SCALAR POSITIVE IMMERSIONS

LUIS A. FLORIT AND BERNHARD HANKE

Abstract. As shown by Gromov-Lawson and Stolz the only obstruction to the existence of positive scalar curvature metrics on closed simply connected manifolds in dimensions at least five appears on spin manifolds, and is given by the non-vanishing of the α-genus of Hitchin.

When unobstructed we shall realise a positive scalar curvature metric by an immersion into Euclidean space whose dimension is uniformly close to the classical Whitney upper bound for smooth immersions. Our main tool is an extrinsic counterpart of the well-known Gromov-Lawson surgery procedure for constructing positive scalar curvature metrics.

1. Introduction

One of the central results in positive scalar curvature geometry [GL80, Sto92] says that a closed simply connected manifold M of dimension $n \geq 5$ admits a Riemannian metric of positive scalar curvature, unless M is spin and Hitchin's α-genus $\alpha(M) \in \mathrm{KO}^{-n}$ is non-zero (see [Hit74]). The purpose of our paper is to apply the ideas behind this result to the classical problem of finding immersions into Euclidean space in low codimensions under certain curvature hypotheses. We are interested here in positive scalar curvature. In this work all manifolds and maps between manifolds are assumed to be smooth, if not stated otherwise.

Definition 1.1. We say that an immersion $f : M \to \mathbb{R}^N$ is *scalar positive* if the Riemannian metric induced on the manifold M by f has positive scalar curvature.

The classical Nash isometric embedding theorem [Nas56] implies that a compact Riemannian n-manifold M of positive scalar curvature admits an isometric, hence scalar positive, immersion into Euclidean space whose dimension depends quadratically on n. Our main result shows that, in the cases mentioned before, this dimension bound can be improved considerably, if we do not restrict to a specific positive scalar curvature metric on M.

Theorem 1.2. *Let M be a closed simply connected manifold with $\dim M = n \geq 5$. If M is spin, assume further that $\alpha(M) = 0$. Then there exists a scalar positive immersion $M \to \mathbb{R}^{2n-1+\delta(n)}$, where*

$$\delta(n) = \begin{cases} \max\{0, 13 - \beta(n+6)\} \in \{0, \ldots, 12\} & \text{if } M \text{ is spin,} \\ \max\{0, 9 - \beta(n+4)\} \in \{0, \ldots, 8\} & \text{if } M \text{ is not spin.} \end{cases}$$

Here $\beta(m)$ denotes the number of digits 1 in the dyadic expansion of $m \in \mathbb{N}$.

Recall that $2n-1$ is Whitney's classical upper dimension bound for immersions of n-manifolds, $n \geq 2$, into Euclidean space. The dimension bound for scalar positive immersions in Theorem 1.2 increases the Whitney bound by at most twelve, and is in fact equal to the Whitney bound in most dimensions. But the Whitney bound itself is in general not sufficient for realising scalar positive immersions. Indeed, as we will see in Section 3, the normal bundle of such an immersion $M \to \mathbb{R}^N$ splits off the line spanned by the nowhere vanishing mean curvature field. Hence, by [Hir59, Theorem 6.4], the manifold M actually immerses into $\mathbb{R}^{N-1}$ if $\dim M < N - 1$, where for

2010 *Mathematics Subject Classification.* Primary: 53C42, 57R65, 53A07; Secondary: 53C23, 57Q60.

Key words and phrases. Manifolds of positive scalar curvature, scalar positive immersions, extrinsic surgery, Veronese embedding.

The first named author has been supported by CNPq - Brazil.

The second named author has been supported by the Special Priority Program SPP 2026 "Geometry at Infinity" funded by the DFG.

non-compact connected M the assumption $\dim M < N - 1$ can be dropped by [Hir61, Theorem 4.7.]. This observation is exploited in the following example.

Example 1.3. On the one hand, according to Theorem 1.2, for $m \geq 3$, the complex projective space $\mathbb{C}P^m$ admits a scalar positive immersion into $\mathbb{R}^{4m+11}$. [1] On the other hand, for $m := 2^\ell$, $\ell \geq 1$, $\mathbb{C}P^m$ does not immerse into $\mathbb{R}^{4m-2}$ by [SS63, Theorem 4]. Hence it does not admit a scalar positive immersion into $\mathbb{R}^{4m-1}$.

These considerations lead us to the following interesting open problem.

Question 1.4. Let M be a closed manifold admitting both a positive scalar curvature metric and an immersion into $\mathbb{R}^N$. Does M admit a scalar positive immersion into $\mathbb{R}^{N+1}$?

Remark 1.5. The corresponding question for non-compact connected manifolds has an affirmative answer due to Gromov's h-principle. For more details see Proposition 3.5 below. In particular, using [Hir61, Theorem 4.7.], this result implies that a non-compact connected parallelizable manifold M of dimension $n \geq 2$ admits a scalar positive immersion into $\mathbb{R}^{n+1}$, but clearly not into $\mathbb{R}^n$.

The main ingredient for our proof of Theorem 1.2 is the following extrinsic version of the surgery result proved independently by Gromov-Lawson and Schoen-Yau [GL80, SY79].

Theorem 1.6. *Let M be an n-dimensional manifold, and assume that $\hat{M}$ is obtained from M by a surgery along an embedded sphere $S^d \subset M$ of codimension $n - d \geq 3$. If M admits a scalar positive immersion $f \colon M \to \mathbb{R}^N$ with $N \geq n + d + 2$, then so does $\hat{M}$. Moreover, this immersion can be constructed in such a way that it coincides with f outside an arbitrarily small neighborhood of S^d in M.*

Our paper is organized as follows. In Section 2 we construct scalar positive immersions of total spaces of fibre bundles whose fibres are equipped with positive scalar curvature metrics, using a variation of the well-known fibrewise shrinking process in Riemannian submersions with scalar positive fibres. In Example 2.7 this leads to scalar positive immersions of total spaces of $\mathbb{C}P^2$-, and $\mathbb{H}P^2$-bundles, from which the scalar positive immersions in Theorem 1.2 will ultimately be constructed by extrinsic surgeries in codimensions at least 3. In Sections 3 and 4, which form the technical core of our paper, we study the two types of local deformations near the surgery sphere that we need for the extrinsic surgery process in Theorem 1.6. At first, we use the local flexibility lemma proven by Bär and the second named author [BH] to bring the given scalar positive immersion into a particularly convenient form near the surgery sphere; see Proposition 3.4. In Proposition 4.12 we construct appropriate bending profiles required for the extrinsic surgery. Some parallels and differences between our deformation and the one in [GL80] will be pointed out in Remark 4.13. After these preparations the proofs of Theorems 1.6 and 1.2 are completed in Section 5

Acknowledgments: The second named author is grateful to IMPA, Rio de Janeiro, and to the Courant Center (NYU), New York, for their hospitality when parts of this research were carried out. Also he wishes to thank Misha Gromov for a number of stimulating remarks.

2. Scalar positive immersions via normal bundle scaling

In this section we construct scalar positive immersions for total spaces of a large class of fibre bundles. On the one hand this will provide scalar positive immersions of the "surgery handles" appearing in the extrinsic surgery in Section 5. On the other hand in Example 2.7 we obtain scalar positive immersions of total spaces of fibre bundles from which the manifolds M in Theorem 1.2 can be obtained by extrinsic surgeries in codimensions at least 3.

Let us begin by establishing the basic setting for this section.

[1]For $m \geq 5$ this improves the embedding dimension $m^2 + 2m$ of the isometric Veronese embedding of $\mathbb{C}P^m$ with the Fubini-Study metric.

Assumption 2.1. Let B be a compact ℓ-dimensional manifold, possibly with boundary, and let $\Psi : E \to B$ be a Euclidean vector bundle of rank m. Furthermore let $X \subset \mathbb{R}^m$ be a compact submanifold with induced metric h, and $\pi : V \to B$ a sub-fibre bundle of Ψ with fibre X in the sense that, around each point in B, there exists an isometric vector bundle trivialisation $\Psi : E|_U \overset{\cong}{\to} U \times \mathbb{R}^m$ satisfying

$$\Psi(V \cap E|_U) = U \times X \subset U \times \mathbb{R}^m \,. \tag{1}$$

In particular, the structure group of π reduces to the isometry group of h.

Definition 2.2. We say that a Riemannian metric g on the total space E is *compatible* with the Euclidean structure of E if there is an orthogonal decomposition

$$g|_B = g_B \oplus \langle \cdot\,, \cdot \rangle_E \,, \tag{2}$$

where g_B is the metric on $B \cong B \times \{0\} \subset E$ induced by g, and $\langle \cdot\,, \cdot \rangle_E$ is the given bundle metric on E considered as a subbundle of $TE|_B$.

For $0 < \lambda \leq 1$ we denote by $\lambda V \subset E$ the image of the fibrewise dilation of V by λ.

Proposition 2.3. *Assume* $\mathrm{scal}_h > 0$ *and let g be a Riemannian metric on E which is compatible with the Euclidean structure of E. Then there exists $0 < \lambda_0 \leq 1$ such that, for all $0 < \lambda \leq \lambda_0$, the induced metric on $\lambda V \subset (E, g)$ has positive scalar curvature.*

Proof. Without loss of generality we can assume that $X \subset \mathbb{R}^m$ is contained in the closed unit ball $D^m_1 \subset \mathbb{R}^m$. Consider an open subset $U \subset B$ together with an isometric bundle trivialisation $\Psi : E|_U \overset{\cong}{\to} U \times \mathbb{R}^m$ satisfying (1), and a local manifold chart $\phi : U \to \phi(U) \subset \mathbb{R}^\ell$. Setting $n = \ell + m$, we obtain a manifold chart

$$\Phi : E|_U \overset{\Psi}{\cong} U \times \mathbb{R}^m \overset{\phi\times \mathrm{id}}{\cong} \phi(U) \times \mathbb{R}^m \subset \mathbb{R}^n \,.$$

Fix standard coordinates $(x^1, \dots, x^\ell)$ and $(x^{\ell+1}, \dots, x^n)$ on $\mathbb{R}^\ell$ and $\mathbb{R}^m$. With respect to Φ write on $E|_U$ the smooth functions $g_{ij} = g(\partial_{x^i}, \partial_{x^j}) : \phi(U) \times \mathbb{R}^m \to \mathbb{R}$, where $1 \leq i, j \leq n$.

For $0 < \lambda \leq 1$ we now consider the metrics g_λ and $\tilde{g}_\lambda$ on $\frac{1}{\lambda}\phi(U) \times \mathbb{R}^m$ given by

$$g_\lambda(x) := \sum_{ij} g_{ij}(\lambda x^1, \dots, \lambda x^N)\, dx^i dx^j \,, \quad \tilde{g}_\lambda(x) := \sum_{i,j} g(\lambda x^1, \dots, \lambda x^\ell, 0, \dots, 0)\, dx^i dx^j \,. \tag{3}$$

We have an isometry

$$\left(\frac{1}{\lambda}\phi(U) \times \mathbb{R}^m, g_\lambda\right) \overset{\alpha}{\cong} \left(E|_U, \frac{1}{\lambda^2}\, g\right), \qquad \alpha(x, y) = \Phi^{-1}(\lambda x, \lambda y).$$

By (2) and since Ψ is an isometric bundle trivialisation, we furthermore have another isometry

$$\left(\frac{1}{\lambda}\phi(U) \times \mathbb{R}^m, \tilde{g}_\lambda\right) \overset{\beta}{\cong} \left(U \times \mathbb{R}^m, \frac{1}{\lambda^2}\,(g_B)_U \oplus g_{eucl}\right), \qquad \beta(x, y) = \big(\phi^{-1}(\lambda x), y\big).$$

Since $\phi(U) \subset \mathbb{R}^m$ is relatively compact all partial derivatives of $g_{ij} : \phi(U) \times D^m_1 \to \mathbb{R}$, $1 \leq i, j \leq n$, are uniformly norm bounded, and hence the chain rule shows that

$$\lim_{\lambda \to 0} \|g_\lambda - \tilde{g}_\lambda\|_{C^2_\lambda} = 0 \,, \tag{4}$$

where C^2_λ denotes the C^2-norm of C^2-bounded smooth sections of $T^*V_\lambda \otimes T^*V_\lambda \to V_\lambda$ with respect to the framing $dx^i dx^j$, setting $V_\lambda := \frac{1}{\lambda}\phi(U) \times D^m_1$. Using that

$$\lim_{\lambda \to 0} \| \,\mathrm{scal}_{\frac{1}{\lambda^2} g_B} \|_{C^0(B)} = 0,$$

and $\mathrm{scal}_h > 0$, we conclude with (4) and the isometry β that there exists $0 < \lambda_0 \leq 1$ such that, for all $0 < \lambda \leq \lambda_0$, the metric g_λ restricts to a positive scalar curvature metric on $\frac{1}{\lambda}\phi(U) \times X$. Here we use the fact that, since $X \subset \mathbb{R}^m$ is compact, it can be covered by finitely many submanifold charts in such a way that (4) also holds for the restricted metrics on $\frac{1}{\lambda}\phi(U) \times X$. Using the isometry α this shows that $\frac{1}{\lambda^2} g$, and hence also g, restrict to positive scalar curvature metrics on $\lambda V|_U$ for $0 < \lambda \leq \lambda_0$.

We finish the proof using the fact that S can be covered by finitely many coordinate neighborhoods U. □

Given an immersion $g : M \to N$, we denote by ν_g its normal bundle, while for an embedded submanifold $S \subset M$ its normal bundle will be denoted by ν_S^M.

Example 2.4. Let M be a Riemannian manifold, let $S \subset M$ be a compact submanifold of codimension at least 3 and let $\rho_0 > 0$ such that the normal exponential map $\exp^\perp : \nu_S^M \to M$ restricts to a diffeomorphism $\{|\eta| < \rho_0\} \approx U_{\rho_0}(S)$ of the open ρ_0-disc bundle in ν_S^M to the open ρ_0-neighborhood of S in M. Since $\exp^\perp$ induces a metric on $\{|\eta| < \rho_0\} \subset \nu_S^M$ which is compatible with the Euclidean structure of ν_S^M, Proposition 2.3 implies that there exists $0 < \rho < \rho_0$ such that, for all $0 < \rho' \leq \rho$, the induced metric on the normal spherical ρ'-tube $\exp^\perp(\{|\eta| = \rho'\}) \subset M$ is of positive scalar curvature.

This statement also appears at the beginning of the proof of [GL80, Lemma 2] for trivial ν_S^M. However, this proof employs a limit process which converges to a singular metric for $\varepsilon \to 0$ and hence, in our opinion, is not well justified. Our proof of Proposition 2.3 elaborates on the argument given in [GL80].

Our purpose now is to apply Proposition 2.3 to certain sub-fibre bundles of Euclidean vector bundles. In order to do this we first need the following.

Lemma 2.5. *Let $E_1, E_2 \to B$ be Euclidean vector bundles and let $\psi : E_1 \to E_2$ be an injective vector bundle homomorphism. Then ψ can be deformed through injective vector bundle homomorphisms into a vector bundle homomorphism $\psi' : E_1 \to E_2$ which induces an isometry between E_1 and the Euclidean subbundle $\psi'(E_1) \subset E_2$.*

Proof. Let $r_1 \leq r_2$ be the ranks of E_1 and E_2. Let $\mathrm{Inj}(r_1, r_2) \subset \mathbb{R}^{r_2 \times r_1}$ denote the space of matrices of maximal rank r_1 and $\mathrm{Iso}(r_1, r_2) \subset \mathrm{Inj}(r_1, r_2)$ denote the subspace of matrices whose columns form an orthonormal family of vectors in $\mathbb{R}^{r_2}$. The inclusion $\mathrm{Iso}(r_1, r_2) \subset \mathrm{Inj}(r_1, r_2)$ is a strong deformation retract by the Gram-Schmidt process. Hence the required deformation can be constructed inductively over a cellular decomposition of B by standard obstruction theory. □

Proposition 2.6 (Normal bundle scaling)**.** *Under Assumption 2.1 suppose furthermore that* $\mathrm{scal}_h > 0$ *and that there exists an immersion $F : E \to \mathbb{R}^N$ of the total space of Ψ. Then there exists a scalar positive immersion $f : V \to \mathbb{R}^N$.*

Proof. Let $g = F|_B$ and $\tau : B \times \mathbb{R}^N \to \nu_g$ be the fibrewise orthogonal projection onto its normal bundle. Since F is an immersion we obtain an injective vector bundle homomorphism $\psi : E \to \nu_g$ which over $q \in B$ is given by $\psi_q : E_q \xrightarrow{d_q F} \mathbb{R}^N \xrightarrow{\tau} (\nu_g)_q$. By Lemma 2.5 we can deform ψ into a fibrewise isometric vector bundle homomorphism $\psi' : E \to \nu_g$.

Now choose $\rho_0 > 0$ such that $\chi : \nu_g \to \mathbb{R}^N$, $\chi(q, \omega) := q + \omega$, restricts to an immersion $\{|\eta| < \rho_0\} \to \mathbb{R}^N$. Notice that the metric on $\{|\eta| < \rho_0\} \subset E$ induced by $\chi \circ \psi'$ is compatible with the Euclidean structure of E. By Proposition 2.3 we find $\rho > 0$ with $\rho V \subset \{|\eta| < \rho_0\} \subset E$ and such that the composition $f : V \xrightarrow{\rho\,\cdot} \rho V \xrightarrow{\chi \circ \psi'} \mathbb{R}^N$ is a scalar positive immersion. □

Example 2.7. Let B be a closed ℓ-dimensional manifold and let $V \to B$ be a fibre bundle with fibre $X = \mathbb{C}P^2$ and structure group $G = \mathrm{U}(3) \rtimes \mathbb{Z}/2$, where $\mathbb{Z}/2$ acts by complex conjugation on $\mathrm{U}(3)$ in the semidirect product. Now, as in [Tai68, (2.13)], consider the well-known Veronese isometric embedding $\mathbb{C}P^2 \to \mathrm{H}(3, \mathbb{C}) := \{A \in \mathbb{C}^{3\times 3} \mid A^* = A, \mathrm{tr}(A) = 1\} \cong \mathbb{R}^8$, which is induced by the map $\mathbb{C}^3 \supset S^5 \to \mathrm{H}(3, \mathbb{C})$ given by

$$(x_0, x_1, x_2) \mapsto \begin{pmatrix} |x_0|^2 & x_0\overline{x_1} & x_0\overline{x_2} \\ x_1\overline{x_0} & |x_1|^2 & x_1\overline{x_2} \\ x_2\overline{x_0} & x_2\overline{x_1} & |x_2|^2 \end{pmatrix}. \tag{5}$$

The embedding (5) is equivariant with respect to the Lie group homomorphism $\psi : G \to \mathrm{O}(8)$, defined by $\psi(X, 1)(A) = XAX^*$ and $\psi(X, -1)(A) = X\overline{A}X^*$ for $(X, \pm 1) \in \mathrm{U}(3) \rtimes \mathbb{Z}/2$. Let $P \to B$

be the G-principal bundle of $V \to B$. Setting $E := P \times_\psi \mathbb{R}^8$, we hence realize $V \to B$ as a sub-fibre bundle of $E \to B$ in the sense of Assumption 2.1.

By Cohen's Immersion Theorem [Coh85] applied to the sphere bundle of $E \oplus \mathbb{R}$, there exists an immersion $E \to \mathbb{R}^N$ with $N = 2(\ell + 8) - \beta(\ell + 8)$, where $\beta(m)$ stands for the number of ones in the dyadic expansion of m. From Proposition 2.6 we conclude that there exists a scalar positive immersion $V \to \mathbb{R}^N$.

A similar construction applies to fibre bundles $V \to B$ with fibre $X = \mathbb{H}P^2$ and structure group $G = \mathrm{Sp}(3) = \{X \in \mathbb{H}^{3\times 3} \mid X^* X = \mathrm{id}\}$. Formula (5) defines an isometric embedding $\mathbb{H}P^2 \to \mathrm{H}(3, \mathbb{H}) := \{A \in \mathbb{H}^{3\times 3} \mid A^* = A, \mathrm{tr}(A) = 1\} \cong \mathbb{R}^{14}$ which is equivariant with respect to the Lie group homomorphism $\psi : G \to \mathrm{O}(14)$, $\psi(X)(A) = XAX^*$. Hence, in this case, we obtain a scalar positive immersion $V \to \mathbb{R}^N$ with $N = 2(\ell + 14) - \beta(\ell + 14)$.

While these examples are the relevant ones for the proof of Theorem 1.2 in Section 5, it is clear that the previous construction applies to the total spaces of many other fibre bundles.

3. Local deformation I: Normally spherical immersions

Most of the remaining part of the paper will be devoted to the implementation of the extrinsic surgery process, following the spirit of [GL80]. The purpose of this section is to show how to deform an immersion near a compact submanifold in order to take it into a normally spherical shape around it while preserving positive scalar curvature.

We first fix some notation. Let $f : M \to \mathbb{R}^N$ be an immersion of an n-dimensional manifold. Its differential $f_* = df$ identifies TM with a subbundle of $f^*(T\mathbb{R}^N) \cong M \times \mathbb{R}^N$ whose orthogonal complement with respect to the Euclidean metric on $\mathbb{R}^N$ is the normal bundle ν_f of f. We denote the induced fibre metrics and fibre norms on bundles constructed from TM and ν_f by $\langle \cdot, \cdot \rangle_f$ and $| \cdot |_f$, where we suppress the subscript if the immersion f is obvious from the context. We denote by $\alpha_f \in \Gamma(T^*M \otimes T^*M \otimes \nu_f)$ the second fundamental form of f. Hence, $\mathrm{tr}(\alpha_f) \in \Gamma(\nu_f)$ is the (unnormalized) mean curvature field of f, while, by the Gauss equation,

$$\mathrm{scal}_f = |\,\mathrm{tr}(\alpha_f)|^2 - |\alpha_f|^2 : M \to \mathbb{R} \tag{6}$$

is the (unnormalized) scalar curvature of (the metric induced by) f. In particular, if in addition f is scalar positive, its mean curvature nowhere vanishes and we obtain the unit normal field

$$\xi := \mathrm{tr}(\alpha_f)/|\,\mathrm{tr}(\alpha_f)| \in \Gamma(\nu_f)\,. \tag{7}$$

The field ξ points in the direction along which f will be deformed. Intuitively, deforming f in the direction of ξ increases the mean curvature faster than the second fundamental form (see the proof of Lemma 3.2), therefore increasing the scalar curvature by (6).

From now on assume that f is scalar positive, and let $S \subset M$ be a closed embedded submanifold of codimension $k = n - \dim S$. For $\rho > 0$ we set

$$U_\rho(S) := \{p \in M \mid d(p, S) < \rho\} \subset M,$$

where d refers to the induced Riemannian distance on M. In this section we fix $\rho_0 > 0$ such that the normal exponential map $\exp^\perp : \nu_S^M \to M$ of the normal bundle of S in M induces a diffeomorphism

$$\exp^\perp : \{|\eta| < \rho_0\} \xrightarrow{\approx} U_{\rho_0}(S)\,.$$

Hence we write $p \in U_{\rho_0}(S)$ in polar coordinates (q, ω, s), where $q \in S$, $\omega \in (\nu_S^M)_q$, $|\omega|_f = 1$, $s \in [0, \rho_0)$ and $p = \exp_q^\perp(s\,\omega)$.

We define smooth maps $F_\tau, G_\tau : U_{\rho_0}(S) \to \mathbb{R}^N$ by

$$\begin{aligned} F_\tau(p) &:= f(p) + \frac{1}{2}\,\tau\, s^2\, \xi(q) \quad \text{for } \tau \geq 0, \\ G_\tau(p) &:= f(q) + \tau^{-1} \sin(\tau s)\, \omega + \tau^{-1}(1 - \cos(\tau s))\, \xi(q) \quad \text{for } \tau > 0\,. \end{aligned}$$

The map G_τ is smooth at $s = 0$ since, setting $\hat{S}^k(1/\tau) := \{|x - \tau^{-1} e_{k+1}| = \tau^{-1}\} \subset \mathbb{R}^{k+1}$ with $e_{k+1} = (0, \ldots, 0, 1)$, the map $\mathbb{R}^k \to \mathbb{R}^{k+1}$ defined in polar coordinates by

$$(u, s) \mapsto \tau^{-1} \sin(\tau s)\, u + \tau^{-1}(1 - \cos(\tau s))\, e_{k+1}$$

can be interpreted as the (smooth) exponential map $\exp_0 : \mathbb{R}^k = T_0\hat{S}^k(1/\tau) \to S^k(1/\tau) \subset \mathbb{R}^{k+1}$.

Observe that

$$F_\tau|_S = f|_S = G_\tau|_S \quad \text{and} \quad dF_\tau|_S = df|_S = dG_\tau|_S\,. \tag{8}$$

Hence F_τ and G_τ restrict to immersions $U \to \mathbb{R}^N$ on some neighborhood $S \subset U \subset U_{\rho_0}(S) \subset M$, and the identifications of $TM|_S$ with a subbundle of $S \times \mathbb{R}^N$ coincide for the immersions f, F_τ and G_τ. The same holds for the normal bundles ν_f, ν_{F_τ} and ν_{G_τ} restricted to S. In particular, the second fundamental forms of F_τ and G_τ restrict to smooth sections of $T^*M|_S \otimes T^*M|_S \otimes (\nu_f)|_S \to S$.

Our aim in this section is to prove in Proposition 3.4 below that, for large τ, the scalar positive immersion f can be globally deformed, through scalar positive immersions, to bring it into the *normally spherical shape* G_τ near S. This deformation will be constructed near S by first applying the deformation $F_{t\tau}$, $t \in [0, 1]$, which creates a large curvature contribution in the direction ξ, and then linearly interpolating between the resulting immersion and G_τ. Using the local flexibility lemma [BH, Theorem 1] this local deformation near S can be extended to a global deformation of scalar positive immersions $M \to \mathbb{R}^N$.

Proposition 3.4 essentially depends on the next three computational lemmas. To state the first one, for $q \in S$ and $X \in T_qM$, let $X^\top \in T_qS$ and $X^\perp \in (\nu_S^M)_q$ denote the orthogonal projections, and notice that these coincide for our three immersions f, F_τ and G_τ because of (8).

Lemma 3.1. *For $q \in S$ and $X, Y \in T_qM$ we have that*

$$\alpha_{F_\tau}(X, Y) = \alpha_f(X, Y) + \tau\,\langle X^\perp, Y^\perp\rangle_f\,\xi(q)\,, \tag{9}$$

$$\alpha_{G_\tau}(X, Y) = \alpha_f(X^\top, Y^\top) + \alpha_f(X^\top, Y^\perp) + \alpha_f(X^\perp, Y^\top) + \tau\,\langle X^\perp, Y^\perp\rangle_f\,\xi(q)\,. \tag{10}$$

Proof. First assume $X \in T_qS$, let $\beta : (-\varepsilon, \varepsilon) \to S$ be a smooth curve through q with $\beta'(0) = X$, and let $\hat{Y} : (-\varepsilon, \varepsilon) \to TM \subset M \times \mathbb{R}^N$ be a vector field along β with $\hat{Y}(0) = Y$. By (8) both $\alpha_{F_\tau}(X, Y)$ and $\alpha_{G_\tau}(X, Y)$ are equal to the orthogonal projection of $\hat{Y}'(0) \in \mathbb{R}^N$ onto $(\nu_f)_q$, and are hence equal to $\alpha_f(X, Y)$. This and the symmetry of second fundamental forms show that for proving Lemma 3.1 we can restrict to the case $X, Y \in (\nu_S^M)_q$, and by polarization and bilinearity we can further restrict to the case $X = Y = \omega \in (\nu_S^M)_q$, $|\omega| = 1$.

Let $\beta : (-\varepsilon, \varepsilon) \to (\nu_S^M)_q \subset \mathbb{R}^N$ be the curve $\beta(s) = s\,\omega$. Then $\beta'(0) = \omega$, and $(F_\tau \circ \beta)''(0) = (f \circ \beta)''(0) + \tau\,\xi(q)$ and $(G_\tau \circ \beta)''(0) = \tau\,\xi(q)$. This gives (9) and (10) after projection onto $(\nu_f)_q$. □

Lemma 3.2. *Along S we have* $\mathrm{scal}_{F_\tau} > 0$ *for all* $\tau \geq 0$.

Proof. We work along S throughout. As $\mathrm{tr}(\alpha_f)$ is a positive multiple of ξ, (9) implies that

$$|\,\mathrm{tr}(\alpha_{F_\tau})|^2 = \big(|\,\mathrm{tr}(\alpha_f)| + \tau k\big)^2 = |\,\mathrm{tr}(\alpha_f)|^2 + 2|\,\mathrm{tr}(\alpha_f)|\tau k + \tau^2 k^2\,.$$

Furthermore, by the triangle inequality,

$$|\alpha_{F_\tau}|^2 \leq \big(|\alpha_f| + \tau\sqrt{k}\big)^2 = |\alpha_f|^2 + 2|\alpha_f|\tau\sqrt{k} + \tau^2 k\,.$$

Since $|\,\mathrm{tr}(\alpha_f)| > |\alpha_f|$ by our assumption $\mathrm{scal}_f > 0$, for $\tau \geq 0$ the Gauss equation gives us

$$\mathrm{scal}_{F_\tau} = |\,\mathrm{tr}(\alpha_{F_\tau})|^2 - |\alpha_{F_\tau}|^2 \geq \mathrm{scal}_f + \tau^2(k^2 - k) \geq \mathrm{scal}_f > 0\,.$$ □

Lemma 3.3. *If $k \geq 2$, there exists $\tau_0 > 0$ such that, for all $\tau \geq \tau_0$ and $t \in [0, 1]$, it holds that* $\mathrm{scal}_{(1-t)F_\tau + tG_\tau} > 0$ *along S.*

Proof. By (9) and (10) there exists $C > 0$, which only depends on the restriction of α_f to S, such that, for all $\tau > 0$, $q \in S$, and $X, Y \in T_qM$, we get

$$\big|(1-t)\,\alpha_{F_\tau}(X, Y) + t\,\alpha_{G_\tau}(X, Y) - \tau\langle X^\perp, Y^\perp\rangle\,\xi(q)\big| \leq C|X||Y|\,.$$

Hence, by the triangle inequality,

$$\big|\,\mathrm{tr}\big((1-t)\,\alpha_{F_\tau} + t\,\alpha_{G_\tau}\big)\big| \geq k\tau - nC\,.$$

Similarly,

$$\left|(1-t)\,\alpha_{F_\tau}+t\,\alpha_{G_\tau}\right|\le\sqrt{k}\tau+\sqrt{n^2C^2}=\sqrt{k}\tau+nC\,.$$

Assuming that $k\tau\ge nC$, the Gauss equation hence implies that, along S,

$$\mathrm{scal}_{(1-t)F_\tau+tG_\tau}\ge(k\tau-nC)^2-(\sqrt{k}\tau+nC)^2=(k^2-k)\tau^2-2(k+\sqrt{k})\tau nC.$$

Since $k\ge 2$, there exists $\tau_0\ge nC/k$ such that the last expression is positive for all $\tau\ge\tau_0$. □

We finally have all the ingredients to prove the main result of this section.

Proposition 3.4 (Normally spherical immersions)**.** *If $k\ge 2$, there exists $\tau_0>0$ such that, for all $\tau\ge\tau_0$, there exist $0<\rho\le\rho_0$ and a continuous family of scalar positive immersions $f_t:M\to\mathbb{R}^N$, $t\in[0,1]$, with $f_0=f$, $f_t|_{M\setminus U_{\rho_0}(S)}=f|_{M\setminus U_{\rho_0}(S)}$ for all $t\in[0,1]$, and*

$$f_t|_{U_\rho(S)}=\begin{cases}F_{2t\tau}|_{U_\rho(S)} & \text{for } 0\le t\le 1/2\,,\\ \big((2-2t)F_\tau+(2t-1)G_\tau\big)|_{U_\rho(S)} & \text{for } 1/2\le t\le 1.\end{cases}$$

In particular, $f_1|_{M\setminus U_{\rho_0}(S)}=f|_{M\setminus U_{\rho_0}(S)}$ and $f_1|_{U_\rho(S)}=G_\tau|_{U_\rho(S)}$.

Proof. Choose τ_0 as in Lemma 3.3 and let $\tau\ge\tau_0$. By (8), Lemma 3.2 and Lemma 3.3, there exists an open neighborhood $S\subset U\subset U_{\rho_0}(S)\subset M$ such that, for $t\in[0,1]$, the maps $F_{t\tau}$ and $(1-t)F_\tau+tG_\tau$ restrict to scalar positive immersions $U\to\mathbb{R}^N$ whose 1-jets along S do not depend on t. Since being a scalar positive immersion defines an open partial differential relation on the 2-jets of maps $M\to\mathbb{R}^N$, the claim follows from the local flexibility lemma [BH, Theorem 1]. □

With the help of Gromov's h-principle for open, Diff-invariant partial differential relations over open manifolds, see [Gro86], the computations in this section can also be used to justify Remark 1.5:

Proposition 3.5. *Let M be a non-compact connected manifold of dimension at least 2 admitting an immersion into $\mathbb{R}^N$. Then there exists a scalar positive immersion of M into $\mathbb{R}^{N+1}$.*

Proof. Consider the trivial vector bundle $X=M\times\mathbb{R}^{N+1}\to M$ and the bundle $X^{(2)}\to M$ of 2-jets of smooth maps $M\to\mathbb{R}^{N+1}$. For a smooth map $f:X\to\mathbb{R}^{N+1}$ we denote by $j^2f:M\to X^{(2)}$ its second order jet map. Recall that for $p\in M$ the value $j^2f(p)\in(X^{(2)})_p$ only depends on the restriction of f to some neighborhood of p. Being a scalar positive immersion $M\to\mathbb{R}^{N+1}$ defines an open, $\mathrm{Diff}(M)$-invariant partial differential relation $\mathscr{R}\subset X^{(2)}$.

Let $g:M\to\mathbb{R}^N$ be an immersion. For a positive continuous map $\tau:M\to\mathbb{R}$ consider the continuous section $g_\tau:M\to X^{(2)}$,

$$p\mapsto\big(j^2g(p),\tau(p)\,j^2\big(x\mapsto d(p,x)^2\big)(p)\big)\,.$$

Since $\dim M\ge 2$ the computation in the proof of Lemma 3.2 implies that, for each compact $W\subset M$, there exists $\tau_0\in(0,\infty)$ such that, if $\tau\ge\tau_0$ on W, we have $g_\tau(W)\subset\mathscr{R}$, that is g_τ formally solves $\mathscr{R}$ over W. By a partition of unity subordinate to a locally finite cover of M by relatively compact open subsets we hence find $\tau:M\to\mathbb{R}$ such that g_τ formally solves $\mathscr{R}$ over M. Now Gromov's h-principle implies that there exists a smooth map $f:M\to\mathbb{R}^{N+1}$ solving $\mathscr{R}$. □

4. Local deformation II: Bending profiles

The aim of this section is to show that a scalar positive immersion which is normally spherical near a compact submanifold as in Proposition 3.4 can be further deformed, again through scalar positive immersions, into a shape proper to add a surgery handle.

As in the previous section let $f:M\to\mathbb{R}^N$ be a scalar positive immersion, $n=\dim M$, and let $S\subset M$ be a closed embedded submanifold with codimension k and normal bundle ν_S^M. If $E\to B$ is a Euclidean vector bundle and $\rho>0$, we denote by $D_\rho(E)=\{v\in E\mid |v|\le\rho\}\to B$ the closed ρ-disc bundle and by $S_\rho(E)=\{v\in E\mid |v|=\rho\}\to B$ the ρ-sphere bundle of E. Points in $S_1(\nu_S^M)_q$ are written as (q,ω) to emphasize the restriction to normal vectors of norm one.

Since S is compact and ξ in (7) is normal to f, we find $0 < \rho_0 \leq 1$ such that the map $S_1(\nu_S^M) \times D_{\rho_0}(\mathbb{R}^2) \to \mathbb{R}^N$,

$$(q, \omega, a, b) \mapsto f(q) + a\,\omega + b\,\xi(q)\,,$$

is an immersion. In the remainder of this section we fix this ρ_0.

Let $I \subset \mathbb{R}$ be a compact interval and $\gamma : I \to \mathbb{R}^2$, $\gamma(s) = (a(s), b(s))$, be a unit speed smooth curve with $a(s) \neq 0$ for $s \in I$.

Definition 4.1. For $0 < \rho \leq \rho_0$, we say that γ is of *extent* ρ, if $|\gamma(s)| \leq \rho$ for all $s \in I$.

For γ of extent $\rho \leq \rho_0$ we consider the compact manifold with boundary

$$\Sigma = \Sigma_\gamma := S_1(\nu_S^M) \times I,$$

together with the immersion $F_\gamma : \Sigma \to \mathbb{R}^N$ along the *bending profile* γ given by

$$F_\gamma(q, \omega, s) := f(q) + a(s)\,\omega + b(s)\,\xi(q)\,. \tag{11}$$

In this section we will first derive a lower bound for scal_{F_γ} in terms of γ; see Proposition 4.8. This requires some preparation, which we shall again split into a number of lemmas. After solving a pertinent ODE for γ in Lemma 4.9, Proposition 4.12 provides the bending profiles required for the extrinsic surgery in Section 5.

The submersion $\pi : \Sigma \to S$, $\pi(q, \omega, s) = q$, induces an orthogonal direct sum decomposition of $T\Sigma$ into vertical and horizontal subbundles,

$$\mathscr{V} = \ker d\pi \subset T\Sigma\,, \quad \mathscr{H} = \mathscr{V}^{\perp_{F_\gamma}} \subset T\Sigma\,.$$

For $X \in T\Sigma$ we denote by $\mathscr{V}X$ and $\mathscr{H}X$ its vertical and horizontal components. Note that for $p = (q, \omega, s) \in \Sigma$ we have an orthogonal splitting with respect to the metric induced by F_γ,

$$\mathscr{V}_p = \omega^\perp \oplus \mathrm{span}\{\partial_s\} \subset (\nu_S^M)_q \oplus T_s I\,. \tag{12}$$

Now consider an isometric local bundle trivialisation

$$\Psi : \nu_S^M|_W \xrightarrow{\cong} W \times \mathbb{R}^k,$$

where $W \subset S$. For $p = (q, \omega, s) \in \Sigma$, $q \in W$, and $\omega \in \mathbb{S}^{k-1} \subset (\nu_S^M)_q \cong \mathbb{R}^k$, we hence obtain a direct sum decomposition

$$T_p\Sigma = T_qS \oplus T_\omega \mathbb{S}^{k-1} \oplus T_s I, \tag{13}$$

with $T_\omega \mathbb{S}^{k-1} = \omega^\perp \subset \mathbb{R}^k$. Note that T_qS is not in general orthogonal to $\mathscr{V}_p$.

For $X \in T_qS$ we denote by $X_p \in T_p\Sigma$ the element $(X, 0, 0)$ with respect to the decomposition (13) and by $\mathscr{H}_pX \in \mathscr{H}_p$ the unique element satisfying $d_p\pi(\mathscr{H}_pX) = X$. Note that $\mathscr{H}_pX = \mathscr{H}X_p$, and in particular the horizontal component of X_p is independent from Ψ.

Lemma 4.2. *There exists $C \geq 0$ depending on Ψ such that, for every curve γ of extent $0 < \rho \leq \rho_0$, it holds that*

$$\left|\alpha_{F_\gamma}(X_p, Y_p)\right|_{F_\gamma} \leq C|X|_f|Y|_f\,, \tag{14}$$

$$\left|\alpha_{F_\gamma}(X_p, V)\right|_{F_\gamma} \leq C|X|_f|V|_{F_\gamma}\,, \tag{15}$$

$$\left|\langle d_pF_\gamma(X_p), Z\rangle\right| \leq \rho C|X|_f|Z|\,, \tag{16}$$

for every $p = (q, \omega, s) \in W \times \mathbb{S}^{k-1} \times I$, $X, Y \in T_qS$, $V \in \mathscr{V}_p$ and $Z \in (\nu_S^M)_q \oplus \mathbb{R}\xi(q) \subset \mathbb{R}^N$. In particular,

$$|\mathscr{H}_pX|_{F_\gamma} \geq (1 - \rho C)|X|_f\,. \tag{17}$$

Proof. For $\eta \in \mathbb{R}^k$ we define $\hat\eta : W \to \mathbb{R}^N$ as $\hat\eta(q) := \Psi_q^{-1}(\eta) \in (\nu_S^M)_q \subset \mathbb{R}^N$. Hence the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^k$ yields an orthonormal framing $\{\hat e_1, \ldots, \hat e_n\}$ of $(\nu_S^M)|_W$. Let $(x^1, \ldots, x^n)$ be local coordinates on a subset $W' \subset W$. In the following $C \geq 0$ denotes a constant which only depends on norm bounds of partial derivatives of order at most 2 of f, ξ and $\hat e_1, \ldots, \hat e_n$ on W' with respect to these coordinates.

Let $X = \partial_i$, $Y = \partial_j$, $1 \le i, j \le n$, be coordinate vector fields. Since a and b are norm bounded by 1 (recall $\rho_0 \le 1$) and $|\omega|_f = 1$, we obtain that

$$|\alpha_{F_\gamma}(X_p, Y_p)|_{F_\gamma} \le |\partial_i\partial_j F_\gamma(\,\cdot\,, \omega, s)| \le |\partial_i\partial_j f| + |a(s)||\partial_i\partial_j\hat\omega| + |b(s)||\partial_i\partial_j\xi| \le C\,.$$

This implies (14) on W' as the vector fields ∂_i have positive lower f-norm bounds on W'.

Now let $X = \partial_i$, $V \in T_\omega\mathbb{S}^{k-1} = \omega^\perp \subset \mathbb{R}^k$, which we consider as vector field in $T\Sigma$ along S by (13), and note that for $q \in S$ we have $\partial_V F_\gamma(q, \,\cdot\,, S) = \hat V(q)$. Hence,

$$|\alpha_{F_\gamma}(X_p, V)|_{F_\gamma} \le |a(s)||\partial_i\hat V| \le C|a(s)||V|_f = C|V|_{F_\gamma}\,.$$

Moreover, for $V = \partial_s \in T_sI$ we get, using that γ is of unit speed,

$$|\alpha_{F_\gamma}(X_p, V)|_{F_\gamma} \le |\partial_i\partial_s F_\gamma(\,\cdot\,, \omega, \,\cdot\,)| = |a'(s)||\partial_i\hat\omega| + |b'(s)||\partial_i\xi| \le C\,,$$

which concludes the proof of (15) on W'.

Estimate (16) holds on W' for $X = \partial_i$. This uses $d_qf(X) \perp Z$, which implies

$$\langle d_pF_\gamma(X_p), Z\rangle = \langle a(s)\partial_i\hat\omega + b(s)\partial_i\xi, Z\rangle \le \big(|a(s)||\partial_i\hat\omega| + |b(s)||\partial_i\xi|\big)|Z| \le \rho C|Z|\,.$$

The last assertion follows from this, as $dF_\gamma(\mathscr{V}_p) \subset (\nu_S^M)_q \oplus \mathbb{R}\xi(q)$.

Finally, the relatively compact subset $W \subset S$ can be covered by finitely many coordinate charts W' such that Lemma 4.2 holds on W. □

For each $p = (q, \omega, s) \in \Sigma$ we define the unit vector field $N : \Sigma \to S_1\big(\nu_S^M \oplus \mathbb{R}\xi\big)$ by

$$N(p) := -b'(s)\omega + a'(s)\xi(q) \in \mathbb{R}^N\,,$$

and write its corresponding orthogonal projections as

$$N = N^\top + N^\perp \;\in\; dF_\gamma(T\Sigma) \oplus^\perp \nu_{F_\gamma} = F_\gamma^*(T\mathbb{R}^N).$$

For $q \in S$ we define $\Sigma_q := \pi^{-1}(q) = \{q\} \times S_1(\nu_S^M)_q \times I \subset \Sigma$, and observe that F_γ restricts to an embedding

$$\Sigma_q \hookrightarrow E_q := (\nu_S^M)_q \oplus \mathbb{R}\,\xi(q) \cong \mathbb{R}^{k+1} \subset \mathbb{R}^N\,,$$

whose image is the revolution hypersurface with meridian γ and axis $\mathbb{R}\,\xi(q)$. This embedding has $N_q(\omega, s) = -b'(s)\omega + a'(s)\xi(q) = N(p)$ as unit normal vector field, that is to say the Gauss map.

Lemma 4.3. *There exists $0 < \rho \le \rho_0$ such that, for all γ of extent ρ, we have $|N^\perp| \ge 1/2$.*

Proof. We fix a local isometric bundle trivialisation $(\nu_S^M)|_W \cong W \times \mathbb{R}^k$ at $p = (q, \omega, s)$ for $q \in W$, and work with the induced splitting (13). Since $N(p) \perp dF_\gamma(\mathscr{V}_p)$ we obtain

$$|N^\top(p)| = \max\{\big|\langle dF_\gamma(X_p), N(p)\rangle\big| \mid X \in T_qS\,,\, |\mathscr{H}_pX|_{F_\gamma} = 1\}\,.$$

Pick $X \in T_qS$ for which this maximum is attained. By (16) and (17) we have $\langle dF_\gamma(X_p), N(p)\rangle \le \rho C|X|_f$ and $(1 - \rho C)|X|_f \le |\mathscr{H}_pX|_{F_\gamma} = 1$. Depending on W we therefore find $0 < \rho \le \rho_0$ such that for all γ of extent ρ we have $|N^\top(p)| \le 1/2$, and hence $|N^\perp(p)| \ge 1/2$.

Now, the compact set S can be covered by finitely many such bundle charts and Lemma 4.3 follows. □

Given a plane curve $\gamma(s) = (a(s), b(s)) : I \to \mathbb{R}^2$ as above we define the smooth functions $\kappa, \sigma : I \to \mathbb{R}$ by

$$\kappa := a'\,b'' - a''\,b'\,, \quad \sigma := b'/a\,. \tag{18}$$

Notice that κ is the curvature of γ with respect to its unit normal $(-b', a') \in \mathbb{R}^2$.

Lemma 4.4. *With respect to the direct sum decomposition $\mathscr{V}_p = \omega^\perp \oplus \operatorname{span}\{\partial_s\}$ we have*

$$(\alpha_{F_\gamma})|_{\mathscr{V}_p\times\mathscr{V}_p} = \big(\sigma(s)\,\langle\,\cdot\,,\cdot\,\rangle_{\omega^\perp} + \kappa(s)\,ds^2\big)\,N^\perp(p)\,.$$

Proof. By a standard computation the second fundamental form α_q of the embedding $\Sigma_q \hookrightarrow E_q$ is given, with respect to the orthogonal decomposition $T_{(\omega,s)}\Sigma_q = \omega^\perp \oplus \operatorname{span}\{\partial_s\}$, by

$$\alpha_q = \big(\sigma(s)\,\langle\,\cdot\,,\cdot\,\rangle_{\omega^\perp} + \kappa(s)\,ds^2\big)\,N(p)\,.$$

The assertion follows from (12) and the definition of $N^\perp(p)$. □

Definition 4.5. A unit speed curve $\gamma : I \to \mathbb{R}^2$ with $a(s) \neq 0$ for all s is called *controlled*, if $\frac{2-k}{4}\,\sigma \leq \kappa \leq \sigma$. (Recall that k is the codimension of S in M.)

Remark 4.6. If γ is controlled we have that $\sigma \geq 0$ and $\max\{|\kappa|, \sigma\} \leq n\sigma$.

Lemma 4.7. *There exist constants $C \geq 0$ and $0 < \rho \leq \rho_0$ with the following property. If γ is controlled and of extent ρ, then, for $p = (q, \omega, s) \in \Sigma$, $X, Y \in T_qS$ and $V \in \mathscr{V}_p$, we have*

$$\big|\alpha_{F_\gamma}(\mathscr{H}_pX, \mathscr{H}_pY)\big|_{F_\gamma} \leq C\big(1 + \rho\sigma\big)\,|X|_f\,|Y|_f\,, \tag{19}$$

$$\big|\alpha_{F_\gamma}(\mathscr{H}_pX, V)\big|_{F_\gamma} \leq C\big(1 + \rho\sigma\big)\,|X|_f\,|V|_{F_\gamma}\,. \tag{20}$$

Proof. Fix a local bundle trivialisation $(\nu_S^M)|_W \cong W \times \mathbb{R}^k$ and choose $C \geq 0$ such that Lemma 4.2 holds. Let $X, Y \in T_qX$. By Remark 4.6 we thus obtain

$$\big|\alpha_{F_\gamma}(\mathscr{H}_pX, \mathscr{H}_pY)\big|_{F_\gamma} \leq \big(C + 2\rho C^2 + \rho^2C^2n\sigma\big)|X|_f|Y|_f\,.$$

Since $\rho \leq 1$ we get (19) with C replaced by $2nC^2$. Estimate (20) is implied analogously by

$$\big|\alpha_{F_\gamma}(\mathscr{H}_pX, V)\big|_{F_\gamma} \leq \big(C + \rho Cn\sigma\big)|X|_f|V|_{F_\gamma}\,.$$ □

Proposition 4.8. *There exist constants $C \geq 0$ and $0 < \rho \leq \rho_0$ with the following property. If γ is controlled and of extent ρ, and $k \geq 3$, then*

$$\mathrm{scal}_{F_\gamma} \geq \frac{(k-1)(k-2)}{16}\sigma^2 - C\sigma - C\,. \tag{21}$$

In particular, there exists a constant $\sigma_0 > 0$ such that F_γ is scalar positive once $\sigma \geq \sigma_0$.

Proof. With respect to the direct sum decomposition $T_p\Sigma = \mathscr{V}_p \oplus \mathscr{H}_p$, write

$$\alpha_{F_\gamma} = \begin{pmatrix} \Delta & B \\ B^T & Q \end{pmatrix},$$

where $\Delta = (\alpha_{F_\gamma})|_{\mathscr{V}_p \times \mathscr{V}_p}$ was computed in Lemma 4.4. The Gauss equation together with the triangle and Cauchy-Schwarz inequalities hence imply that

$$\begin{aligned}\mathrm{scal}_{F_\gamma}(p) &= |\operatorname{tr}(\alpha_{F_\gamma})|^2 - |\alpha_{F_\gamma}|^2 \\ &= \big|\operatorname{tr}(\Delta) + \operatorname{tr}(Q)\big|^2 - |\Delta|^2 - 2|B|^2 - |Q|^2 \\ &\geq \big(|\operatorname{tr}(\Delta)|^2 - |\Delta|^2\big) - 2|\operatorname{tr}(\Delta)||\operatorname{tr}(Q)| + |\operatorname{tr}(Q)|^2 - 2|B|^2 - |Q|^2\,.\end{aligned}$$

By (17) and Lemma 4.3 we find $0 < \rho \leq \rho_0$ such that, for all γ of extent ρ, $p = (q, \omega, s) \in \Sigma$ and $X \in T_qX$, we have

$$|\mathscr{H}_pX|_{F_\gamma} \geq |X|_f/2\,, \quad 1/2 \leq |N^\perp(p)| \leq 1\,. \tag{22}$$

Since γ is controlled we get $2\kappa \geq -\frac{k-2}{2}\,\sigma$, and hence

$$|\operatorname{tr}(\Delta)|^2 - |\Delta|^2 = (k-1)((k-2)\sigma^2 + 2\kappa\sigma)|N^\perp(p)|^2 \geq \frac{(k-1)(k-2)}{8}\sigma^2\,.$$

Using (22), Remark 4.6 and Lemma 4.7 we see that the entries of $Q \in (\mathbb{R}^N)^{(n-k)\times(n-k)}$ are norm bounded by $4C(1 + \rho\sigma)$, the ones for $B \in (\mathbb{R}^N)^{k\times(n-k)}$ are norm bounded by $2C(1 + \rho\sigma)$, and the ones for Δ are norm bounded by $n\sigma$. Hence $2|\operatorname{tr}(\Delta)||\operatorname{tr}(Q)| + 2|B|^2 + |Q|^2$ is bounded by a quadratic polynomial in σ and passing to a smaller ρ we can assume that the coefficient of σ^2 is bounded by $\frac{(k-1)(k-2)}{16}$, which is a positive number as $k \geq 3$. This completes the proof of Proposition 4.8 for an appropriate C. □

Lemma 4.9. *Fix $k \geq 3$. Let $(x, y) \in \mathbb{R}^2$ with $x > 0$, and let $(u, v) \in S^1$ with $u, v > 0$. Set $\lambda := \frac{k-2}{4} > 0$. Then there exists $-\frac{\pi\,x}{2\lambda v} < R < 0$ and a unit speed curve $\gamma = (a, b) : [R, 0] \to \mathbb{R}^2$ with the following properties:*

i) $\gamma(0) = (x, y)$ *and* $a(s) > 0$ *for* $s \in [R, 0]$*;*
ii) $\gamma'(R) = (0, 1)$ *and* $\gamma'(0) = (u, v)$*. In particular* $\sigma(0) = v/x$*;*
iii) $\kappa = -\lambda\,\sigma$*, with* κ *and* σ *as in* (18)*.*

Proof. Consider a maximal solution $\gamma : I \to \mathbb{R}^2$, $\gamma(s) = (a(s), b(s))$, $0 \in I \subset \mathbb{R}$, of the system of second order nonlinear ordinary differential equations

$$\begin{pmatrix} a'' \\ b'' \end{pmatrix} = -\lambda \frac{b'}{a} \begin{pmatrix} -b' \\ a' \end{pmatrix}, \tag{23}$$

with initial conditions $\gamma(0) = (x, y)$ and $\gamma'(0) = (u, v)$. Then, $\langle \gamma', \gamma'' \rangle = 0$ and thus γ has unit speed. Furthermore the quantity $z(s) := b'(s)a(s)^\lambda$ is preserved along γ since

$$z' = b''a^\lambda + \lambda b' a^{\lambda-1} a' = -\lambda a' w a^\lambda + \lambda a' b' a^{\lambda-1} = 0\,.$$

Therefore $z(s) = z(0) > 0$ for all s, which shows that $b' > 0$ and that $a > 0$ is bounded from below by a positive constant since $|b'| \le 1$. This implies that (a, a', b') stays in a compact subset of $\mathbb{R}_+ \times \mathbb{R}^2$. We conclude that $I = \mathbb{R}$. Moreover, by the Frenet equation the curve γ satisfies (ii).

As $b' > 0$ we obtain a continuous function $\theta : \mathbb{R} \to (0, \pi)$ which measures the angle in counter-clockwise direction between $(1, 0) \in \mathbb{R}^2$ and $\gamma'(s)$, that is, $\cos(\theta) = a'$ and $\sin(\theta) = b'$. Note that for the given γ we have

$$\theta' = \kappa = -\lambda\sigma\,.$$

Since z is constant along γ and $b' > 0$, we know that b' and hence $\sigma = b'/a$ are decreasing on the subset $\{a' > 0\} \subset \mathbb{R}$. Combining this with $\sigma(0) = b'(0)/a(0) = v/x$ we conclude that $\theta' \le -\frac{\lambda v}{x}$ on the maximal interval $(R, 0]$, $R < 0$, with $a' > 0$.

Since $a'(0) = u > 0$ and $\theta(0) \in (0, \pi/2)$, we get $-\frac{\pi x}{2\lambda v} < R < 0$ and $\theta(R) = \pi/2$. This implies $a'(R) = 0$ and hence $b'(R) = 1$, since γ has unit speed. □

Remark 4.10. This proof is inspired by [EF, Lemma 3.13], but we preferred to solve a differential equation for γ instead of writing $a = h(b)$ and solving a differential equation for h.

Corollary 4.11. *Let $\sigma_0 > 0$ and $0 < \rho \le \rho_0$ be chosen as in Proposition 4.8. Let $0 < \rho' \le \rho$, and let $(x, y) \in \mathbb{R}^2$ with $x > 0$, and $(u, v) \in S^1$ with $u, v > 0$, that satisfy $|(x, y)| \le \rho'$ and $v/x \ge \max\{\sigma_0, \frac{\pi}{2\lambda\rho'}\}$. Then the curve $\gamma : [R, 0] \to \mathbb{R}^2$ constructed in Lemma 4.9 is controlled and of extent $2\rho'$. Moreover, $\mathrm{scal}_{F_\gamma} > 0$ on Σ_γ.*

Proof. The curve γ is controlled by (iii) and of extent $2\rho'$, since $|\gamma(0)| < \rho'$, $|R| < \frac{\pi x}{2\lambda v} \le \rho'$ and γ is of unit speed. It follows from the proof of Lemma 4.9 that σ is decreasing on $(R, 0]$. As $\sigma(0) = v/x \ge \sigma_0$, this implies $\sigma \ge \sigma_0$ on $[R, 0]$, and hence $\mathrm{scal}_{F_\gamma} > 0$ on Σ by Proposition 4.8. □

Finally, we are able to prove the main result of this section. Roughly speaking it says that we can choose scalar positive bending profiles which interpolate between the normally spherical immersions near S resulting from Proposition 3.4 and scalar positive immersions which are "parallel" to the normal field ξ. This is done by means of a suitable bending profile γ as in Figure 1, and is an essential ingredient for completing the scalar positive extrinsic surgery in Section 5.

Proposition 4.12 (Construction of bending profiles)**.** *There exists $0 < \rho \le \rho_0$ such that, for all $0 < \rho' \le \rho/2$, there exists $\tau_0 > 0$ with the following property. For all $\tau \ge \tau_0$ and all $0 < \rho'' \le \min\{\rho', \frac{\pi}{2\tau}\}$, there exists a unit speed curve $\gamma = (a, b) : [R, 0] \to \mathbb{R}^2$ with $a > 0$, satisfying:*

i) $|\gamma(s)| \le 2\rho'$ for all s;
ii) the immersion $F_\gamma : \Sigma \to \mathbb{R}^N$ is scalar positive;
iii) $\gamma(s) = \tau^{-1}\big(\sin(\tau(\rho'' + s)), 1 - \cos(\tau(\rho'' + s))\big)$ near $s = 0$;
iv) $\gamma(s) = (a(R), b(R) + s)$ near $s = R$.

Proof. We claim that the assertion holds for ρ as in Proposition 4.8. Let $0 < \rho' \le \rho/2$ and set $\tau_0 := \max\{\sigma_0, \frac{\pi}{2\lambda\rho'}\}$, where σ_0 is also taken from Proposition 4.8. Fix $\tau \ge \tau_0$ and $0 < \rho'' \le \min\{\rho', \frac{\pi}{2\tau}\}$.

For $(x, y) := \tau^{-1}\big(\sin(\tau\rho'')), 1 - \cos(\tau\rho'')\big)$ and $(u, v) = (\cos(\tau\rho''), \sin(\tau\rho''))$ Corollary 4.11 applies, since $\rho'' \le \rho'$ and $v/x = \tau \ge \tau_0 \ge \max\{\sigma_0, \frac{\pi}{2\lambda\rho'}\}$. The resulting curve $\gamma : [R, 0] \to \mathbb{R}^2$ satisfies (i) and (ii). The proof will be completed once we deform γ near 0 in such a way that (iii) holds as well, (iv) being analogous.

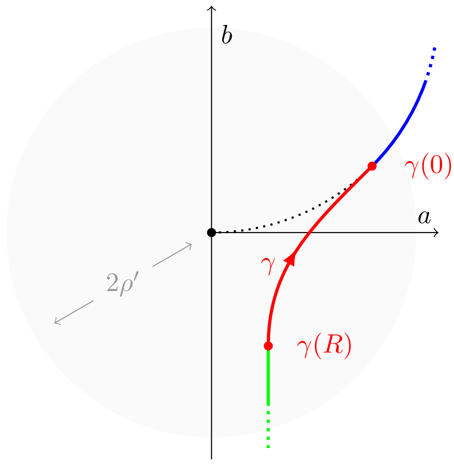

FIGURE 1. In red: the bending profile in Proposition 4.12.

In order to do this, first set $\varepsilon := \min\{|R/2|, \rho''/2\}$ and consider the continuous family $\gamma_t : (-\varepsilon, 0] \to \mathbb{R}^2$ given by

$$\gamma_t(s) = (a_t(s), b_t(s)) := (1-t)\,\gamma(s) + t\,\tau^{-1}\big(\sin(\tau(\rho''+s)), 1-\cos(\tau(\rho''+s))\big)\,.$$

Then $j^1\gamma_t(0)$ is constant in t by Lemma 4.9 (i) and (ii), and hence the same holds for $\sigma_t(0) := \frac{b_t'(0)}{a_t(0)}$. For the curvature $\kappa_t(0)$ of γ_t at $s=0$ we obtain a linear interpolation

$$\kappa_t(0) = (1-t)\,\kappa(0) + t\,\tau = -(1-t)\,\lambda\sigma_t(0) + t\,\sigma_t(0)\,.$$

In particular $-\lambda\sigma_t(0) \le \kappa_t(0) \le \sigma_t(0)$ for all t, and thus γ_t is controlled and of unit speed at $s=0$, for all $t \in [0,1]$.

Proposition 4.8 shows that $\mathrm{scal}_{F_{\gamma_t}} > 0$ along $S_1(\nu_S^M) \times \{0\} \subset \Sigma$ for $t \in [0,1]$. For sufficiently small $\varepsilon > 0$ this implies that $F_{\gamma_t} : S_1(\nu_S^M) \times (-\varepsilon, 0] \to \mathbb{R}^N$, $t \in [0,1]$, is a scalar positive immersion. Since F_γ being a scalar positive immersion defines an open partial differential relation on the 2-jet $j^2\gamma$, and $j^1\gamma_t(0)$ is constant in t, we can apply the local flexibility lemma [BH, Theorem 1] in order to find $0 < \varepsilon_0 < \varepsilon$ and a deformation $\gamma_t : [R, 0] \to \mathbb{R}^2$, $t \in [0,1]$ with $\gamma_0 = \gamma$, such that γ_t coincides with γ_t on $(-\varepsilon_0, 0]$, γ_t is constant in t on $[R, -\varepsilon]$, and the immersion $f_{\gamma_t} : \Sigma \to \mathbb{R}^N$ is scalar positive for all t. We now replace γ by γ_1, achieving (iii). □

Remark 4.13. We can draw some parallels between our computation and the one in [GL80] by considering the embedding $\tilde{F}_\gamma : \Sigma \hookrightarrow M \times \mathbb{R}$,

$$(q, \omega, s) \mapsto \big(\exp^\perp(a(s)\,\omega), b(s)\big) \subset M \times \mathbb{R},$$

which is defined if the extent of γ is smaller than the normal injectivity radius of $S \subset M$. Roughly speaking, the normal field ξ in (11) is replaced by the normal field $\partial_t \in \Gamma(T(M \times \mathbb{R})|_M)$. Since, contrary to ξ, the field ∂_t is parallel, the computations for $\tilde{F}_\gamma$ simplify considerably. Indeed, with respect to the metric induced by the embedding $\tilde{F}_\gamma$, the Gauss lemma for $\exp^\perp$ implies that $T_qS = \mathscr{H}_p$ in the direct sum decomposition (13). In particular, $\mathscr{H}_pX = X$ for all $X \in T_qS$, and $N(p) = N^\perp(p)$. Hence the entries of the matrices B and Q in the proof of Proposition 4.8 are norm bounded independently of γ, and our computation essentially specializes to [GL80, Equation (1') on p. 429]; compare also with [EF, Lemma 3.10].

Note that contrary to [GL80, Equation (1')] Proposition 4.8 does not yield a positive lower bound for scal_{F_γ} if $\sigma = 0$. This corresponds to the fact that the target of F_γ is flat $\mathbb{R}^N$, whereas the one for $\tilde{F}_\gamma$ is scalar positive. Hence in our extrinsic setting the "initial stage" of the bending process requires a different approach than in [GL80], which is provided by our Proposition 3.4, based on the local flexibility lemma [BH, Theorem 1].

5. Extrinsic scalar positive surgery

Here we bring together the previous constructions in order to perform the extrinsic surgery. At the end of this section we give the proofs of our two main results in the Introduction.

Assumption 5.1. Let $f : M \to \mathbb{R}^N$ be a scalar positive immersion, $n = \dim M$, and let $S^d \subset M$ be an embedded sphere of codimension $n - d = k \geq 3$ with trivial normal bundle together with an orthonormal frame $\{e_1, \ldots, e_k\}$ of ν_S^M. Furthermore, let $0 < \varepsilon < 1$ and let $F : D^{d+1}_{1+\varepsilon} \to \mathbb{R}^N$ be an immersion of the closed $(1+\varepsilon)$-disc in $\mathbb{R}^{d+1}$, together with an orthonormal family $\{E_1, \ldots, E_k\}$ of smooth sections of ν_F which are compatible with f and $\{e_1, \ldots, e_k\}$ in the following sense: for each $q \in S^d = S_1(\mathbb{R}^{d+1}) \subset D^{d+1}_{1+\varepsilon}$ and $r \in [1-\varepsilon, 1+\varepsilon]$,

$$F(rq) = f(q) + (r-1)\xi(q)\,, \quad E_i(rq) = e_i(q) \text{ for } i = 1, \ldots, k\,.$$

By Proposition 2.3 we can fix $0 < \lambda_0 \leq 1$ such that, for all $0 < \lambda \leq \lambda_0$, the map $\mathscr{F}_\lambda : D^{d+1}_{1+\varepsilon} \times \mathbb{S}^{k-1} \to \mathbb{R}^N$ given by

$$\mathscr{F}_\lambda\big(q, v_1, \ldots, v_k\big) := F(q) + \lambda \sum_{i=1}^{k} v_i E_i(q),$$

is a scalar positive immersion.

Proposition 5.2. *There exist $\tau > 0$ and $\rho > 0$ with the following properties:*

i) The normal exponential map $\exp^\perp : \nu_S^M \to M$ induces a diffeomorphism $D_\rho(\nu_S^M) \approx \overline{U_\rho(S)}$;

ii) We can deform f through scalar positive immersion into a scalar positive immersion $f_1 : M \to \mathbb{R}^N$ such that, for $(q, \omega, s) \in D_\rho(\nu_S^M)$,

$$f_1(q, \omega, s) = f(q) + \tau^{-1}\sin(\tau s)\omega + \tau^{-1}(1 - \cos(\tau s))\xi(q)\,;$$

iii) There exists $R < 0$ and a unit speed curve $\gamma = (a, b) : [R, 0] \to \mathbb{R}^2$ satisfying

$$\gamma(s) = \begin{cases} \tau^{-1}\big(\sin(\tau(\rho + s)), 1 - \cos(\tau(\rho + s))\big) & \text{near } 0, \\ (a(R), b(R) + s) & \text{near } R, \end{cases}$$

with $0 < a(R) < \lambda_0$ and $-\varepsilon < b(R) < \varepsilon$, such that the map $F_\gamma : S_1(\nu_S^M) \times [R, 0] \to \mathbb{R}^N$ given by $F(q, \omega, s) = f(q) + a(s)\omega + b(s)\xi(q)$ is a scalar positive immersion.

Proof. Choose ρ as in Proposition 4.12 and set $\rho' := \min\{\rho/2, \lambda_0/2, \varepsilon/2\}$. For this ρ', let τ_0 be chosen as in Proposition 4.12. By Proposition 3.4 there exists $\tau \geq \tau_0$ together with some $0 < \rho'' \leq \min\{\rho', \frac{\pi}{2\tau}\}$ such that f can be deformed into f_1 in such a way that (ii) holds for $(q, \omega, s) \in D_{\rho''}(\nu_S^M)$. Furthermore, by Proposition 4.12, we find γ such that (iii) holds with ρ replaced by ρ''. We conclude that all assertions of Proposition 5.2 hold for $\rho = \rho''$. □

With ρ as in Proposition 5.2 we now consider the smooth manifold

$$\hat{M} := M \setminus D_\rho(\nu_S^M) \bigcup \Sigma_\gamma \bigcup D^{d+1}_{1+b(R)} \times \mathbb{S}^{k-1}, \tag{24}$$

where we glue

$$\partial\big(M \setminus D_\rho(\nu_S^M)\big) = S_\rho(\nu_S^M) \quad \rightleftharpoons \quad S_1(\nu_S^M) \times \{0\} \subset \partial\Sigma_\gamma$$

with the canonical map $S_\rho(\nu_S^M) \approx S_1(\nu_S^M)$, and

$$S_1(\nu_S^M) \times \{R\} \subset \partial\Sigma_\gamma \quad \rightleftharpoons \quad \partial\big(D^{d+1}_{1+b(R)} \times \mathbb{S}^{k-1}\big)$$

with the map

$$S_1(\nu_S^M) \times \{R\} \approx S^d \times \mathbb{S}^{k-1} = \partial\big(D^{d+1}_{1+b(R)} \times \mathbb{S}^{k-1}\big),$$

which is induced by the frame $\{e_1, \ldots, e_k\}$. As usual we say that the manifold $\hat{M}$ is obtained from M by a *surgery* along $S^d \subset M$ (with respect to the normal framing $\{e_1, \ldots, e_k\}$). By Proposition 5.2 (ii) and (iii), the maps f_1 on $(M \setminus D_\rho(\nu_S^M))$, F_γ on Σ_γ, and $\mathscr{F}_{a(R)}$ on $D^{d+1}_{1+b(R)} \times \mathbb{S}^{k-1}$, are compatible at the glueing regions for $\hat{M}$ and define a scalar positive smooth immersion $\hat{f}$:

$\hat{M} \to \mathbb{R}^N$. In terms of bending profiles near $S \subset M$ the images of the first, second and third pieces in (24) under $\hat{f}$ correspond to the blue, red and green pieces in Figure 1.

We finish by proving our main results.

Proof of Theorem 1.6. Since $2d+1 \leq N$, the immersion $f|_{S^d} : S^d \to \mathbb{R}^N$ extends to an immersion $F : D^{d+1}_{1+\varepsilon} \to \mathbb{R}^N$ by [Sma59, Theorem B]. We can easily arrange it in such a way that, for some $0 < \varepsilon < 1$, we have $F(rq) = f(q) + (r-1)\xi(q)$ for all $r \in [1-\varepsilon, 1+\varepsilon]$.

The manifold $D^{d+1}_{1+\varepsilon}$ is contractible and hence the normal bundle $\nu_F \to D^{d+1}_{1+\varepsilon}$, which is of rank $N-d-1$, is trivial. Since the Stiefel manifold $V_{n-d}(\mathbb{R}^{N-d-1})$ of orthonormal $(n-d)$-frames in $\mathbb{R}^{N-d-1}$ is $(N-n-2)$-connected and $d \leq N-n-2$, $\{e_1, \ldots, e_k\}$ extends to an orthonormal family of sections $\{E_1, \ldots, E_k\}$ of ν_F such that all the requirements of Assumption 5.1 hold.

The last assertion follows since ρ_0 in Proposition 3.4 can be chosen arbitrarily small. □

Proof of Theorem 1.2. Assume M is spin. Since $\alpha(M) = 0$, by [KS93, Proposition 3.3] M is spin bordant to the total space of a fibre bundle $\mathbb{H}P^2 \hookrightarrow V \to B$ over a closed spin manifold B, with structure group equal to the isometry group PSp(3) of $\mathbb{H}P^2$. Using $\dim B = n-8$ there exists a scalar positive immersion $V \to \mathbb{R}^{2n-1+\delta(n)}$, compare Example 2.7. Since M and V are spin bordant, M is simply connected, and $\dim M \geq 5$, we can obtain M from V by a finite number of surgeries in codimensions at least 3, using Smale's handle cancellation technique. Theorem 1.2 now follows from Theorem 1.6.

If M is not spin, Führing [Füh, Theorem 1.2] used the methods of [Sto92] and [KS93] to show that M is oriented bordant to the total space of a fibre bundle $\mathbb{C}P^2 \hookrightarrow V \to B$ over a closed oriented manifold B, with structure group equal to the isometry group $\mathrm{PU}(3) \rtimes \mathbb{Z}/2$ of $\mathbb{C}P^2$. Using $\dim B = n-4$ there exists a scalar positive immersion $V \to \mathbb{R}^{2n-1+\delta(n)}$, compare Example 2.7. Since M and V are oriented bordant, M is simply connected and not spin, and $\dim M \geq 5$, we can obtain M from V by a finite number of surgeries in codimensions at least three. Hence Theorem 1.2 again follows from Theorem 1.6. □

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IMPA, Est. Dona Castorina, 110, Jardim Botanico 22460-320, Rio de Janeiro, RJ, BRAZIL
E-mail address: luis@impa.br

Universität Augsburg, Institut für Mathematik, 86135 Augsburg, Germany
E-mail address: hanke@math.uni-augsburg.de