mathbf{H}^{n+1}(-1)$. Equation (15) yields

$$A_k^g = A_h^f \circ D, \quad (28)$$

for D extended to TM as $D|_{\mathcal{U}} = 0$. On the other hand, we easily get taking derivatives of (7) and using (3) that

$$\text{Id} = \rho(A_\eta^f - A_k^g),$$

where $\rho = 1/\lambda$ (see (1)) is considered as a function on V^2 . Therefore,

$$\rho A_h^f \circ D = \rho A_\eta^f - \text{Id}. \quad (29)$$

Consider the map $\beta: M^n \rightarrow \mathbf{R}^{n+2}$, defined as

$$\beta = f + \rho\eta.$$

Then, β can also be viewed as defined along V^2 since

$$\beta_* Z = f_* Z + \rho \widetilde{\nabla}_Z \eta = Z - \rho A_\eta^f Z = 0, \quad \forall Z \in \mathcal{U}.$$

Our purpose is to describe f parametrically as

$$f = \beta - \rho\eta, \quad (30)$$

where β and η will be expressed in terms of data generated by h . Since the images by f of the leaves of \mathcal{U} are spheres, we have from (30) that η , when restricted to a leaf, parametrizes part of a round $(n-2)$ -dimensional sphere contained in some $(n-1)$ -dimensional affine subspace of \mathbf{R}^{n+2} . Fix a leaf L of \mathcal{U} , $x_0 \in L$, $Y \in \mathcal{U}^\perp(x_0)$ and set $y = \pi_* Y$. For any $Z \in TL$, we have

$$Z\langle \eta, h_* y \rangle = Z\langle \eta, h_* \pi_* Y \rangle = \langle \widetilde{\nabla}_Z \eta, h_* \pi_* Y \rangle = \langle A_\eta^f Z, A_h^f Y \rangle = \frac{1}{\rho} \langle A_h^f Z, Y \rangle = 0.$$

Therefore, the $h_*(T_{\pi(x_0)}V)$ -component of $\eta|_L$ is constant. It follows easily that there exists $\sigma \in TV$ so that η can be written as

$$\eta = h_* \sigma + \Phi + \kappa\psi$$

where $\Phi \in \mathcal{N} \subset TS^{n+1}(1)$, $\kappa \in C^\infty(V)$ and $\psi(x)$ is a parametrization of the unit sphere in the fiber $\mathcal{N}([x])$ when x varies along L . Using $\|\eta\| = 1$, we easily obtain that $\Phi = 0$. Hence,

$$\eta = h_*\sigma - \sqrt{1 - \|h_*\sigma\|^2} \psi. \quad (31)$$

Let $\phi(x): \mathcal{U}^\perp(x) \rightarrow T_{[x]}V$ be the isomorphism $\phi(x) = \pi_*|_{\mathcal{U}^\perp(x)}$. From (29), we have for $X \in \mathcal{U}^\perp(x)$,

$$\begin{aligned} \beta_*X &= X + \rho \widetilde{\nabla}_X \eta + X(\rho)\eta \\ &= (\text{Id} - \rho A_\eta^f)X - \rho \langle \widetilde{\nabla}_X h, \eta \rangle h + X(\rho)\eta \\ &= -\rho A_h^f DX - \rho \langle \widetilde{\nabla}_X h, \eta \rangle h + X(\rho)\eta. \end{aligned}$$

Since h, ρ and β can be viewed as maps on V^2 , the above and (31) yield

$$\beta_*\phi X = \rho h_*\phi DX - \rho \langle \phi X, \sigma \rangle h + (\phi X(\rho) - \rho \langle \phi DX, \sigma \rangle) \eta. \quad (32)$$

By (20), a tensor field Ω on V^2 is defined so that

$$\Omega \circ \phi = \phi \circ D.$$

Notice that $\Theta = \Omega^{-1}$. Only η in (32) is not constant along the leaves of \mathcal{U} . Hence, this equation is equivalent to

$$\sigma = \frac{1}{\rho} \Theta^* \text{grad } \rho, \quad (33)$$

and

$$\beta_*y = \rho h_*\Theta^{-1}y - \langle \Theta y, \text{grad } \rho \rangle h, \quad \forall y \in TV. \quad (34)$$

In conjugate coordinates, (34) takes the form

$$\begin{cases} \beta_u = \rho \theta h_u - \frac{\rho_u}{\theta} h \\ \beta_v = -\frac{\rho}{\theta} h_v + \theta \rho_v h. \end{cases} \quad (35)$$

Set $\tau = -\theta^2$. An easy computation shows that equations (8), (10) and (26) are the integrability conditions for (35). Therefore, system (35) is completely integrable if and only if h is a surface of first or second type and ρ verifies (26). The proof follows from (30), (31), (33) and (35). The direct statement is straightforward. ■

Remarks 11. 1) A long but straightforward calculation using (28) and that f and g are isometric yields

$$\langle x, y \rangle' = \langle \Theta^{-1}x, \Theta^{-1}y \rangle, \quad \forall x, y \in TV,$$

where we denote by $\langle \cdot, \cdot \rangle'$ and ∇'' the metric and connection induced on V^2 by k . Therefore,

$$\nabla''_x y = \Theta \nabla'_x \Theta^{-1} y.$$

Then,

$$\Theta^{-1} \text{grad}' \rho = \Theta^* \text{grad} \rho$$

and

$$\text{Hess}'_\rho(\partial_u, \partial_v) - \rho \langle \partial_u, \partial_v \rangle' = \frac{1}{\theta} \{ \rho_{uv} + \theta^2 \Gamma^2 \rho_v + \frac{1}{\theta^2} \Gamma^1 \rho_u + \rho \langle \partial_u, \partial_v \rangle \}.$$

Hence, equations (25) and (26) are equivalent. Also, the radius of the umbilical spheres in both ambient spaces (c.f. (7) and (27)) are equal, as they must be, since f and g are isometric, that is,

$$s := \sqrt{\rho^2 - \|\Theta^* \text{grad} \rho\|^2} = \sqrt{\rho^2 - \|\text{grad}' \rho\|^2}.$$

2) It is easy to see that the regular points of the Euclidean parametrization are the ones for which the operator $P - sB_w$ is nonsingular. Here P is defined as

$$\langle Px, y \rangle = \langle \rho \Theta^{-1}x - \nabla'_x(\Theta^* \text{grad} \rho), y \rangle,$$

and B_w stands for the second fundamental form of h in $\mathbf{S}^{n+1}(1)$ for the normal direction w .

We conclude the section presenting a large family of examples generated by a class of spherical surfaces of first type already considered by Cartan ([Ca₁]).

Examples 12. Given a spherical surface $h: V^2 \rightarrow \mathbf{S}^{n+1}(1) \subset \mathbf{R}^{n+2}$ of first type, define $H: V^2 \rightarrow \mathbf{R}^{n+2}$ by $H = \sqrt{U - V} h$. From (9) and (12) we have that, componentwise, any surface of first type is produced giving a set of functions (h^1, \dots, h^{m+1}) , with $\sum_{j=1}^{m+1} (h^j)^2 = 1$, all satisfying the same differential equation:

$$h_{uv}^j - \frac{V'}{2(U - V)} h_u^j + \frac{U'}{2(U - V)} h_v^j + F h^j = 0, \quad 1 \leq j \leq m + 1, \quad (36)$$

where $U = U(u)$, $V = V(v)$ and $F = F(u, v)$ are arbitrary smooth functions. A straightforward computation shows that equation (36) takes the form

$$H_{uv} + (\Gamma^1 \Gamma^2 + F) H = 0, \quad F = \langle \partial_u, \partial_v \rangle.$$

Let us analyze the case when

$$\Gamma^1 \Gamma^2 + F = 0. \tag{37}$$

In that situation, there exist two regular curves $\alpha_j: I_j \rightarrow \mathbf{R}^{n+2}$ so that

$$H(u, v) = \alpha_1(u) + \alpha_2(v).$$

Hence,

$$U(u) - V(v) = \|\alpha_1\|^2 + \|\alpha_2\|^2 + 2\langle \alpha_1, \alpha_2 \rangle.$$

In particular, $\langle \alpha_1'(u), \alpha_2'(v) \rangle = 0$ along $I_1 \times I_2$. Therefore, there exist affine orthogonal subspaces $E_j \subset \mathbf{R}^{n+2}$ such that $\alpha_j(I_j) \subset E_j$, $1 \leq j \leq 2$, and

$$U(u) = \|\alpha_1\|^2, \quad V(v) = -\|\alpha_2\|^2.$$

Hence, surfaces of first type satisfying the additional condition (37) have the expression

$$h(u, v) = \frac{(\alpha_1(u), \alpha_2(v))}{\sqrt{\|\alpha_1(u)\|^2 + \|\alpha_2(v)\|^2}}.$$

Assume, in addition, that the α_j 's are spherical curves, i.e., $\|\alpha_j\| = c_j$, $c_j \in \mathbf{R}_+$ with $c_1^2 + c_2^2 = 1$. Then, $F = \Gamma^1 = \Gamma^2 = 0$, $\tau = -k^2$ is constant and

$$\rho(u, v) = \rho_1(u) + \rho_2(v).$$

We conclude that

$$\begin{aligned} \varphi(w) &= \frac{1}{k} \left(k^2 \rho \alpha_1 - (1 + k^2) \int \rho_1' \alpha_1 du, -\rho \alpha_2 - (1 + k^2) \int \rho_2' \alpha_2 dv \right) \\ &\quad + \phi + \sqrt{\rho^2 - \|\phi\|^2} w, \end{aligned}$$

where

$$\phi = \left(\frac{\rho_1' \alpha_1'}{k \|\alpha_1'\|^2}, -\frac{k \rho_2' \alpha_2'}{\|\alpha_2'\|^2} \right),$$

parametrizes a 1-parameter family of generic conformally flat submanifolds.

The above set of examples contains very simple ones obtained by taking $\rho = 1$. This yields a 1-parameter family of immersions of the unit normal bundle \mathcal{N}_1 of h in $\mathbf{S}^{n+1}(1)$, given by

$$\varphi(w) = \left(k\alpha_1, -\frac{1}{k}\alpha_2 \right) + w, \quad k \in \mathbf{R}_+.$$

These examples are even simpler if the α_j 's are taken to be circles.

§5 The flat case

Our goal in this section is to parametrically describe all flat Euclidean submanifolds in codimension 2 which cannot be obtained as compositions. Arguments here will be quite sketchy in regard to their similarity with the ones in [CD].

We assume that $f: M^n \rightarrow \mathbf{R}^{n+2}$ is flat, 1-regular and nowhere a composition. The first assumption means that the first normal spaces $N_1^f(x) \subset T_{f(x)}^\perp M$, i.e., the subspaces spanned by the second fundamental form at each point, form a subbundle of $T_f^\perp M$. By Theorem 1 in [DT₂] we have that $\dim N_1^f = 2$. As in [CD], we conclude that in a neighborhood of each point there exists an orthonormal normal local basis $\{\xi, \eta\}$ for globally defined line bundles in $T_f^\perp M$, such that

$$\text{rank} A_\xi = \text{rank} A_\eta = 1$$

and, on an open dense subset,

$$\ker A_\xi \not\subset \ker \psi, \quad \ker A_\eta \not\subset \ker \psi, \tag{38}$$

where ψ is the 1-form defined as

$$\psi(X) = \langle \nabla_X^\perp \xi, \eta \rangle.$$

We say that a 1-regular flat submanifold $f: M^n \rightarrow \mathbf{R}^{n+2}$ is *generic* when $\dim N_1^f = 2$ and condition (38) holds everywhere. So f is nowhere a composition.

By Codazzi's equation,

$$A_\xi[X, W] = -\psi(X)A_\eta W, \quad \forall X \in \Delta,$$

where $0 \neq W \in \ker A_\xi \cap \Delta^\perp$. From $\text{Im}A_\xi \neq \text{Im}A_\eta$ and $A_\eta W \neq 0$, we get

$$\Delta \subset \ker \psi. \quad (39)$$

It follows easily from $\ker A_\xi \not\subset \ker \psi$ that $h: V^2 \rightarrow \mathbf{S}^{n+1}(1)$, defined by $\xi = h \circ \pi$, is an immersion. Here, V^2 is the quotient space of leaves of relative nullity of M^n with projection $\pi: M^n \rightarrow V^2$ endowed with the metric induced by h .

For all $X \in \Delta$ and $Y \in TM$, we have that

$$0 = \langle A_\xi X, Y \rangle = -\langle \xi_* Y, f_* X \rangle.$$

Hence, $\Delta(x) \subset T_{h([x])}^\perp V$. Set $\tilde{\gamma} = \langle f, \xi \rangle$. Then,

$$X(\tilde{\gamma}) = \langle f, \xi_* X \rangle = 0, \quad \forall X \in \Delta.$$

Therefore, $\tilde{\gamma}$ is constant along the leaves of Δ . Let $\delta \in T_h^\perp V$ be a unit vector field orthogonal to Δ . Similar to Proposition 3.5 of [CD], we can define a parametrization φ of f along the subbundle $N_\delta = \{\beta \in T_h^\perp V: \langle \beta, \delta \rangle = 0\}$ by

$$\varphi(\beta) = (\gamma h + \text{grad} \gamma + \theta \delta)(y) + \beta, \quad y = \pi(\beta), \quad (40)$$

where γ is defined by $\tilde{\gamma} = \gamma \circ \pi$. Notice that $\theta \in C^\infty(V)$ since $f_* X(x) = X(x) \in N_\delta([x])$ for all $X \in \Delta$.

Let P be the symmetric tensor field on V^2 defined as $P = \text{Hess}_\gamma + \gamma \text{Id}$. We have that $\varphi_* Y = Y$ for any $Y \in N_\delta(\pi(\beta))$ in the fiber at $\pi(\beta) \in V^2$. If $T \in T_\beta N_\delta$ with $\pi_* T = z$, a calculation similar to that of Proposition 3.9 in [CD] gives

$$\varphi_* T(\beta) = h_*(P - \theta B_\delta - B_\beta)z + (\langle B_\delta \text{grad} \gamma + \text{grad} \theta, z \rangle - \langle \nabla_z^\perp \delta, \beta \rangle)\delta + \mu \quad (41)$$

where $\mu \in N_\delta(\pi(\beta))$ and B_δ denotes the second fundamental form of h in the normal direction δ . Set $\phi = \pi_*|_{\Delta^\perp}$. From $W \in \ker A_\xi$, we get $\eta = \lambda \xi_* W = h_* \phi \lambda W$, for some function $\lambda \neq 0$. Since η is constant along the leaves of relative nullity, it follows that there exists a unit vector field $w \in TV$ such that

$$\eta = h_* w \circ \pi. \quad (42)$$

By (41), this is equivalent to

$$B_\beta w = 0, \quad \forall \beta \in N_\delta, \quad \text{and} \quad (P - \theta B_\delta) w = 0. \quad (43)$$

But,

$$A_\eta \lambda W = \lambda^2 A_{\xi_* W} W = \langle B_\delta w, w \rangle \delta \circ \pi + \epsilon W^\perp, \quad \epsilon \in C^\infty(V),$$

where W^\perp is a unit generator of $\text{Im} A_\xi$. From $\dim N_f^1 = 2$, we conclude that

$$\langle B_\delta w, w \rangle \neq 0. \quad (44)$$

Since $\text{rank} A_\eta = 1$ and $\dim N_f^1 = 2$, we have that there exists a vector field $V \in \Delta^\perp$, linearly independent with W , such that

$$A_{\xi_* W} V = 0.$$

As in [CD], using (41) and (42), we easily conclude that there exists a vector field $v \in TV$, linearly independent with w ($v \circ \pi = \phi V$), verifying

$$\nabla'_v w = 0, \quad \alpha_h(v, w) = 0. \quad (45)$$

By (45), equations (43) are equivalent to

$$\delta = \frac{\alpha_h(w, w)}{\|\alpha_h(w, w)\|}, \quad \theta = \frac{\langle Pw, w \rangle}{\|\alpha_h(w, w)\|}, \quad \text{and} \quad \langle Pv, w \rangle = 0. \quad (46)$$

Extending the definition in [CD], we call $h: V^2 \rightarrow \mathbf{S}^{n+1}(1)$ a *surface of type C* when there exists a conjugate coordinate system (u, v) such that

$$\Gamma^1 = 0 \quad \text{and} \quad \alpha_h(\partial_u, \partial_v) \neq 0.$$

Theorem 13. *Let $h: V^2 \rightarrow \mathbf{S}^{n+1}(1)$ be a surface of type C, and let γ be any solution of equation*

$$\text{Hess}_\gamma(\partial_u, \partial_v) + \langle \partial_u, \partial_v \rangle \gamma = 0.$$

Let $\delta \in T_h^\perp V$ and $\theta \in C^\infty(V)$ be given by

$$\delta = \frac{\alpha_h(\partial_v, \partial_v)}{\|\alpha_h(\partial_v, \partial_v)\|}, \quad \theta = \frac{\langle P\partial_v, \partial_v \rangle}{\|\alpha_h(\partial_v, \partial_v)\|}.$$

Then, on the open subset of regular points, the map $\varphi: N_\delta \rightarrow \mathbf{R}^{n+2}$, defined on $N_\delta = \{\beta \in T_h^\perp V: \langle \delta, \beta \rangle = 0\}$ and given by

$$\varphi(\beta) = \gamma h + \text{grad}\gamma + \theta\delta + \beta,$$

is a parametrization of a generic n -dimensional flat submanifold of \mathbf{R}^{n+2} . Conversely, any such submanifold can be locally parametrized this way.

Proof: Follows using (40), (44), (45) and (46). ■

Remarks 14. 1) The regular points of φ are characterized by conditions:

$$\langle B_\delta \text{grad}\gamma + \text{grad}\theta, \partial_v \rangle \neq \langle \nabla_{\partial_v}^\perp \delta, \beta \rangle, \quad \langle (P - \theta B_\delta - B_\beta) \partial_u, \partial_u \rangle \neq 0.$$

2) As in [CD], we have that $R_f^\perp = 0 \iff \nabla'_w w = 0$.

References

- [AD₁] Asperti, A. and Dajczer, M., *Conformally flat Riemannian manifolds as hypersurfaces of the light cone*. Canadian Math. Bull. **32** (1989), 281–285.
- [AD₂] Asperti, A. and Dajczer, M., *N -dimensional submanifolds of R^{N+1} and S^{N+2}* . Illinois J. of Math. **28** (1984), 621–645.
- [CD] do Carmo, M. and Dajczer, M., *Local isometric immersions of R^2 into R^4* . J. reine angew. Math. **442** (1993), 205–219.
- [Ca₁] Cartan, E., *La déformation des hypersurfaces dans l'espace euclidien réel a n dimensions*. Bull. Soc. Math. France **44** (1916), 65–99.
- [Ca₂] Cartan, E., *Sur certains hypersurfaces de l'espace conforme réel a cinq dimensions*. Bull. Soc. Math. France **46** (1918), 84–105.
- [CDM] do Carmo, M., Dajczer, M. and Mercuri, F., *Compact conformally flat hypersurfaces*. Trans. Amer. Math. Soc. **288** (1985), 189–203.
- [CY] Chen, B. and Yano, Y., *Special conformally flat spaces and canal hypersurfaces*. Tôhoku Mat. J. **25** (1973), 177–184.
- [Da] Dajczer, M., “Submanifolds and isometric immersions”. Math. Lec. Series **13**, Publish or Perish Inc. Houston, 1990.
- [DG₁] Dajczer, M. and Gromoll, D., *Gauss parametrizations and rigidity aspects of submanifolds*. J. Differential Geometry **22** (1985), 1–12.
- [DG₂] Dajczer, M. and Gromoll, D., *Isometric deformations of compact Euclidean submanifolds in codimension 2*. To appear in Duke Math. J.
- [DT₁] Dajczer, M. and Tojeiro, R., *On compositions of isometric immersions*. J. Diff. Geometry **36** (1992), 1–18.
- [DT₂] Dajczer, M. and Tojeiro, R., *Submanifolds with nonparallel first normal bundle*. Canad. Math. Bull. **37** (1994), 330–337.
- [Ki] Kitagawa, Y., *Umbilics of conformally flat submanifolds*. Tôhoku Math. J. **32** (1980), 433–438.

- [Mo₁] Moore, J., *Conformally flat submanifolds of Euclidean space*. Math. Ann. **225** (1977), 89–97.
- [Mo₂] Moore, J., *Submanifolds of constant positive curvature I*. Duke Math. J. **44** (1977), 449–484.
- [MM] Moore, J. and Morvan, J., *Sous-varietés conformément plates de codimension quatre*. C. R. Acad. Sc. Paris **287** (1978), 655–657.
- [No₁] Noronha, M., *Conformally flat immersions in codimension two*. Geometriae Dedicata **23** (1987), 115–130.
- [No₂] Noronha, M., *Conformally flat immersions and flatness of the normal connection*. Pacific J. of Math. **138** (1989), 145–150.
- [Sb] Sbrana, V., *Sulla varietà ad $n - 1$ dimensioni deformabili nello spazio euclideo ad n dimensioni*. Rend. Circ. Mat. Palermo **27** (1909), 1–45.

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