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Curvature homogeneous hypersurfaces in space forms

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ABSTRACT

We provide a classification of curvature homogeneous hypersurfaces in space forms by classifying the ones in S^4 and H^4. In higher dimensions, besides the isoparametric and the constant curvature ones, there is a single one in H^5. Besides the obvious examples, we show that there exists an isolated hypersurface with a circle of symmetries and a one parameter family admitting no continuous symmetries. Outside the set of minimal points, which only exists in the case of S^4, every example is, locally and up to covers, of this form.

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1. Introduction

Since the early days of differential geometry we know that a metric on a manifold defines a natural notion of curvature, collected in what today is called its curvature tensor. In particular, isometries preserve this curvature, and thus the curvature of a homogeneous space should be somehow constant along the manifold. This logically leads to the question of whether a Riemannian manifold whose curvature tensor is "the same" at every point is in fact homogeneous. More generally, a natural problem in Riemannian geometry is to what extent, and even in which sense, "the curvature determines the metric". This question is more subtle than it seems at first glance, having several answers and open aspects.

More precisely, a Riemannian manifold M^n is called *curvature homogeneous* if, for any pair of points $p, q \in M$, there exists a linear isometry $J_{pq} : T_pM \to T_qM$ that preserves the curvature tensor R, i.e., $J_{pq}^* R_q = R_p$. In dimension $n = 3$ this is equivalent to the condition that the eigenvalues of the Ricci tensor are constant. I. Singer showed in [15] that, if sufficiently many covariant derivatives of R match, then the metric is in fact homogeneous. He then asked whether it is sufficient to only require this condition for R. Since then, this has been an active research area and the question turns out to be surprisingly subtle.

The first complete non-homogeneous examples were given by K. Sekigawa and H. Takagi in [14,16], where they showed that such examples exist in any odd dimension and depend on several functions of one variable. On the other hand, there are also several known obstructions. In [18] it was shown that, if the curvature tensor R is that of an irreducible symmetric space, then the metric itself must be locally symmetric. In the above examples, R was that of $\mathbb{H}^2 \times \mathbb{R}^{n-2}$ and these were in fact classified in [3,12], where it was also shown that the fundamental group must be a free group. In contrast, in dimension 3 any curvature tensor can be realized locally, see [2,4]. But in higher dimensions, the condition that R needs to satisfy seems to be quite strong. Surprisingly, the only known compact non-homogeneous examples are the Ferus-Karcher-Münzner isoparametric hypersurfaces (i.e., the eigenvalues of the shape operator are constant), see [11].

It is natural to attack this kind of problem in the context of submanifolds in space forms since, by the Gauss equation, more control is gained on the structure of the curvature tensor. This is particularly true for hypersurfaces where the curvature tensor, which in general is a very complicated algebraic object, is determined just by an endomorphism of the tangent bundle, namely, the shape operator.

In [17] K. Tsukada studied the problem of classifying curvature homogeneous hypersurfaces M^n in the simply connected space form $\mathbb{Q}_c^{n+1}$ of constant curvature c. He showed that any such hypersurface is isoparametric, or has constant curvature c, or has rank two, that is, the rank of its shape operator is two everywhere; see Proposition 2. Observe that constant scalar curvature and curvature homogeneous are equivalent notions for rank two submanifolds in space forms by the Gauss equation.

There are two natural examples of 3-dimensional complete rank two hypersurfaces with constant scalar curvature, $f_c : \Lambda \to \mathbb{Q}_c^4$, $c = \pm 1$. The first one f_1 is the unit normal bundle Λ of the Veronese surface $\mathbb{RP}_{1/3} \subset \mathbb{S}^4$, which is one of Cartan's isoparametric hypersurfaces with three different principal curvatures. This hypersurface is not only isoparametric, but homogeneous. The second one f_{-1} is the unit normal bundle Λ of the flat torus in the De-Sitter space, $g = (g_0, 1)\colon T^2 \to \mathbb{S}^4_{-1} \subset \mathbb{R}^{4,1}$, where $g_0\colon T^2 \to \mathbb{S}^3(\sqrt{2}) \subset \mathbb{R}^4 \times \{0\} \subset \mathbb{R}^{4,1}$ is the minimal equivariant flat Clifford torus; see Section 4.1. This hypersurface has a two parameter family of symmetries induced by the symmetries of the Clifford torus.

If $n \geq 4$ or $n = 3$ and $c = 0$ the condition on the hypersurface is quite rigid. Tsukada showed that, in this case, besides the obvious Euclidean cylinders over constant curvature surfaces, there is only one rank two example, a complete hypersurface in the hyperbolic 5-space $\mathbb{H}^5$. We will give a simpler proof of this fact in Section 7 together with a more geometric description of this example closely related to both f_c's.

On the other hand, the case $n = 3$ and $c \neq 0$ remained an open problem, see e.g. [2] p.255 and [8]. Our purpose in this paper is to answer this question. Recall that a rank two hypersurface in $\mathbb{Q}_c^4$ is foliated by special geodesics, the so called relative nullity leaves, tangent to the kernel of the shape operator.

Theorem 1. *Let $\mathcal{M}$ be the set of immersed rank two hypersurfaces in $\mathbb{Q}_c^4$, $c = \pm 1$, whose induced metric has constant scalar curvature. Then $\mathcal{M}$ contains f_c as the only complete example, an isolated hypersurface $\hat{f}_c$ with a circle of symmetries, and a one parameter family of hypersurfaces admitting no continuous symmetries. Moreover, up to a covering, any connected hypersurface in $\mathcal{M}$ is an open subset of one of these, provided it has no leaf of relative nullity of minimal points in the case $c = 1$.*

To prove Theorem 1 we will make use of the Gauss Parametrization that we recall in Section 2.1, which is a powerful tool to study hypersurfaces of constant rank in space forms. Our hypersurfaces will then be the unit normal bundles of their polar surfaces in $\mathbb{S}_c^4$, in fact globally since we will show that the relative nullity geodesics are complete. These surfaces are characterized by the property that all shape operators along unit normal directions have the same non-zero determinant, and thus have constant Gaussian curvature. Our work will then reduce to classifying such surfaces. Topologically, the polar surfaces of the one parameter family of hypersurfaces with no continuous symmetries in Theorem 1 are diffeomorphic to a pair of pants if $c = 1$, or to either a cylinder or a plane if $c = -1$; see Section 5. Moreover, the hypothesis on the minimal points for $c = 1$ is equivalent to asking for the polar surface to have no minimal points.

This approach also allows us to get simple explicit parametrizations for $\hat{f}_c$ in Theorem 1 as follows. Set $r_0 := \arccos(\sqrt{2/3})$, and $D^2(r_0) \subset \mathbb{R}^2$ the 2-disk of radius r_0 with polar coordinates (r, θ) if $c = 1$. Then, up to congruences, $\hat{f}_1 = \hat{f}_1(r, \theta, \alpha)\colon D^2(r_0) \times S^1 \to \mathbb{S}^4$ and $\hat{f}_{-1} = \hat{f}_{-1}(r, \theta, \alpha)\colon (r_0, \pi - r_0) \times S^1 \times \mathbb{R} \to \mathbb{H}^4$ have the unified expression

$$\hat{f}_c = \begin{pmatrix} \cos_c(\alpha)\sin(2\theta)\cos(2r) - \sin_c(\alpha)\cos(2\theta)\cos(r)\sqrt{(3\cos(2r)-1)/2c} \\ \cos_c(\alpha)\cos(2\theta)\cos(2r) + \sin_c(\alpha)\sin(2\theta)\cos(r)\sqrt{(3\cos(2r)-1)/2c} \\ \cos_c(\alpha)\ \sin(\theta)\ \sin(2r) - 2\sin_c(\alpha)\cos(\theta)\sin(r)\sqrt{(3\cos(2r)-1)/2c} \\ \cos_c(\alpha)\ \cos(\theta)\ \sin(2r) + 2\sin_c(\alpha)\sin(\theta)\sin(r)\sqrt{(3\cos(2r)-1)/2c} \\ (3/2)\sin_c(\alpha)(\cos(2r)-1) \end{pmatrix}, \tag{1}$$

where $\sin_c$ and $\cos_c$ stand for sin and cos if $c = 1$, sinh and cosh if $c = -1$. We will see that at the boundary $\hat{f}_1$ has a 2-torus as singular set (where its mean curvature is unbounded), and at the boundary $\hat{f}_{-1}$ has two singular 2-cylinders. In addition, from (1) it is not hard to check that $\hat{f}_c$ is algebraic; see Section 6.

In a forthcoming paper [5], a more complete description of the corresponding polar surfaces will be provided. When $c = 1$, it will be shown that there exists a 1-parameter family of real-analytic mappings $\overline{g}_a : S^2 \to S^4$ for $0 \leq a \leq 1$ such that the polar surface of a hypersurface as described in Theorem 1 is congruent to an open subset of $\overline{g}_a(S^2)$ for some a. The map $\overline{g}_a$ is a topological embedding and is an immersion except along the equator in S^2, where its differential has rank 1. When $a = 0$, the image $\overline{g}_0(S^2)$ has a rotational symmetry and is congruent to the algebraic surface described by (18). Its only minimal points are the two 'poles' of the rotational symmetry. When $a > 0$, the image $\overline{g}_a(S^2)$ has an 8-fold discrete symmetry group, and it contains exactly four distinct minimal points (at which the surface $\overline{g}_a(S^2)$ is smooth). At present, it is not known whether the compact surface $\overline{g}_a(S^2)$ is algebraic when $a > 0$. Meanwhile, when $c = -1$, a correspondingly complete description will be given of the polar surfaces of the hypersurfaces described by Theorem 1. Again, it turns out that there is a 1-parameter family of such polar surfaces up to congruence, and, except for one particular value of the parameter, the analytically-completed surfaces have similar singularity properties, while, for the exceptional value, the singular structure is quite different. Again, it is not known at present whether these surfaces are algebraic.

The paper is organized as follows. In Section 2 we reduce the classification to hypersurfaces of $\mathbb{Q}_c^4$ and explain the Gauss parametrization. We then convert our problem to a classification of the corresponding polar surface. In Section 3 we discuss the structure equations of the polar surface and in Section 4 the compatibility condition that needs to be satisfied in order to find local solutions. In Section 5 we determine the maximal domain of the polar surface, and its topological type. Finally, in Section 6 we discuss the example with rotational symmetry, and in Section 7 give a simple description of the Tsukada example.

2. Preliminaries

Let M^n be a curvature homogeneous Riemannian manifold with curvature tensor R. Let $f : M^n \to \mathbb{Q}_c^{n+p}$ be an isometric immersion with second fundamental form α into the simply connected space form $\mathbb{Q}_c^{n+p}$ of curvature c. Fix $x_0 \in M^n$. Then, for each $x \in M$, there is a linear isometry $J_x : T_xM \to T_{x_0}M$ such that

$$R_x = J_x^* R_{x_0}. \tag{2}$$

We say that such an f is *weakly isoparametric* if for each $x \in M$ there exists another linear isometry $\hat{J}_x : T_x^\perp M \to T_{x_0}^\perp M$ such that

$$\hat{J}_x \circ \alpha_x = J_x^* \alpha_{x_0}. \tag{3}$$

Notice that, by the Gauss equation, (3) implies (2).

For each $x \in M$, define the bilinear map

$$\beta_x : T_x M \times T_x M \to W_x := T_x^\perp M \times T_{x_0}^\perp M$$

as $\beta_x = (\alpha_x, J_x^* \alpha_{x_0})$. Again by Gauss equation, M^n is curvature homogeneous (with respect to J) if and only if β_x is *flat*, that is,

$$\langle \beta_x(X,Y), \beta_x(U,V)\rangle = \langle \beta_x(X,V), \beta_x(U,Y)\rangle, \quad \forall\ X, Y, U, V \in T_x M,$$

where the inner product on W_x is the natural indefinite one of type (p,p), namely, $\langle\, , \rangle = \langle\, , \rangle_{T_x^\perp M} - \langle\, , \rangle_{T_{x_0}^\perp M}$. It turns out that f is weakly isoparametric at x if and only if β_x is *null*, i.e.,

$$\langle \beta_x(X,Y), \beta_x(U,V)\rangle = 0, \quad \forall\ X, Y, U, V \in T_x M.$$

Indeed, for the converse just observe that the expression in (3) serves as a good definition of $\hat{J}_x$ between the images of α_x and $J_x^* \alpha_{x_0}$, which can afterwards be extended by linearity as a linear isometry.

Deciding when a flat bilinear map is null is a key point in isometric rigidity problems of submanifolds. Theorem 3 in [9] ensures that, if not null, a symmetric flat bilinear form must have a highly degenerate component, at least if $p \le 5$. More precisely, W_x decomposes orthogonally as

$$W_x = W_0 \oplus^\perp W_1$$

and β decomposes accordingly as $\beta = \beta_0 + \beta_1$, where β_0 is null and β_1 has nullity of dimension $\nu_x \ge n - \dim W_1$. In particular, if the codimension p is equal to 1 we have $\nu_x \ge n - 2$. We conclude the following (see Theorem 2.3 in [17]):

Proposition 2. *A hypersurface in $\mathbb{Q}_c^{n+1}$ is curvature homogeneous if and only if it is isoparametric, or has constant sectional curvature c, or has rank two with constant scalar curvature.*

Remark 3. The case of constant curvature c is well understood, since the set of such nowhere totally geodesic hypersurfaces can be naturally parametrized by the set of regular smooth curves in $\mathbb{Q}_c^{n+1}$ using the Gauss Parametrization; see Section 2.1. The isoparametric case was completely classified by E. Cartan for $c \leq 0$, while the $c > 0$ case in full generality still remains a well-known open problem.

In view of this, we concentrate from now on to the general task of describing constant scalar curvature rank two hypersurfaces in space forms $\mathbb{Q}_c^{n+1}$, of any dimension. Since the Euclidean case was solved in [17], we restrict ourselves to the cases $c = \pm 1$.

2.1. The Gauss parametrization for rank two hypersurfaces

The Gauss parametrization is a powerful tool to work with hypersurfaces with constant rank in space forms, as in our situation. It was created by Sbrana in [13] with the purpose of classifying nonflat locally isometrically deformable Euclidean hypersurfaces, which also have constant rank two. The tool was studied in further detail in [10], and we briefly describe it next.

Fix $c = \pm 1$ and let $\mathbb{E}^n$ denote the corresponding Euclidean space $\mathbb{R}^n$ or the Lorentzian space $\mathbb{R}^{n-1,1}$, i.e., $\mathbb{R}^n$ with the metric $\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_{n-1}^2 + c\,\mathrm{d}x_n^2$. Let $f : M^n \to \mathbb{Q}_c^{n+1} \subset \mathbb{E}^{n+2}$, $n \geq 3$, be a rank two connected orientable hypersurface and Δ^{n-2} its totally geodesic relative nullity foliation, namely, the integral leaves of the kernel of its second fundamental form. Consider the map $\hat{g} : M^n \to \mathbb{S}_c^{n+1} := \{x \in \mathbb{E}^{n+2} : \langle x, x\rangle = 1\}$ such that $\{f, \hat{g}\}$ is an oriented pseudo-orthonormal normal frame of f seen in $\mathbb{E}^{n+2}$, namely, $\langle \hat{g}, f\rangle = 0$ and $\langle \hat{g}(x), f_{*x}v\rangle = 0$ for all $x \in M^n, v \in T_xM$. If we take the (local) leaf space

$$\pi : M^n \to V^2 := M^n/\Delta,$$

the map $\hat{g}$ descends to the quotient. That is, there is an immersion called the *polar map* of f given by

$$g : V^2 \to \mathbb{S}_c^{n+1} \quad \text{with} \quad g \circ \pi = \hat{g}.$$

We fix on V^2 the metric induced by g, which is Riemannian since

$$\Delta^\perp(w) = g_{*p}(T_pV), \quad p = \pi(w). \tag{4}$$

It turns out that, locally, $f(M^n)$ can be seen as the unit normal bundle Λ of g,

$$\Lambda := \{w \in T_g^\perp V \subset T\mathbb{S}_c^{n+1} : p \in V^2,\ \langle w, w\rangle = c\},$$

that is, as the image of the map $\hat{f} : \Lambda \to \mathbb{Q}_c^{n+1}$ that sees each $w \in \Lambda$ as an element in $\mathbb{Q}_c^{n+1} \subset \mathbb{E}^{n+2}$ under parallel translation,

$$\hat{f}(w) = w. \tag{5}$$

The leaves of relative nullity of f are then identified to (open subsets of) the fibers of Λ as a bundle. Denote by A_w the shape operator of g in the direction $w \in \Lambda_p \subset T_{g(p)}^\perp V$,

$p = \pi(w)$. It is easy to check that the regular points of the *Gauss parametrization* $\hat{f}$ in (5) are the vectors $w \in \Lambda$ such that A_w is invertible. Moreover, using the identification in (4), we have that the shape operator of f restricted to $\Delta^\perp(w)$ in the direction $\hat{g}$ is just A_w^{-1}. Conversely, given any surface $g : V^2 \to \mathbb{S}_c^{n+1}$, the map (5) gives a rank two hypersurface in $\mathbb{Q}_c^{n+1}$, when restricted to the open subset of its regular points as described above; see [10] for details.

In view of this construction and the Gauss equation we are able to transfer our problem to the polar map g:

Proposition 4. *For $c = \pm 1$, consider a connected orientable rank two hypersurface $f : M^n \to \mathbb{Q}_c^{n+1}$ with polar map $g : V^2 \to \mathbb{S}_c^{n+1}$. Then f is curvature homogeneous if and only if the map $w \in \Lambda \mapsto \det A_w$ is a non-zero constant.*

Clearly, if this last map is constant along a small segment of a relative nullity geodesic γ, then it must be constant along the whole γ, and in particular $\hat{f}$ in (5) must be regular along all of γ. We conclude that we can assume from now on that all these geodesics are complete, even though we do not ask for the hypersurface M^n itself to be complete. In particular, M^n becomes the total space of the bundle $\mathbb{Q}_c^{n-1} \to M^n \xrightarrow{\pi} V^2$, and we conclude the following.

Corollary 5. *For f as above we have that $f(M^n) = \hat{f}(\Lambda)$. Conversely, if a surface g satisfies the property of the last proposition, then $\hat{f}$ in (5) gives globally a curvature homogeneous rank two immersion defined on the whole unit normal bundle Λ of g.*

Proof. For each $p \in V^2$, since along an open subset of Λ_p the map $w \in \Lambda_p \mapsto \det A_w$ in Proposition 4 is a non-zero constant, then this property holds over the whole leaf Λ_p, and therefore the map $\hat{f}$ is an immersion over all of Λ, still preserving the property that $\det A_w$ is a non-zero constant. Therefore $\hat{f}$ extends the original immersion f and is a curvature homogeneous regular hypersurface defined over the whole bundle Λ. ∎

As a consequence we conclude that our problem is low dimensional (the case $n = 4$ and $c = -1$ will be completely classified in Section 7):

Corollary 6. *Either $n = 3$ for $c = 1$, or $3 \leq n \leq 4$ for $c = -1$.*

Proof. Fix $p \in V^2$ and let E^3 be the vector space of self-adjoint endomorphisms of $T_pV \cong \mathbb{R}^2$. A maximal subspace of non-singular elements in E^3, excluding 0, has dimension 2. Consider the linear map $\varphi : T_p^\perp V \to E^3$, $\varphi(w) = A_w$. For $c = 1$, clearly φ has trivial kernel since $\varphi(\Lambda_p) \subset \mathrm{Iso}(T_pV)$. For $c = -1$, φ has kernel at most one dimensional since $\det \circ \varphi$ is a non-zero constant (see e.g. (19) in Section 7). The proof follows from $n = \dim\ker\varphi + \dim\mathrm{Im}\,\varphi + 1 \leq \dim\ker\varphi + 3$. ∎

Remark 7. As already pointed out, the problem for hypersurfaces in Euclidean space is simpler and completely understood. It turns out that the only examples are $(n-2)$-cylinders over surfaces with constant Gaussian curvature in $\mathbb{R}^3$, which are themselves classified. This can also be easily obtained using the Gauss parametrization; see Theorem 3.4 in [10].

Example 8. A well-known example of the situation in Proposition 4 is the minimal Veronese surface $g_1 : \mathbb{RP}^2_{1/3} \to \mathbb{S}^4 \subset \mathbb{R}^5$. It has the property that, given any orthonormal local tangent frame $\{e_1, e_2\}$ of $T\mathbb{RP}^2_{1/3}$, there exists a unique orthonormal normal frame $\{\xi_1, \xi_2\}$ such that

$$A_{\xi_1} = a\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_{\xi_2} = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad a > 0. \tag{6}$$

A point on a surface in $\mathbb{S}^4$ whose second fundamental form satisfies (6) will be called *Veronese-like*. For g_1 we have that $a = 1/\sqrt{3}$, and g_1 is the only surface in $\mathbb{S}^4$ which is Veronese-like everywhere. In fact, E. Cartan in [7] classified all isoparametric hypersurfaces in space forms with 3 different principal curvatures, the unit normal bundle of g_1 being the only one with rank two.

The next lemma will be needed in the following sections. It is convenient to call a basis $\{\xi_1, \xi_2\}$ of $\mathbb{E}^2$ orthonormal if $\langle \xi_i, \xi_j \rangle = \delta_{ij}$ for $c = 1$, as usual, while $\epsilon := \langle \xi_1, \xi_1 \rangle = -\langle \xi_2, \xi_2 \rangle = \pm 1$, $\langle \xi_1, \xi_2 \rangle = 0$ if $c = -1$.

Lemma 9. *Let $g : V^2 \to \mathbb{S}^4_c \subset \mathbb{E}^5$ be an isometric immersion such that $\det A_w \neq 0$ is constant for all $w \in \Lambda$. Then, locally around each non-minimal point of g, there exists an orthonormal tangent frame $\{e_1, e_2\}$, an orthonormal normal frame $\{\xi_1, \xi_2\}$, a constant $a > 0$, and a smooth function $h > 0$ on V^2, with $h > 1$ if $c = 1$, such that, in those frames,*

$$A_{\xi_1} = a\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_{\xi_2} = a\begin{pmatrix} h & 0 \\ 0 & -c/h \end{pmatrix}. \tag{7}$$

Moreover, the Gaussian curvature of V^2 is constant $1 - 2\epsilon a^2$, and outside the minimal points all this data is unique up to signs and permutations of e_1 and e_2.

Proof. Let L be the line bundle $L = \{\xi \in T_g^\perp V : \operatorname{tr} A_\xi = 0\}$. First, we claim that if $c = -1$ then L is not light-like. To see this, assume otherwise, take a generator η_1 of L and complete it to a basis $\{\eta_1, \eta_2\}$ such that $\langle \eta_1, \eta_1 \rangle = \langle \eta_2, \eta_2 \rangle = 0$ and $\langle \eta_1, \eta_2 \rangle = 1$. Then Λ can be written as $\Lambda = \{\eta_t = (t^{-1}\eta_1 - t\eta_2)/\sqrt{2} : 0 \neq t \in \mathbb{R}\}$. Write $A_{\eta_1} = a\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in some orthonormal basis, and $A_{\eta_2} = a\begin{pmatrix} x & y \\ y & z \end{pmatrix}$. Thus, $2a^{-2}\det A_{\eta_t} = t^2(xz - y^2) + 2y - 1/t^2$, which is not independent of t.

Now, choose $\xi_1 \in L$ with $\langle \xi_1, \xi_1 \rangle = \epsilon = \pm 1$ and complete it to an orthonormal normal frame $\{\xi_1, \xi_2\}$, that is, $\langle \xi_1, \xi_2 \rangle = 0$ and $\langle \xi_2, \xi_2 \rangle = \epsilon c$, where of course $\epsilon = 1$ if $c = 1$. We can then write $\Lambda = \{\xi_t = C_t \xi_1 + S_t \xi_2 : t \in I \subset \mathbb{R}\}$, where C_t and S_t are smooth functions of t satisfying $c\, C_t^2 + S_t^2 = \epsilon$. In an orthonormal tangent frame of isotropic vectors for A_{ξ_1} we have that

$$A_{\xi_1} = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \quad A_{\xi_2} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}.$$

Hence, $\det A_{\xi_t} = (\gamma\alpha - \beta^2 + ca^2) S_t^2 - 2a\beta S_t C_t - c\epsilon a^2$, which must be a non–zero constant. Therefore, $a \neq 0$ is constant, $\beta = 0$, and $\gamma\alpha = -ca^2$. The lemma now follows easily. ■

Remark 10. Notice that the set Σ of minimal points of g is nonempty only if $c = 1$ and it represents those points for which $h \to 1$ in (7). Therefore all minimal points are Veronese-like. Hence, the problem with the minimal points is that *any* pair of orthogonal tangent directions provides the same normal form (7), thus they are not unique and the special frames in Lemma 9 may not extend smoothly or continuously to the minimal points, even if isolated.

Remark 11. Since the shape operator of f restricted to $\Delta^\perp$ at the point $\xi_t \in \Lambda$ is $A_{\xi_t}^{-1}$, its mean curvature is $c\epsilon S_t(h^2 - c)/ha$. Thus, in terms of the Gauss parametrization, the set of minimal points of (a maximal) hypersurface f, for $c = 1$, is $\Lambda|_\Sigma$ together with the two surfaces $\{\pm\xi_1(p) : p \in V^2\} \subset \Lambda$. In particular, $\Lambda|_\Sigma$ corresponds to the set of leaves of relative nullity of f contained in its set of minimal points. Therefore, the exclusion of this set in Theorem 1 is equivalent to the exclusion of the minimal points of g.

Remark 12. Since V^2 has constant Gaussian curvature it has many local isometries. Yet, since h and the frames in Lemma 9 are unique, any continuous family of (extrinsic) symmetries preserving V^2 cannot fix points in V^2.

3. Reduction of the structure equations

In this section we compute the structure equations of the polar map $g : V^2 \to \mathbb{S}_c^4 \subset \mathbb{E}^5$ of our hypersurface in $\mathbb{Q}_c^4$, namely, a nowhere minimal Riemannian surface as in Lemma 9. For this section, and the remainder of the paper, the reader may find it useful to verify our long but straightforward computations using the Maple file [1].

Following the notations in Lemma 9, extend the tangent frame $\{e_1, e_2\}$ with $e_0 := g$, $e_3 = \xi_1$, $e_4 = \xi_2$. This is an orthonormal frame of $\mathbb{E}^5$, since $\langle e_i, e_j \rangle = 0$ if $i \neq j$ and

$$\langle e_0, e_0 \rangle = \langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = 1, \quad \langle e_3, e_3 \rangle = c\langle e_4, e_4 \rangle = \epsilon = \pm 1,$$

with $c = \pm 1$, and $\epsilon = 1$ if $c = 1$. Set

$$\mathrm{d}e_i = \sum_j e_j \eta_{ji}, \quad \text{with} \quad \eta_{ji}\langle e_i, e_i\rangle = -\eta_{ij}\langle e_j, e_j\rangle, \ \ 0 \le i, j \le 4.$$

Hence $\eta_{30} = \eta_{40} = 0$, and $\omega_1 := \eta_{10}$, $\omega_2 := \eta_{20}$ must be linearly independent. The associated tangent and normal connection 1-forms are $\omega := \eta_{21}$ and $\mu := \eta_{43}$, respectively. Lemma 9 is then equivalent to

$$\eta_{13} = \epsilon a\,\omega_2, \ \eta_{14} = \epsilon cah\,\omega_1, \ \eta_{23} = \epsilon a\,\omega_1, \ \eta_{24} = -\epsilon a h^{-1}\omega_2,$$

with $a > 0$ constant and $h > 0$ smooth on V, with $h > 1$ if $c = 1$ since we exclude minimal points. Putting the above together gives

$$\eta = \begin{pmatrix} 0 & -\omega_1 & -\omega_2 & 0 & 0 \\ \omega_1 & 0 & -\omega & -a\,\omega_2 & -ah\,\omega_1 \\ \omega_2 & \omega & 0 & -a\,\omega_1 & ach^{-1}\omega_2 \\ 0 & \epsilon a\,\omega_2 & \epsilon a\,\omega_1 & 0 & -c\mu \\ 0 & \epsilon cah\,\omega_1 & -\epsilon a h^{-1}\omega_2 & \mu & 0 \end{pmatrix}. \tag{8}$$

For convenience call $t_0 = h$ and write

$$\mathrm{d}t_0 = t_0(t_1\,\omega_1 + t_2\,\omega_2)$$

for certain smooth functions t_1, t_2. Recall that the structure equations are

$$\mathrm{d}\eta_{ji} = -\sum_k \eta_{jk} \wedge \eta_{ki}. \tag{9}$$

These for $j = 3, 4$ and $i = 1, 2$ are the Codazzi equations. It is easy to check using (8) that they are equivalent to the determination of the tangent and normal connections with the above data:

$$\omega = -\frac{t_0^2 t_2}{t_0^2 - c}\,\omega_1 + \frac{c t_1}{t_0^2 - c}\,\omega_2, \qquad \mu = \frac{2c t_0^3 t_2}{t_0^2 - c}\,\omega_1 - \frac{2c t_1}{t_0(t_0^2 - c)}\,\omega_2. \tag{10}$$

For example, since $\eta_{31} = a\eta_{20}$, we get $a\omega_1 \wedge \omega - a t_0 \mu \wedge \omega_1 = a\omega \wedge \omega_1$, i.e., $\omega_1 \wedge (2\omega + t_0\mu) = 0$.

Now define the functions t_{rs}, $1 \le r, s \le 2$ by

$$\mathrm{d}t_i = t_{i1}\,\omega_1 + t_{i2}\,\omega_2.$$

The structure equations (9) for $(j, i) = (1, 2)$ and $(3, 4)$ can be solved for t_{11} and t_{22} in terms of the others as

$$t_{11} = \frac{c\epsilon a^2(5t_0^4 - 4ct_0^2 - 1) + 2c(t_0^4(t_2^2 - 1) - 2t_1^2) + 2t_0^2(2t_1^2 + 1)}{2(t_0^2 - c)}, \tag{11}$$

$$t_{22} = \frac{\epsilon a^2(5 - 4ct_0^2 - t_0^4) + 2ct_0^2(2t_2^2 + 1) - 2(2t_0^4 t_2^2 - t_1^2 + 1)}{2t_0^2(t_0^2 - c)}. \tag{12}$$

Since $0 = \mathrm{d}(\mathrm{d}(\log t_0)) = (t_1t_2 + t_{12} - t_{21})\,\omega_1 \wedge \omega_2$, we express t_{12} and t_{21} in terms of a new function t_3 as

$$t_{12} = \frac{t_3}{t_0} - t_1t_2\frac{t_0^2}{t_0^2 - c}, \qquad t_{21} = \frac{t_3}{t_0} - t_1t_2\frac{c}{t_0^2 - c}.$$

Moreover, the identities $\mathrm{d}(\mathrm{d}t_1) = \mathrm{d}(\mathrm{d}t_2) = 0$ are equivalent to

$$\mathrm{d}t_3 = \big(ct_0^3t_2(9\epsilon a^2 - 4) + 6t_1t_3\big)\,\omega_1 + \big(ct_1(9\epsilon a^2 - 4) - 4t_0t_2t_3\big)\,\omega_2.$$

4. Compatibility analysis

At this point, we have, on V^2, two 1-forms ω_1 and ω_2 which satisfy

$$\mathrm{d}\omega_1 = -\frac{t_0^2t_2}{t_0^2 - c}\,\omega_1 \wedge \omega_2 \quad \text{and} \quad \mathrm{d}\omega_2 = \frac{ct_1}{t_0^2 - c}\,\omega_1 \wedge \omega_2, \tag{13}$$

and four functions t_0, t_1, t_2, and t_3, whose exterior derivatives are expressed explicitly in terms of ω_1, ω_2 and t_0, t_1, t_2, and t_3. In addition, it is easy to check that the structure equations are satisfied by our choices.

By a theorem of Élie Cartan [6], if these explicit formulae for the exterior derivatives imply that $\mathrm{d}(\mathrm{d}t_k) = 0$ for $k = 0, 1, 2, 3$, then local solutions exist in the following sense: for every set of constants $r = (r_0, r_1, r_2, r_3)$, with $r_0 \geq 1$ and $r_0 > 1$ if $c = 1$, there exist a surface V_r and a point $p_r \in V_r$ such that, on V_r, there exist a coframing ω_1, ω_2 and smooth functions t_0, t_1, t_2, and t_3 such that $t_k(p_r) = r_k$. Moreover, such a surface V_r is unique up to local diffeomorphism fixing p_r. Thus, if the $\mathrm{d}^2 = 0$ identity were to hold formally for this system, there would be a 4-parameter family of germs of 'solution manifolds' to these differential equations. However, it turns out that $\mathrm{d}^2 = 0$ is not an identity for this system.

Of course, we know that we must have $\mathrm{d}(\mathrm{d}\omega_1) = \mathrm{d}(\mathrm{d}\omega_2) = 0$ and $\mathrm{d}(\mathrm{d}t_0) = \mathrm{d}(\mathrm{d}t_1) = \mathrm{d}(\mathrm{d}t_2) = 0$, because we used those equations to find the formula for t_{ij}, but we have not checked whether $\mathrm{d}(\mathrm{d}t_3)$ vanishes. In fact, it turns out that the above formulae imply

$$\mathrm{d}(\mathrm{d}t_3) = -\frac{R_0[a, t_0, t_1, t_2, t_3]}{2ct_0}\,\omega_1 \wedge \omega_2,$$

where

$$R_0[a, t_0, t_1, t_2, t_3] = 20ct_3^2 - (9\epsilon a^2 - 4)(\epsilon a^2(t_0^4 + 10ct_0^2 + 1) - 12(t_0^4t_2^2 + t_1^2) - 4ct_0^2).$$

Notice that, if $\epsilon = 1$ and $a = 2/3$, then the vanishing of R_0 is equivalent to the vanishing of t_3. Consequently, we will obtain a system satisfying Cartan's Conditions when $\epsilon = 1$ and $a = 2/3$ by setting $t_3 = 0$. We have shown:

Proposition 13. *If $\epsilon = 1$ and $a = 2/3$ there exists precisely a 3-parameter family of germs of non-minimal surfaces g as in Lemma 9 for both $c = 1$ and $c = -1$.*

We now rule out the remaining cases.

Proposition 14. *Let g be a non-minimal surface as in Lemma 9. If either $\epsilon = -1$ or $a \neq 2/3$, then $c = -1$ and $h \equiv 1$ is constant.*

Proof. Let R_0 be the polynomial in $a, t_0, \ldots, t_3$ defined above. This polynomial vanishes on every solution to the structure equations, and hence its exterior derivative does as well. Compute $\mathrm{d}(R_0)$ using the formulae for the derivatives of the t_k. This will be a 1-form that is a linear combination of ω_1 and ω_2 with coefficients that are rational functions of $a, t_0, \ldots, t_3$ with denominators that are products of powers of t_0 and $t_0^2 - c$. Let R_1 be the numerator of the coefficient of ω_1 in $\mathrm{d}(R_0)$ and let R_2 be the numerator of the coefficient of ω_2 in $\mathrm{d}(R_0)$. Then R_1 and R_2 are polynomials in $a, t_0, \ldots, t_3$ that vanish on all solutions of the structure equations.

Continuing, let R_{11} be the numerator of the coefficient of ω_1 in $\mathrm{d}(R_1)$ and let R_{12} be the numerator of the coefficient of ω_2 in $\mathrm{d}(R_1)$, when these coefficients are expressed as rational functions of $a, t_0, \ldots, t_3$ with denominators that are products of powers of t_0 and $t_0^2 - c$.

In this way, we generate a sequence of polynomials $R_0, R_1, R_2, R_{11}, \ldots$. Consider the ideal F in the polynomial ring $\mathbb{R}[a, t_0, \ldots, t_3]$ generated by the 15 polynomials

$$R_0, R_1, R_2, R_{11}, R_{12}, R_{21}, R_{22}, R_{111}, \ldots, R_{222}.$$

Let B be the Groebner basis of this ideal computed using the pure lexicographical order $t_3 > t_2 > t_1 > t_0 > a$. Then B is an ordered list with 39 elements. The fourth element of B factors as

$$B_4 = (t_0^2 - c)(9\epsilon a^2 - 4)^2 P(a, t_0),$$

where $P(a, t_0)$ is an irreducible polynomial of degree 16 in a and t_0 (the reader can verify this claim using [1]). Now, B_4 being in the ideal F must vanish on any solution of the structure equations. Since $t_0^2 \neq c$, it follows that either $a = 2/3$ and $\epsilon = 1$, or else $P(a, t_0) = 0$.

However, if $P(a, t_0)$ vanishes identically on the solution, then t_0 must be a root of a nontrivial polynomial with constant coefficients and hence t_0 must be constant. Since $\mathrm{d}t_0$ would then vanish identically, it would then follow that t_1 and t_2, and hence t_{11}, t_{12}, t_{21} and t_{22} would vanish identically, but this is clearly impossible unless $t_0 \equiv 1$ by (11) and (12). ■

Corollary 15. *The Ricci eigenvalues of the hypersurfaces in Theorem 1 are constant. For* f_1 *they are* $\{2, -1, -1\}$*, for* f_{-1} *they are* $\{-2, -4, -4\}$*, while for all the others they are* $\{2c, 2c - 9/4, 2c - 9/4\}$*.*

Proof. We only check for the last case, since f_1 is well-known, and the principal curvatures of f_{-1} will be computed in the next section.

Since our 3 manifold has relative nullity Δ of dimension 1, its corresponding Ricci eigenvalue is $2c$. Now, any vector in $\Delta^\perp$ is then a Ricci eigenvector with eigenvalue equal to $c + (c - a^{-2}) = 2c - 9/4$. ■

4.1. The case $c = -1$ and $h \equiv 1$

In this case, we have that $t_0 = h \equiv 1$, and then $t_i = t_{ij} = t_3 = 0$ and, by (13), $\mathrm{d}\omega_i = 0$ for $i, j = 1, 2$. In addition, $\omega = \mu = 0$, so V^2 is a flat surface with flat normal bundle. In particular, $\epsilon = 1$ and $a = 1/\sqrt{2}$ by the second part of Lemma 9. Since $\mathrm{d}\eta = -\eta \wedge \eta$, we conclude from Maurer-Cartan Fundamental Lemma that there exists a unique (up to left translations) solution $G : \tilde{V}^2 \to \mathrm{SO}(4,1)$ of the system $\mathrm{d}G = G\eta$ defined on the universal cover $\tilde{V}^2$ of V^2. In our situation, this is just $G = e^\gamma$, where $\eta = \mathrm{d}\gamma$ and

$$\gamma(x,y) = \begin{pmatrix} 0 & -\sqrt{2} & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix} \frac{x}{\sqrt{2}} + \begin{pmatrix} 0 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \sqrt{2} & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix} \frac{y}{\sqrt{2}}.$$

Then $g = e_0(G)$ is the flat two torus $g : T^2 := \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{S}^4_{-1} \subset \mathbb{R}^{4,1}$ given by

$$g(x,y) := \begin{pmatrix} 2\cos(x)\cos(y) - 1 \\ -\sqrt{2}\sin(x)\cos(y) \\ -\sqrt{2}\cos(x)\sin(y) \\ \sqrt{2}\sin(x)\sin(y) \\ -\sqrt{2}\cos(x)\cos(y) + \sqrt{2} \end{pmatrix},$$

whose induced metric is twice the canonical one. It is easy to check that g satisfies Lemma 9 with $h \equiv 1$ and $a = 1/\sqrt{2}$.

Observe now that g is also contained in the hyperplane $x_1 + \sqrt{2}x_5 = 1$. In fact, after a change of orthonormal basis g can be written as

$$g = (g_0, 1) \colon T^2 \to \mathbb{S}^3(\sqrt{2}) \times \mathbb{R} \subset \mathbb{R}^4 \times \mathbb{R} = \mathbb{R}^{4,1}, \tag{14}$$

where $g_0 \colon T^2 \to \mathbb{S}^3(\sqrt{2}) \subset \mathbb{R}^4$ is the standard minimal equivariant flat Clifford torus,

$$g_0(x,y) = \sqrt{2}\,(\cos(x)\cos(y), \cos(x)\sin(y), \sin(x)\cos(y), \sin(x)\sin(y)).$$

Then, g is an $\mathrm{Iso}(T^2)$-equivariant isoparametric surface in codimension two in the De-Sitter space $\mathbb{S}^4_{-1}$, i.e., it has parallel second fundamental form. The corresponding non-isoparametric complete curvature homogeneous hypersurface $f_{-1} : \Lambda = T^2 \times \mathbb{R} \to \mathbb{H}^4$ is thus given by

$$f_{-1}(x, y, t) = \sinh(t)\xi_1 + \cosh(t)\xi_2 = \frac{1}{\sqrt{2}} \Big(\cosh(t)g_0 + \sqrt{2}\sinh(t)\xi, 2\cosh(t)\Big), \quad (15)$$

where $\xi = (g_0)_{xy}/\sqrt{2}$ is the Gauss map of g_0. The equivariant isometries of g_0 induce a two-parameter family of extrinsic symmetries of f_{-1}. The principal curvatures of f_{-1} are $\{\cosh(t) + \sinh(t), \cosh(t) - \sinh(t), 0\}$.

5. Existence of solutions

In this section we compute the maximal surfaces in Proposition 13.

As already seen, in this case we must have $\epsilon = 1$, $a = 2/3$ and $t_3 = 0$, and therefore our system becomes

$$\begin{pmatrix} \mathrm{d}t_0 \\ \mathrm{d}t_1 \\ \mathrm{d}t_2 \end{pmatrix} = \begin{pmatrix} t_0 t_1 & t_0 t_2 \\ \frac{2(t_0^2-c)(9t_1^2+1)+ct_0^4(9t_2^2+1)-t_0^2}{9(t_0^2-c)} & -\frac{t_0^2 t_1 t_2}{t_0^2-c} \\ -\frac{ct_1t_2}{t_0^2-c} & \frac{-2t_0^2(t_0^2-c)(9t_2^2+1)+9t_1^2+1-ct_0^2}{9t_0^2(t_0^2-c)} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix},$$

together with the given formulae for $\mathrm{d}\omega_1$ and $\mathrm{d}\omega_2$ in (13). Now, as one can verify, one has the identity $\mathrm{d}(\mathrm{d}t_k) = 0$ for $k = 0, 1, 2$, so Cartan's Theorem suffices to prove existence of a one parameter family of surfaces since two degrees of freedom come from moving the base point over the surface.

Now, one can, in this case, prove existence without having to quote Cartan's Theorem, at the price of doing some further computation. In fact, there are other advantages to doing an explicit computation, as will be seen.

Let us write the above equation in the form

$$t_0^2(t_0^2 - c)(\mathrm{d}t_0, \mathrm{d}t_1, \mathrm{d}t_2) = P[t_0, t_1, t_2]\, \omega_1 + Q[t_0, t_1, t_2]\, \omega_2,$$

where $P[t_0, t_1, t_2]$ and $Q[t_0, t_1, t_2]$ are $\mathbb{R}^3$-valued polynomials in t_0, t_1, t_2. Define the $\mathbb{R}^3$-valued polynomial $N[u_0, u_1, u_2]$ as

$$t_0^2(t_0^2 - c)N[t_0, t_1, t_2] = P[t_0, t_1, t_2] \times Q[t_0, t_1, t_2].$$

Notice that the entries of N have no common factor by the definitions of P and Q.

Consider the 1-form θ on $\mathbb{R}^3$ defined by

$$\theta = \langle N[u], \mathrm{d}u \rangle,$$

where $[u] = [u_0, u_1, u_2]$ and $\mathrm{d}u = (\mathrm{d}u_0, \mathrm{d}u_1, \mathrm{d}u_2)$. Calculation shows that θ vanishes only along the two curves

$$C_1 = \{u_2 = 0, 9u_1^2 = (2{u_0}^2 + c)({u_0}^2 - c)\},$$
$$C_2 = \{u_1 = 0, 9u_2^2 = (2{u_0}^{-2} + c)({u_0}^{-2} - c)\},$$

and these two curves only intersect when $c = 1$ and do so at the points $(u_0, u_1, u_2) = (\pm 1, 0, 0)$. Moreover, one computes that $\theta \wedge \mathrm{d}\theta = 0$, i.e., the distribution

$$D = \ker \theta$$

on $\mathbb{R}^3_+ = \{u \in \mathbb{R}^3 : u_0 > 0\}$ satisfies Frobenius integrability, so that its leaves foliate $\mathbb{R}^3_+ \setminus (C_1 \cup C_2)$. In fact, a calculation allows one to find a first integral. Indeed, setting

$$L := \frac{u_0^4(u_0^2(9u_2^2 + 1) + c(9u_1^2 + 1))^2}{(u_0^4(9u_2^2 + 1) + cu_0^2 + (9u_1^2 + 1))^3} \tag{16}$$

one gets that $\theta \wedge \mathrm{d}L = 0$. Note that $0 \leq L \leq 4/27$, with $L = 4/27$ only on $C_1 \cup C_2$. Moreover, $L = 0$ only when $c = -1$ and on the hypersurface $\Omega = \{u_0^2 = (9u_1^2 + 1)/(9u_2^2 + 1)\} \subset \mathbb{R}^3_+$, which is homeomorphic to a plane. For any other value $0 < R < 4/27$, $L^{-1}(R)$ is a smooth integral surface of D which cannot intersect the plane $u_0 = 0$.

Notice also that L is invariant under the transformation

$$\varphi(u_0, u_1, u_2) = (1/u_0, u_2, u_1),$$

and that φ interchanges C_1 and C_2. This corresponds to an arbitrary choice between $h \geq 1$ and $h \leq 1$, and the corresponding swap of the elements of the tangent frame in Lemma 9.

For $c = 1$, let $\Pi \subset \mathbb{R}^3_+$ be the plane $u_0 = 1$ and $\Sigma \subset V$ the set of minimal points of g. For $c = -1$, set both sets Π and Σ as empty.

If $c = 1$, all 2-dimensional leaves of D intersect Π transversally since θ is nonvanishing when pulling back to Π. In fact, given $r \geq 0$, if $R := (9r^2+2)^2/(9r^2+3)^3$ the intersection $\Pi \cap L^{-1}(R)$ is the circle $\mathcal{C}_r$ of radius r centered at the origin, with $r \to 0$ as $R \to 4/27$ and $r \to +\infty$ as $R \to 0$. Each 2-dimensional leaf of D is a union of 2 pair of pants glued at their 'waistline' $\mathcal{C}_r$ (rotated 90° from being aligned with the 'legs' of the opposite pair), that are interchanged by φ, and which then becomes a tube over the connected curve $C_1 \cup C_2$; see the picture on the left in Fig. 1.

If $c = -1$, each 2-dimensional leaf $L^{-1}(R)$ for $0 < R < 4/27$ has two connected components separated by Ω, each of which is diffeomorphic to a cylinder as a tube around one of the disjoint curves C_1 or C_2; see the picture on the right in Fig. 1.

Let V^* be a connected component of $V \setminus \Sigma$. By construction, since N is perpendicular to both P and Q, the function $t = (t_0, t_1, t_2) : V^* \to \mathbb{R}^3_+ \setminus \Pi$ pulls back θ to zero, i.e., it maps V^* onto a leaf of D. Because $N[u_0, u_1, u_2]$ does not vanish outside $C_1 \cup C_2$, it

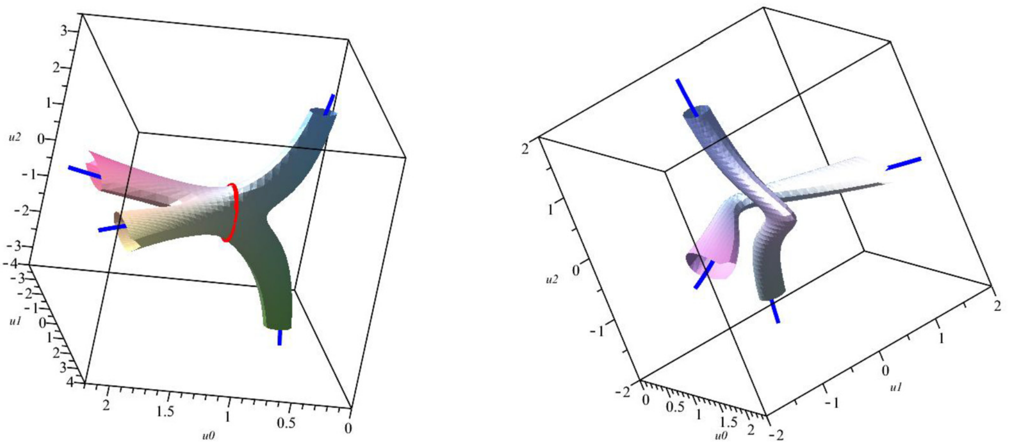

Fig. 1. Leaves of the foliations D for $c = 1$ and $c = -1$.

follows that the map $t : V^* \to \mathbb{R}^3_+ \setminus \Pi$ is an immersion unless its image lies in either C_1 or C_2. Thus, unless $t(V^*) \subset C_1 \cup C_2$, one can regard V^*, up to a covering, as an open set in a leaf of D.

Conversely, if $V \subset \mathbb{R}^3_+$ is a 2-dimensional leaf of D then u_0, u_1, and u_2 restricted to a connected component V^* of $V \setminus \Sigma$ define functions $0 < t_0$, t_1 and t_2, with $t_0 \neq 1$ if $c = 1$, such that the differential of $t = (t_0, t_1, t_2)$ satisfies $\langle N[t_0, t_1, t_2], \mathrm{d}t\rangle = 0$. It follows that there will be unique 1-forms ψ_1 and ψ_2 on V^* satisfying $\mathrm{d}t = P[t]\psi_1 + Q[t]\psi_2$. Setting $\omega_i = t_0^2(t_0^2 - c)\psi_i$, $i = 1, 2$, then defines a coframe on V^*. One can verify that this coframe satisfies (13). In particular, now defining the various η_{ab}, $0 \leq a, b \leq 4$ using their formulae given above in terms of the ω_i and t_0, t_1, t_2, the 1-form η satisfies $\mathrm{d}\eta = -\eta \wedge \eta$. By Maurer-Cartan Fundamental Lemma, there will be a mapping G from the simply connected cover $\tilde{V}^*$ of V^* into $\mathrm{SO}_c(5)$, where $\mathrm{SO}_c(5) = \mathrm{SO}(5)$ if $c = 1$, or $\mathrm{SO}(4,1)$ if $c = -1$, such that $G^{-1}\mathrm{d}G = \eta$. The resulting mapping $g = e_0(G)\colon \tilde{V}^* \to \mathbb{S}^4_c$ will then give an immersion of $\tilde{V}^*$ onto $\mathbb{S}^4_c$ as a surface satisfying Lemma 9 with $a = 2/3$ and $\epsilon = 1$. Notice that, by the above discussion, $\tilde{V}^*$ is homeomorphic to the universal cover of a pair of pants if $c = 1$, and to a plane if $c = -1$.

Since there is a 1-parameter family of 2-dimensional leaves of D, these give a 1-parameter family of these surfaces in $\mathbb{S}^4_c$ that have no continuous symmetries (since the map t is an immersion and it should be invariant by all symmetries, see Remark 12), and every such connected surface in $\mathbb{S}^4_c$ without continuous symmetries is, locally, an open set in one of these surfaces.

It turns out that none of these surfaces is complete:

Proposition 16. *The only complete surfaces g as in Lemma 9 are the Veronese surface and the torus in (14). In particular, there is no rank two complete curvature homogeneous hypersurface in $\mathbb{Q}^4_c$ besides f_c.*

Proof. Assume such a complete surface g different from the Veronese and the one in (14) exists. Since $a = 2/3$ and $\epsilon = 1$, the Gaussian curvature of g is constant $1/9 > 0$. Hence the surface is diffeomorphic to either $\mathbb{S}^2$ or $\mathbb{RP}^2$.

For $c = -1$, since there are no minimal points in g we have a global coframe ω_1, ω_2 on $\mathbb{S}^2$ which is obviously impossible.

For $c = 1$, a computation shows that the square of the mean curvature vector of g, namely, $H = (h^2 - 1)^2/h^2$, is a superharmonic function, since

$$h^4 \Delta H/2 = (4h^6 + h^4 + 2h^2 + 1)t_1^2 + h^2(h^6 + 2h^4 + h^2 + 4)t_2^2 + (h^4 - 1)^2/9 \geq 0.$$

Thus H and h are constant. By the above $h = 1$, g is minimal and therefore the Veronese surface. ■

Remark 17. In [5] the global topology of these surfaces will be addressed. In particular, for $c = 1$, it will be shown that Σ is a smooth isolated minimal point in V and that the structure equations can be extended smoothly to the circles $\mathcal{C}_r$.

6. The rotationally symmetric case

In this section we analyze the remaining case, namely, when $t(V)$ lies in one of the curves C_1 and C_2.

We first claim that we may assume that $t(V) \subset C_1$. Indeed, for $c = -1$, the case $t(V) \subset C_2$ is completely analogous, since it corresponds to reversing the roles between e_1 and e_2 (and thus between h and $-c/h$) in Lemma 9, namely, the φ-invariance above. In particular, both cases give isometric surfaces. For $c = 1$, the curve C_2 is empty if $t_0 > 1$ by the sign of the right hand side polynomial defining the curves.

Now, since $t_2 = 0$ and $t_1^2 = (2t_0^2 + c)(t_0^2 - c)/9$, the structure equations are

$$\mathrm{d}\omega_1 = 0, \quad \mathrm{d}\omega_2 = \frac{ct_1}{t_0^2 - c}\,\omega_1 \wedge \omega_2, \quad \mathrm{d}t_0 = t_0 t_1\,\omega_1, \quad \mathrm{d}t_1 = \frac{1}{9}t_0^2(4t_0^2 - c)\,\omega_1.$$

Notice that the last one is a consequence of the third one and the above formula for t_1^2. These can be easily solved for certain coordinates r and θ on V as

$$\omega_1 = 3\mathrm{d}r, \quad \omega_2 = 3\sin(r)\mathrm{d}\theta, \quad t_0 = \sqrt{\frac{2c}{3\cos(2r) - 1}}, \quad t_1 = \frac{\sin(2r)}{3\cos(2r) - 1}.$$

Set $r_0 = \arccos(\sqrt{2/3}\,)$ and $r_1 = \pi - r_0$. A maximal domain of the chart is $0 < r < r_0$ if $c = 1$ and $r_0 < r < r_1$ if $c = -1$, namely, $V = D^2(r_0)$ is a disk of radius r_0 if $c = 1$ and the annulus $V = (r_0, r_1) \times S^1$ if $c = -1$. Moreover, the surface becomes singular as $r \to r_i$ where its mean curvature vector field is unbounded. If $c = 1$, then $r \to 0$ if and only if $t_0 \to 1$, that is, the origin is the only minimal point of V, and one can verify that it is a smooth point.

Using these formulae in (8) and (10) we get $\eta = \eta_1(r)\mathrm{d}r + \eta_2(r)\mathrm{d}\theta$, where

$$\eta_1(r) = \begin{pmatrix} 0 & -3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & -2t_0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 2ct_0 & 0 & 0 & 0 \end{pmatrix}, \ \eta_2(r) = \begin{pmatrix} 0 & 0 & -3S & 0 & 0 \\ 0 & 0 & -C & -2S & 0 \\ 3S & C & 0 & 0 & 2cS/t_0 \\ 0 & 2S & 0 & 0 & 2cC/t_0 \\ 0 & 0 & -2S/t_0 & -2C/t_0 & 0 \end{pmatrix},$$

where S and C stand for $\sin(r)$ and $\cos(r)$ for clarity. It is easy to verify that the structure equation $\mathrm{d}\eta = -\eta \wedge \eta$, or equivalently $[\eta_1, \eta_2] = -\eta_2'$, is satisfied. Maurer-Cartan Fundamental Lemma thus implies that there is a map $G_c : V \to \mathrm{SO}_c(5)$ such that $G_c^{-1}\mathrm{d}G_c = \eta$. Then $\hat{g}_c = e_0(G_c) : V \to \mathbb{S}^4_c$ gives an immersion whose image is a surface in $\mathbb{S}^4_c$ as in Lemma 9. Observe that $\hat{g}_c$ has a 1-parameter symmetry group induced by translations in θ, since η is invariant under them. Notice also that the system $G_c^{-1}\mathrm{d}G_c = \eta$ is equivalent to

$$\frac{\partial G_c}{\partial r} = G_c\,\eta_1(r), \quad \frac{\partial G_c}{\partial \theta} = G_c\,\eta_2(r).$$

The first equation (or equivariance) implies that $G_c(r,\theta) = e^{\theta H}T(r)$ and $T' = T\eta_1$, with $H \in \mathfrak{so}_c(5)$. By the second equation, $H = T(r)\eta_2(r)T(r)^{-1}$ does not depend on r and gives us H. In addition, since $\eta_1 = \eta_{11} \oplus \eta_{12}$ is reducible in the $\{e_0, e_1, e_4\}$ and $\{e_2, e_3\}$ subspaces, the problem becomes an ODE in $Gl(5,\mathbb{R})$ of the form $T_1' = T_1\eta_{11}$ by taking an initial value in $\mathrm{SO}_c(5)$. This is easily and explicitly integrable, giving G whose first column is (congruent to) the surface

$$\hat{g}_c = \begin{pmatrix} 3\sin(\theta)\sin(r)\cos(2r) \\ 3\cos(\theta)\sin(r)\cos(2r) \\ (3/2)\sin(2\theta)\sin(r)\sin(2r) \\ (3/2)\cos(2\theta)\sin(r)\sin(2r) \\ ((3\cos(2r)-1)/2c)^{3/2} \end{pmatrix}. \tag{17}$$

As a subset in $\mathbb{R}^5$, $\hat{g}_c(V)$ is cut out by three polynomial equations, so it is contained in a singular algebraic surface $\mathcal{V}_c$, the intersection of three polynomials of degrees 2, 3, and 6. The parametrization above only gives half of $\mathcal{V}_c$, the half for which the fifth coordinate is greater than or equal to zero. The other half is got by replacing the fifth coordinate with its negative. The values for which $r \to r_i$ are closed torus knots in the Clifford torus in $\mathbb{S}^3$ where $\mathcal{V}_c$ has 'creases', and it is smooth everywhere else. It is here that the mean curvature of $\hat{g}_c$ goes to infinity, similarly to the rim of the tractroid in $\mathbb{R}^3$ with constant curvature -1. The points $(0,0,0,0,\pm 1)$ for $c = 1$ are the minimal smooth points. Therefore, the maximal V is a disk for $c = 1$, and an annulus for $c = -1$, with torus knots as boundaries. Moreover, making the substitution $u = \sqrt{3}\sin(\theta)\sin(r)$, $v = \sqrt{3}\cos(\theta)\sin(r)$ and $w = \pm\sqrt{(3\cos(2r)-1)/2}$ in (17), we see that for $c = 1$ the full $\mathcal{V}_1$ is smoothly parametrized by the unit sphere $u^2 + v^2 + w^2 = 1$ in the form

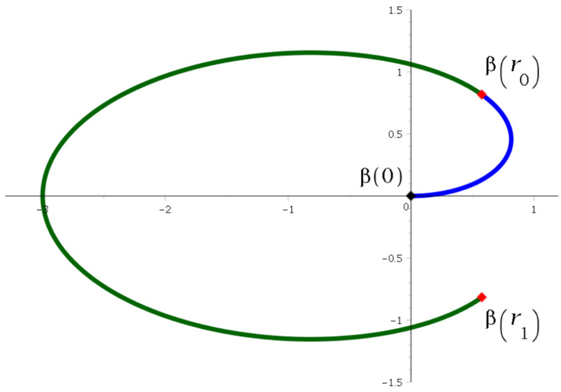

Fig. 2. The curve β, in blue for $c = 1$ and green for $c = -1$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$Y(u,v,w) = \frac{1}{\sqrt{3}} \begin{pmatrix} u\,(1+2w^2) \\ v\,(1+2w^2) \\ 2uv\,\sqrt{2+w^2} \\ (u^2-v^2)\,\sqrt{2+w^2} \\ \sqrt{3}\,w^3 \end{pmatrix}. \tag{18}$$

The embedding Y is a smooth immersion away from the circle $w = 0$, which corresponds to the crease.

Notice also that, by (17), $\hat{g}_c$ can also be constructed as a specific $\mathbb{S}^1$-orbit in $\mathbb{R}^4$ of a piece of the algebraic plane curve

$$(x^2+4y^2)^3 - 9(x^2+4y^2)^2 + 81y^2 = 0,$$

parametrized by $\beta(r) = 3\sin(r)(\cos(2r), \sin(2r)/2)$, and then simply adding as a fifth coordinate $\sqrt{c(1-\|\beta\|^2)}$ to place it in $\mathbb{S}^4_c$. From this we easily see that $\hat{g}_c$ is embedded; see Fig. 2.

We can also use the map G_c to give an explicit parametrization of $\hat{f}_c$ in Theorem 1 by taking $\hat{f}_c = \hat{f}_c(r,\theta,\alpha) = \cos_c(\alpha)e_3(G) + \sin_c(\alpha)e_4(G)$, giving us the explicit expression (1) in the Introduction. The image of $\hat{f}_c$ is also contained in an algebraic hypersurface of $\mathbb{Q}^4_c$, namely, the intersection of $\mathbb{Q}^4_c \subset \mathbb{E}^5$ with the 0-level set of the polynomial

$$64x_5^4(R+1) - \left(x_5^2(R^2-4R-8) - 27c\,(x_1(x_3^2-x_4^2) + 2x_2x_3x_4)^2\right)^2,$$

where $R = 8x_1^2 + 8x_2^2 - x_3^2 - x_4^2$.

7. The unique example in $\mathbb{H}^5$

Here we show how the Gauss parametrization can be used to obtain a simpler and more direct proof of Tsukada's theorem, which states that there is a unique rank two curvature homogeneous hypersurface in $\mathbb{H}^5$. We will also recover its basic properties, showing in particular that it is closely related to both f_c's in the Introduction.

The polar map of such a hypersurface is a surface in the De-Sitter space, $g : V^2 \to \mathbb{S}^5_{-1} \subset \mathbb{R}^{5,1}$, i.e.,

$$\langle f, f\rangle = -1, \quad \langle f, g\rangle = 0, \quad \langle g, g\rangle = 1, \quad \langle \mathrm{d}f, g\rangle = 0.$$

By Proposition 4 we know that $\det A_w \neq 0$ is constant for every w in an open subset of Λ.

Choose a orthonormal normal frame $\{\xi_0, \xi_1, \xi_2\}$ of $T_g^\perp V$ with $-\langle \xi_0, \xi_0\rangle = \langle \xi_1, \xi_1\rangle = \langle \xi_2, \xi_2\rangle = 1$. We call the respective shape operators A, B, C for short, and we can assume that $\operatorname{tr} B = 0$. Write $w = \cosh(r)\xi_0 + \sinh(r)(\cos(t)\xi_1 + \sin(t)\xi_2) \in \Lambda$ for certain $(r, t) \in W \subset \mathbb{R}^2$, W open, and thus $a^2 = -\det A_w$ is constant. Therefore,

$$a\, A_w = \cosh(r)A + \sinh(r)B_t, \quad B_t = \cos(t)B + \sin(t)C.$$

Since $a \neq 0$ it easily follows that A is invertible, and hence

$$\cosh(r)^2 \det A + \sinh(r)^2 \det B_t + \cosh(r)\sinh(r)\operatorname{tr}(A^{-1}B_t)\det A = -1. \tag{19}$$

This is equivalent to $\det A = -\det B_t = 1$, and $\operatorname{tr}(A^{-1}B_t) = 0$. By Lemma 9, the pair $\{B, C\}$ has the special normal form (7) for $a = c = 1$. Since $\operatorname{tr}(A^{-1}B_t) = 0$, in this tangent frame A must have the form $Ae_1 = he_1$, $Ae_2 = h^{-1}e_2$, up to a possible change of the sign of ξ_0. Finally, replacing ξ_0 by $\frac{1+h^2}{2h}\xi_0 + \frac{1-h^2}{2h}\xi_2$ and ξ_2 by $\frac{1-h^2}{2h}\xi_0 + \frac{1+h^2}{2h}\xi_2$, we can assume that $h = 1$ and $A = I$. We conclude that V^2 has constant curvature $1 - 3a^2$, and that, in a fixed orthonormal basis $\{e_1, e_2\}$ of V^2, the second fundamental form of g is unique and satisfies $A_{\xi_0} = aI$, with A_{ξ_1}, A_{ξ_2} as in (6) in some orthonormal normal frame that we still call $\{\xi_0, \xi_1, \xi_2\}$.

We can now easily compute the normal connection 1-forms w^i_j, that is, $\nabla^\perp_X \xi_j = \sum_{i=1}^3 w^i_j(X)\xi_i$. Noticing that $(-1)^{\delta^j_0} w^i_j + w^j_i = 0$, set $w_{i+j} = w^j_i$ for $i < j$. The Codazzi equations are as usual $[DA_{\xi_j}]^* = -(-1)^{\delta^j_0}\sum_i w^i_j \circ JA_{\xi_i}$, where $[DA] = \nabla_{e_1}A(e_2) - \nabla_{e_2}A(e_1) - A[e_1, e_2]$ and J is given by $Je_1 = e_2$, $Je_2 = -e_1$. In our case, if $\beta = \langle \nabla_\bullet e_1, e_2\rangle$,

$$w_1 \circ JB + w_2 \circ JC = 0, \quad 2\beta \circ B = w_1 \circ J - w_3 \circ JC, \quad 2\beta \circ C = w_2 \circ J + w_3 \circ JB,$$

which determines the normal connection and is independent of a. Using that $-BC = CB = J$ and $B^2 = C^2 = I$ we easily see that $w_1 = w_2 = 0$, $w_3 = 2\beta$. In particular, ξ_0 is normal parallel and $\mathrm{d}w_3 = 2(3a^2 - 1)dvol$ since V^2 has constant curvature $1 - 3a^2$. Furthermore, the Ricci equation implies that

$$2a^2 = -\langle [A_{\xi_1}, A_{\xi_2}]e_1, e_2\rangle = -\langle R^\perp(e_1, e_2)\xi_1, \xi_2\rangle = -\mathrm{d}w_3(e_1, e_2) = 2(1 - 3a^2).$$

We conclude that $a^2 = 1/4$, V^2 is locally isometric to $\mathbb{S}^2_{1/4}$, and g is unique.

Now, since all shape operators of the minimal Veronese embedding $g_1 : \mathbb{RP}^2_{1/3} \to \mathbb{S}^4$ are conjugate to $3^{-1/2}B$, and the shape operator in the normal parallel direction ξ_0 is $I/2$, it is easy to get an explicit expression for g,

$$g = \frac{1}{\sqrt{3}}(2g_1, 1) : \mathbb{RP}^2_{1/4} \to \mathbb{S}^5_{-1} \subset \mathbb{R}^{5,1}.$$

Once we computed g we can finally recover f. The normal bundle of g in $\mathbb{R}^{5,1}$ is $\text{span}\{g, \xi_0 = (g_1, 2)/\sqrt{3}\,\} \oplus \nu$, where ν stands for the normal bundle of g_1 and ν_1 its unit normal bundle. So $\Lambda = \{c\xi_0 + s(\xi, 0) : \xi \in \nu_1, c^2 - s^2 = 1\}$, and therefore $f : M^4 = \Lambda = \nu_1 \times \mathbb{R} \to \mathbb{H}^5 \subset \mathbb{R}^{5,1}$ is

$$f(\xi_x, s) = \frac{1}{\sqrt{3}} \left(\cosh(s) g_1(x) + \sqrt{3}\sinh(s)\xi_x, 2\cosh(s) \right).$$

Since ν_1 as a hypersurface in $\mathbb{S}^4$ is SO(3)-equivariant, so is f, with SO(3) acting on $\mathbb{R}^5 \times \{0\} \subset \mathbb{R}^{5,1}$. Notice that f is clearly complete since ν_1 is compact and $\|f_*\partial_s\| = 1$. Compare the above expression for f with the one for f_{-1} in (15).

References

[1] R. Bryant, L. Florit, W. Ziller, Maple file to follow the computations in the paper "Curvature homogeneous hypersurfaces in space forms", https://luis.impa.br/papers/CurvHomHypPart1.mws.

[2] E. Boeckx, O. Kowalski, L. Vanhecke, Riemannian Manifolds of Conullity Two, World Scientific Publishing Co., 1996, pp. xviii+300.

[3] T. Brooks, 3-manifolds with constant Ricci eigenvalues $(\lambda, \lambda, 0)$, Geom. Dedic. 219 (2024).

[4] R. Bryant, On the geometry of curvature-homogeneous Riemannian three-manifolds, in preparation.

[5] R. Bryant, The global structure of curvature homogeneous hypersurfaces in 4-dimensional space forms, in preparation.

[6] É. Cartan, Sur la structure des groupes inifinis de transformations, Ann. Éc. Norm. 21 (1904) 153–206.

[7] É. Cartan, Sur quelque familles remarquables d'hypersurfaces, C.R. Congr. Math. Liège (1939) 30–41 (also in Oeuvres Complètes, Partie III, vol. 2, 1481-1492).

[8] G. Calvaruso, R. Marinosci, D. Perrone, Three-dimensional curvature homogeneous hypersurfaces, Arch. Math. 36 (2000) 269–278.

[9] M. Dajczer, L. Florit, Compositions of isometric immersions in higher codimension, Manuscr. Math. 105 (2001) 507–517.

[10] M. Dajczer, D. Gromoll, Gauss parametrizations and rigidity aspects of submanifolds, J. Differ. Geom. 22 (1985) 1–12.

[11] D. Ferus, H. Karcher, H.F. Münzner, Clifford algebras and new isoparametric hypersurfaces, Math. Z. 177 (1981) 479–502.

[12] J. van Hook, On the geometry of conullity two manifolds, Differ. Geom. Appl. 92 (2024).

[13] V. Sbrana, Sulla varietá ad $n - 1$ dimensioni deformabili nello spazio euclideo ad n dimensioni, Rend. Circ. Mat. Palermo 27 (1909) 1–45.

[14] K. Sekigawa, On the Riemannian manifolds of the form $B_f \times F^n$, Kodai Math. Semin. Rep. 26 (1975) 343–347.

[15] I.M. Singer, Infnitesimally homogeneous spaces, Commun. Pure Appl. Math. 13 (1960) 685–697.

[16] H. Takagi, On curvature homogeneity of Riemannian manifolds, Tohoku Math. J. 26 (1974) 581–585.

[17] K. Tsukada, Curvature homogeneous hypersurfaces immersed in a real space form, Tohoku Math. J. 40 (1988) 221–244.

[18] F. Ticerri, L. Vanhecke, Varietes riemanniennes dont le tenseur de courbure est celui d'un espace symetrique irreductible, C. R. Acad. Sci. Paris 302 (1986) 233–235.