Submanifold Theory: class guide

Luis A. Florit (luis@impa.br, office 404)

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Prerequisites: Basics about manifolds, tensors, at least up to page 12 here. A bit of Riemannian geometry, fundamental group and covering maps.

Bibliography: [**DT**], [**dC**], [**ON**], [**Pe**], [**Sp**], [**KN**]....

DO ALL THE EXERCISES IN [**DT**] !!!!

Clickable index

§1. Notations

Top. manifolds: Hausdorff + countable basis. Partitions of unity.
n-dimensional differentiable manifolds: M^n. Everything is C^∞.
$\mathcal{F}(M) := C^\infty(M,\mathbb{R})$; $\mathcal{F}(M,N) := C^\infty(M,N)$.
(x,U) chart $\Rightarrow$ coordinate vector fields $= \partial_i := \partial/\partial x_i \in \mathfrak{X}(U)$.
Tangent bundle TM, vector fields $\mathfrak{X}(M) := \Gamma(TM) \cong \mathcal{D}(M)$.
Submersions, immersions, embeddings, local diffeomorphisms.
Vector bundles, trivializing charts, transition functions, sections.
Tensor fields $\mathfrak{X}^{r,s}(M)$, k-forms $\Omega^k(M)$, orientation, integration.
Pull-back of a vector bundle $\pi : E \to N$ over N: $f^*(E)$.
Vector fields along a map $f : M \to N \;\Rightarrow \mathfrak{X}_f \cong \Gamma(f^*(TN))$.
f-related vector fields.

Distributions: Definition. Integrable and involutive distributions.

Theorem 1 (Frobenius)**.** A distribution $D \subset TM$ is integrable if and only if it is involutive, i.e., $[X,Y] \in \Gamma(D), \forall X,Y \in \Gamma(D)$.

§2. Riemannian metrics

Gauss, 1827: $M^2 \subset \mathbb{R}^3 \Rightarrow \;\langle\,,\rangle|_{M^2},\; K_M = K_M(\langle\,,\rangle)$, distances, areas, volumes... Non-Euclidean geometries.
Riemann, 1854: $\langle\,,\rangle \Rightarrow K_M$ (relations proved decades later).
Slow development. General Relativity pushed up!
Riemannian metric, Riemannian manifold: $(M^n,\langle\,,\rangle) = M^n$.
$g_{ij} := \langle \partial_i, \partial_j\rangle \in \mathcal{F}(U) \Rightarrow (g_{ij}) \in C^\infty(U, S(n,\mathbb{R}) \cap Gl(n,\mathbb{R}))$.
Isometries, local isometries, isometric immersions.
Product metric. $T_p\mathbb{V} \cong \mathbb{V},\; T\mathbb{V} \cong \mathbb{V}\times\mathbb{V}$.

Examples: $(\mathbb{R}^n, \langle\,,\rangle_{can})$, Euclidean submanifolds. Nash.

Example: (bi-)invariant metrics on Lie groups.

Proposition 2. *Every differentiable manifold admits a Riemannian metric.*

Angles between vectors at a point. Norm.
Riemannian vector bundles: $(E, \langle\,,\rangle)$.
It always exists local orthonormal frames: $\{e_1, \ldots, e_n\}$.
Length of a piecewise differentiable curve $\Rightarrow$ Riem. distance d.
The topology of d coincides with the original one on M.

§3. Linear connections

If $M^n = \mathbb{R}^n$, or even if $M^n \subset \mathbb{R}^N$, there is a natural way to differentiate vector fields. And this depends only on $\langle\,,\rangle$.

Def.: An *affine connection* or a *linear connection* or a *covariant derivative* on M is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

with $\nabla_X Y$ being $\mathbb{R}$-bilinear, tensorial in X and a derivation in Y.

Tensoriality in $X \Rightarrow (\nabla_X Y)(p) = \nabla_{X(p)} Y$ makes sense.
Local oper.: $Y|_U = 0 \Rightarrow (\nabla_X Y)|_U = 0 \Rightarrow (\nabla_X Z)|_U = \nabla^U_{X|_U}(Z|_U)$
$\Rightarrow$ The *Christoffel symbols* Γ^k_{ij} of ∇ in a coordinate system $\Rightarrow$ Christoffel symbols completely determine the connection: all that is needed is to have local basis of sections $\Rightarrow$
Connections on vector bundles: formally exactly the same.
The above property on U is a particular case of the following:

Proposition 3. *(or "Everything I know about connections!") Let ∇ be a linear connection on a vector bundle $\pi : E \to M$. Then, for every smooth map $f : N \to M$, there exists a unique linear connection ∇^f on $f^*(E)$ such that*

$$\nabla^f_Y(\xi \circ f) = \nabla_{f_* Y}\xi, \quad \forall\ Y \in \mathfrak{X}(N), \xi \in \Gamma(E).$$

We will omit the superindex f in ∇^f.
In particular, Proposition 3 holds for any smooth curve $\alpha(t) = \alpha : I \subset \mathbb{R} \to M$, and if $V \in \mathfrak{X}_\alpha$ we denote $V' := \nabla_{\partial_t} V \in \mathfrak{X}_\alpha$.
So, if $\alpha'(0) = v$, $\nabla_v Y = (Y \circ \alpha)'(0)$. But beware of "$\nabla_{\alpha'}\alpha'$"!!

Def.: $V \in \mathfrak{X}_\alpha$ is *parallel* if $V' = 0$. We denote by $\mathfrak{X}''_\alpha$ the set of parallel vector fields along α.

Proposition 4. *Let $\alpha : I \subset \mathbb{R} \to M$ be a piecewise smooth curve, and $t_0 \in I$. Then, for each $v \in T_{\alpha(t_0)}M$, there exists a unique parallel vector field $V_v \in \mathfrak{X}_\alpha$ such that $V_v(t_0) = v$.*

The map $v \mapsto V_v$ is an isomorphism between $T_{\alpha(t_0)}M$ and $\mathfrak{X}''_\alpha$, and the map $(v, t) \mapsto V_v(t)$ is smooth when α is smooth $\Rightarrow$

Def.: The *parallel transport of $v \in T_{\alpha(t)}M$ along α between t and s* is the map $P^\alpha_{ts} : T_{\alpha(t)}M \to T_{\alpha(s)}M$ given by $P^\alpha_{ts}(v) = V_v(s)$.

Notice that $\mathcal{F}(M) = \mathfrak{X}^0(M) = \mathfrak{X}^{0,0}(M)$ and $\mathfrak{X}(M) = \mathfrak{X}^{0,1}(M)$.
Covariant differentiation of 1-forms and tensors: $\forall r, s \geq 0$,

$$\nabla \Rightarrow \begin{cases} \nabla : \mathfrak{X}^r(M) \to \mathfrak{X}^{r+1}(M); \\ \nabla : \mathfrak{X}^{r,s}(M) \to \mathfrak{X}^{r+1,s}(M); \\ \nabla : \mathfrak{X}^{r,s}(E, \hat{\nabla}) \to \mathfrak{X}^{r+1,s}(E, \hat{\nabla}); \end{cases}$$

for any affine vector bundle $(E, \hat{\nabla})$ (in partic., for $E = (TM, \nabla)$).

3.1 The Levi-Civita connection

Def.: A linear connection ∇ on a Riemannian manifold $(M, \langle\, ,\rangle)$ is said to be *compatible* with $\langle\, ,\rangle$ if, for all $X, Y, Z \in \mathfrak{X}(M)$,

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

Exercise. ∇ is compatible with $\langle\, ,\rangle \iff \forall\, V, W \in \mathfrak{X}_\alpha$, $\langle V, W\rangle' = \langle V', W\rangle + \langle V, W'\rangle \iff \forall\, V, W \in \mathfrak{X}''_\alpha$, $\langle V, W\rangle$ is constant $\iff P^\alpha_{ts}$ is an isometry, $\forall\, \alpha, t, s \iff \nabla\langle\, ,\rangle = 0$.

Def.: The tensor $T_\nabla(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ is called the *torsion* of ∇. We say that ∇ is *symmetric* if $T_\nabla = 0$.

Miracle: *Every Riemannian manifold $(M, \langle\, ,\rangle)$ has a unique linear connection that is symmetric and compatible with $\langle\, ,\rangle$, called the Levi-Civita connection of $(M, \langle\, ,\rangle)$.*

This is a consequence of the *Koszul formula*: $\forall\, X, Y, Z \in \mathfrak{X}(M)$,

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle - \langle Y, [X, Z]\rangle + \langle Z, [X, Y]\rangle.$$

Exercise. Verify that this formula defines a linear connection with the desired properties.

This is the only connection that we will work with. In coordinates, if $(g^{ij}) := (g_{ij})^{-1}$,

$$\Gamma^k_{ij} = \frac{1}{2} \sum_r \left(\frac{\partial g_{ir}}{\partial x_j} + \frac{\partial g_{jr}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_r} \right) g^{rk} \ .$$

Exercise. Show that, for $(\mathbb{R}^n, \langle\, ,\rangle_{can})$, $\Gamma^k_{ij} = 0$ and ∇ is the usual vector field derivative.

Exercise. Use Koszul formula to show that the Levi-Civita connection of a bi-invariant metric of a Lie Group satisfies, and is characterized, by the property that $\nabla_X X = 0\ \forall\, X \in \mathfrak{g}$.

Lemma 5. *(Symmetry and Compatibility Lemma) Let N be any manifold, and $f : N \to M$ a smooth map into a Riemannian manifold M. Then:*

- ∇^f *is symmetric, that is,* $\nabla^f_X f_*Y - \nabla^f_Y f_*X = f_*[X,Y]$, $\forall\ X, Y \in \mathfrak{X}(N)$;
- ∇^f *is compatible with the natural metric on* $f^*(TM)$.

Example: $f\colon N \to M$ an isometric immersion $\Rightarrow f^*(TM) = f_*(TN) \oplus^\perp T_f^\perp N \ \Rightarrow \forall\ Z \in \mathfrak{X}_f, Z = Z^\top + Z^\perp \ \Rightarrow$ the relation between the Levi-Civita connections is $f_*\nabla^N_X Y = (\nabla^f_X f_*Y)^\top$.

§4. Geodesics

When do we have minimizing curves? What are those curves? Critical points of the arc-length funct. $L : \Omega_{p,q} \to \mathbb{R}$: *geodesics*:

$$\gamma'' := \nabla_{\frac{d}{dt}}\gamma' = 0.$$

Geodesics = second order nonlinear nice ODE $\Rightarrow$

Proposition 6. $\forall\ v \in TM$, $\exists\ \epsilon > 0$ *and a unique geodesic* $\gamma_v : (-\epsilon, \epsilon) \to M$ *such that* $\gamma_v'(0) = v$ $(\Rightarrow \gamma_v(0) = \pi(v))$.

γ a geodesic $\Rightarrow \|\gamma'\| =$ constant.
γ and $\gamma \circ r$ nonconstant geodesics $\Rightarrow r(t) = at + b$, $a, b \in \mathbb{R} \Rightarrow$ $\gamma_v(at) = \gamma_{av}(t)$; $\gamma_v(t+s) = \gamma_{\gamma_v'(s)}(t) \Rightarrow$ *geodesic field* G of M:

Proposition 7. *There is a unique vector field* $G \in \mathfrak{X}(TM)$ *such that its trajectories are* γ', *where* γ *are geodesics of* M.

The local flux of G is called the *geodesic flow* of M. In particular:

Corollary 8. *For each* $p \in M$, *there is a neighborhood* $U_p \subset M$ *of* p *and positive real numbers* $\delta, \epsilon > 0$ *such that the map*

$$\gamma : T_\epsilon U_p \times (-\delta, \delta) \to M, \quad \gamma(v,t) = \gamma_v(t),$$

is differentiable, where $T_\epsilon U_p := \{v \in TU_p \colon \|v\| < \epsilon\}$.

Since $\gamma_v(at) = \gamma_{av}(t)$, changing ϵ by $\epsilon\delta/2$ we can assume $\delta = 2 \Rightarrow$
We have the *exponential map* of M (terminology from $O(n)$):

$$\exp : T_\epsilon U_p \to M, \quad \exp(v) = \gamma_v(1).$$

$\Rightarrow \exp(tv) = \gamma_v(t) \Rightarrow \exp_p = \exp|_{T_pM} \colon B_\epsilon(0_p) \subset T_pM \to M \Rightarrow$

Proposition 9. *For every* $p \in M$ *there is* $\epsilon > 0$ *such that* $B_\epsilon(p) := \exp_p(B_\epsilon(0_p)) \subset M$ *is open and* $\exp_p \colon B_\epsilon(0_p) \to B_\epsilon(p)$ *is a diffeomorphism.*

An open set $p \in V \subset M$ onto which $\exp_p$ is a diffeomorphism as above is called a *normal neighborhood* of p, and when $V = B_\epsilon(p)$ it is called a *normal* or *geodesic ball* centered at p.

Proposition 9 $\Rightarrow \left(\exp_p|_{B_\epsilon(0_p)}\right)^{-1}$ is a chart of M in $B_\epsilon(p) \Rightarrow$
We always have (local!) *polar coordinates* for any $(M, \langle\, , \rangle)$:

$$\varphi : (0, \epsilon) \times \mathbb{S}^{n-1} \to B_\epsilon(p) \backslash \{p\}, \qquad \varphi(s, v) = \gamma_v(s), \qquad (1)$$

where $\mathbb{S}^{n-1} = \{v \in T_pM : \|v\| = 1\}$ is the unit sphere in T_pM.

Examples: $(\mathbb{R}^n, can)$; $(\mathbb{S}^n, can)$.

Exercise. Show that for a bi-invariant metric on a Lie Group, it holds that $exp_e = exp^G$.

4.1 Geodesics are (local) arc-length minimizers

Lemma 10. *(Gauss' Lemma) Let* $p \in M$ *and* $v \in T_pM$ *such that* $\gamma_v(s)$ *is defined up to time* $s = 1$*. Then,*

$$\langle (\exp_p)_{*v}(v), (\exp_p)_{*v}(w) \rangle = \langle v, w \rangle, \quad \forall\ w \in T_pM.$$

Proof. If $f(s,t) := \gamma_{v+tw}(s) = \exp_p(s(v+tw))$ then, for $t=0$, $f_s = (\exp_p)_{*sv}(v)$, $f_t = (\exp_p)_{*sv}(sw)$ and $\langle f_s, f_t\rangle_s = \langle v, w\rangle$. ∎

Gauss' Lemma $\Rightarrow \mathbb{S}_\epsilon(p) := \partial B_\epsilon(p) \subset M$ is a regular hypersurface of M orthogonal to the geodesics emanating from p, called the *geodesic sphere* of radius ϵ centered at p.
Now, $B_\epsilon(p) := \exp_p(B_\epsilon(0_p)) \subset M$ as in Proposition 9 agrees with the metric ball of (M,d) !!!!! More precisely:

Proposition 11. *Let $B_\epsilon(p) \subset U$ a normal ball centered at $p \in M$. Let $\gamma : [0,a] \to B_\epsilon(p)$ be the geodesic segment with $\gamma(0)=p$, $\gamma(a)=q$. If $c:[0,b]\to M$ is another piecewise differentiable curve joining p and q, then $l(\gamma)\le l(c)$. Moreover, if equality holds, then c is a monotone reparametrization of γ.*

Proof. In polar coordinates, $c(t) = \exp_p(s(t)v(t))$ in $B_\epsilon(p)\backslash\{p\}$, and if $f(s,t) := \exp_p(sv(t)) = \gamma_{v(t)}(s)$, we have that $c' = s' f_s + f_t$. Now, use that $f_s \perp f_t$, by Gauss' Lemma. ∎

Corollary 12. *d is a distance on M, $d_p := d(p,\cdot)$ is differentiable in $B_\epsilon(p)\backslash\{p\}$, and d_p^2 is differentiable in $B_\epsilon(p)$.*

Exercise. Compute $\|\nabla d_p\|$ and the integral curves of ∇d_p inside $B_\epsilon(p)\backslash\{p\}$.

Remark 13. Proposition 11 is LOCAL ONLY, and $\epsilon = \epsilon(p)$: $\mathbb{R}^n$; $\mathbb{S}^n$; $\mathbb{R}^n \setminus \{0\}$.

§5. Curvature

Gauss: $K(M^2 \subset \mathbb{R}^3) = K(\langle\,,\rangle)$. Riemann: $K(\sigma) = K_p(\exp_p(\sigma))$.

Def.: The *curvature tensor* or *Riemann tensor* of M is (sign!)

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

We also call R the (4,0) tensor given by

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle.$$

Curvature tensor $R_{\hat{\nabla}}$ of a vector bundle E with a connection $\hat{\nabla}$: exactly the same.

Proposition 14. *For all $X,Y,Z,W \in \mathfrak{X}(M)$, it holds that:*

- *R is a tensor;*
- *$R(X,Y,Z,W)$ is skew-symmetric in X,Y and in Z,W;*
- *$R(X,Y,Z,W) = R(Z,W,X,Y)$;*
- *$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ (first Bianchi id.);*
- *$R^s_{ijk} = \sum_l \Gamma^l_{ik}\Gamma^s_{jl} - \sum_l \Gamma^l_{jk}\Gamma^s_{il} + \partial_j \Gamma^s_{ik} - \partial_i \Gamma^s_{jk}$ ($\Rightarrow R \cong \partial^2\langle\,,\rangle$).*

Proof. Exercise. ∎

$\langle\,,\rangle \Rightarrow \mathfrak{X}(M) \cong \Omega^1(M)$ and $\langle\,,\rangle$ extends to the tensor algebra $\Rightarrow$ the *curvature operator* $R : \Omega^2(M) \to \Omega^2(M)$ is self-adjoint.

Def.: If $\sigma \subset T_pM$ is a plane, then the *sectional curvature* of M at σ is given by

$$K(\sigma) := \frac{R(u,v,v,u)}{\|u\|^2\|v\|^2 - \langle u,v\rangle^2}, \qquad \sigma = \operatorname{span}\{u,v\}.$$

Proposition 15. *If R and R' are tensors with the symmetries of the curvature tensor and Bianchi such that $R(u,v,v,u) = R'(u,v,v,u)$ for all u, v, then $R = R'$ (i.e., K determines R).*

Corollary 16. *If M has constant sectional curvature $c \in \mathbb{R}$, then $R(X,Y,Z,W) = c(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle)$.*

Def.: The *Ricci tensor* is the symmetric (2,0) tensor given by

$$Ric(X,Y) := \frac{1}{n-1}\mathrm{trace}\, R(X,\cdot,\cdot,Y),$$

and the *Ricci curvature* is $Ric(X) = Ric(X,X)$ for $\|X\| = 1$.

Example: $\mathbb{CP}^n$ as $\mathbb{S}^{2n+1}/\mathbb{S}^1$ has $K(X,Y) = 1 + 3\langle JX,Y\rangle^2$ and $Ric \equiv (n+2)/(n-1)$.

Def.: The *scalar curvature* of M is $\frac{1}{n}\mathrm{trace}\, Ric$.

Lemma 17. *(Compare with Lemma 5) Let $f : U \subset \mathbb{R}^2 \to M$ be a map into a Riemannian manifold and $V \in \mathfrak{X}_f$. Then,*

$$\nabla_{\partial_t}\nabla_{\partial_s}V - \nabla_{\partial_s}\nabla_{\partial_t}V = R(f_*\partial_t, f_*\partial_s)V.$$

Equivalently, $R_{\nabla^f}(\cdot,\cdot)V = R_{\nabla}(f_*\cdot, f_*\cdot)V$, $\forall\, f\colon N \to M$.

Proof. Since R_{∇^f} is a tensor, it is enough to check the lemma for coordinate vector fields on N and for $V = \overline{V} \circ f$, $\overline{V} \in \mathfrak{X}(M)$. ∎

Exercise. Let $\pi : E \to M$ be a vector bundle of rank k with a linear connection ∇. Then ∇ is flat if and only if each $\xi \in E$ has a (unique!) local parallel extension. If M is simply connected, such an extension exists globally and therefore $E \cong M \times \mathbb{R}^k$ is trivial.

§6. Isometric immersions (finally!)

As we have seen in the Example in page 5, if $f : M \to N$ is an isometric immersion $\Rightarrow f^*(TN) = f_*(TM) \oplus^{\perp} T_f^{\perp}M$, and $\nabla_X^M Y = (\nabla_X^f f_* Y)^{\top}$, $\forall X, Y \in TM$. Moreover, we have that

$$\alpha(X,Y) := \left(\nabla_X^f f_* Y\right)^{\perp}$$

is a symmetric tensor, called the *second fundamental form of f*. In addition, $\nabla^{\perp} : TM \times \Gamma(T_f^{\perp}M) \to \Gamma(T_f^{\perp}M)$ given by

$$\nabla_X^{\perp}\eta = \left(\nabla_X^f \eta\right)^{\perp}$$

is a connection in $T_f^{\perp}M$, called the *normal connection of f*. Identifications.

<u>*Exercise*</u>. Show that $\nabla^{\perp}$ is a compatible connection with the induced metric on $T_f^{\perp}M$.

$\alpha(p)$ is the quadratic approximation of $f(M) \subset N$ at $p \in M$.
$\alpha(v,v) = \gamma_v'(0)$: Picture!
$\eta \in T_{f(p)}^{\perp}M \Rightarrow$ (self-adjoint!) *shape operator* $A_\eta : T_pM \to T_pM$.
The Fundamental Equations. Particular case: $K =$ constant $\Rightarrow$ the *Fundamental Theorem of Submanifolds*.
Gauss equation $\Leftrightarrow K(\sigma) = \overline{K}(\sigma) + \langle \alpha(u,u), \alpha(v,v)\rangle - \|\alpha(u,v)\|^2$
$\Rightarrow$ Riemann notion of sectional curvature agrees with ours.

Example: $\mathbb{S}^{n-1}(r) \subset \mathbb{R}^n \Rightarrow K \equiv 1/r^2$ (it *had* to be constant!).

Example: Height functions and the graph of a real function.

Model of the hyperbolic space $\mathbb{H}^n$ as a submanifold of $\mathbb{L}^{n+1}$.

§7. Hypersurfaces

Principal curvatures and directions; mean curvature; Gauss-Kronecker curvature; Gauss map.
Fundamental equations in this simpler case.
Locally convex and strictly convex hypersurfaces.

Proposition 18. *Given a compact Euclidean hypersurface $M^n \subset \mathbb{R}^{n+1}$, for every $0 \neq v \in \mathbb{R}^{n+1}$ there exists $x \in M^n$ such that v is normal to M^n at x and $A_v > 0$.*

Theorem 19. *For a compact Euclidean hypersurface M^n:*
The Gauss-Kronecker curvature never vanishes $\iff$
M is orientable and the Gauss map is a diffeomorphism $\iff$
The second fundamental form is definite everywhere $\iff$
M is a convex hypersurface ($M = \partial B$ for a convex body B).

§8. Totally geodesic and umbilic submanifolds

$\mathbb{Q}^m_{\tilde{c}} \subset \mathbb{Q}^{m+p}_c$ for $\tilde{c} \geq c$.
Axioms of r-planes and r-spheres.

§9. Nullity distributions

The (relative) nullity distribution (Δ) Γ_c and the index of (relative) nullity ($\nu = \dim \Delta$) $\mu_c = \dim \Gamma_c$. $\Gamma = \{X : R(X, ...) = \tilde{R}(X...)\}$ and $\mu = \dim \Gamma$ are extrinsic.

Proposition 20. *For an isometric immersion $f : M \to \tilde{M}$, the following assertions hold:*

i) ν, μ *and* μ_c *are upper semicontinuous. Hence, the subsets where* ν, μ *and* μ_c *attain their minimum values are open, and there is an open and dense subset of* M^n *where* ν *(also* μ *and* μ_c*) is locally constant;*

ii) Δ (Γ_c) *is smooth on any open subset of* M^n *where* ν *(also* μ *and* μ_c*) is constant;*

iii) If $\tilde{M}$ *has constant sectional curvature, then* Δ *is a totally geodesic (hence integrable) distribution on any open subset where* ν *is constant, and the restriction of* f *to each leaf of* Δ *is totally geodesic.*

Remark 21. Γ_c is always an intrinsic totally geodesic foliation where μ_c is constant (why?). Moreover, $\Delta \subset \Gamma$.

Exercise. Every umbilical distribution of a Riemannian manifold is integrable and its leaves are umbilical submanifolds.

§10. Principal Normals and flat normal bundle

Principal and Dupin principal normals. Eigendistributions. Submanifolds with flat normal bundle.

§11. Reduction of codimension

First normal spaces $N_1(x) := \text{span}\{\alpha(X,Y) : X, Y \in T_xM\}$.

Proposition 22. *Let* $f : M^n \to \mathbb{Q}_c^{n+p}$ *be an isometric immersion. Suppose that there exists a parallel normal subbundle* $L^q \subset T^\perp M$ *of rank* $q < p$ *such that* $N_1(x) \subset L^q(x)$ *for all* $x \in M^n$*. Then the codimension of* f *reduces to* q*.*

s-nullities ν_s and ν_s^*. 1-regular isometric immersions.

Proposition 23. *Let $f : M^n \to \mathbb{Q}_c^m$ be a 1-regular isometric immersion such that rank $N_1 = q \leq n-1$. If $\nu_s^*(x) < n-s$ for all $1 \leq s \leq q$ at any point $x \in M^n$, then N_1 is parallel and thus f reduces codimension to q.*

§12. Minimal submanifolds

Let $f_t : M^n \to \bar{M}$ be an isotopy of $f = f_0$. Write $T = f_0' = f_* Z + \eta \in \mathfrak{X}_f$, $Z \in TM$ and $\eta \in T_f^\perp M$. We will denote by $H = \text{trace } \alpha/n \in \Gamma(T_f^\perp M)$ the *mean curvature vector* of f. Then,

$$(dvol_t)'(0) = (-n\langle H, \eta\rangle + div\ Z)\, dvol.$$

Proposition 24. *M^n compact with boundary and $Z|_{\partial M} = 0$, then $V(t) := Vol(f_t(M))$ satisfies $V'(0) = -n\int_M \langle H, \eta\rangle dvol$. In particular, minimal submanifolds are the critical points of the volume functional for compactly supported variations.*

$f : M^n \to \mathbb{R}^m \Rightarrow \Delta f = nH$. Hence, minimal $\Rightarrow$ harmonic $\Rightarrow$ *There are no compact minimal Euclidean submanifolds.* But:

Proposition 25. *A compact minimal Euclidean submanifold with boundary is contained in the interior of the convex hull of its boundary.*

If $f : M^n \to \mathbb{Q}_c^{n+p}$ is minimal, then $Ric_M \leq c$ since

$$Ric_M(X) = c + \frac{n}{n-1}\langle A_H X, X\rangle - \frac{1}{n-1}\sum_{i=1}^{p}\langle A_{\xi_i}^2 X, X\rangle.$$

In particular, $scal_M = c + \frac{n}{n-1}\|H\|^2 - \frac{1}{n(n-1)}\|\alpha\|^2$.

Lemma 26. *Given $F : M^n \to \mathbb{R}^{m+1}$, there exists a minimal $f : M^n \to \mathbb{S}^m_c$ such that $F = inc \circ f \iff \Delta F = -ncF$.*

<u>*Construction*</u>. Let $\mathcal{H}(m,d)$ be the vector space of homogeneous harmonic polynomials of degree d in $(m+1)$ real variables. Then, $\dim \mathcal{H}(m,d) = n+1$, where $n = n(m,d) = \frac{(2d+m-1)(d+m-2)!}{d!(m-1)!} - 1$. Then, $W = W(m,d) = \{f|_{\mathbb{S}^m} \colon f \in \mathcal{H}(m,d)\}$ is contained in (actually, it is equally to) the eigenspace of $\Delta_{\mathbb{S}^m}$ with eigenvalue $\lambda(m,d) = -d(m+d-1)$. Fix $\langle\,,\rangle$ the L^2-inner product on W, and $\{f_0,\dots,f_n\}$ an orthonormal basis of W. Set $G := O(m+1)$,

$$F := (f_0,\dots,f_n) : \mathbb{S}^m \to \mathbb{R}^{n+1}.$$

Since $\langle\,,\rangle$ is invariant under the G-action $A \cdot f = f \circ A$, the basis $\{A\cdot f_0,\dots,A\cdot f_n\}$ is also orthonormal. So there is $\tilde{A} \in O(n+1)$ such that $F \circ A = \tilde{A} \circ F$, and the map $A \mapsto \tilde{A}$ is a group homomorphism (such an F is said to be *G-equivariant*). In particular, G acts isometrically and transitively with the metric induced by F, and the isotropy groups $O(m)$ act transitively on the Grassmannians of each tangent space. Thus, there exists $\tilde{c} > 0$ such that $F^*\langle,\rangle = \tilde{c}\langle,\rangle$, and hence F induces an isometric immersion of $\mathbb{S}^m_{1/\tilde{c}}$ into $\mathbb{R}^{n+1}$ with $\Delta F = -(1/\tilde{c})\lambda(m,d)F$. We conclude by Lemma 26 that there is a minimal equivariant isometric immersion $f : \mathbb{S}^m_{1/\tilde{c}} \to \mathbb{S}^n_c$, $c = \lambda(m,d)/m\tilde{c}$, $F = inc \circ f$. We have constructed the minimal and equivariant *Veronese embeddings*,

$$f : \mathbb{S}^m_r \to \mathbb{S}^n, \quad r = r(m,d) := \frac{m}{d(m+d-1)}$$

(these are embeddings if d is odd, and embeddings of the projective space if d is even). And they are (essentially) *unique!!*

§13. Minimal rigidity of hypersurfaces

Deformability and rigidity. The associated family.

Theorem 27. *Let $f : M^n \to \mathbb{Q}_c^{n+1}$ be a minimal immersion of a Riemannian manifold with $\mu_c \not\geq n-2$. Then, f is rigid among minimal immersions $g : M^n \to \mathbb{Q}_c^{n+p}$, i.e., $g = inc \circ f$.*

Proof. Diagonalize the shape operator of f, $Ae_i = \lambda_i e_i$, and set $\alpha_{ij} := \alpha_g(e_i, e_j)$. By Gauss equation, $\lambda_i^2 = \sum_k \|\alpha_{ik}\|^2$ and

$$(\langle \alpha_{ii}, \alpha_{jj} \rangle - \|\alpha_{ij}\|^2)^2 = \lambda_i^2 \lambda_j^2 = \sum_k \|\alpha_{ik}\|^2 \sum_k \|\alpha_{jk}\|^2$$
$$\geq (\|\alpha_{ii}\|^2 + \|\alpha_{ij}\|^2)(\|\alpha_{jj}\|^2 + \|\alpha_{ij}\|^2) \geq (\langle \alpha_{ii}, \alpha_{jj} \rangle + \|\alpha_{ij}\|^2)^2.$$

So, $\alpha_{ij} \neq 0 \;\Rightarrow \langle \alpha_{ii}, \alpha_{jj} \rangle \leq 0 \Rightarrow \lambda_i \lambda_j \leq -\|\alpha_{ij}\|^2 < 0$. Thus, at a point with $\nu = \mu_c \leq n-3$, there should be a pair with $\alpha_{ij} = 0$. The above equation implies that α_{ii} and α_{jj} are linearly dependent, and $\alpha_{is} = 0$ for $i \neq s \neq j$. Changing the roles of s and j we get $\alpha_{ij} = 0$. We conclude that $(\alpha_g)_{N_g^1} = \pm \alpha_f$. Done, since N_g^1 is parallel (e.g. by Proposition 23) and g is analytic. ∎

§14. Local rigidity and flat bilinear forms

In local coordinates, an isometric immersion is a solution of a nonlinear PDE, so if the codimension is small it should be overdetermined, hence rigidity should be true under generic conditions. Analyze the proof of Theorem 27: It's just Gauss equation!

But: f rigid $\Rightarrow$ Find $\tau : T_f^\perp M \to T_g^\perp M$ satisfying

$$\tau \circ \alpha_f = \alpha_g.$$

Such τ is unique if f is full (or unique in N_1^f), and its parallelism is not hard to see. Now, a necessary condition for the existence such a bundle isometry τ is that

$$\|\alpha_f(X,Y)\| = \|\alpha_g(X,Y)\|, \quad \forall X,Y \in TM. \tag{2}$$

which is equivalent by polarization to that, $\forall\, X, Y, X', Y' \in TM$,

$$\langle \alpha_f(X,Y), \alpha_f(X',Y')\rangle = \langle \alpha_g(X,Y), \alpha_g(X',Y')\rangle.$$

But this is also sufficient(!!): just define τ as $\tau \circ \alpha_f = \alpha_g$ and extend linearly. In other words, we need to understand when the flat bilinear form (FBF) $\beta = (\alpha_f, \alpha_g)$ is null, where

$$\beta = (\alpha_f, \alpha_g) : TM \times TM \to (T_f^\perp M \times T_g^\perp M, \langle\,,\rangle_f - \langle\,,\rangle_g).$$

14.1 Flat bilinear forms

Let $\beta : \mathbb{V} \times \mathbb{V}' \to \mathbb{W}^{p,q}$ a FBF.

Def.: $RE(\beta)$. $S(\beta)$. β_X for $X \in \mathbb{V}$. Isotropic (null) subspaces. $\nu_\beta := \dim N(\beta)$.

Proposition 28. *The subset $RE(\beta)$ is open and dense in V.*

Observe that, by flatness: *if $\beta_X(\mathbb{V}') \subset \mathbb{W}$ is isotropic for all X in a dense subset, then β is null.*

Proposition 29. *For any bilinear form β and $X \in RE(\beta)$, $\beta(\mathbb{V}, \operatorname{Ker}\beta_X) \subset \operatorname{Im}\beta_X$. If β is also flat, then $\beta|_{\mathbb{V}\times\operatorname{Ker}\beta_X}$ is null since $\beta(\mathbb{V}, \operatorname{Ker}\beta_X) \subset \mathcal{U}(X) := \operatorname{Im}\beta_X \cap \operatorname{Im}\beta_X^\perp$.*

Proof. For any $Y \in \mathbb{V}$ and t small, $L_t = \operatorname{Im}\beta_{X+tY} \subset \mathbb{W}$ is a *continuous* family of subspaces that contain $\beta_{X+tY}(\operatorname{Ker}\beta_X) = \beta_Y(\operatorname{Ker}\beta_X)$, which does not depend on t. ∎

Corollary 30. $\beta\colon \mathbb{V}\times\mathbb{V}' \to \mathbb{W}^{p,0}$ *FBF* $\Rightarrow \nu_\beta \geq \dim\mathbb{V}' - \dim\mathbb{W}$.

Theorem 31 (Chern-Kuiper)**.** $M^n \subset \tilde{M}^{n+p} \Rightarrow \nu \leq \mu \leq \nu + p$.

Corollary 32. $M^n \subset \mathbb{R}^{n+p}$ *compact* $\Rightarrow \mu (= \mu_0) \not> p$.

Corollary 33. $M^n \subset \mathbb{R}^{n+p}$ *compact and flat* $\Rightarrow p \geq n$.

14.2 Uniqueness of the normal connection

Proposition 34. *Let $f, f' : M^n \to \mathbb{Q}_c^{n+p}$ be isometric immersions and let $\tau : T_f^\perp M \to T_{f'}^\perp M$ be a vector bundle isometry that preserves the second fundamental forms. Then it also preserves the normal connections on the first normal bundles. In particular, it is parallel if either immersion is full.*

Def.: Type number τ of f. Obs: $\tau \geq 1 \Rightarrow f$ is full.

Remark 35. Allendoerfer: if $\tau \geq 4$, then G $\Rightarrow$ C+R.

§15. Local algebraic rigidity

Lemma 36 (Lorenzian version of Corollary 30)**.** *If β is a FBF with $S(\beta) = \mathbb{W}^{p,1}$ Lorentzian $\Rightarrow \nu_\beta \geq \dim\mathbb{V}' - \dim\mathbb{W}$.*

Theorem 37 (Beez-Killing)**.** *$M \subset \mathbb{Q}_c^{n+1}$ with $\tau \geq 3$ is rigid.*

Corollary 38. *Let $f, f' : M^n \to \mathbb{Q}_c^{n+1}$ be nowhere congruent isometric immersions of a Riemannian manifold with no points of constant sectional curvature c. Then f and f' carry a common relative nullity distribution of rank $n-2$.*

Theorem 39 (Allendoerfer)**.** *Any $f : M^n \to \mathbb{Q}_c^{n+p}$ with $\tau \geq 3$ everywhere is rigid.*

Proof. By Proposition 34, we only need to show that $\beta = \alpha \oplus \alpha'$ is null, since $\tau \geq 3$ implies that f is null.
Let $k := \min\{\dim \mathcal{U}(X)\colon X \in RE(\beta)\}$. Similarly to $RE(\beta)$, $RE^o(\beta) := \{X \in \mathbb{V}\colon \dim \mathcal{U}(X) = k\}$ is also open and dense in $\mathbb{V}$. So, we only need to show that $k = p$.
$\tau \geq 3 \;\Rightarrow \exists\, L^{n-3p} := (\mathrm{span}\{A_{\xi_j} X_i\colon 1 \leq j \leq p, 1 \leq i \leq 3\})^\perp = \cap_{i=1}^3 \mathrm{Ker}\,\alpha_{X_i}$. But $\dim \mathrm{Ker}\,\beta_{X_1} = n$–$\mathrm{rank}\,\beta_{X_1} \geq n$–$2p$+$k$. Proposition 29 $\Rightarrow \dim \mathrm{Ker}\,\beta_{X_1} \cap \mathrm{Ker}\,\beta_{X_2} \geq \dim \mathrm{Ker}\,\beta_{X_1} - \dim \mathcal{U}(X_1) \geq n - 2p$ and similarly $\dim \cap_{i=1}^3 \mathrm{Ker}\,\beta_{X_i} \geq n - 2p - k$. Done, since $\cap_{i=1}^3 \mathrm{Ker}\,\beta_{X_i} \subset L^{n-3p}$. ∎

§16. The Main Lemma

Def.: Nondegenerate FBFs.

Given a FBF $\beta : \mathbb{V}^n \times \mathbb{V}^n \to \mathbb{W}^{p,q}$, set $U := S(\beta) \cap S(\beta)^\perp$, $W = U \oplus \hat{U} \oplus^\perp L$ where $\hat{U}$ is null, $S(\beta) = U \oplus L$, and $\beta = \beta_U + \beta_L$, with β_U null and β_L nondegenerate.

Lemma 40 (The Main Lemma)**.** *β symmetric nondegenerate with* $\min\{p,q\} \leq 5 \;\Rightarrow \nu_\beta \geq n - p - q$.

Theorem 41. *$f\colon M^n \to \mathbb{Q}_c^{n+p}$ with $p \leq 5$ and $\nu_j \leq n - 2j - 1$ for all $1 \leq j \leq p \;\Rightarrow f$ is rigid.*

Proof. Just show that U above has dimension p. Since $\nu_1 \leq n-3$, f is full. ∎

§17. Submanifolds with constant curvature

FBF were introduced by Cartan to study $f\colon M^n_c \to \mathbb{Q}^{n+p}_c$. Moore.

Lemma 42. *$f\colon M^n_c \to \mathbb{Q}^{n+p}_{\tilde{c}}$. $c < \tilde{c} \;\Rightarrow p \geq n-1$. $c > \tilde{c}$ and $p \leq n-2 \;\Rightarrow \alpha_f = \gamma + \sqrt{c-\tilde{c}}\,\langle\ ,\ \rangle\xi$ with γ flat and ξ unit.*

Proof. Compose f with an umbilical inclusion $\mathbb{Q}^{n+p}_{\tilde{c}} \to \mathbb{Q}^{n+p+1}_c$ (the latter Lorentzian if $c > \tilde{c}$) and apply the Main Lemma 40. ∎

A point $x \in M$ where $\alpha_f(x) = \gamma(x) + \sqrt{c-\tilde{c}}\,\langle\ ,\ \rangle\xi(x)$ is called a *weak umbilic* f. Weak umbilic everywhere $\Rightarrow$ composition???

What happens if $c > \tilde{c}$, f free of weak umbilics, and $p = n-1$?

Proposition 43 (Moore). *$\beta : \mathbb{V}^n \times \mathbb{V}^n \to \mathbb{W}^{p,q}$ symmetric FBF, with $q = 0, 1$ and $\nu_\beta = n - p - q$. If $q = 1$, assume further that β is nondegenerate and that there exist a vector $e \in \mathbb{W}$ such that $\langle \beta, e\rangle > 0$. Then, β decomposes uniquely as the direct sum of $p+q$ rank one flat forms.*

Proof. Let's do the case $q=0$ only. We may assume $p=n$, $\nu_\beta=0$. Fix $X_0 \in RE(\beta) \;\Rightarrow \beta_{X_0}$ is an isomorphism $\;\Rightarrow C(Y) := \beta_Y \circ \beta_{X_0}^{-1} \in End(\mathbb{W})$ are all self-adjoint and commuting by flatness $\Rightarrow$ $\exists$ and O.N.B. of $\mathbb{W}$ such that $C(Y)\xi_i = \mu_i(Y)\xi_i$. Set $\beta_i = \langle \beta, \xi_i\rangle$, $\beta_{X_0}X_i = \xi_i \Rightarrow \beta(Y, X_i) = \mu_i(Y)\xi_i$ and $\beta_{iY} = \mu_i(Y)\beta_{iX_0} \Rightarrow \beta_{iX_0} = a_i\mu_i$ for $a_i = \beta_{iX_0}X_0 \Rightarrow a_i\mu_i(X_j) = \beta_{iX_0}X_j = \langle \xi_j, \xi_i\rangle = \delta_{ij} \Rightarrow \{\mu_i\} = \{X_i\}^*$ and $\beta_i = a_i\mu_i \otimes \mu_i$. ∎

Corollary 44. *Let $f\colon M^n_c \to \mathbb{Q}^{2n}_c$ with $\nu = 0 \Rightarrow \exists$ unique basis $\{X_i\}$ of unit vectors and o.n.b. $\{\eta_i\}$ such that $\alpha(X_i, X_j) =$*

$\delta_{ij}\eta_j$. The basis $\{X_i\}$ is orthogonal if and only if $R^\perp = 0$, in which case the $\{\eta_i\}$ are the principal normals of f.

Corollary 45. *Let $f\colon M_c^n \to \mathbb{Q}_{\tilde{c}}^{2n-1}$ with $c \neq \tilde{c}$. If $c > \tilde{c}$ assume that f has no weak umbilics. Then $R^\perp = 0$.*

§18. Nonpositive extrinsic curvature

Asymptotic vectors of α: $A(\alpha) := \{X \in \mathbb{V} : \alpha(X, X) = 0\}$.
As we saw in the proof of Lemma 42, we have:

Lemma 46 (Otsuki)**.** *Let $\alpha : \mathbb{V}^n \times \mathbb{V}^n \to \mathbb{W}^{p,0}$ symmetric such that $K_\alpha \leq 0$ and $A(\alpha) = 0 \ \Rightarrow p \geq n$.*

Corollary 47. *$f\colon M^n \to \mathbb{Q}_c^{n+p}$ with $K_M(x_0) < c \Rightarrow p \geq n-1$.*

Corollary 48. *$f\colon M^n \to \mathbb{R}^{n+p}$ compact such that, $\forall x \in M$ there is $V_x \subset T_xM$ of dimension $\geq m$ with $K(\sigma) < 0 \; \forall \sigma \subset V_x$. Then, $p \geq m$.*

There is a generalization of Lemma 46:

Lemma 49. *Let $\alpha : \mathbb{V}^n \times \mathbb{V}^n \to \mathbb{W}^p$ symmetric. If $p < n$ and $A(\alpha) = 0$, there are $X, Y \in \mathbb{V}$ L.I. such that $\alpha(X, X) = \alpha(Y, Y)$ and $\alpha(X, Y) = 0$.*

Proof. Complexifying, it is equivalent to p quadratic equations $\alpha(Z, Z) = 0$ in $n > p$ variables, which is well-known to always have a nontrivial solution that cannot be real by assumption. ∎

Compactness in Corollary 48 can be relaxed.

Def.: The *Omori-Yau maximum principle for the Hessian* (OYMP for short) is said to hold on a given Riemannian manifold

M if for any function $g \in C^2(M)$ with $g^* := \sup g < +\infty$ there exists a sequence of points $\{x_k\}$ in M satisfying the following: $g(x_k) > g^* - 1/k$, $\|\nabla g(x_k)\| < 1/k$, $\mathrm{Hess}_g(x_k) < 1/k$.

The following result by Pigola-Rigoli-Setti gives conditions for the OYMP to hold in a complete manifold (no proof):

Theorem 50. *Let M be a complete noncompact R.M, $\rho(x) := d(x, x_0)$. If $K_M \geq -\phi^2 \circ \rho$, where $\phi \in C^1([0, +\infty))$ satisfies $\phi(0) > 0$, $\phi' > 0$, $1/\phi \notin L^1$, then M satisfies the OYMP.*

Theorem 51. *Let $f : M^n \to P^m \times \mathbb{R}^\ell$, $2 \leq m \leq 2(n-\ell) - 1$, i.i. between complete R.M. such that $f(M) \subset B_R(o) \times \mathbb{R}^\ell$ with $K_P|_{B_R(o)} \leq b$ and $R < \min\{inj_P(o), \pi/2b\}$ ($\pi/2b = +\infty$ if $b \leq 0$). If $scal_M \geq -C\rho^2(\Pi_{j=1}^N log^{(j)} \circ \rho)^2$ outside of a compact set for certain $C > 0$ and $N \in \mathbb{N}$, then* $\sup K_f \geq c_b^2(R)(= cot...)$.

Proof. May assume $\sup K < +\infty$. Then, $K \geq -C'\rho^2(...)$ also $\Rightarrow$ OYMP by Theorem 50. Find a contradiction like Otsuki. ∎

Corollary 52. *$f : M^n \to N^{n+p}$, $p \leq n-1$, M complete, N Hadamard. Assume that the scalar curvature of M is bounded from below. If $K_f \leq 0$, then $f(M)$ is unbounded.*

18.1 Relative nullity in nonpositive extrinsic curvature

Theorem 53. *$\alpha : \mathbb{V}^n \times \mathbb{V}^n \to \mathbb{W}^{p,0}$ symmetric with $K_\alpha \leq 0$. Then, $\nu_\alpha \leq n - 2p$ (and this estimate is sharp!).*

This follows from a sequence of three propositions:

- $X_0 \in A(\alpha)$, $\hat{V} = \mathrm{Ker}\, \alpha_{X_0}$, $\hat{W} = (\mathrm{Im}\, \alpha_{X_0})^\perp \Rightarrow S(\alpha|_{\hat{V} \times \hat{V}}) \subset \hat{W}$.

Proof. $K_f(X_0 + tY, Z) \leq 0$ for $Z \in \hat{V}$.

- $\exists\, T^m \subset A(\alpha)$ <u>subspace</u> with $m \geq n-p$ ($\Rightarrow$ Otsuki Lemma 46!).

Proof. Inductive Otsuki for $X_i \in A(\alpha_i)$ and $\max \operatorname{rank} \alpha_{X_i}$. ∎

- $\nu_\alpha \geq \dim T - p$.

Proof. Let $T \oplus T' = \mathbb{V}$, $\beta := \alpha|_{T' \times T}$, $Y_0 \in RE(\beta)$. Use that $K(Y_0 + tZ, Y + sZ') \leq 0$ for $Z' \in \operatorname{Ker} \beta_{Y_0} \subset T$, $Z \in T$, $Y \in T'$ is affine in s $\Rightarrow \beta(Y, \operatorname{Ker} \beta_{Y_0}) \subseteq \beta_{Y_0}(T')^\perp$ $\Rightarrow \operatorname{Ker} \beta_{Y_0} \subset \Delta_\alpha$. ∎

Many corollaries (no proofs):

Corollary 54. *M^n complete and finite volume, $K \leq c < 0$. $f\colon M^n \to N_c^{n+p}$ for $2p < n$ $\Rightarrow f$ totally geodesic.*

Corollary 55. *$f\colon M^{2n} \to \mathbb{Q}_c^{2n+p}$ Kahler. If $\exists\, x_0 \in M$ with $K(x_0) \leq c \neq 0$ $\Rightarrow p \geq n$.*

Special cases of Thm. 53: $R^\perp = 0$, $\nu_f = n - 2p$ and $\nu_f = n - 2p + 1$.

Corollary 56. *$M^n \subset \mathbb{R}^{n+p}$, $p \leq n/2$, $K \leq 0$ and $Ric < 0 \Rightarrow 2p = n$, local and global product of p surfaces $K < 0$ in $\mathbb{R}^3$.*

What happens with $\nu_f = n - 2p$ for $M^n \subset \mathbb{Q}_c^{n+p}$ if $c \neq 0$??

§19. The relative nullity

Splitting tensor $C : D \times D^\perp \to D^\perp$ of a distribution D.
C is symmetric $\iff$ $D^\perp$ is integrable.
C_T is a multiple of the identity $\forall T$ $\iff$ $D^\perp$ is umbilic.
$C = 0$ $\iff$ $D^\perp$ is totally geodesic.

Proposition 57. *Let $f\colon M^n \to \mathbb{Q}_c^{n+p}$ and $D \subset \Delta$ a totally geodesic distribution. Then, $\forall \xi \in T^\perp M, S, T \in D, X, Y \in D^\perp$, and $\gamma' \subset D$ geodesic with parallel transport P_γ, we have:*

1. $\nabla_T C_S = C_S C_T + C_{\nabla_T S} + c\langle T, S\rangle I$;

2. $C'_{\gamma'} = C^2_{\gamma'} + cI$ *(Riccati !!)*;

3. $P_\gamma^{-1} \circ C_{\gamma'} \circ P_\gamma^{-1} = (\sin(t)I + \cos(t)B)(\cos(t)I - \sin(t)B)^{-1}$ *for e.g.* $c = 1$, *where* $B = C_{\gamma'}(\gamma(0))$;

4. $(\nabla_X C_T)Y - (\nabla_Y C_T)X = C_{(\nabla_X T)_D}Y - C_{(\nabla_Y T)_D}X$;

5. $\nabla_T A_\xi = A_\xi C_T + A_{\nabla^\perp_T \xi}$ *(*$A_\bullet$ *restricted to* $D^\perp$*)*;

6. $A'_\xi = A_\xi C_{\gamma'}$, *if* ξ *is parallel along* $\gamma' \subset D$;

7. $A_\xi C_T = C^t_T A_\xi$;

8. *Both* $\operatorname{Ker} A_\xi$ *and* $\operatorname{Im} A_\xi$ *are parallel along* γ *if* ξ *also is.*

Proposition 58. *If* $D^{k\perp}$ *totally geodesic, then* $c = 0$ *and* f *is a* k*-cylinder. If* $D^{k\perp}$ *umbilical, then* f *is locally a generalized cone over* $g\colon M^{n-\nu} \to \mathbb{Q}^{n+p-\nu}_{\tilde{c}}$, *and* $\mathbb{Q}^{n+p-\nu}_{\tilde{c}} \subset \mathbb{Q}^{n+p}_c$ *umbilic.*

19.1 Completeness of the relative nullity

Proposition 59. *Let* $f\colon M^n \to \mathbb{Q}^{n+p}_c$, *and* $U \subset M$ *an open subset where* $\nu_f = s > 0$. *If* $\gamma\colon [0, b] \to M$ *is a geodesic such that* $\gamma([0, b))$ *is contained in a leaf of* Δ *in* U, *then* $\Delta(\gamma(b)) = P_\gamma(\Delta(\gamma(0))$ *and* $\nu_f(\gamma(b)) = s$. *Moreover,* $C_{\gamma'}$ *extends smoothly to* $\gamma(b)$ *and Proposition 57 items 2, 6, 7 and 8 hold on* $[0, b]$.

Proof. Consider the solution of $J' + CJ = 0$, $J(0) = I$, for $C := C_{\gamma'} \Rightarrow J'' + cJ = 0 \Rightarrow J$ extends to b as an endomorphism of $P_{0,b}(\Delta(\gamma(0))$. If $Z, Y \in \mathfrak{X}''_\gamma$ with $Y \in \Delta^\perp \Rightarrow \alpha(JY, Z)' = 0$ $\Rightarrow J$ invertible in $[0, b]$, $P_{0,b}(\Delta(\gamma(0)) = \Delta(\gamma(b))$ and C extends to b since $C = -J'J^{-1}$. ∎

Corollary 60 (**!!!!**). *The minimum relative nullity distribution is complete if M is complete.*

Corollary 61. *Propositions 57,59 and Corollary 60 hold for the intersection of the relative nullities of a finite number of immersions (since $(\alpha_1, \alpha_2, ...)$ is Codazzi).*

19.2 The spherical case

Theorem 62. *M^n complete, $f\colon M^n \to \mathbb{S}^{n+p}$ with $\nu > 0$. Then, at any point in $U = \{\nu = \nu_0 > 0\}$ where is ν is minimal, and for any normal direction at that point, the numbers of positive and negative principal curvatures are equal.*

Proof. Let $\gamma\colon \mathbb{R} \to U$ a geodesic in a leaf of Δ, ξ normal parallel along γ. By Proposition 57.6, $\operatorname{Ker} A_\xi$ is parallel along γ $\Rightarrow$ $\operatorname{rank} A_\xi$ is constant $\Rightarrow$ so are the number of positive and negative eigenvalues. But the antipodal map I of $\mathbb{S}^{n+p}$ leaves U invariant $\Rightarrow \exists\, \tau \in \operatorname{Iso}(U)$ such that $f \circ \tau = I \circ f|_U$. But $inc_* \xi$ is constant in $\mathbb{R}^{n+p+1}$ along γ, so $\xi \circ \tau = -I_* \xi$ and thus $A_{\xi \circ \tau} \circ \tau_* = -\tau_* A_\xi$. ∎

Corollary 63. *Either $Ric < 1$ in U or f is totally geodesic.*

Corollary 64. *$f : \mathbb{S}^n \to \mathbb{S}^{n+p}$ with $p \le n-1 \Rightarrow f$ is totally geodesic. In particular, it is rigid.*

Example 65. The product isometric immersion $F\colon \mathbb{R}^{n+1} \to \mathbb{R}^{2n+2}$ given by $F(t) = \frac{1}{\sqrt{n+1}} e^{i\sqrt{n+1}t}$ induces $f = F|_{\mathbb{S}^n}\colon \mathbb{S}^n \to \mathbb{S}^{2n+1}$ which is <u>not</u> totally geodesic. And what about $p = n$???

19.3 The Euclidean case

Theorem 66 (Extrinsic splitting theorem of Harman)**.** *M^n complete with $Ric_M \geq 0$ and $f\colon M^n \to \mathbb{R}^{n+p}$ containing r independent lines $\Rightarrow$ f is an r-cylinder ($\Rightarrow$ f is an ν_0-cylinder).*

Proof. Let $g : \mathbb{R}^2 \to \mathbb{R}^m$ i.i. containing a straight line L. Write $g(x, y) = (u(x, y), v(x, y))$ with $u(0, y) = 0 \in \mathbb{R}^{m-1}$, $v(0, y) = y$. But $v(x, y) = y$: indeed, if $p = (x, y)$ and $c = v(p) - y$, let $q = (0, y + \lambda c) \in L$ with $\lambda >> 1$ so that $d(p, q) - d((0, y), q) \leq |c|/2$. So, $(\lambda + 1)|c| \leq d(g(p), g(q)) \leq d(p, q) \leq (\lambda + 1/2)|c|$, and thus $c = 0$. Since $1 = \|g_y\|^2 = \|u_y\|^2 + 1$, we get $g(x, y) = (u(x), y)$. Now, by the splitting theorem, $M = N \times \mathbb{R}^{\nu_0}$. Fix $x_0 \in N$ and take $\gamma \subset N$ a geodesic with $\gamma(0) = x_0$. Set $f_\gamma : R \times \mathbb{R}^{\nu_0} \to \mathbb{R}^N$, $f_\gamma(s, v) = f(\gamma(s), v)$. By the above $f(\{x\} \times \mathbb{R}^{\nu_0})$ are parallel, so $f(x, v) = (g(x), r(x) + A(x)v) \in \mathbb{R}^{n+p-\nu_0} \times \mathbb{R}^{\nu_0}$, with $A(x) \in O(\nu_0)$. But $\langle f_x, f_v \rangle = 0 \;\Rightarrow r_x = A_x = 0$. ∎

§20. The Gauss Parametrization

Let $f : M^n \to \mathbb{R}^{n+1}$ a submanifold with constant relative nullity $n - k$ and Gauss map $\eta : M^n \to \mathbb{S}^n$. Let $\gamma = \langle f, \eta \rangle$ be the *support function* of f. Then, locally, we have a projection $\pi : M^n \mapsto V^k := M^n/\Delta$ onto the leaf space, which is a k-dimensional manifold. We thus have that η and γ depend only on their projections, i.e., $\eta = h \circ \pi$ and $\gamma = r \circ \pi$, for certain

$$h : V^k \to \mathbb{S}^n \;\text{ and }\; r : V^k \to \mathbb{R} : \quad \text{the } \textit{Gauss data} \text{ of } f.$$

Therefore, at regular points, $f(M)$ is locally parametrized by

$$\Psi : M \cong T_h^{\perp} V \to \mathbb{R}^{n+1}, \quad \Psi \circ \xi = rh + \nabla r + \xi, \;\; \xi \in \Gamma(T^{\perp} V).$$

Observe that, if $w = \xi(x) \in T_{h(x)}^{\perp} V$ and since $h_{*x}(T_x V)$ and $\Psi_{*w}(\Delta^{\perp}(w))$ are parallel, we identify $T_x V$ with $\Delta^{\perp}(w)$ with the isometry $j = j_w \colon T_{h(x)} V \to \Delta^{\perp}(w)$ given by $h_{*x} = \Psi_{*w} \circ j_w$. Set

$$P_w := rI + \mathrm{Hess}_r - A_w^h \;\; \in \;\; EndSim(T_x V).$$

Hence, $j_w \circ P_w = (\xi_{*x})_{\Delta^{\perp}(w)}$ and $\pi_* \circ j_w = P_w^{-1}$, that conjugates all operators. Note that $\nabla_{\Delta} j = 0$ and set $\hat{\xi} = \xi \circ \pi \in \Delta$. Thus:

- $\Delta(w) = T_{h(x)}^{\perp} V$ and $\Delta^{\perp}(w) = T_{h(x)} V$ by construction;
- $(\xi_* X)_{\Delta^{\perp}(w)} \neq 0$ and $\|\Psi_*(\xi_* X)_{\Delta^{\perp}(w)}\| = \|P_w X\|_h \;\; \forall X \neq 0$;
- w is a regular point of $\Psi \iff P_w$ is invertible;
- The shape operator of f in $\Delta^{\perp}(w)$ is given by $A = -(P_w)^{-1}$;
- The splitting tensor of Δ for $\hat{\xi}$ is $C_{\hat{\xi}} = A_{\xi}^h P_w^{-1}$;
- The connections are related by $\langle \nabla_{P_w X} Y, Z \rangle_M = \langle \nabla_X^h Y, Z \rangle_h$;
- The normal connection $\nabla^{\perp}$ of h is related to the Levi-Civita connection of M along Δ by $\langle \nabla_{P_w X} \hat{\xi}, \hat{\eta} \rangle_M = \langle \nabla_X^{\perp} \xi, \eta \rangle$.

Corollary 67. *$\Delta^{\perp}$ is integrable $\iff$ h has flat normal bundle and $[\mathrm{Hess}_r, A_w] = 0$ $\forall w \in T^{\perp} V$.*

Corollary 68. *Any submanifold $h : V^k \to \mathbb{S}^n$ is a Gauss map. The set of hypersurfaces with Gauss map h is parametrized by $\mathcal{F}(V^k)$.*

Corollary 69. *In $\mathbb{Q}_c^{n+1}$ we also have Gauss parametrization.*

Corollary 70. *f is a cylinder $\iff$ h reduces codimension.*

Corollary 71. *$f\colon M^n \to \mathbb{R}^{n+1}$ with $\mu \equiv n-2$ and complete leaves of Δ along which the mean curvature of f does not change sign. Then, $h : V^2 \to \mathbb{S}^n$ is minimal and f is a cylinder over $g : N^{2+\epsilon} \to \mathbb{R}^{3+\epsilon}$ and $\nu_g = \epsilon$, where $\epsilon = 0, 1$.*

The last one and Theorem 27 give:

Corollary 72. *$f\colon M^n \to \mathbb{R}^{n+1}$ complete minimal without euclidean factors, $n \geq 4 \Rightarrow f$ is rigid in $\mathbb{R}^{n+p}$ among minimal.*

Corollary 73. *$f\colon M^n \to \mathbb{R}^{n+1}$ with $\mu \equiv n-2$ and $scal_M$ constant. Then f is locally a cylinder over a surface. If in addition M^n is complete $\Rightarrow f(M) = \mathbb{S}_c^2 \times \mathbb{R}^{n-2} \subset \mathbb{R}^3 \times \mathbb{R}^{n-2}$.*

Proof. Need to prove that h is totally geodesic (global part then follows from Hilbert's $\mathbb{H}^2 \not\subset \mathbb{R}^3$ and the rigidity of $\mathbb{S}^2 \subset \mathbb{R}^3$). Otherwise $\Rightarrow \nu_h = 1$ and $K_{V^2} = 1$ in some open subset $U \subset V^2$. Let $\{X, Y\}$ o.n.b. of TU and $Y \in \Delta_h$. So, $\nabla_Y X = \nabla_Y Y = 0$. Moreover $YY(r) + r = 0$ and $YX(r) = scal_M^{-1} \neq 0$ is constant $\Rightarrow 0 = X(r) + XYY(r) = 2\langle \nabla_{[X,Y]} \nabla r, Y\rangle = 2\langle \nabla_X Y, X\rangle YX(r)$ $\Rightarrow \nabla Y = 0 \Rightarrow K_{V^2} = 0$, contradiction. ∎

§21. Homogeneous hypersurfaces

Def.: $f\colon M^n \to \mathbb{Q}_c^{n+1}$ is *isoparametric* if it has constant principal curvatures ($\lambda_1 < \cdots < \lambda_g$ with multiplicity m_i).

Lemma 74. *$f\colon M^n \to \mathbb{Q}_c^{n+1}$ with M^n homogeneous $\Rightarrow$ Either $\tau \leq 1$ or τ is constant. If $\tau \geq 3$, then f is isoparametric.*

Theorem 75 (Cartan fund. formula)**.** $\forall i$, $\sum_{j\neq i}^{g} m_j \frac{c+\lambda_i\lambda_j}{\lambda_i-\lambda_j} = 0.$

Proof. $Ae_i = \lambda_i e_i$, $E_i = \mathrm{Ker}\,(A-\lambda_i I)$, $\Gamma_{ij}^k = \langle \nabla_{e_i} e_j, e_k\rangle = -\Gamma_{ik}^j$. Codazzi: $(\lambda_i - \lambda_k)\Gamma_{ij}^k = (\lambda_j - \lambda_k)\Gamma_{ji}^k \Rightarrow E_i$ totally geodesic. $\Rightarrow$ Gauss: $c + \lambda_i\lambda_j = \sum_k(\Gamma_{ij}^k\Gamma_{ji}^k + \Gamma_{ij}^k\Gamma_{ki}^j + \Gamma_{ji}^k\Gamma_{kj}^i) = 2\sum_k \Gamma_{ij}^k\Gamma_{ji}^k = \sum_{i\neq k\neq j} \frac{(\lambda_i-\lambda_j)^2}{(\lambda_1-\lambda_k)(\lambda_j-\lambda_k)}(\Gamma_{ki}^j)^2$. Now just sum. ∎

Corollary 76. *If $f\colon M^n \to \mathbb{R}^{n+1}$ is isoparametric, then $f(M) \subset \mathbb{S}_c^k \times \mathbb{R}^{n-k}$ for some $0 \leq k \leq n$.*

Proof. Let i in Theorem 75 s.t. λ_i is the smallest positive one $\Rightarrow$ all others are 0, $E_{\lambda_i} = \Delta^\perp$ is tot. geod., done by Proposition 58. ∎

Remark 77. Similar result holds for $\mathbb{H}^{n+1}$, also proved by Cartan. So the only interesting case is for $\mathbb{S}^{n+1}$. Münzner showed that g =1, 2, 3, 4 or 6, and for g odd all multiplicities are equal, while if g is even $m_i = m_{i+2}$ (i mod g). Using representations of Clifford algebras, Ferus-Karcher-Münzner gave a beautiful construction of a large family with $g = 4$. Lots of things are understood, but the full classification is still an important open problem.

Theorem 78. *$f\colon M^n \to \mathbb{R}^{n+1}$ with M^n homogeneous $\Rightarrow f$ is either a complete cylinder over a plane curve, or $\mathbb{S}_c^k \times \mathbb{R}^{n-k}$.*

§22. Immersions of Riemannian products

Orthogonal nets. Ex: Riemannian products. Adapted tensors.

Theorem 79 (Moore)**.** *If the second fundamental form of $f : M_1 \times M_2 \to \mathbb{R}^m$ is adapted, then f is an extrinsic product.*

Proof. By taking lifts of vector fields in each factors we see that $f_*T_xM_1 \perp f_*T_yM_2 \;\forall x, y \in M_1 \times M_2 \Rightarrow V_1 \perp V_2$, where

$$V_i := \operatorname{span}\{f_{*(x_1,x_2)}(T_{x_i}M_i) : (x_1, x_2) \in M_1 \times M_2\}.$$

If $P_i : \mathbb{R}^m \to V_i$ is the orthogonal projection and $x_i \in M_i$ is fixed, then $f_i = P_i \circ f \circ \iota_j$ is an isometric immersion of M_i, for the natural injection $\iota_j : M_j \to M$ based on x_i, and $P_i \circ f = f_i \circ \pi_i$ $\Rightarrow f = v_0 + P_1 \circ f + P_2 \circ f = v_0 + f_1 \circ \pi_1 + f_2 \circ \pi_2$. ∎

Remark 80. The decomposition is unique (if f_i is substantial).

Corollary 81. *Same in $\mathbb{S}^m$. Almost the same in $\mathbb{H}^m$.*

22.1 Splitting under a curvature condition

Theorem 82. *Let $f : M^n = \times_{i=0}^{k} M_i^{n_i} \to \mathbb{R}^{n+k}$ such that the set of flat points of M_j has empty interior, $\forall 1 \leq j \leq k$. Then, $M_0^{n_0}$ is flat and $f(M)$ is an open subset of a n_0-cylinder over an extrinsic product of k hypersurfaces.*

Proof. Fix $1 \leq j \neq j' \leq k$, and let $\sigma_j = \operatorname{span}\{e_{2j-1}, e_{2j}\} \subset T_{x_j}M_j$ with $k_j := K(\sigma_j) \neq 0$ and $\eta_j := \operatorname{span}\alpha(e_{2j}, e_{2j}) \neq 0$. By the Gauss equation,

$$\beta := \alpha \oplus B_1 \oplus \cdots \oplus B_k : TM \times TM \to T_x^{\perp}M \oplus \mathbb{R}^k = W^{2k,0}$$

is flat, where $B_i = \sqrt{|k_j|}(e^{2j-1} \otimes e^{2j-1} - sign(k_j)e^{2j} \otimes e^{2j})$. By Proposition 43, there is a basis $\{e'_1, \ldots, e'_{2k}\}$ of $\operatorname{span}\{e_1, \ldots, e_{2k}\}$ such that $\beta(e'_r, e'_s) = 0$, $\forall 1 \leq r \neq s \leq 2k$. In particular, $\operatorname{Ker} B_i$ and $\operatorname{Im} B_i$ are spanned by vectors in this basis. Hence, up to order, $e'_{2j-1}, e'_{2j} \in \sigma_j$, and $\alpha(\sigma_j, \sigma_{j'}) = 0$, $\alpha(\sigma_j, \sigma_j) = \eta_j$. Therefore, $\alpha(\Gamma_j^{\perp}, \Gamma_{j'}^{\perp}) = 0$, $\alpha(\Gamma_j^{\perp}, \Gamma_j^{\perp}) = \eta_j$. Now is just Gauss equation:

Indeed, $\{\eta_1, \ldots, \eta_k\}$ is an o.n.b. of directions of $T_x^\perp M$. Since $\Gamma_j \subset \Gamma$, $\langle \alpha(\Gamma_j, \Gamma_j^\perp), \alpha(\Gamma_{j'}^\perp, \Gamma_{j'}^\perp)\rangle = 0$ and then $\alpha(\Gamma_j, \Gamma_j^\perp) \subset \eta_j$. But if $X \in \Gamma_j^\perp$, there is $Y \in \Gamma_j^\perp$ such that $\alpha(X, Y) = 0$ and $0 \neq \alpha(Y, Y) \in \eta_j$. So $\alpha(\Gamma_j, X), \alpha(Y, Y)\rangle = 0$, and then $\alpha(\Gamma_j, T_{x_j}M_j) = 0$, $\alpha(T_{x_j}M_j, T_{x_j}M_j) = \eta_j$, $\alpha(T_{x_j}M_j, T_{x_{j'}}M_{j'}) = 0$. Analogously we get $\alpha(T_{x_0}M_0, T_{x_j}M_j) = \alpha(T_{x_0}M_0, T_{x_0}M_0) = 0$ and so $T_{x_0}M_0 \subset \Delta(x)$ and $\alpha(x)$ is adapted. Therefore α is everywhere adapted to the product structure and the result follows from Proposition 58. ∎

Remark 83. Similar in $\mathbb{Q}_c^m$. For $k = 2$, we can say a bit more.

22.2 Splitting under an algebraic condition

Theorem 84. *$f \colon M^n = \times_{i=1}^k M_i^{n_i} \to \mathbb{Q}_c^{n+p}$ with $\nu_s < n - 2s$ $\forall\ 1 \leq s \leq p \ \Rightarrow f$ is an extrinsic product.*

Proof. Follows from the above and the next lemma. ∎

Lemma 85. *Let $\beta : V^n \times V^n \to W^{p,0}$ symmetric, $V = V_1 \oplus V_2$ and $R_\beta(V_1, V_i, V_j, V_2) = 0$ for $1 \leq i, j \leq 2$. If $\nu_s < n - 2s$ $\forall 1 \leq s \leq p$, then $\beta(V_1, V_2) = 0$.*

Proof. Let $\beta^{ij} := \beta|_{V_i \times V_j} : V_i \times V_j \to S_{ij}$ surjective and $L^{ij} := \operatorname{Ker} \beta^{ij} \subset V_i$. But β^{12} is flat, so $\beta(L^{12}, V_1) \perp S_{12}^s \Rightarrow \beta(L^{12}, V) \perp S_{12}^s$. Analogously, $\beta(V, L^{21}) \perp S_{12}^s \ \Rightarrow \beta(L^{12} \oplus L^{21}, V) \perp S_{12}^s$, with $\dim(L^{12} \oplus L^{21}) \geq \dim V_2 - s + \dim V_1 - s = n - 2s \ \Rightarrow s = 0$. ∎

22.3 Splitting under a global condition

The following is a generalization of Chern-Kuiper Theorem 31:

Theorem 86. *$f\colon M^n \to \mathbb{R}^{n+p}$. If $\Gamma^\perp(x) = T_1 \oplus^\perp \cdots \oplus^\perp T_s$ with all T_i's being R-invariant then $\nu(x) \le \mu(x) \le \nu(x)+p-s$.*

Proof. If $S = (\Gamma \cap \Delta^\perp) \oplus \operatorname{span}\{Y_1, \dots, Y_s\}$ for $Y_i \in T_i \Rightarrow R(S,S) = 0$. But if $Z \in RE(\alpha|_{TM\times S})$ then $\operatorname{Ker}(\alpha_Z|_S) = 0$ since $S \cap \Delta = 0$. Hence, $p \ge \dim S = \mu - \nu + s$. ∎

Corollary 87. *Let $f : M^n = \times_{i=1}^k M_i^{n_i} \to \mathbb{R}^{n+k}$. If $\alpha_f(x)$ is <u>not</u> adapted, then $\mu(x) \ge \nu(x) \ge \mu(x) - r(x)$, where $r(x)$ is the number of factors that are flat at x.*

Theorem 88. *Let $M_i^{n_i}$ be compact with $n_i \ge 2$. Then, every $f\colon M^n = \times_{i=1}^p M_i \to \mathbb{R}^{n+p}$ splits as a product of hypersurfaces.*

Proof. Let $U \subset M$ be the open subset where α_f is not adapted, and $U_0 \subset U$ where the relative nullity of $f|_U$ is minimum $\nu_0 \Rightarrow \nu_0 \ge \mu - r \ge r > 0$ by Corollary 87. Since M^n is compact and U is open, by Proposition 59 a maximal geodesic in a leaf of Δ in U_0 has to leave U. At the end point α is adapted, hence it is adapted inside U by Proposition 57.6; see also the proof of Proposition 59. So $U = \emptyset$ and the result follows from Theorem 79. ∎

Questions: If $f_i\colon M_i^{n_i} \to \mathbb{R}^{n_i+p_i}$ with p_i the minimal codimension, is $q = p_1 + p_2$ the minimal codimension for an is.im. $f\colon M_1^{n_1} \times M_2^{n_2} \to \mathbb{R}^{n_1+n_2+q}$? If yes, is it necessarily a product?

Remark 89. Similar results to those in this section exist for *warped products*; see the book.

§23. Conformal immersions

General philosophy: $\mathbb{Q}_c^m \cong \mathbb{R}^m$; conformal immersions in $\mathbb{R}^m \cong$ isometric immersions in the *light cone* $\mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$.
If $w \in \mathbb{V}^{m+1}$, $\mathbb{E}^m = \mathbb{E}_w := \{v \in \mathbb{V}^{m+1} : \langle v, w\rangle = 1\} \cong \mathbb{R}^m$: if $C : \mathbb{R}^m \to \operatorname{span}\{v, w\}^\perp \subset \mathbb{L}^{m+2}$ is a linear isometry, then

$$\psi(x) = v + Cx + \frac{1}{2}\|x\|^2 w$$

is an isometric embedding with $\psi(\mathbb{R}^m) = \mathbb{E}^m$. In fact, if $z \in \mathbb{L}^{m+1}$ with $\langle z, z\rangle = -1/c$, then $\mathbb{Q}_c^m \cong \{u \in \mathbb{L}^{m+1} : \langle u, z\rangle = -1/c\}$.
The set of transformations $(v, w, C) \mapsto (v', w', C')$ is $\mathbb{O}_1(m+2)$.
If $f : M^n \to \mathbb{V}^{m+1}$, then $f \in T_f^\perp M$ and $A_f^f = -I$. In particular, $T_\psi \mathbb{R}^m = \operatorname{span}\{\psi, w\}$, $A_w^\psi = 0$, and $\alpha_\psi = -\langle\ ,\ \rangle w$.

23.1 The conformal Gauss parametrization

Let $f\colon M^n \to \mathbb{R}^{n+1}$ orientable with Gauss map η and a Dupin principal curvature $\lambda \neq 0$ of multiplicity $n-k$. Since the corresponding eigendistribution E_λ is umbilic (integrable), we have the leaf space $V^k := M^n/E_\lambda$ and a submersion $\pi : M^n \to V^k$. The map

$$h = f + \lambda\eta$$

is constant along the leaves of E_λ, hence it descends to the quotient and we have an immersion $g : V^k \to \mathbb{R}^{n+1}$ and a function $r \in \mathcal{F}(V)$ given by

$$g \circ \pi = h, \quad r \circ \pi = \lambda^{-1}.$$

We endow V^k with the metric induced by g. In particular, $f = g \circ \pi - (r \circ \pi)\, \eta$. Since η is normal to f, $\eta^\top = (\nabla r) \circ \pi$ and therefore, by dimension reasons, we can parametrize f over the unit normal bundle $T_1^\perp V$ of g by

$$f \circ \xi = g - r \left(\nabla r + \sqrt{1 - \|\nabla r\|^2}\, \xi \right), \quad \xi \in \Gamma(T_1^\perp V).$$

23.2 Light cone representative

Conformal structure. Pull back.

Proposition 90. *Let M^n be a Riemannian manifold, and $f \colon M^n \to \mathbb{R}^{n+p} \cong \mathbb{E}_w^{n+p}$ a conformal immersion with conformal factor φ. Then, $\hat{f} := \varphi^{-1}\, \psi \circ f : M^n \to \mathbb{V}^{n+p+1}$ is an isometric immersion. Conversely, if $\hat{f} : M^n \to \mathbb{V}^{n+p+1} \backslash \mathbb{R}w$ is an isometric immersion, then $f := \psi^{-1}(\langle \hat{f}, w\rangle^{-1} \hat{f}) : M^n \to \mathbb{R}^{n+p}$ is a conformal immersion with conformal factor $\varphi = \langle \hat{f}, w\rangle^{-1}$. We call $\hat{f}$ the isometric light cone representative of f .*

Exercise. If $R(u) := u - 2\langle u, z\rangle z \in \mathbb{O}_1(m+2)$ is the reflection with respect to the space-like vector z with $\langle z, w\rangle \neq 0$ in $\mathbb{E}_w^m$, then $\hat{R}$ is the inversion with respect to the hypersphere $\mathbb{E}_w^m \cap \{z\}^\perp$.

It turns out that $\mathbb{R}^m$ is locally rigid in $\mathbb{V}^{m+1}$ for $m \geq 3$:

Theorem 91. *If $F : U \subset \mathbb{R}^m \to \mathbb{V}^{m+1}$ is an isometric immersion with $m \geq 3$, then $F = \psi|_U$ for some (v, w, C).*

Proof. Let X, Y parallel vector fields in $\mathbb{R}^m$. So $\tilde{\nabla}_X (i \circ F)_* Y = \alpha(X, Y) = -\langle X, Y\rangle \eta + \langle BX, Y\rangle F$, for a parallel normal η with $\langle \eta, \eta\rangle = 0$, $\langle \eta, f\rangle = 1$. Now, $K_U \equiv 0$ and $m \geq 3 \Rightarrow B = 0$. But $\tilde{\nabla}_X \eta = -BX = 0 \Rightarrow \eta$ is a constant vector and $F(U) \subset \mathbb{E}_\eta^{m+1}$. ∎

Corollary 92. *For any conformal map $f : U \subset \mathbb{R}^m \to \mathbb{R}^m$, there is $T \in \mathbb{O}_1(m+2)$ such that $\hat{f} = T \circ \psi|_U$.*

Conformal congruence can be regarded as a special case of isometric congruence (so we can use the isometric methods!):

Proposition 93. *f',f: $M^n \to \mathbb{R}^{n+p}$ are conformally congruent $\iff$ $\hat{f}'$ and $\hat{f}$ are isometrically congruent.*

Proof. Observe that the conformal factor of a composition $i \circ j$ satisfies $\varphi_{i \circ j} = \varphi_j \varphi_i \circ j$. If $f' = \mathcal{T} \circ f$ for a conformal diffeo $\mathcal{T}$ of $\mathbb{R}^{n+p}$ $\Rightarrow \hat{\mathcal{T}} = T \circ \psi$ for $T \in \mathbb{O}_1(n+p+2)$, and $\hat{f}' = \varphi_{\mathcal{T} \circ f}^{-1}\, \psi \circ \mathcal{T} \circ f = \varphi_f^{-1}(\varphi_{\mathcal{T}}^{-1} \psi \circ \mathcal{T}) \circ f = \varphi_f^{-1}\, T \circ \psi \circ f = T \circ \hat{f}$. ∎

Remark 94. See the book for equations relating the second fundamental forms, normal connections, etc between a conformal immersion and its light-cone representative, and the Fundamental Theorem in Moebius geometry. Not surprisingly, by Proposition 93 many isometric results have natural conformal counterparts, that usually can be proved adapting isometric methods.

§24. Sbrana-Cartan deformable hypersurfaces

Let Δ an integrable distribution on M, and $L = M/\Delta$ the (local) space of leves with projection $\pi : M \to L$. A vector field $X \in \mathfrak{X}(M)$ is called *projectable* if there is $\bar{X} \in \mathfrak{X}(L)$ π-related to X. Equivalently, the horizontal lift $\bar{X}^h$ of $\bar{X}$ agrees with $X_{\Delta^\perp}$.

Lemma 95. *$X \in \mathfrak{X}(M)$ is projectable $\iff$ $[X, \Delta] \subset \Delta$.*

Proof. Use the usual flux formula for the Lie bracket: $[X, T] = \lim_{t \to 0} \frac{1}{t}(X \circ \varphi_t - \varphi_{t*} X)$, where ${\varphi_t}' = T \circ \varphi_t$ ($\Rightarrow \pi \circ \varphi_t = \pi$). ∎

Lemma 96. *$S \subset End(\mathbb{R}^2)$ a subspace, and $D \in End(\mathbb{R}^2)$, $D \notin span\{I\}$ such that $[D, C] = 0\ \forall\, C \in S \Rightarrow \dim S \leq 2$.*

Proof. There is $C \in S$ symmetric such that $C \neq aI$. Then D and all elements in S diagonalize in the same basis. ∎

Definition 97. We say that a nowhere flat Euclidean hypersurface f is a *Sbrana-Cartan hypersurface* if there is another $\hat{f}$ nowhere congruent to f (i.e., f is *locally deformable*).

Example: Associated family of a minimal rank 2 hypersurface.

Nowhere flat surfaces in $\mathbb{R}^3$ are locally deformable, but no classification exists. Surface-like hypersurfaces $\iff \operatorname{Im} C \subset \operatorname{span}\{I\}$. Nadirashvili's example.

From now on in this section, assume that $f\colon M^n \to \mathbb{R}^{n+1}$ is a nowhere surface-like Sbrana-Cartan hypesurface with deformation $\hat{f}$, and A and $\hat{A}$ their shape operators in $\Delta^\perp$. Then:

(a) $\hat{\Delta} = \Gamma = \Delta$ agree and are intrinsic (since $\hat{\nu} = \mu = \nu \equiv n-2$).

(b) Hence, the splitting tensor C of Δ is the same and intrinsic!

(c) Gauss $\iff$ $D := A^{-1}\hat{A} \in End(\Delta^\perp)$ satisfies $\det D = 1$.

(d) Noncongruent $\iff$ $D \notin \operatorname{span}\{I\}$ on an open dense $U \subset M^n$.

(e) $[D, C_T] = 0\ \ \forall T \in \Delta$ (by Proposition 57.7).

(f) $\nabla_\Delta D = 0$ (by the last and Codazzi).

(g) $\dim(\operatorname{span}\{I\} + \operatorname{Im} C) = 2$ a.e. on U (by (e)) $\Rightarrow$

(h) Up to sign, $\exists!\, J \in End(\Delta^\perp)$ such that $J^2 = \epsilon I$, $\epsilon = 1, 0, -1$, $\|J\| = 1$, satisfying $\mathrm{span}\{I\} \neq \mathrm{Im}\, C \subset \mathrm{span}\{I, J\}$.

(i) $AJ = J^t A$ (by (h) and Section 20).

(j) (e)+(h) $\Rightarrow D \in \mathrm{span}\{I, J\}$ (same computation as (g)) $\Rightarrow$

(k) $\nabla_\Delta J = 0$ (by (f), since also $J \in \mathrm{span}\{I, D\}$).

Def.: A Riemannian manifold M^n with $\mu \equiv n-2$ is called *parabolic* $(\epsilon = 0)$, *hyperbolic* $(\epsilon = 1)$, *elliptic* $(\epsilon = -1)$ if there is $J \in End(\Gamma^\perp)$ satisfying (h)+(k) ($\Rightarrow M^n$ is nowhere surfacelike).

Proposition 98. *Both D and J project to $V^2 := M^n/\Delta$, i.e., $\exists\, \bar{D}, \bar{J}$ such that $\bar{D} \circ \pi_* = \pi_* \circ D$ and $\bar{J} \circ \pi_* = \pi_* \circ J$ on $\Delta^\perp$.*

Proof. Since D is parallel along Δ it projects, since $[D\bar{X}^h, \Delta]_{\Delta^\perp} = D[\bar{X}^h, \Delta]_{\Delta^\perp} = 0$. Same for J. ∎

Parabolic, hyperbolic and elliptic surfaces: existence of real or complex conjugate coordinates, first normal space of dimension 2.

Set

$$\mathcal{D}_M := \{f : M^n \to \mathbb{R}^{n+1}\}/\text{congruence}.$$

Proposition 99. *$f\colon M^n \to \mathbb{R}^{n+1}$ rank two nowhere surface-like. Then, M^n is parabolic (resp hyperbolic, elliptic) w.r.t. J $\iff$ the Gauss data is parabolic (resp. hyperbolic, elliptic) w.r.t. $\bar{J}$. In particular, every member of $\mathcal{D}_M$ is parabolic (resp. hyperbolic, elliptic).*

Proof. We have $g_* A\bar{X}^h = -h_*\pi_*\bar{X}^h = -h_*\bar{X}$. So $g_* AJ\bar{X}^h = -h_*\pi_* J\bar{X}^h = -h_*\bar{J}\bar{X}$. We use now Section 20 (and omit j_w):

by the symmetry of AJ in (i), $J^t = -\bar{J}$, and since $A = -P_w^{-1}$, we have $\bar{J}^t P_w = P_w \bar{J}$. ∎

Corollary 100. *By (hi) and Proposition 99, the Gauss data is parabolic, hyperbolic or elliptic with respect to $\bar{J}$.*

We first deal with the easiest parabolic case:

Proposition 101. *M^n is parabolic $\iff$ f is ruled. In this case $\mathcal{D}_M = \mathbb{R}$ and every $g \in \mathcal{D}_M$ is ruled with the rullings of f.*

Def.: Gauss data (h, r) of *first and second species* (with conjugate coordinate system (u, v)): hyperbolic (resp. elliptic) h and r, such that

$$\tau(\Gamma^v_v - 2\Gamma^u\Gamma^v) = (\Gamma^u_u - 2\Gamma^u\Gamma^v)$$

(resp. $\mathrm{Im}\,(\rho(\Gamma_z - 2\Gamma\bar{\Gamma})) = 0$).

Proposition 102. *Assume M^n is hyperbolic or elliptic. Then, $\hat{A}$ is Codazzi $\iff$ $\bar{D}$ is Codazzi $\iff$ the Gauss data is of first or second species.*

Finally, we can give the complete classification:

Theorem 103. *Let M^n be any Riemannian manifold. Then, each connected component U of an open dense subset of M^n falls, even locally, exactly into one of these categories:*

i) $\mathcal{D}_U = \emptyset$, i.e., U is not even locally a Eucl. hypersurface;

ii) U is rigid, i.e., $\mathcal{D}_U$ is a point;

iii) U is flat, and $\mathcal{D}_U = \mathcal{F}(\mathbb{R}, \mathbb{S}^n) \times \mathcal{F}(\mathbb{R})$;

iv) U is nonflat surface-like and $\mathcal{D}_U$ is the one of the surface;

v) U is parabolic and ruled, $\mathcal{D}_U = \mathcal{F}(\mathbb{R})$, and every element in $\mathcal{D}_U$ is ruled with same rulings as U;

vi) The Gauss data is of first species, and $\mathcal{D}_U = \mathbb{R}$;

vii) The Gauss data is of second species, and $\mathcal{D}_U = \mathbb{Z}_2$.

Case (vi) is called the *continuous type* (e.g., g minimal), while case (vii) is called the *discrete type* (e.g....?????????).

§25. Intersections

These were the first known Sbrana-Cartan hypersurfaces of the discrete type. They can be obtained intersecting two flat hypersurfaces, or, better, as in [**FF**] as rank two hyperbolic submanifolds in codimension two that extend flat in two ways. We briefly describe this work.

Let $f\colon M^n \to \mathbb{R}^{n+2}$ be a hyperbolic rank two submanifold. It is easy to see that f has a (hyperbolic) *polar surface* that "integrates" its normal bundle, i.e, there is $g : V^2 = M/\Delta \to \mathbb{R}^{n+2}$ such that

$$g_{*[x]}(T_{[x]}V) = T^{\perp}_{f(x)}M \quad \forall\, x \in M^n.$$

In [**FF**] it was shown that f extends as a flat hypersurface in two different ways $\iff$ $\Gamma^u = \Gamma^v = 0$, i.e., if

$$g(u, v) = c_1(u) + c_2(v),$$

with c_1', c_1'', c_2', c_2'' pointwise L.I.. The *shared dimension* $I(c_1, c_2) \in \mathbb{N}_0$ is the smallest integer k for which there is an orthogonal decomposition in affine subspaces, $\mathbb{R}^{n+2} = \mathbb{V}_1 \oplus^{\perp} \mathbb{V}^k \oplus^{\perp} \mathbb{V}_2$, satisfying that $\mathrm{span}(c_i) \subset \mathbb{V}_i \oplus^{\perp} \mathbb{V}^k$, $i = 1, 2$. It turns out that:

- $I(c_1, c_2) = 0 \iff M^n$ is flat;
- $I(c_1, c_2) = 1 \iff M^n$ is of the continuous type;
- $I(c_1, c_2) \geq 2 \iff M^n$ is of the discrete type.

Corollary 104. *Crazy collage of the different types.*

§26. Genuine rigidity

Higher codimensions - rigidity and compositions: particular cases of *isometric extensions.* Algebraic rigidity results like Theorem 37, Theorem 39 and Theorem 41 are *not* geometric in nature. As such, they should be understood as "generic" without much usefulness for classification. Now we search for some geometry.

We say that a pair $f\colon M^n \to \mathbb{R}^{n+p}$ and $\hat{f}\colon M^n \to \mathbb{R}^{n+q}$ of isometric immersions *extends isometrically* when there are an isometric embedding $j\colon M^n \hookrightarrow N^m$ into a Riemannian manifold N^m with $m > n$ and isometric immersions $F\colon N^m \to \mathbb{R}^{n+p}$ and $\hat{F}\colon N^m \to \mathbb{R}^{n+q}$ such that $f = F \circ j$ and $\hat{f} = \hat{F} \circ j$. In other terms, the following diagram commutes:

$$\begin{tikzcd} & & \mathbb{R}^{n+p} \\ M^n \arrow[urr, "f"] \arrow[r, hook, "j"] \arrow[drr, "\hat{f}"'] & N^m \arrow[ur, "F"] \arrow[dr, "\hat{F}"'] & \\ & & \mathbb{R}^{n+q} \end{tikzcd} \tag{1}$$

We want to *discard* deformations $\hat{f}$ that arise in this way, since the deformation problem essentially depends on the codimension, not on the dimension. This gives rise to the following:

Def.: We say that $\hat{f}\colon M^n \to \mathbb{R}^{n+q}$ is a *genuine deformation* of a given $f\colon M^n \to \mathbb{R}^{n+p}$ (or that $\{f_\lambda\colon M^n \to \mathbb{R}^{n+p_\lambda}\}$ is a *genuine set*) if $\nexists U \subset M^n$ s.t. $f|_U$ and $\hat{f}|_U$ extend isometrically.

Def.: We say that $f\colon M^n \to \mathbb{R}^{n+p}$ is *genuinely rigid in* $\mathbb{R}^{n+q}$ if every $\hat{f}\colon M^n \to \mathbb{R}^{n+q}$ is *nowhere* a genuine deformation of f.

Motivation of the next: structure of the second fundamental forms and normal connections when a pair extends isometrically.

D^d-ruled submanifolds, mutually ruled (sets!), ruled extensions.

Lemma 105. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ be a D-ruled submanifold. $\Rightarrow L_D := S(\alpha|_{D\times TM})$ is parallel along D (constant dimension).*

Let $f\colon M^n \to \mathbb{R}^{n+p}$ and $\hat{f}\colon M^n \to \mathbb{R}^{n+q}$, and τ a parallel vector bundle isometry that preserves second fundamental forms,

$$\tau : L^\ell \subset T_f^\perp M \to \hat{L}^\ell \subset T_{\hat{f}}^\perp M. \qquad (3)$$

Equivalently, the induced v.b. isometry $\bar{\tau}$ is parallel, where

$$\bar{\tau} = Id \oplus \tau\colon\ f_*TM \oplus L \to \hat{f}_*TM \oplus \hat{L}.$$

Define $\phi_\tau\colon TM \times (TM \oplus L) \to (L^\perp \times \hat{L}^\perp, \langle\,,\rangle_{L^\perp} - \langle\,,\rangle_{\hat{L}^\perp})$ by

$$\phi_\tau(X,\eta) = \left((\tilde{\nabla}_X \eta)_{L^\perp}, (\tilde{\nabla}_X \bar{\tau}\eta)_{\hat{L}^\perp}\right).$$

Proposition 106. *The bilinear form ϕ_τ is Codazzi and flat.*

Proof. Exercise. ∎

Notice that $\alpha_{L^\perp} \oplus \hat{\alpha}_{\hat{L}^\perp} = \phi_\tau|_{TM \times TM}$. Assume that the subspaces

$$D = D_\tau := \mathcal{N}(\alpha_{L^\perp} \oplus \hat{\alpha}_{\hat{L}^\perp}) \subset TM,$$

$$\Delta = \Delta_\tau := \mathcal{N}_r(\phi_\tau) \subset TM \oplus L$$

have constant dimensions $d_\tau \leq \nu_\tau$ respectively (observe that $\Delta \cap TM = D$). It follows that $\bar{\tau}\,|_\Delta \colon \Delta \to \hat{\Delta}$ is a parallel vector bundle isometry, and hence, we can identify $\hat{\Delta}$ with Δ.

Corollary 107. *If L and $\hat{L}$ are parallel along D (in the normal connections), then $D \subset \Delta$ is integrable.*

Corollary 108. *The space* $\mathrm{Im}\,(\phi_\tau)$ *is parallel along the leaves of Δ_τ. In particular, both the nullity space and the light cone bundle of $\langle\,,\rangle|_{\mathrm{Im}\,(\phi_\tau)^\perp}$ are smooth and parallel along these leaves on any open subset where they have constant dimension.*

Let $\pi \colon \Lambda \to M^n$ the vector bundle $\Lambda := D^\perp \subset \Delta \subset TM \oplus L$, and consider the extensions $F : \Lambda \to \mathbb{R}^{n+p}$ and $\hat{F} : \Lambda \to \mathbb{R}^{n+q}$,

$$F \circ \xi = f \circ \pi + \xi, \quad \hat{F} \circ \xi = \hat{f} \circ \pi + \bar{\tau}\xi, \quad \forall \xi \in \Gamma(\Lambda), \qquad (4)$$

restricted to a neighborhood N of $M^n \cong 0 \subset \Lambda$ to get immersions. Still denote by $L^\perp$ and Δ the natural corresponding bundles over Λ (i.e., $L^\perp(\xi_x) = L^\perp(x)$, or, equivalently, $\pi^*(L^\perp)$).

The following is the main result on isometric extensions:

Theorem 109. *If L and $\hat{L}$ are parallel along D, then F and $\hat{F}$ are isometric $\pi^*(\Delta)$-ruled extensions of f and $\hat{f}$. Moreover, there are orthogonal splittings*

$$T_F^\perp N = \mathcal{L} \oplus^\perp \pi^*(L^\perp), \quad T_{\hat{F}}^\perp N = \hat{\mathcal{L}} \oplus^\perp \pi^*(\hat{L}^\perp),$$

and a parallel vector bundle isometry $\mathcal{T} : \mathcal{L} \to \hat{\mathcal{L}}$ that preserves second fundamental forms such that $\pi^*(\Delta) = \Delta_{\mathcal{T}}$. In addition, $\mathcal{L}$ and $\hat{\mathcal{L}}$ are parallel along $\pi^*(\Delta)$.

Proof. Let's argue first for F, $\hat{F}$ being similar. Observe first that $\operatorname{Im} F_* \subset \pi^*(T_f M \oplus L)$ which is parallel along D. Thus, $\pi^*(L^\perp) \subset T_F^\perp \Lambda$, and

$$T_F \Lambda \oplus^\perp \mathcal{L} = \pi^*(T_f M \oplus L), \quad \mathcal{L}^\perp = \pi^*(L^\perp). \tag{5}$$

Now, $\tilde{\nabla}_D \Delta \subset \Delta$, since $\tilde{\nabla}_D \Delta \subset TM \oplus L$ by definition, and $\tilde{\nabla}_X \tilde{\nabla}_D \Delta = \tilde{\nabla}_D \tilde{\nabla}_X \Delta + \tilde{\nabla}_{[X,D]} \Delta \in TM \oplus L$. Since D is integrable, this easily implies that F and is $\pi^*(\Delta)$-ruled. (We can set $V := M/D$ and define F over the bundle $\pi^*(\Delta) \to V$ instead.) Take $\eta \in \Gamma(L^\perp)$, $Z \in \Gamma(\Delta)$. Thus, since $\pi^*(\Delta) = \pi_*^{-1}(D)$,

$$(\tilde{\nabla}_{Z \circ \pi} \eta \circ \pi)_{T_F \Lambda \oplus \mathcal{L}} = (\tilde{\nabla}_Z \eta \circ \pi)_{T_f M \oplus L} = (\tilde{\nabla}_{\pi_* Z}\, \eta)_{T_f M \oplus L} = 0.$$

This proves that $\pi^*(\Delta) = \Delta_{\mathcal{T}}$ and the last assertion.
From (5) we have that $\bar{\mathcal{T}} = \pi^*(\bar{\tau})$ is parallel, and the extensions are isometric since $\hat{F}_* = \bar{\mathcal{T}} \circ F_*$, which concludes the proof. ∎

Corollary 110. *The extensions F and $\hat{F}$ are trivial $\iff$ f and $\hat{f}$ are D-ruled.*

Corollary 111. *$\{f, \hat{f}\}$ is genuine $\Rightarrow$ f and $\hat{f}$ are mutually D-ruled and we have:*

$$\begin{tikzcd}[column sep=small]
T_f^\perp M = L_D \oplus L_D^\perp \arrow[dd, "\begin{array}{c}\mathcal{T}_D : \\ (\nabla^\perp)_{L_D} = (\hat{\nabla}^\perp)_{\hat{L}_D} \\ \alpha_{L_D} = \hat{\alpha}_{\hat{L}_D}\end{array}"'] \arrow[dr] & & L_D := \text{span}\,\{\alpha(D, TM)\} \\
& D = \mathcal{N}\left(\alpha_{L_D^\perp} \oplus \hat{\alpha}_{\hat{L}_D^\perp}\right) \text{ are rulings!!} & \\
T_{\hat{f}}^\perp M = \hat{L}_D \oplus \hat{L}_D^\perp \arrow[ur] & & \hat{L}_D := \text{span}\,\{\hat{\alpha}(D, TM)\}
\end{tikzcd}$$

There are always isometries such as τ as in (3), e.g., $\tau = 0$! In this case $D^d = \Delta_f \cap \Delta_{\hat{f}}$, where the result is obvious. But we have no estimate on d for $\tau = 0$, since ϕ_0 may be degenerate. But in [**DF**] we explicitly constructed another τ, L, and D^d for which the rulings are big:

Theorem 112. *Let $f \colon M^n \to \mathbb{R}^{n+p}$ and $\hat{f} \colon M^n \to \mathbb{R}^{n+q}$ a genuine pair with $p + q < n$ and $\min\{p, q\} \leq 5$. Then, $\{f, \hat{f}\}$ are mutually D^d-ruled a.e., with $d \geq n - p - q + 3 \dim L_D$. Moreover, the isometry τ_D is parallel and preserves second fundamental forms.*

Just the proof of this estimate took 5 pages and it is quite delicate. But it generalizes all known result about compositions, rigidity with s-nullities, etc, i.e., the ones we studied so far.

26.1 Better s-nullities: $\bar{\nu}_s$

Since L_D is always parallel along D by Codazzi equation, Corollary 110 screams to use the s-nullity of another bilinear form instead of the ones for the second fundamental form. Indeed, given $V^s \subset T^\perp M$ a normal subbundle of rank s of an isometric immersion $f : M^n \to \mathbb{R}^{n+p}$, define as in [**FG**] the tensor

$$\phi_V : TM \times (TM \oplus V^\perp) \to V, \quad \phi_V(X, v) = (\tilde{\nabla}_X v)_V.$$

Since ϕ_V is Codazzi, its left nullity

$$\mathcal{N}_l(\phi_V) = \{X \in \mathcal{N}(\alpha_V) \colon \nabla^\perp_X V \subset V\}$$

is integrable where it has constant dimension (compare with the right nullity $\Delta_\tau = \mathcal{N}_r(\phi_\tau)$ in Theorem 109). Set

$$\bar{\nu}^f_s := \max_{V^s \subset T^\perp M} \dim \mathcal{N}_l(\phi_V).$$

Since D is totally geodesic, L_D is parallel along D. So Corollary 110 + Theorem 112 imply that $D = \mathcal{N}_l(\phi_\tau) \subset \mathcal{N}_l(\phi_{L_D^\perp}) \Rightarrow$

Corollary 113. *Vet $f \colon M^n \to \mathbb{R}^{n+p}$, and $q \in \mathbb{N}$ such that $\min\{p, q\} \leq 5$. If $\bar{\nu}^f_s < n + 2p - q - 3s$ almost everywhere for all $1 \leq s \leq p$, then f is genuinely rigid in $\mathbb{R}^{n+q}$.*

Remark 114. This result is stronger than all the ones with s-nullities cited before (and probably all the ones not cited too...):

- $\bar{\nu}^f_s \leq \nu^f_s$ since $\mathcal{N}_l(\phi_V) \subset \mathcal{N}(\alpha_V)$;
- The bound on $\bar{\nu}^f_s$ is weaker than the usual one for ν^f_s by $p - s$;
- $\mathcal{N}_l(\phi_V)$ is always integrable, and 'almost' totally geodesic;
- We can require in the definition of $\bar{\nu}^f_s$ to $\mathcal{N}_l(\phi_V)$ be totally geodesic, or asymptotic, since the leaves of D are rulings.

§27. Global rigidity

Global rigidity results in submanifold theory are *way* more scarce than local ones, the most beautiful of which is Sacksteder's:

Theorem 115. *A compact Euclidean hypersurface is rigid provided its set of non-totally geodesic points is connected. (We can understand if not). Same for complete bounded.*

Proof. By Proposition 106, $\beta := \phi_0$ is flat and Codazzi. Hence, Propositions 57, 59 and Corollary 60 hold for $\Delta_0 = \mathcal{N}(\beta)$ (Corollary 61). Now use the spirit of the proof of Theorem 88 to show that $A = \pm\hat{A}$ everywhere. ∎

In [**DG**] the codimension two case was solved by showing that, giving a pair of is.ims., along each connected component of an open dense subset, the immersions are either congruent or extend isometrically to flat hypersurfaces, or to singular Sbrana-Cartan hypersurfaces. That singularities are necessary was proved much later in [**FF**] (also in the flat case, filling a gap in [**DG**]). In other words, compact codimension two Euclidean submanifolds are *singularly genuinely rigid*, and singularities are needed!

§28. Singular genuine rigidity ([FG])

As we saw in Theorem 109, given $\tau : L \to \hat{L}$ we get isometric (possibly trivial) ruled extensions as in (4). In particular, this holds for $\tau = 0$, in which case $\phi_0 = \beta := (\alpha, \hat{\alpha})$. The key distribution here was thus $\Delta_0 = \operatorname{Ker}\beta$. The extensions in (4) are then obviously isometric since $\hat{F}_* = \bar{\mathcal{T}} \circ F_*$. This is a sufficient condition, but not a necessary one (!!!). Indeed, a tautology:

Proposition 116. *Let $f, \hat{f}$ and $\tau : L \to \hat{L}$ parallel that preserves second fundamental forms. Let $\Lambda \subset TM \oplus L$ be any subbundle. Then, F and $\hat{F}$ in (4) are isometric $\iff$*

$$\phi_\tau(TM, \Lambda) \subset L^\perp \oplus \hat{L}^\perp \ \ \textit{is null.} \tag{6}$$

Of course this holds if $\Lambda \subset \Delta_\tau$ as before, but it has two very important advantages:

- No Main Lemma! In particular, no need for $\min\{p, q\} \leq 5$ in an analogous to Theorem 112.
- Null subspaces are much, Much, <u>MUCH</u> easier to get than nullities due to Proposition 29. Thanks J.D.Moore!

Observe that Proposition 116 holds even if Λ is not transversal to M^n. In this case, F and $\hat{F}$ are not immersions along $M^n \subset \Lambda$. Actually, the only problem to extend is when $\Lambda = D \subset TM$, otherwise we just take a subbundle of Λ transversal to M^n.
If f is D-ruled, then F is not an immersion, it has constant rank equal to n, and $F(D) = f(M)$. But what if not?

Def.: We say that $F = F_{\Lambda,f}$ in (4) is a *singular extension of* f if it is an immersion in some open neighborhood of M^n (the 0-section of Λ), except of course at M^n itself.

Def.: We say that $\hat{f}$ is a *strongly genuine deformation of* f, or that $\{f, \hat{f}\}$ is a *strongly genuine pair*, if there is no open subset U where $f|_U$ and $\hat{f}|_U$ singularly extend isometrically.

Def.: Given $q \in \mathbb{N}$, the isometric immersion f is said to be *singularly genuinely rigid* in $\mathbb{R}^{n+q}$ if, for any isometric immersion $\hat{f}\colon M^n \to \mathbb{R}^{n+q}$, singularly extend isometrically a.e..

We say that $F = F_{\Lambda,f}$ *nowhere induces a singular extension of* f if, for every open subset $U \subset M^n$ and every subbundle $\Lambda' \subset \Lambda|_U$, the restriction of $F|_{\Lambda'}$ is not a singular extension of $f|_U$.

The key point is that this only happens when f is $\bar{\Lambda}$-ruled a.e. (recall that here $\Lambda \subset TM$ is not necessarily integrable!):

Proposition 117. *Let $\Lambda \subset TM$ any smooth distribution. Then, $F_{\Lambda,f}$ nowhere induces a singular extension of f $\iff$ f is $\bar{\Lambda}$-ruled a.e..*

Proof. We only need to prove the direct statement. SPG, $\Lambda \subset TM$ with $\operatorname{rank}\Lambda = 1$. So we may parametrize $F(x,t) = f(x) + tX(x)$. Then, $\forall p \in M$ there is $(p_m, t_n) \to (p, 0)$ with $t_m \neq 0$ such that $\operatorname{rank} F_*(p_m, t_n) = n$. Define the tensors on M by $K = \nabla_\bullet X$ and $H_t = I + tK$. Hence, there is $Y_m \in T_{p_m}M$ such that $F_{*(p_m,t_m)}Y_m = X(p_m)$, i.e., $H_{t_m}Y_m = X(p_m)$ and

$$\alpha(X(p_m), H_{t_m}^{-1}X(p_m)) = 0. \tag{7}$$

Consider a precompact open neighborhood $U \subset M^n$ of p, so $\|\alpha\| < c$ and $\|K\| < c$ for some constant $c > 1$. Hence for $t \in I = (-\frac{1}{c^2}, \frac{1}{c^2})$ we have that H_t is invertible on U, and

$$H_t^{-1} = \sum_{i\geq 0}(-t)^i K^i,$$

since $H_t(\sum_{i=0}^{N}(-t)^i K^i) = Id - (-t)^{N+1}K^{N+1}$.
We claim that $\alpha(X, S_X) = 0$ on M^n, where S_X is the K-invariant

subspace generated by X, i.e., $S_X = \operatorname{span}\{X, KX, K^2X, \dots\}$. If otherwise, set $j := \min\{k \in \mathbb{N} : \alpha(X(q), K^k(X(q))) \neq 0,\ q \in M^n\}$ and take $p \in M^n$ such that $\alpha(X(p), K^j(X(p))) \neq 0$. By (7),

$$\sum_{i\geq 0}(-t_m)^i\alpha(X(p_m), K^{j+i}(X(p_m))) = 0.$$

Taking $m \to \infty$ we get $\alpha(X(p), K^j(X(p))) = 0$, a contradiction. Now, since $\alpha(X, S_X) = 0$ on M^n, for any $t \in I$ and $p \in U$ we get $F_{*(p,t)}(H_t^{-1}(X)) = X$ since $H_t^{-1}(X) \in S_X$. It follows that $\operatorname{rank}(F_*) = n$ in all $U \times I$, and therefore $F(U \times I) = f(U)$. Hence a segment of the line generated by X is contained in $f(U)$. ∎

We have thus shown:

Theorem 118. *Let $\hat{f}\colon M^n \to \mathbb{R}^{n+q}$ be a strongly genuine deformation of $f\colon M^n \to \mathbb{R}^{n+p}$ and $\tau\colon L^\ell \subset T_f^\perp M \to \hat{L}^\ell \subset T_{\hat{f}}^\perp M$ a parallel vector bundle isometry that preserves second fundamental forms. Let $D \subset TM \oplus L^\ell$ be a subbundle such that $\phi_\tau(TM, D)$ is a null subset. Then $D \subset TM$ and, along each connected component of an open dense subset of M^n, f and $\hat{f}$ are mutually $\bar{D}$-ruled.*

From Proposition 29 we immediately get the following, where

$$i(\phi_\tau)(x) := \max\{\operatorname{rank}(\phi_\tau(X, \cdot))\colon X \in T_xM\}.$$

Corollary 119. *Under the assumptions of Theorem 118, along each connected component of an open dense subset of M^n, $i(\phi_\tau)$ is constant and f and $\hat{f}$ are mutually $\bar{D}^d_Y$-ruled for any smooth vector field $Y \in Re(\phi_\tau)$, where $D^d_Y := \operatorname{Ker}(\phi^Y_\tau) \subset TM$.*

In particular, we conclude that f and $\hat{f}$ are mutually d-ruled with $d = n + \ell - i(\phi_\tau) \geq n - p - q + 3\ell$.

By allowing singular extensions we recover all the corollaries in [**DF**], even without the technical restrictions on the codimensions required there due to the Main Lemma. For example:

Corollary 120. *Any $M^n \subset \mathbb{R}^{n+p}$ with positive Ricci curvature is singularly genuinely rigid in $\mathbb{R}^{n+q}$, for every $q < n - p$.*

Now, we did all this to apply to global rigidity. So...?

Theorem 121. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ and $\hat{f}\colon M^n \to \mathbb{R}^{n+q}$ be isometric immersions of a compact Riemannian manifold with $p+q < n$. Then, along each connected component of an open dense subset of M^n, either f and $\hat{f}$ singularly extend isometrically, or f and $\hat{f}$ are mutually d-ruled, with $d \geq n-p-q+3$.*

This is an immediate consequence of Corollary 119 and the next.

Lemma 122. *Under the assumptions of Theorem 121, at each point of M^n either $i(\beta) \geq p + q - 3$, or $S(\beta)^\perp$ is indefinite. The last possibility holds globally if* $\min\{p, q\} \leq 5$.

Proof. Let W be the complement, i.e., where either $S(\beta)^\perp$ is definite, or $i(\beta) \leq p + q - 2$ and $\min\{p, q\} \geq 6$. First, $\nu_0 > 0$ on W since this is the easy part of the Main Lemma where no hypothesis is needed. Then, use Sacksteder's trick: not only ν_0, but also $i(\beta)$ (by the proof of Proposition 59), are constant along a geodesic in Δ_0. ∎

In particular, for $p+q \leq 4$, Theorem 121 easily unifies Sacksteder and Dajczer-Gromoll Theorems above, states that the only way to isometrically immerse a compact Euclidean hypersurface in codimension 3 is through compositions (which in turn were classified in [**FG**]), and provides a global version of the main result in [**DFT**]:

Corollary 123. *Any compact isometrically immersed submanifold M^n of $\mathbb{R}^{n+p}$ is singularly genuinely rigid in $\mathbb{R}^{n+q}$ for all $q < min\{5, n\} - p$.*

Proof. The only case left is the $(n-1)$-ruled one, which is not hard. Or you can attack it directly. ∎

From Theorem 121 we get the following topological criteria for singular genuine rigidity in line with the rigidity question proposed by M. Gromov in *Partial Differential Relations* p.259 and answered in without any *a priori* assumption on the codimensions:

Corollary 124. *M^n a compact manifold whose k-th Pontrjagin class $[p_k] \neq 0$ for some $k > \frac{3}{4}(p+q-3)$. Then, any analytic immersion $f\colon M^n \to \mathbb{R}^{n+p}$ (with the induced metric) is singularly genuinely rigid in $\mathbb{R}^{n+q}$ in the C^∞-category.*

§29. Real Kahler submanifolds

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