Riemannian Geometry: class guide

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Prerequisites: Basics about manifolds and tensors, at least up to page 12 here. Fundamental group and covering maps.

Bliography: [**CE**], [**dC**], [**Me**], [**ON**], [**Pe**], [**Sp**], [**KN**], [**Es**], [**Ri**].

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§1. Notations

Top. manifolds: Hausdorff + countable basis. Partitions of unity.
n-dimensional differentiable manifolds: M^n. Everything is C^∞.
$\mathcal{F}(M) := C^\infty(M, \mathbb{R})$; $\mathcal{F}(M,N) := C^\infty(M,N)$.
(x,U) chart $\Rightarrow$ coordinate vector fields $= \partial_i := \partial/\partial x_i \in \mathfrak{X}(U)$.
Tangent bundle TM, vector fields $\mathfrak{X}(M) := \Gamma(TM) \cong \mathcal{D}(M)$.
Submersions, immersions, embeddings, local diffeomorphisms.
Vector bundles, trivializing charts, transition functions, sections.
Tensor fields $\mathfrak{X}^{r,s}(M)$, k-forms $\Omega^k(M)$, orientation, integration.
Pull-back of a vector bundle $\pi : E \to N$ over N: $f^*(E)$.
Vector fields along a map $f : M \to N \Rightarrow \mathfrak{X}_f \cong \Gamma(f^*(TN))$.
f-related vector fields.

Example: Lie Groups G, L_g, R_g; $\mathfrak{g} := T_eG$ is an algebra;
Integral curve γ of $X \in \mathfrak{g}$ through e is a homomorphism $\Rightarrow$
$exp^G : \mathfrak{g} \to G$, $exp^G(X) := \gamma(1) \Rightarrow exp^G(tX) = \gamma(t)$.

§2. Geometry = Measurement of the Earth

Geography: Protagoras (481BC - 411BC): Earth should be somehow curved, since boats "sank" at the horizon. Anaximander (610BC - 546BC): Imagined Earth as a "column" floating in the center of the universe, "without resting on anything, but without falling". Pythagoras (570 BC - 495 BC): Believed a spherical Earth, and so Aristotles. By 350BC, every illustrated Greek believed in a spherical Earth. Eratosthenes (276BC - 194BC), measured the Earth circumference in 'stadia'. He computed the angle as "a fiftieth of a circle." Total error $< 16.3\%$. Columbus knew

Eratosthenes measurement (!!!) But cited Strabo (63BC - 23BC) and Ptolomy (100AC - 170AC), who wrongly computed 29000km instead of 40000km. Eratosthenes also measured the angle of the Earth axis with respect to the ecliptic, and its distance to the Sun.

§3. Riemannian metrics

Gauss, 1827: $M^2 \subset \mathbb{R}^3 \Rightarrow \ \langle\,,\rangle|_{M^2},\ K_M = K_M(\langle\,,\rangle)$, distances, areas, volumes... Non-Euclidean geometries.
Riemann, 1854: $\langle\,,\rangle \Rightarrow K_M$ (relations proved decades later).
Slow development. General Relativity pushed up!
Riemannian metric, Riemannian manifold: $(M^n, \langle\,,\rangle) = M^n$.
$g_{ij} := \langle \partial_i, \partial_j \rangle \in \mathcal{F}(U) \Rightarrow (g_{ij}) \in C^\infty(U, S(n,\mathbb{R}) \cap Gl(n,\mathbb{R}))$.
Isometries, local isometries, isometric immersions.
Product metric. $T_p\mathbb{V} \cong \mathbb{V},\ T\mathbb{V} \cong \mathbb{V} \times \mathbb{V}$.

Examples: $(\mathbb{R}^n, \langle\,,\rangle_{can})$, Euclidean submanifolds. Nash.

Example: (bi-)invariant metrics on Lie groups.

Proposition 1. *Every differentiable manifold admits a Riemannian metric.*

Angles between vectors at a point. Norm.
It always exists <u>local</u> orthonormal frames: $\{e_1, \ldots, e_n\}$. $\Rightarrow$

Proposition 2. *Given an <u>oriented</u> Riemannian manifold M^n, there exists a unique volume form $dvol \in \Omega^n(M^n)$ such that $dvol(e_1, \ldots, e_n) = 1$ for any positively oriented orthonormal basis $\{e_1, \ldots, e_n\}$ at any point.*

If $\partial_i = \sum_j C_{ij} e_j \Rightarrow (g_{ij}) = CC^t \Rightarrow dvol(\partial_1, \cdots, \partial_n) = \det(C) \Rightarrow$

$$dvol|_U = \sqrt{\det(g_{ij})}\, dx_1 \wedge \cdots \wedge dx_n.$$

So, we can "integrate functions". Volume of (compact) sets.
Riemannian vector bundles: $(E, \langle\, , \rangle)$.

§4. Distance

Length of a piecewise differentiable curve $\Rightarrow$ Riem. distance d.
The topology of d coincides with the original one on M.

§5. Linear connections

If $M^n = \mathbb{R}^n$, or even if $M^n \subset \mathbb{R}^N$, there is a natural way to differentiate vector fields. And this depends only on $\langle\, , \rangle$.

Def.: An *affine connection* or a *linear connection* or a *covariant derivative* on M is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

with $\nabla_X Y$ being $\mathbb{R}$-bilinear, tensorial in X and a derivation in Y.

Tensoriality in $X \Rightarrow (\nabla_X Y)(p) = \nabla_{X(p)} Y$ makes sense.
Local oper.: $Y|_U = 0 \Rightarrow (\nabla_X Y)|_U = 0 \Rightarrow (\nabla_X Z)|_U = \nabla^U_{X|_U}(Z|_U)$
$\Rightarrow$ The *Christoffel symbols* Γ^k_{ij} of ∇ in a coordinate system $\Rightarrow$
Christoffel symbols completely determine the connection: all that is needed is to have <u>local basis of sections</u> $\Rightarrow$
Connections on vector bundles: formally exactly the same.
The above property on U is a particular case of the following:

Proposition 3. *(or "everything I know about connections.") Let* ∇ *be a (linear) connection on a vector bundle* E *over* M*. Then, for every* $f\colon N \to M$*, there exists a unique connection* $\nabla^f : \mathfrak{X}(N) \times \Gamma(f^*(E)) \to \Gamma(f^*(E))$ *on* $f^*(E)$ *such that*

$$\nabla^f_Y(\xi \circ f) = \nabla_{f_* Y}\xi, \quad \forall\ Y \in \mathfrak{X}(N),\ \xi \in \Gamma(E).$$

Exercise. Give meaning and prove that $g^*(f^*(E)) = (f \circ g)^*(E)$ and $(\nabla^f)^g = \nabla^{f \circ g}$.

We will omit the superindex f in ∇^f.
In particular, Proposition 3 holds for any smooth curve $\alpha(t) = \alpha : I \subset \mathbb{R} \to M$, and if $V \in \mathfrak{X}_\alpha$ we denote $V' := \nabla_{\partial_t} V \in \mathfrak{X}_\alpha$.
So, if $\alpha'(0) = v$, $\nabla_v Y = (Y \circ \alpha)'(0)$. But beware of "$\nabla_{\alpha'}\alpha'$"!!

Def.: $V \in \mathfrak{X}_\alpha$ is *parallel* if $V' = 0$. We denote by $\mathfrak{X}''_\alpha$ the set of parallel vector fields along α.

Proposition 4. *Let* $\alpha : I \subset \mathbb{R} \to M$ *be a piecewise smooth curve, and* $t_0 \in I$*. Then, for each* $v \in T_{\alpha(t_0)}M$*, there exists a unique parallel vector field* $V_v \in \mathfrak{X}_\alpha$ *such that* $V_v(t_0) = v$*.*

The map $v \mapsto V_v$ is an isomorphism between $T_{\alpha(t_0)}M$ and $\mathfrak{X}''_\alpha$, and the map $(v, t) \mapsto V_v(t)$ is smooth when α is smooth $\Rightarrow$

Def.: The *parallel transport of* $v \in T_{\alpha(t)}M$ *along* α *between* t *and* s is the map $P^\alpha_{ts} : T_{\alpha(t)}M \to T_{\alpha(s)}M$ given by $P^\alpha_{ts}(v) = V_v(s)$.

Notice that $\mathcal{F}(M) = \mathfrak{X}^0(M) = \mathfrak{X}^{0,0}(M)$ and $\mathfrak{X}(M) = \mathfrak{X}^{0,1}(M)$.
Covariant differentiation of 1-forms and tensors: $\forall r, s \geq 0$,

$$\nabla \Rightarrow \begin{cases} \nabla : \mathfrak{X}^r(M) \to \mathfrak{X}^{r+1}(M); \\ \nabla : \mathfrak{X}^{r,s}(M) \to \mathfrak{X}^{r+1,s}(M); \\ \nabla : \mathfrak{X}^{r,s}(E, \hat{\nabla}) \to \mathfrak{X}^{r+1,s}(E, \hat{\nabla}); \end{cases}$$

for any affine vector bundle $(E, \hat{\nabla})$ (in partic., for $E = (TM, \nabla)$).

§6. The Levi-Civita connection !

Def.: A linear connection ∇ on a Riemannian manifold $(M, \langle\,,\rangle)$ is said to be *compatible* with $\langle\,,\rangle$ if, for all $X, Y, Z \in \mathfrak{X}(M)$,

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

<u>*Exercise*</u>. ∇ is compatible with $\langle\,,\rangle \iff \forall\, V, W \in \mathfrak{X}_\alpha,\ \langle V, W\rangle' = \langle V', W\rangle + \langle V, W'\rangle \iff \forall\, V, W \in \mathfrak{X}''_\alpha, \langle V, W\rangle$ is constant $\iff P^\alpha_{ts}$ is an isometry, $\forall\, \alpha, t, s \iff \nabla\langle\,,\rangle = 0$.

Def.: The tensor $T_\nabla(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ is called the *torsion* of ∇. We say that ∇ is *symmetric* if $T_\nabla = 0$.

Miracle: *Every Riemannian manifold $(M, \langle\,,\rangle)$ <u>has</u> a <u>unique</u> linear connection that is symmetric and compatible with $\langle\,,\rangle$, called the <u>Levi-Civita connection</u> of $(M, \langle\,,\rangle)$.*

This is a consequence of the *Koszul formula*: $\forall\, X, Y, Z \in \mathfrak{X}(M)$,

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle - \langle Y, [X, Z]\rangle + \langle Z, [X, Y]\rangle.$$

<u>*Exercise*</u>. Verify that this formula defines a linear connection with the desired properties.

This is the only connection that we will work with. In coordinates, if $(g^{ij}) := (g_{ij})^{-1}$,

$$\Gamma^k_{ij} = \frac{1}{2}\sum_r \left(\frac{\partial g_{ir}}{\partial x_j} + \frac{\partial g_{jr}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_r}\right) g^{rk}\ .$$

<u>*Exercise*</u>. Show that, for $(\mathbb{R}^n, \langle\,,\rangle_{can})$, $\Gamma^k_{ij} = 0$ and ∇ is the usual vector field derivative.

<u>*Exercise*</u>. Use Koszul formula to show that the Levi-Civita connection of a bi-invariant metric of a Lie Group satisfies, and is characterized, by the property that $\nabla_X X = 0\ \forall\, X \in \mathfrak{g}$.

Lemma 5. *(Symmetry and Compatibility Lemma) Let N be any manifold, and $f : N \to M$ a smooth map into a Riemannian manifold M. Then:*

- ∇^f *is symmetric, that is,* $\nabla^f_X f_*Y - \nabla^f_Y f_*X = f_*[X,Y]$, $\forall\ X, Y \in \mathfrak{X}(N)$;
- ∇^f *is compatible with the natural metric on* $f^*(TM)$.

Example: $f\colon N \to M$ an isometric immersion $\Rightarrow f^*(TM) = f_*(TN) \oplus^\perp T_f^\perp N \ \Rightarrow \forall\ Z \in \mathfrak{X}_f, Z = Z^\top + Z^\perp \ \Rightarrow$ the relation between the Levi-Civita connections is $f_*\nabla^N_X Y = (\nabla^f_X f_*Y)^\top$.

Remark 6. $f : N \to M \Rightarrow \mathfrak{X}_f = T_f(\mathcal{F}(N, M))$ *(check for* $f(N) \subset$ *chart of* M*).*

§7. Geodesics !!

When do we have minimizing curves? What are those curves? The Brachistochrone problem and the Calculus of Variations.

Galileo, 1638: wrong solution (circle) in the *Discorsi*. Johann Bernoulli posed the problem in 1696 and gave 6 months to solve it (he already knew the solution was a cycloid). Leibniz asked for more time for 'foreign mathematicians' to attack the problem. They tempted Newton, who didn't like to be teased 'by foreigners', but solved the problem in less than half a day. The Royal Society published Newton's solution anonymously, but there is a legend of Johann Bernoulli claiming in awe with the solution in his hands: *"I recognize the lion by his paw."*

Critical points of the arc-length funct. $L : \Omega_{p,q} \to \mathbb{R}$: *geodesics*:

$$\gamma'' := \nabla_{\frac{d}{dt}}\gamma' = 0.$$

Geodesics = second order nonlinear nice ODE $\Rightarrow$

Proposition 7. $\forall\ v \in TM$, $\exists\ \epsilon > 0$ *and a unique geodesic* $\gamma_v : (-\epsilon, \epsilon) \to M$ *such that* $\gamma_v'(0) = v$ $(\Rightarrow \gamma_v(0) = \pi(v))$.

γ a geodesic $\Rightarrow \|\gamma'\| =$ constant.
γ and $\gamma \circ r$ nonconstant geodesics $\Rightarrow r(t) = at + b$, $a, b \in \mathbb{R} \Rightarrow$
$\gamma_v(at) = \gamma_{av}(t)$; $\gamma_v(t+s) = \gamma_{\gamma_v'(s)}(t) \Rightarrow$ *geodesic field* G of M:

Proposition 8. *There is a unique vector field $G \in \mathfrak{X}(TM)$ such that its trajectories are γ', where γ are geodesics of M.*

The local flux of G is called the *geodesic flow* of M. In particular:

Corollary 9. *For each $p \in M$, there is a neighborhood $U_p \subset M$ of p and positive real numbers $\delta, \epsilon > 0$ such that the map*

$$\gamma : T_\epsilon U_p \times (-\delta, \delta) \to M, \quad \gamma(v, t) = \gamma_v(t),$$

is differentiable, where $T_\epsilon U_p := \{v \in TU_p \colon \|v\| < \epsilon\}$.

Since $\gamma_v(at) = \gamma_{av}(t)$, changing ϵ by $\epsilon\delta/2$ we can assume $\delta = 2 \Rightarrow$ We have the *exponential map* of M (terminology from $O(n)$):

$$\exp : T_\epsilon U_p \to M, \quad \exp(v) = \gamma_v(1).$$

$\Rightarrow \exp(tv) = \gamma_v(t) \Rightarrow \exp_p = \exp|_{T_pM} \colon B_\epsilon(0_p) \subset T_pM \to M \Rightarrow$

Proposition 10. *For every $p \in M$ there is $\epsilon > 0$ such that $B_\epsilon(p) := \exp_p(B_\epsilon(0_p)) \subset M$ is open and $\exp_p \colon B_\epsilon(0_p) \to B_\epsilon(p)$ is a diffeomorphism.*

An open set $p \in V \subset M$ onto which $\exp_p$ is a diffeomorphism as above is called a *normal neighborhood* of p, and when $V = B_\epsilon(p)$ it is called a *normal* or *geodesic ball* centered at p.

Proposition 10 $\Rightarrow \left(\exp_p|_{B_\epsilon(0_p)}\right)^{-1}$ is a chart of M in $B_\epsilon(p) \Rightarrow$

We always have (local!) <u>*polar coordinates*</u> for any $(M, \langle\, , \rangle)$:

$$\varphi : (0, \epsilon) \times \mathbb{S}^{n-1} \to B_\epsilon(p)\backslash\{p\}, \qquad \varphi(s, v) = \gamma_v(s), \qquad (1)$$

where $\mathbb{S}^{n-1} = \{v \in T_pM : \|v\| = 1\}$ is the unit sphere in T_pM.

Examples: $(\mathbb{R}^n, can)$; $(\mathbb{S}^n, can)$.

<u>*Exercise*</u>. Show that for a bi-invariant metric on a Lie Group, it holds that $exp_e = exp^G$.

§8. Geodesics are (local) arc-length minimizers

Lemma 11. *(Gauss' Lemma) Let $p \in M$ and $v \in T_pM$ such that $\gamma_v(s)$ is defined up to time $s = 1$. Then,*

$$\langle (\exp_p)_{*v}(v), (\exp_p)_{*v}(w) \rangle = \langle v, w \rangle, \quad \forall\ w \in T_pM.$$

Proof: If $f(s, t) := \gamma_{v+tw}(s) = \exp_p(s(v + tw))$ then, for $t = 0$, $f_s = (\exp_p)_{*sv}(v)$, $f_t = (\exp_p)_{*sv}(sw)$ and $\langle f_s, f_t \rangle_s = \langle v, w \rangle$. ∎

Gauss' Lemma $\Rightarrow \mathbb{S}_\epsilon(p) := \partial B_\epsilon(p) \subset M$ is a regular hypersurface of M orthogonal to the geodesics emanating from p, called the <u>*geodesic sphere*</u> of radius ϵ centered at p.
Now, $B_\epsilon(p) := \exp_p(B_\epsilon(0_p)) \subset M$ as in Proposition 10 agrees with the metric ball of (M, d) !!!!! More precisely:

Proposition 12. *Let $B_\epsilon(p) \subset U$ a normal ball centered at $p \in M$. Let $\gamma : [0, a] \to B_\epsilon(p)$ be the geodesic segment with $\gamma(0) = p$, $\gamma(a) = q$. If $c : [0, b] \to M$ is another piecewise differentiable curve joining p and q, then $l(\gamma) \leq l(c)$. Moreover, if equality holds, then c is a monotone reparametrization of γ.*

Proof: In polar coordinates, $c(t) = \exp_p(s(t)v(t))$ in $B_\epsilon(p)\backslash\{p\}$, and if $f(s,t) := \exp_p(sv(t)) = \gamma_{v(t)}(s)$, we have that $c' = s' f_s + f_t$. Now, use that $f_s \perp f_t$, by Gauss' Lemma. ∎

Corollary 13. *d is a distance on M, $d_p := d(p, \cdot)$ is differentiable in $B_\epsilon(p)\backslash\{p\}$, and d_p^2 is differentiable in $B_\epsilon(p)$.*

Exercise. Compute $\|\nabla d_p\|$ and the integral curves of ∇d_p inside $B_\epsilon(p)\backslash\{p\}$.

Remark 14. Proposition 12 is LOCAL ONLY, and $\epsilon = \epsilon(p)$: $\mathbb{R}^n$; $\mathbb{S}^n$; $\mathbb{R}^n \setminus \{0\}$.

§9. Geodesics: convex neighborhoods

Problem: a normal ball $B_\epsilon(p)$ may not be a *convex set*, like in $\mathbb{S}^n$. But it is a *strongly convex set* for ϵ small enough!

Proposition 15. *For each $p \in M$, there is an open neighborhood W of p and $\delta > 0$ such that, for all $q \in W$, $B_\delta(q)$ is a normal ball around q and $W \subset B_\delta(q)$ (e.g., $W = B_{\delta/2}(p)$). That is, W is a normal neighborhood of all of its points.*

Proof: Following the notations in Corollary 9, consider $F : T_\epsilon U_p \to M \times M$, $F(v) = (\pi(v), \exp(v))$ for the usual bundle projection $\pi : TM \to M \Rightarrow F_{*0_p} = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \Rightarrow F : T_\delta U'_p \to \mathcal{W}$ is a diffeo, with $p \in U'_p$ and $F(0_p) = (p,p) \in \mathcal{W} \subset M \times M$. Now take any $W \subset M$ with $(p,p) \in W \times W \subset \mathcal{W}$. ∎

W as Proposition 15 is called a *totally normal neighborhood*.

Remark 16. The proof shows that, $\forall q, q' \in W, \exists!$ geodesic γ_v joining q and q' with $l(\gamma_v) < \delta$. Moreover, $v = v(q, q')$ is a differentiable function, so γ_v depends differentiably of q and q'.

Corollary 17. *If a piecewise differentiable curve $c : [a, b] \to M$ p.b.a.l. realizes the distance between $c(a)$ and $c(b)$, then c is a geodesic. In particular, c is regular (see Proposition 12).*

Lemma 18. *Given $p \in M$, there exists an $\epsilon > 0$ such that, for all $0 < r < \epsilon$, every geodesic γ tangent to $\mathbb{S}_r(p)$ at $\gamma(0)$ stays outside of $B_r(p)$ around 0.*

Proof: Let W and δ as in Proposition 15, and consider $\gamma : (-\delta, \delta) \times T_1W \to M$, $\gamma(t, v) = \gamma_v(t)$. If $w(t, v) := \exp_p^{-1}(\gamma_v(t))$, then $F(t, v) := \|w(t, v)\|^2 = d^2(p, \gamma_v(t))$ for $|t| < \delta$. Observe that for $q = p$, $\partial^2 F/\partial t^2(0, v) = 2$, and hence $\partial^2 F/\partial t^2(0, v) > 0$ for $q \in W$ close to p and all unit $v \in T_qM$. But for $B_s(p) \subset W$ and $v \in T_q(\mathbb{S}_s(p))$, by Gauss Lemma $\partial F/\partial t(0, v) = 0$. Therefore, $t = 0$ is a local minimum of $F(\cdot, v)$ for $v \in T_q(\mathbb{S}_s(p))$. ∎

Proposition 19. *For every $p \in M$, there is $\alpha > 0$ such that $B_\alpha(p)$ is strongly convex.*

Proof: Take $\alpha < \epsilon/2$ for ϵ as in Lemma 18 in such a way that $B_\epsilon(p) \subset W$ for any W as in Proposition 15. ∎

What we have shown can be summarized as follows:

Theorem 20. *For all $p \in M$, there is $\epsilon_0 > 0$ such that, for every $0 < \epsilon < \epsilon_0$, $B_\epsilon(p)$ is a totally normal and strongly convex neighborhood. In particular, for every $q \neq q' \in \overline{B_\epsilon(p)}$,* *there exists a unique minimizing (p.b.a.l.) piecewise differentiable curve joining q and q', which is a smooth geodesic segment (whose interior is) contained in $B_\epsilon(p)$, and that depends differentiably on q and q'.*

§10. Curvature !!

Gauss: $K(M^2 \subset \mathbb{R}^3) = K(\langle\,,\rangle)$. Riemann: $K(\sigma) = K_p(\exp_p(\sigma))$.

Def.: The <u>*curvature tensor*</u> or <u>*Riemann tensor*</u> of M is (sign!)

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

We also call R the (4,0) tensor given by

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle.$$

Curvature tensor $R_{\hat{\nabla}}$ of a vector bundle E with a connection $\hat{\nabla}$: exactly the same.

Proposition 21. *For all $X, Y, Z, W \in \mathfrak{X}(M)$, it holds that:*

- *R is a tensor;*
- *$R(X,Y,Z,W)$ is skew-symmetric in X, Y and in Z, W;*
- *$R(X,Y,Z,W) = R(Z,W,X,Y)$;*
- *$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ (first Bianchi id.);*
- *$R^s_{ijk} = \sum_l \Gamma^l_{ik}\Gamma^s_{jl} - \sum_l \Gamma^l_{jk}\Gamma^s_{il} + \partial_j \Gamma^s_{ik} - \partial_i \Gamma^s_{jk}$ ($\Rightarrow R \cong \partial^2\langle\,,\rangle$).*

Proof: Exercise. ∎

$\langle\,,\rangle \Rightarrow \mathfrak{X}(M) \cong \Omega^1(M)$ and $\langle\,,\rangle$ extends to the tensor algebra $\Rightarrow$ the *curvature operator* $R : \Omega^2(M) \to \Omega^2(M)$ is self-adjoint.

Def.: If $\sigma \subset T_pM$ is a plane, then the *sectional curvature* of M in σ is given by

$$K(\sigma) := \frac{R(u,v,v,u)}{\|u\|^2\|v\|^2 - \langle u,v\rangle^2}, \qquad \sigma = \mathrm{span}\{u,v\}.$$

Proposition 22. *If R and R' are tensors with the symmetries of the curvature tensor + Bianchi such that $R(u,v,v,u) = R'(u,v,v,u)$ for all u, v, then $R = R'$ ($\Rightarrow K$ determines R).*

Corollary 23. *If M has constant sectional curvature $c \in \mathbb{R}$, then $R(X,Y,Z,W) = c(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle)$.*

Def.: The *Ricci tensor* is the symmetric (2,0) tensor given by

$$Ric(X,Y) := \frac{1}{n-1}\mathrm{trace}\, R(X,\cdot,\cdot,Y),$$

and the *Ricci curvature* is $Ric(X) = Ric(X,X)$ for $\|X\| = 1$.

Example: $\mathbb{CP}^n$ as $\mathbb{S}^{2n+1}/\mathbb{S}^1$ has $K(X,Y) = 1 + 3\langle JX,Y\rangle^2$ and $Ric \equiv (n+2)/(n-1)$.

Def.: The *scalar curvature* of M is $\frac{1}{n}\mathrm{trace}\, Ric$.

Lemma 24. *(Compare with Lemma 5) Let $f : U \subset \mathbb{R}^2 \to M$ be a map into a Riemannian manifold and $V \in \mathfrak{X}_f$. Then,*

$$\nabla_{\partial_t}\nabla_{\partial_s}V - \nabla_{\partial_s}\nabla_{\partial_t}V = R(f_*\partial_t, f_*\partial_s)V.$$

Equivalently, $R_{\nabla^f}(\cdot,\cdot)V = R_{\nabla}(f_*\cdot, f_*\cdot)V$, $\forall\, f\colon N \to M$.

Proof: Since R_{∇^f} is a tensor, it is enough to check the lemma for coordinate vector fields on N and for $V = \overline{V} \circ f$, $\overline{V} \in \mathfrak{X}(M)$. ∎

§11. Jacobi fields

There's a strong relationship between geodesics and curvature, since curvature measures how fast geodesics come apart. The same tool to prove this is used also to understand the singularities of the exponential map: the Jacobi fields.
Given a variation of a geodesic γ by geodesics, the variational vector field $J \in \mathfrak{X}_\gamma$ satisfies the *Jacobi equation*, i.e.,

$$J'' + R(J, \gamma')\gamma' = 0.$$

A vector field along a geodesic γ satisfying the Jacobi equation above is called a *Jacobi field*: $\mathfrak{X}_\gamma^J = \{J \in \mathfrak{X}_\gamma \colon J'' = R(\gamma', J)\gamma'\}$.
The Jacobi equation is a second order linear ODE (take a parallel frame if not convinced) $\Rightarrow \forall$ geodesic γ and every $u, v \in T_{\gamma(t_0)}M$, there exists a unique $J \in \mathfrak{X}_\gamma^J$ such that $J(t_0) = u$, $J'(t_0) = v$.

Remark 25. $\gamma'(t), t\gamma'(t) \in \mathfrak{X}_\gamma^J$, $\langle J, \gamma' \rangle'' = 0 \;\Rightarrow$ WLG, $J \perp \gamma$.

Proposition 26. *Let $\gamma(s)$ a geodesic, $v = \gamma'(0) \in T_pM$, and $J \in \mathfrak{X}_\gamma^J$ with $J(0) = 0, J'(0) = w \Rightarrow J(t) = d(\exp_p)_{*tv}(tw)$, and there is a variation ξ of γ by geodesics such that $J = \xi_t(0, \cdot)$.*

Example: If $K = c =$ constant $\Rightarrow J(t) = s_c(t)w(t)$, where $w \in \mathfrak{X}_\gamma''$ and $s_c(t) = \sin(t), t, \sinh(t)$ according to $c = 1, 0, -1$.

Proposition 27. *With the notations of Proposition 26,*

$$\|J(t)\|^2 = t^2\|w\|^2 - \frac{1}{3}\langle R(w, v)v, w\rangle t^4 + O(t^4).$$

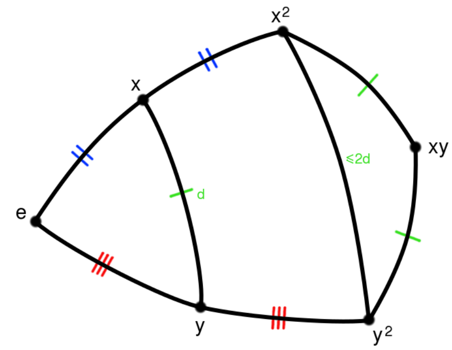
Exercise. Show that $d(\gamma_v(t), \gamma_w(t)) = \|v - w\|t - \frac{1}{6}\frac{\langle R(w,v)v,w\rangle}{\|v-w\|}t^3 + O(t^4)$; see eq.(9) in [**Me**].

Corollary 28. *If in addition $v \perp w$, $\|v\| = \|w\| = 1$, then*

$$\|J(t)\| = t - \frac{1}{6}K(v,w)t^3 + O(t^3).$$

OBS: Geometric relation between geodesics and curvature!!!

Exercise. Prove that a bi-invariant metric on a Lie group has $K \geq 0$ justifying the following diagram:



§12. Conjugate points

Conjugate points and their multiplicity = singularities of $\exp_p$. $C(p)$ = locus of the *first conjugate points to p*.

Example: $\mathbb{S}^n$.

NCP manifolds.

Proposition 29. *If $p' = \gamma(t_0)$ is not conjugate to $p = \gamma(0)$ along $\gamma \Rightarrow \forall\ v \in T_pM$, $\forall\ v' \in T_{p'}M$, there exists a unique $J \in \mathfrak{X}^J_\gamma$ such that $J(0) = v$ and $J(t_0) = v'$. In particular, if $\{J_1, \ldots, J_{n-1}\}$ is a basis of the space of Jacobi fields orthogonal to γ vanishing at 0, then $\{J_1(t_0), \ldots, J_{n-1}(t_0)\}$ is a basis of $\gamma'(t_0)^\perp \subset T_{p'}M$.*

This is useful to construct special bases of vector fields along geodesics.

§13. Isometric immersions

As we have seen in the Example in page 5, if $f : M \to N$ is an isometric immersion $\Rightarrow f^*(TN) = f_*(TM) \oplus^\perp T_f^\perp M$, and $\nabla^M_X Y = (\nabla^f_X f_* Y)^\top$, $\forall X, Y \in TM$. Moreover, we have that

$$\alpha(X,Y) := \left(\nabla^f_X f_* Y\right)^\perp$$

is a symmetric tensor, called the *second fundamental form of* f. In addition, $\nabla^\perp : TM \times \Gamma(T_f^\perp M) \to \Gamma(T_f^\perp M)$ given by

$$\nabla^\perp_X \eta = \left(\nabla^f_X \eta\right)^\perp$$

is a connection in $T_f^\perp M$, called the *normal connection of* f.
Identifications.

Exercise. Show that $\nabla^\perp$ is a connection, and is compatible with the induced metric on $T_f^\perp M$.

$\alpha(p)$ is the quadratic approximation of $f(M) \subset N$ at $p \in M$.
Picture!
$\eta \in T^\perp_{f(p)} M \Rightarrow$ (self-adjoint!) *shape operator* $A_\eta : T_pM \to T_pM$.
Hypersurfaces: Principal curvatures and directions; mean curvature; Gauss-Kronecker curvature; Gauss map.
The Fundamental Equations. Particular case: K = constant $\Rightarrow$ the *Fundamental Theorem of Submanifolds.*
Gauss equation $\Leftrightarrow K(\sigma) = \overline{K}(\sigma) + \langle \alpha(u,u), \alpha(v,v) \rangle - \|\alpha(u,v)\|^2$
$\Rightarrow$ Riemann notion of sectional curvature agrees with ours.

Example: $\mathbb{S}^{n-1}(r) \subset \mathbb{R}^n \Rightarrow K \equiv 1/r^2$ (it *had* to be constant!).

Model of the hyperbolic space $\mathbb{H}^n$ as a submanifold of $\mathbb{L}^{n+1}$.

§14. An interesting example: the geodesic spheres

If γ is a unit geodesic, $p = \gamma(0)$, we consider the shape operator $A(s) = -A_{\gamma'(s)} \in \mathrm{End}(T_{\gamma(s)}M)$ with respect to the unit inward normal at $\gamma(s)$ of a small geodesic sphere of radius s centered at p, then $AJ = J'$ for any $J \in \mathfrak{X}_\gamma^J$ with $J(0) = 0$ and $J \perp \gamma$. In particular: $K \equiv 0 \Rightarrow A(s) = s^{-1}I$; $K \equiv 1 \Rightarrow A(s) = \cot(s)I$.

Exercise. Show that $A = -\mathrm{Hess}_{d_p}|_{\gamma^\perp}$, and $\lim_{s\to 0} sA(s) = Id$ (Sug: use normal coordinates).

If $R_X := R(\cdot, X)X$, then $AJ = J' \Rightarrow$

$$A' + A^2 + R_{\gamma'} = 0 \tag{2}$$

This is known as the *Riccati equation*, and has the same information as the Jacobi equation. Moreover, it implies that: *if we can compare the curvature of two manifolds, we can also compare the shape of geodesic balls* (like $s^{-1}I < \cot(s)I$ above). We will see this in Section 25 and Section 29.

Global Riemannian Geometry

§15. Completeness and the Hopf-Rinow Theorem

Until now, only local stuff. We have problems: Geodesics not defined in $\mathbb{R}$; domain of the exponential map may be strange; far away points may not have geodesics joining them; even if they do, may not be minimizing; the manifolds may have "holes"; (M, d) may not be complete... All these problems have the same solution!

Def.: M is (geod.) <u>*complete*</u> if all geodesics are defined in $\mathbb{R}$.

Proposition 30. *M complete $\Rightarrow$ M is non-extendible.*

Lemma 31. *If $q \notin B_\epsilon(p)$ normal $\Rightarrow d(q, \partial B_\epsilon(p)) = d(q,p) - \epsilon$.*

Theorem 32. *(H-R) Let $(M, \langle\, , \rangle)$ be a connected Riemannian manifold, and $p_0 \in M$. The following are equivalent:*

a) $\exp_{p_0}$ *is defined in $T_{p_0}M$;*

b) Closed bounded subsets of M are compact;

c) (M, d) is a complete metric space;

d) $(M, \langle\, , \rangle)$ is (geodesically) complete;

e) There is a sequence of compact sets $K_n \subset K_{n+1} \subset M$, $\cup_n K_n = M$ such that if $p_n \notin K_n\ \forall n \Rightarrow \lim_{n\to+\infty} d(p_0, p_n) = +\infty$.

In addition, any of these is equivalent to the following:
f) $\forall p, q \in M$, there is a minimizing geodesic joining p and q.

Corollary 33. *M compact $\Rightarrow$ M is complete $\forall \langle\, , \rangle$.*

Corollary 34. *If $S \subset M$ is a closed embedded submanifold of a complete Riemannian manifold M, then S is complete.*

§16. Quick review of covering spaces (see [**Ha**])

Group actions, proper discontinuous group actions, quotients.

Def.: A *covering map* is a surjective local diffeo $\pi : \tilde{M} \to M$ such that $\forall p \in M$, $\exists\, p \in U_p \subset M$ for which $\pi^{-1}(U_p) = \mathbin{\dot\cup}_\lambda V_\lambda$, where each $\pi|_{V_\lambda} : V_\lambda \to U_p$ is a diffeomorphism.

Example: $\pi(\theta) = e^{2\pi i\theta}$ is a covering map from $\mathbb{R}$ to $\mathbb{S}^1 \subset \mathbb{C}$, but $\pi|_{(-1,1)}$ is not.

Proposition 35. *A surjective local diffeomorphism π is a covering map $\Leftrightarrow$ π lifts curves: $\forall p' \in \pi^{-1}(p), \forall c : I \to M$ with $c(0) = p$, $\exists !\ \tilde{c} : I \to \tilde{M}$ such that $\tilde{c}(0) = p'$ and $\pi \circ \tilde{c} = c$.*

Def.: *Homotopic loops* at $p_0 \in M$.

Def.: $\pi_1(M) = \pi_1(M, p_0) =$ *fundamental group* of M.

Def.: M is *simply connected* if $\pi_1(M) = 0$.

Proposition 36. *If $\sigma_1, \sigma_2 : I \to M$ are homotopic, then $\tilde{\sigma}_1(1) = \tilde{\sigma}_2(1)$. The converse holds if $\tilde{M}$ is simply connected.*

Def.: $\mathrm{Deck}(\pi) := \{g \in \mathrm{Diff}(\tilde{M}) : \pi \circ g = \pi\}$, the *deck group.*

$\mathrm{Deck}(\pi)$ acts properly discontinuously on $\tilde{M}$, transitively on the fibers if $\pi_1(\tilde{M}) = 0$, and $\tilde{M}/\mathrm{Deck}(\pi) \cong M$.

Corollary 37. *$\tilde{M}$ simply connected $\Rightarrow$ $j : \pi_1(M) \to \mathrm{Deck}(\pi)$ given by $j([\sigma]) = g$ where $g(\tilde{\sigma}(0)) = \tilde{\sigma}(1)$ is an isomorphism.*

Proposition 38. *For any manifold M there exists a unique (up to diffeomorphisms) simply connected manifold $\tilde{M}$ covering M, called the <u>universal cover</u> of M.*

<u>*Exercise*</u>. $\forall G \subset \pi_1(M)$ subgroup $\Rightarrow \exists\ \pi' : \tilde{M} \to M'$ with $\pi_1(M') = G$. Particular case: $G = \{g \in \mathrm{Deck}(\pi) : g$ preserves orientation$\}$ has index 2 $\Rightarrow$ oriented double covering.

Proposition 39. *If M is compact and $f : M \to M'$ is a surjective local diffeomorphism, then f is a covering map.*

<u>*Exercise*</u>. Give a counterexample to Proposition 39 when M is only complete.

§17. Hadamard manifolds

Lemma 40. *M complete, $f : M \to N$ local diffeo such that $\|f_* v\| \geq \epsilon > 0 \; \forall \; v \in T_1 M \Rightarrow f$ is a covering map ($\Rightarrow$ Pr.39.)*

Proof: f has the lifting property ($\Rightarrow f$ is surjective). ∎

Def.: A point $p \in M$ is called a *pole* if $C(p) = \emptyset$.

Theorem 41. *(Hadamard) M complete simply connected with a pole $p \Rightarrow \exp_p$ is a diffeomorphism ($\Rightarrow M \cong \mathbb{R}^n$!!).*

Lemma 42. *$K \leq 0 \Rightarrow C(p) = \emptyset \; \forall \, p \in M$ (M is said NCP).*

Proof: $\|J\|^{2''} \geq 0$ for $0 \neq J \in \mathfrak{X}^J_\gamma$ with $J(0) = 0$. ∎

Def.: M is a *Hadamard manifold* if it is complete, simply connected and $K \leq 0$.

Corollary 43. *(Hadamard) M Hadamard $\Rightarrow \exp_p$ is a diffeomorphism, $\forall p \in M$.*

Remark 44. M compact has $NCP \not\Rightarrow K \leq 0$. But is there *some* metric on M with $K \leq 0$?? This is a deep open problem!

§18. Manifolds with constant sectional curvature

These are the "simplest" spaces: lots of (local) isometries; congruencies; rigid motions: *geometric postulates.*
We can always assume that $K \equiv -1, 0, 1$: $\mathbb{Q}^n_c = \mathbb{S}^n, \mathbb{R}^n, \mathbb{H}^n$ are complete, connected and simply connected. And they are unique!

Any isometry is locally constructed as i, ϕ, f like in the following:

Theorem 45. *(Cartan) Given $p \in M^n$ and $\hat{p} \in \hat{M}^n$, let $i\colon T_pM \to T_{\hat{p}}\hat{M}$ be a linear isometry. Let V_p a star shaped normal neighborhood of p such that $\exp_{\hat{p}}$ is defined in $\hat{V}_{\hat{p}} := i(\exp_p^{-1}(V_p))$. Define*

$$f = \exp_{\hat{p}} \circ i \circ \exp_p^{-1}|_{V_p} : V_p \to \hat{V}_{\hat{p}}.$$

Let $\phi : TV_p \to TV_{\hat{p}}$ be the natural bundle isometry defined using radial parallel transports and i, that is,

$$\phi(P^{0,t}_{\gamma_v}(w)) = P^{0,t}_{\hat{\gamma}_{iv}}(iw), \quad \forall v, w \in T_pM.$$

If $\phi^*\hat{R} = R$, then f is a local isometry with $f_{*p} = i$ and $f_* = \phi$.

Proof: Observe that $f_*J = \hat{J}$ for Jacobi fields along corresponding radial geodesics γ_v and $\hat{\gamma}_{iv}$ such that $J(0) = 0$, $\hat{J}(0) = 0$, $\hat{J}'(0) = iJ'(0)$. Since ϕ is parallel in radial directions, ϕJ is Jacobi: $(\phi J)'' = \phi J'' = -\phi R_{\gamma_v'} J = -\hat{R}_{\hat{\gamma}_{iv}'}(\phi J)$. Since $\phi|_{T_pM} = i$, then $\hat{J} = \phi J$ and the result follows since ϕ is a bundle isometry. ∎

Remark 46. $\phi^*\hat{R} = R \Leftrightarrow K(\gamma_v', \cdot) = \hat{K}(\hat{\gamma}_{iv}', \phi(\cdot))\ \forall v \in T_pM$.

Corollary 47. *If M^n and $\hat{M}^n$ have the same constant sectional curvature, then $\forall p \in M$, $\forall \hat{p} \in \hat{M}$, $\forall i \in \mathrm{Iso}(T_pM, T_{\hat{p}}\hat{M})$ there exists an isometry $f\colon V_p \to \hat{V}_{\hat{p}}$ with $f(p) = \hat{p}$ and $f_{*p} = i$.*

Remark 48. This holds in particular for $\hat{M} = M$: *spaces of constant curvature are rich (the richest!) in isometries.*

Let $\pi : \tilde{M} \to M$ be a covering map. Given a metric $\langle\, , \rangle$ in M, $\pi^*\langle\, , \rangle$ is called the *covering metric* on $\tilde{M} \Rightarrow \mathrm{Deck}(\pi) \subset \mathrm{Iso}(\tilde{M})$. Conversely, given a metric in $\tilde{M}$, if $\Gamma \subset \mathrm{Iso}(\tilde{M})$ acts properly discontinuous (called a *crystallographic group* when $\tilde{M} = \mathbb{R}^n$), $M := \tilde{M}/\Gamma$ is naturally a Riemannian manifold and the projection π is a local isometry. Moreover, $\tilde{M}$ is complete or has constant $K \Leftrightarrow$ same for M. In particular, $\mathbb{Q}^n_c/\Gamma$ is a *space form*: connected complete with constant sectional curvature $K \equiv c$.

Theorem 49. *(Hopf-Killing) If M^n is a space form, then its universal cover (with the covering metric) is isometric to $\mathbb{Q}^n_c$, and M^n is isometric to $\mathbb{Q}^n_c/\Gamma$, with $\pi_1(M) \cong \Gamma \subset \mathrm{Iso}(\mathbb{Q}^n_c)$.*

Therefore, the classification of space forms is purely an algebraic problem (solved for $c > 0$ in the 60's, well understood for $c = 0$, wide open for $c < 0$).

Corollary 50. *M^{2n} complete with $K \equiv 1 \Rightarrow M^{2n}$ isometric to $\mathbb{S}^{2n}$ or $\mathbb{RP}^{2n}$.*

Remark 51. $\mathbb{R}^n/\mathbb{Z}^n$ is not isometric to $\mathbb{R}^n/2\mathbb{Z}^n$, and two 3-dimensional *lens spaces* $L^3(p,q)$ and $L^3(p,q')$ are not even homeomorphic if $q \neq \pm q'^{\pm 1} \bmod (p)$. In particular, the isomorphism type of $\pi_1(M)$ does not determine the space form. However, it does if $c < 0$, $n \geq 3$ and M^n has finite volume (Mostow rigidity theorem), or if $c > 0$, $n = 3$, and $\pi_1(M^3)$ is not cyclic.

Remark 52. *Does the curvature determine the metric?* More precisely: If f is a diffeo with $f^*\hat{K} = K$, is f an isometry? This is false if $n = 2$ (just take the flow of a generic vector field orthogonal to the gradient of the curvature), or if M^n contains an open subset with constant curvature. However, we have:

If M^n has nowhere constant sectional curvature and $n \geq 4$, then any curvature preserving diffeomorphism is an isometry. For $n = 3$ it is true if M^3 is compact. (Kulkarni-Yau). See here.

Exercise. Read from the book the classification of $\mathrm{Iso}(\mathbb{H}^n)$.

§19. Geodesics as minimizers: Variations of energy

We already know that geodesics are the critical points of the arc-length functional $L(c)$ when restricted to piecewise differentiable (p.d. from now on) curves $c : [0, a] \to M$ p.p.a.l.. To understand when a geodesic is an actual minimizer, we will take second derivatives. But it is easier to work with the *energy functional*:

$$E(c) := \frac{1}{2} \int_0^a \|c'(t)\|^2 dt.$$

Cauchy-Schwarz $\Rightarrow L(c)^2 \leq 2aE(c)$, with $= \Leftrightarrow c$ is p.p.a.l.

Def.: $\Omega_{p,q} = \Omega^a_{p,q} := \{c : [0, a] \to M \ p.d. : c(0) = p, c(a) = q\}$.

Proposition 53. *If $\gamma : [0, a] \to M$ is a minimizing geodesic between $p = \gamma(0)$ and $q = \gamma(a)$, then $E(\gamma) \leq E(c)$ for every $c \in \Omega_{p,q}$, with equality $\Leftrightarrow c$ is a minimizing geodesic.*

Proof: $2aE(\gamma) = L(\gamma)^2 \leq L(c)^2 \leq 2aE(c)$. ∎

That is, E is not only easier to work with than L, but it also takes into account the parametrization. So let's try to minimize E.

Def.: *Variation* $c(s, t)$ of a curve $c = c(0, \cdot)$: $c(s, t) \in C^0$ and there is a partition $0 = t_0 < t_1 < \cdots < t_{m+1} = a$ of $[0, a]$ such that $c|_{(-\epsilon,\epsilon)\times[t_i,t_{i+1}]} \in C^\infty$ (notice that this implies that $c_{ss}(0, \cdot) \in C^0$).

Let $c = c_0 : [0, a] \to M$ be a p.d. curve, $V \in \mathfrak{X}_c$ ($\Rightarrow V \in C^0$), and $c(s, \cdot)$ a variation of c with variational vector field V. For $E(s) = E(c(s, \cdot))$ we have:

Proposition 54. *(Formula for the first variation of energy)*

$$E'(0) = -\int_0^a \langle V(t), c''(t)\rangle dt \ + \ \langle V, c'\rangle|_0^a \ + \ \sum_{i=1}^m \langle V(t_i), c'(t_i^-) - c'(t_i^+)\rangle.$$

Corollary 55. *c is a geodesic $\Leftrightarrow$ c is a critical point of E for proper variations (i.e., for $E|_{\Omega_{c(0),c(a)}}$).*

<u>*Exercise*</u>. Given N and N' two compact submanifolds of a complete Riemannian manifold $\Rightarrow$ there exists a minimizing geodesic γ between N and N'. For such a γ, $\gamma \perp N$ and $\gamma \perp N'$.

Proposition 56. *(Formula for the second variation of E) If $\gamma(t)$ is a geodesic and $f(s,t)$ a variation of γ with variational vector field V, then* (*recall that $R_v := R(\cdot, v)v$*)

$$\begin{aligned} E''(0) &= -\int_0^a \langle V, V'' + R_{\gamma'}V\rangle dt + \sum_{i=1}^m \langle V(t_i), V'(t_i^-) - V'(t_i^+)\rangle + \langle V, V'\rangle|_0^a + \langle \gamma', \nabla_{\partial_s} f_s(0,\cdot)\rangle|_0^a \\ &= I_a(V,V) + \langle \gamma', \nabla_{\partial_s} f_s(0,\cdot)\rangle|_0^a \ , \end{aligned}$$

where $I_a(V,W) := \int_0^a (\langle V', W'\rangle - \langle R_{\gamma'}V, W\rangle)dt$ is the <u>index form</u> of γ.

Corollary 57. *(Jacobi) If a geodesic γ has a conjugate point $\gamma(b)$ to $\gamma(0)$ $\Rightarrow I_{b+\delta} \not\geq 0$ $\Rightarrow \gamma$ does not minimize after b.*

Proof: Let $0 \neq J \in \mathfrak{X}_\gamma^J$, $J(0) = 0$, $J(b) = 0$, $\delta > 0$ and choose any $Z \in \mathfrak{X}_\gamma$ with $Z|_{[0,b-\delta]} = 0$, $Z(b+\delta) = 0$ and $\langle Z(b), J'(b)\rangle < 0$. Define $V_\epsilon \in \mathfrak{X}_\gamma$ as $V_\epsilon = J + \epsilon Z$ in $[0,b]$ and $V_\epsilon = \epsilon Z$ in $[b, b+\delta]$. Then, $I_{b+\delta}(V_\epsilon, V_\epsilon) = 2\epsilon I_b(J,Z) + \epsilon^2 I_{b+\delta}(Z,Z) = 2\epsilon\langle Z(b), J'(b)\rangle + \epsilon^2 I_{b+\delta}(Z,Z) < 0$ for $\epsilon > 0$ small enough. ∎

Remark 58. If the variation is proper, $E''(0) = I_a(V,V)$ only depends on V, and hence I_a is actually the Hessian of $E|_{\Omega_{\gamma(0),\gamma(a)}}$ at its critical point γ ($\forall f\colon M \to N \Rightarrow T_f(\mathcal{F}(M,N)) = \mathfrak{X}_f$).

§20. Application: The Bonnet-Myers Theorem

Theorem 59. *If M is complete with Ric $\geq 1/k^2 > 0$, then M is compact, and* $\operatorname{diam}(M) \leq \pi k$*. In particular, its universal cover is compact and hence $\#\pi_1(M) < \infty$.*

Remark 60. This is false for $K > 0$ (paraboloid). But the curvature bound can be relaxed asking for slow decay at infinity.

Remark 61. The estimate in diam is sharp: $\mathbb{S}^n_k$. And there's *rigidity* (!!): *If* $\operatorname{diam}(M) = \pi k$, *then* $M^n = \mathbb{S}^n_k$ (Corollary 96).

§21. Application: The Synge-Weinstein Theorem

Theorem 62. (*Weinstein*) *M^n compact and oriented with $K > 0$. If $f \in \operatorname{Iso}(M^n)$ preserves (resp.reverses) the orientation of M^n if n is even (resp.odd), then f has a fixed point.*

Proof: Let $g(x) := d(x, f(x))^2$ and assume $g(p) = \min g > 0$. If γ is a unit minimizing geodesic between p and $f(p)$, then $f(\gamma) = \gamma$. So, $(P^\gamma)^{-1} \circ f_{*p}$ fixes some vector $v \in \gamma'(0)^\perp \Rightarrow f \circ \gamma_v = \gamma_{f_* v}$. Now the second variation for $c_s(t) = \exp_{\gamma(t)}(sP^\gamma_{0t} v)$ says that 0 is a strict maximum of $E(s) \Rightarrow g(\gamma_v(s))^2 \leq L(c_s)^2 \leq 2g(p)E(c_s) < 2g(p)E(\gamma) = L(\gamma)^2 = g(p)^2$, a contradiction. ∎

Remark 63. Weinstein Theorem 62 is still true for conformal diffeomorphisms, but it is not known for diffeomorphisms. If this were also true, then $\mathbb{S}^2 \times \mathbb{S}^2$ would not admit a metric with $K > 0$ ($f = (-Id, -Id)$): this is the well known *Hopf conjecture*, one of the most important open conjectures in Riemannian geometry!

Corollary 64. *(Synge) If M^n is compact with $K > 0$, then:*

a) If n is even, then $\pi_1(M^n) = 0$ if M^n orientable, while $\pi_1(M^n) = \mathbb{Z}_2$ if M^n is nonorientable (see Corollary 50);

b) If n is odd, then M^n is orientable.

Remark 65. $\mathbb{RP}^2$ and $\mathbb{RP}^3$ show that the 3 hypothesis in Corollary 64 (a) and (b) are necessary. Yet, compactness is not since for noncompact M^n the soul of its universal cover is a unique point, hence fixed by Deck(π).

Remark 66. B-M and S-W theorems are quite deep:
• Compact manifolds with $K \geq 0$ abound: products of compact manifolds with $K \geq 0$; compact Lie groups G with bi-invariant metrics; homogeneous spaces G/H; biquotients $G//H$; etc.
• OTOH, very few examples are know with $K > 0$: aside from CROSSES ($\mathbb{S}^n, \mathbb{RP}^n, \mathbb{CP}^n, \mathbb{HP}^n, Ca^2$), Eschenburg spaces E^7_p and Bazaikin spaces B^{13}_q for infinite many $p, q \in \mathbb{Z}^5$, only a handful of examples are known, and only in dimensions 6, 7, 12 and 24.
• However, very few obstructions are known for $K > 0$ that do not hold already for $K \geq 0$ and Theorem 59 and Theorem 62 are the most important. In fact: *there is no known obstruction that distinguishes the class of compact simply connected manifolds which admit $K \geq 0$ from the ones that admit $K > 0$* !!

§22. The Index Lemma

We show next that *Jacobi fields are the unique minimizers of the index form* (up to the first conjugate point):

Lemma 67. *(Index lemma). Let $\gamma : [0, a] \to M$ be a geodesic without conjugate points to $\gamma(0)$. Let $V \in \mathfrak{X}_\gamma$ p.d. with $V \perp \gamma'$ and $V(0) = 0$. Consider $t_0 \in (0, a]$ and $J \in \mathfrak{X}^J_\gamma$ the unique Jacobi field such that $J(0) = 0$ and $J(t_0) = V(t_0)$. Then, $I_{t_0}(J, J) \le I_{t_0}(V, V)$, and equality holds $\Leftrightarrow V = J$ in $[0, t_0]$.*

Proof: $\{J_1, \dots, J_{n-1}\}$ basis of $\{J \in \mathfrak{X}^J_\gamma : J \perp \gamma, J(0) = 0\}$, and write $V = \sum f_i J_i$ on $(0, t_0]$.
Claim: $\{f_i\}$ extend C^∞ to 0: If $J_i(t) = tA_i(t) \Rightarrow A_i(0) = J_i'(0)$ are L.I. $\Rightarrow V = \sum g_i A_i$ with g_i p.d. on $[0, t_0]$ and $g_i(0) = 0$ $\Rightarrow g_i(t) = th_i(t)$ where $h_i(t) = \int_0^1 g_i'(ts)ds \Rightarrow f_i = h_i|_{(0,t_0]}$.
But $\langle V', V'\rangle - \langle R_{\gamma'}V, V\rangle = \| \sum f_i' J_i\|^2 + \langle \sum f_i J_i, \sum f_i J_i'\rangle'$ since $\langle J_i, J_j'\rangle = \langle J_i', J_j\rangle$, so $I_{t_0}(V, V) = I_{t_0}(J, J) + \int_0^{t_0} \| \sum f_i' J_i\|^2$. ∎

§23. The Rauch comparison Theorem

Two goals: refine the idea of Bonnet-Myers, and make a global version of Proposition 27: compare Jacobi fields when there is comparison of curvature (we can only expect this NCP). As an inspiration, an old ODE result that will be used in Theorem 93:

Theorem 68. *(Sturm) Let $K, \tilde{K}, f, \tilde{f} : [0, a] \to \mathbb{R}$ satisfying $f'' + Kf = 0$ and $\tilde{f}'' + \tilde{K}\tilde{f} = 0$, with $f(0) = \tilde{f}(0) = 0$ and $f'(0) = \tilde{f}'(0) > 0$. If $\tilde{f} > 0$ in $(0, a]$ and $\tilde{K} \ge K$, then $f/\tilde{f}$ is nondecreasing (and hence $\tilde{f} \le f$). Moreover, if $f(r) = \tilde{f}(r)$ for some $r \in (0, a]$, then $\tilde{K} = K$ and $f = \tilde{f}$ in $[0, r]$.*

Proof: Since $(f'\tilde{f} - f\tilde{f}')(t) = \int_0^t (\tilde{K} - K) f\tilde{f} \Rightarrow f$ does not vanish before $\tilde{f}$ (if $f > 0$ in $(0, r)$ and $f(r) = 0 < \tilde{f}(r) \Rightarrow f'(r) < 0$ contradicting the above equality) $\Rightarrow f/\tilde{f}$ is increasing. ∎

Theorem 69. *(Rauch Comparison) Let $\gamma\colon [0, a] \to M^n$, $\tilde{\gamma}\colon [0, a] \to \tilde{M}^{n+p}$ be geodesics, and $J \in \mathfrak{X}_\gamma^J$ and $\tilde{J} \in \mathfrak{X}_{\tilde{\gamma}}^J$ with comparable initial conditions, i.e., $\|\gamma'\| = \|\tilde{\gamma}'\|$, $J(0) = 0$, $\tilde{J}(0) = 0$, $\langle J'(0), \gamma'(0)\rangle = \langle \tilde{J}'(0), \tilde{\gamma}'(0)\rangle$, and $\|J'(0)\| = \|\tilde{J}'(0)\|$. Assume that $\tilde{\gamma}$ has no conjugate points and that, on $(0, a]$, $K(\gamma', J) \leq \tilde{K}(\tilde{\gamma}', \bullet)$. Then, $\|J\|/\|\tilde{J}\|$ is non-decreasing and, in particular, $\|J\| \geq \|\tilde{J}\|$. Moreover, if $\|\tilde{J}(r)\| = \|J(r)\|$ for some $r \in (0, a]$, then $K(\gamma', J) = \tilde{K}(\tilde{\gamma}', \tilde{J})$ on $(0, r]$.*

Proof: We may assume $0 \neq J \perp \gamma'$, $0 \neq \tilde{J} \perp \tilde{\gamma}'$. If $f := \|J\|^2$ and $\tilde{f} := \|\tilde{J}\|^2$, $g := f/\tilde{f}$ is well defined in $(0, a]$ and $g(0^+) = 1$. So it is enough to see that $g' \geq 0$, or, equivalently, $\tilde{f}'(r)/\tilde{f}(r) \leq f'(r)/f(r)$ when $f(r) \neq 0$. Since $U := J/\sqrt{f(r)}$ and $\tilde{U} := \tilde{J}/\sqrt{\tilde{f}(r)}$ are Jacobi fields, by the hypothesis on the curvature and the Index Lemma 67, $\tilde{f}'(r)/\tilde{f}(r) = 2\tilde{I}_r(\tilde{U}, \tilde{U}) \leq 2\tilde{I}_r(\phi U, \phi U) \leq 2I_r(U, U) = f'(r)/f(r)$, where $\phi : \mathfrak{X}_\gamma \to \mathfrak{X}_{\tilde{\gamma}}$ is any parallel isometry (with the image) with $\phi(\gamma') = \tilde{\gamma}'$ and $\phi(U(r)) = \tilde{U}(r)$. Equality $\Rightarrow$ on $(0, r]$: $g \equiv 1$, $\tilde{I}_r(\phi U, \phi U) = I_r(U, U)$, $\tilde{U} = \phi U$, and so $K(\gamma', J) = \tilde{K}(\tilde{\gamma}', \tilde{J})$. ∎

Corollary 70. *If $K \geq 1/k^2$ (resp. $K \leq 1/k^2$) for some $k > 0$, then the distance d between two consecutive conjugate points along any geodesic satisfies that $d \leq \pi k$ (resp. $d \geq \pi k$).*

Remark 71. According to Section 14, $AJ = J'$ along geodesics without conjugate points, so the inequality $\tilde{f}'/\tilde{f} \leq f'/f$ in the proof above is equivalent to $\tilde{A} \le A$. In fact, Rauch Theorem 69 is equivalent to a Sturm-type comparison for the general Riccati equation (2); see Theorem 3.1 pg.12 due to J. Eschenburg here.

Exercise. Prove the Sturm comparison Theorem using Rauch Theorem 69.

§24. An application to submanifold theory

Theorem 72. *(Moore) Let M^n be a compact submanifold of a Hadamard manifold $\tilde{M}^{n+p}$ with $K \le \tilde{K} + c \le 0$ for certain $c \ge 0$. Then, $p \ge n$.*

Proof: Fix $\tilde{q}_0 \notin M$, $q \in M$ realizing the maximum distance to $\tilde{q}_0$, γ a unit minimizing geodesic between $\tilde{q}_0 = \gamma(0)$ and $q = \gamma(\ell)$, $v \in T_qM$ unitary and $\hat{c}(s)$ a curve in M with $\hat{c}'(0) = v$. If $c(s) = \exp_{\tilde{q}_0}^{-1}(\hat{c}(s))$, for the variation $\gamma_{c'(s)}(t)$ of γ we have that $0 \ge E''(0) = I_\ell(J,J) + \langle \alpha(v,v), \gamma'(\ell)\rangle$, with $J(\ell) = v$. Comparing $\tilde{M}$ with $\mathbb{Q}^{n+p}_{-c}$ we have $I_\ell(J,J) \ge \tilde{I}_\ell(\tilde{J},\tilde{J}) > \sqrt{c} \Rightarrow \|\alpha(v,v)\|^2 \ge \langle \alpha(v,v), \gamma'(\ell)\rangle^2 > c$. Now apply Otsuki's Lemma. ∎

Remark 73. Simply connectedness of $\tilde{M}$ is essential ($T^n \subset T^{n+1}$), as well as compactness of M (catenoid in $\mathbb{R}^3$; even bounded minimal surfaces exist), but $\mathbb{H}^2 \not\subset \mathbb{R}^3$ (Hilbert). The nonexistence of an is.im. $\mathbb{H}^n \subset \mathbb{R}^{2n-1}$ is a famous century old open conjecture.

§25. Applications: comparing geometries!! :o))

As in Cartan's Theorem 45, take $p \in M^n$, $\tilde{p} \in \tilde{M}^n$, $i\colon T_pM \to T_{\tilde{p}}\tilde{M}$ a linear isometry and $r > 0$ such that $B_r(p) \subset M$ is a normal ball and $\exp_{\tilde{p}}$ is non-singular in $B_r(0_{\tilde{p}}) \subset T_{\tilde{p}}\tilde{M}$. For the map $f := \exp_{\tilde{p}} \circ i \circ \exp_p^{-1}|_{B_r(p)}\colon B_r(p) \subset M \to B_r(\tilde{p}) \subset \tilde{M}$ we have:

Proposition 74. *If* $\tilde{K}(\tilde{\gamma}'_{iv}(t),\cdot) \geq K(\gamma'_v(t),\cdot)\ \forall\, v \in T_pM$, $\|v\| = 1, |t| < r \Rightarrow f$ *is a* <u>*contraction*</u>*:* $\|f_*\| \leq 1$*. In particular, if* $c : I \to B_r(p)$ *is any p.d. curve, then* $L(f \circ c) \leq L(c)$*, and, if* $B_r(p)$ *is convex, then* f *is also a metric contraction, i.e.,*

$$\tilde{d}(f(x), f(y)) \leq d(x,y) \quad \forall\, x, y \in B_r(p).$$

<u>*Exercise*</u>. Check that Corollary 47 follows immediately from Proposition 74.

Corollary 75. *If* $K(\gamma'_v(t),\cdot) = k$ *is constant* $\forall\, v \in T_{p_0}M$, $\|v\| = 1$, $|t| < r \ \Rightarrow K \equiv k$ *in* $B_r(p_0)$ *(see Remark 46).*

Remark 76. Proposition 74 is the local version of Toponogov Theorem 99.

§26. Index Lemma and Rauch Thm for focal points

Focal points are generalizations of conjugate points: given $p \in N \subset M$, a normal variation by geodesics of a geodesic γ emanating orthogonally from p gives rise to $J \in \mathfrak{X}^J_\gamma$ such that

$$J(0) \in T_pN \quad \text{and} \quad J'(0) + A_{\gamma'(0)}J(0) \in T_p^\perp N, \qquad (3)$$

and conversely, by considering $\gamma_s(t) = \exp_{\alpha(s)}(t\eta(s))$, where $\eta \in T^\perp_\alpha N$, $\alpha'(0) = J(0)$, $\eta(0) = \gamma'(0)$ and $\eta'(0) = J'(0)$.

<u>*Exercise*</u>. See the details in the book.

Def.: $q \in M$ is a *focal point* of a submanifold $N \subset M$ if there is a geodesic γ orthogonal to N at $\gamma(0) \in N$ with $q = \gamma(r)$, and $0 \neq J \in \mathfrak{X}^J_\gamma$ as in (3) such that $J(r) = 0$. The *focal set* $F(N)$ *of* N is the union of its focal points.

Examples: $\mathbb{S}^n \subset \mathbb{S}^{n+1}$, $F(\mathbb{S}^n) = \pm N$. $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, $F(\mathbb{S}^n) = \{0\}$.

Def.: The *normal exponential map* of N is $\exp^\perp : T^\perp N \to M$.

Proposition 77. *The focal points of $N \subset M$ are precisely the singularities of* $\exp^\perp : T^\perp N \to M$*.*

<u>*Exercise*</u>. See the details in the book.

<u>*Exercise*</u>. Compute the focal points of $N^n \subset \mathbb{R}^{n+1}$ in terms of its principal curvatures.

Analogously to Theorem 41, the following holds: *If M is complete and $N \subset M$ is closed and without focal points, then* $\exp^\perp : T^\perp N \to M$ *is a covering map.* (Hermann).

Def.: A geodesic $\gamma : [0, a] \to M$ is *free of focal points* if $N_\epsilon = \exp_{\gamma(0)}(B_\epsilon(0_p) \cap \gamma'(0)^\perp)$ has no focal points along γ (equivalently, $0 \neq J \in \mathfrak{X}^J_\gamma$ with $J \perp \gamma$ and $J'(0) = 0 \Rightarrow J(t) \neq 0$ $\forall t \in [0, a]$).

Making slight modifications in their proofs, we have: *Both the Index Lemma 67 and Rauch Theorem 69 hold for geodesics free of focal points.*

<u>*Exercise*</u>. Prove the last assertion without looking at the book.

Def.: We say that M *has no focal points* (NFP) if no embedded geodesic $\gamma(-\epsilon, \epsilon) \subset M$ has focal points (as a submanifold).

Proposition 78. *$K \leq 0 \Rightarrow NFP \Rightarrow NCP$. In fact:*

i) $K \leq 0 \Leftrightarrow \|J\|^{2''} \geq 0,\ \forall J \in \mathfrak{X}^J_\gamma$;

ii) NFP $\Leftrightarrow \|J(t)\|^{2'} > 0,\ \forall t > 0, 0 \neq J \in \mathfrak{X}^J_\gamma$ *with* $J(0) = 0$;

iii) NCP $\Leftrightarrow \|J(t)\|^2 > 0,\ \forall t > 0, 0 \neq J \in \mathfrak{X}^J_\gamma$ *with* $J(0) = 0$;

Remark 79. $NCP \not\Rightarrow NFP \not\Rightarrow K \leq 0$ for complete metrics. But what about plain differentiable manifolds *admitting* such metrics? Two important open problems: it is not known if $\mathcal{M}_C^n \subset \mathcal{M}_F^n$, or if $\mathcal{M}_F^n \subset \mathcal{M}_0^n$, for $\mathcal{M}_0^n = \{M^n : \exists \langle\,,\rangle \text{ with } K \leq 0\}$, $\mathcal{M}_F^n = \{M^n : \exists\ NFP\ \langle\,,\rangle\}$ and $\mathcal{M}_C^n = \{M^n : \exists\ NCP\ \langle\,,\rangle\}$.

§27. The Morse Index Theorem

Given a geodesic $\gamma\colon [0,a] \to M$, consider $\mathcal{V}_a$ the set of p.d. vector fields along γ that vanish at 0 and a (i.e., $\mathcal{V}_a = T_\gamma \Omega_{\gamma(0),\gamma(a)}$).
For proper variations of γ, $\mathrm{Hess}_E = I_a$ where $I_a\colon \mathcal{V}_a \times \mathcal{V}_a \to \mathbb{R}$.

Def.: The *nullity* of I_a is $\nu(I_a) := \dim \mathrm{Ker}\,(I_a)$, while its *index* is $i(I_a) := \max\{\dim L\colon I_a|_{L\times L} < 0\}$. ($\gamma$ minimizing $\Rightarrow i(I_a) = 0$).

The purpose now is to show that $i(I_a) = \#$ of conjugate points along γ. We will reduce the problem to a finite dimensional one.

Proposition 80. $\mathrm{Ker}\,(I_a) = \mathcal{V}_a \cap \mathfrak{X}_\gamma^J$. *I.e., I_a is degenerate $\Leftrightarrow \gamma(a)$ is conjugate to $\gamma(0)$ along γ, with $\nu(I_a)$ as multiplicity.*

Proof: Immediate from the two expressions in Proposition 56. ∎

Let $0 = t_0 < t_1 < \cdots < t_k = a$ be a *normal subdivision* of $[0,a]$ ($\gamma([t_i, t_{i+1}])$ is contained in a totally normal neighborhood).
Define

$$\mathcal{V}_a^+ := \{V \in \mathcal{V}_a : V(t_i) = 0, i = 0, \dots, k\},$$

$$\mathcal{V}_a^- := \{V \in \mathcal{V}_a : V|_{[t_i,t_{i+1}]} \text{ is Jacobi}\} \Rightarrow \dim \mathcal{V}_a^- = n^{k-1} < +\infty.$$

Proposition 81. $\mathcal{V}_a = \mathcal{V}_a^+ \oplus \mathcal{V}_a^-$, $I_a|_{\mathcal{V}_a^+\times\mathcal{V}_a^-} = 0$, $I_a|_{\mathcal{V}_a^+\times\mathcal{V}_a^+} > 0$.

Proof: Proposition 56 + $\gamma|_{[t_i,t_{i+1}]}$ minimizing + Proposition 80. ∎

Corollary 82. $i(I_a)=i(I_a|_{\mathcal{V}_a^-\times\mathcal{V}_a^-})<+\infty$, $\nu(I_a)=\nu(I_a|_{\mathcal{V}_a^-\times\mathcal{V}_a^-})$.

Theorem 83. *(Morse) $i(I_a) < +\infty$ is equal to the number of conjugate points (with multiplicities) to $\gamma(0)$ along γ in $[0, a)$.*

Proof: Take $t \in (0, a)$ and choose the normal partition such that $t \in (t_i, t_{i+1})$. Consider $\varphi_t : S := T_{\gamma(t_1)}M \times \cdots \times T_{\gamma(t_i)}M \to \mathcal{V}_t^-$, $\varphi_t^{-1}(V) = (V(t_1), \ldots, V(t_i))$, and work with $\hat{I}_t = \varphi_t^* I_t : S \times S \to \mathbb{R}$, that also depends continuously on t (since the vector $\left(d(\exp_{\gamma(t)})_{-(t-t_i)\gamma'(t)}\right)^{-1}(v_0/(t - t_i))$ depends continuously on t as long as no conjugate points appear). Set $i(t) := i(\hat{I}_t)$ and $\nu(t) := \nu(\hat{I}_t)$. By continuity, $i(t+\epsilon) \leq i(t)+\nu(t)$ for all $|\epsilon|$ small enough. But by the Index Lemma 67 we have that $\hat{I}_t > \hat{I}_{t+\epsilon}$, and then $i(t + \epsilon) \geq i(t) + \nu(t)$ if $\epsilon > 0$. Then, $i(t)$ is increasing and $i(t + \epsilon) = i(t) + \nu(t)$. ∎

Corollary 84. *(Jacobi) Let $\gamma\colon [0, a] \to M$ be a geodesic such that $q = \gamma(a)$ is not conjugate to $p = \gamma(0)$ along γ. Then, γ has no conjugate points $\Leftrightarrow$ γ is a strict local minimum of $E|_{\Omega_{p,q}}$. In particular, γ minimizing $\Rightarrow$ γ has no conjugate points (compare with Corollary 57).*

Corollary 85. *The set of conjugate points to $\gamma(0)$ along γ is discrete.*

§28. The cut locus

Given M complete, $p \in M$ and $v \in \mathbb{S}^{n-1}(0_p) \subset T_pM$, define $\rho(v) = \rho_p(v) := \sup\{t > 0 : d(p, \gamma_v(t)) = t\} \in (0, +\infty]$. If $\rho(v) < +\infty$, $\gamma_v(\rho(v))$ is called the *cut point of p along γ*. The *cut locus $C_m(p)$ of p* is the union of its cut points.

$i(p) := d(p, C_m(p)) \in (0, +\infty]$ is the *injectivity radius at* p. $i(M) := \inf_{p \in M} i(p) \in [0, +\infty]$ is the *injectivity radius of* M.

Proposition 86. *Let γ be a minimizing geodesic between p and q. Then, q is the cut point of p along γ if and only if either q is the first conjugate point of p along γ, or there exists another minimizing geodesic between p and q.*

Corollary 87. $q \in C_m(p) \Leftrightarrow p \in C_m(q)$.

Corollary 88. $q \in M \setminus C_m(p) \Rightarrow$ *there exists a unique minimizing geodesic between p and q.*

Examples: $C(p)$ and $C_m(p)$: $\mathbb{S}^n, \mathbb{RP}^n, \mathbb{S}^1 \times \mathbb{S}^1, \mathbb{S}^1 \times \mathbb{R}$, ellipsoid.

Proposition 89. $\rho : T_1 M \to (0, +\infty]$ *is continuous.*

Proof: Continuity of d + Proposition 86 using the funcion F in Proposition 15, since $F_{*v} = \begin{pmatrix} I & 0 \\ * & d(exp_p)_v \end{pmatrix}$ for $p = \pi(v)$. ∎

Corollary 90. $C_m(p)$ *is closed.*

Corollary 91. M *is compact* $\Leftrightarrow \rho$ *is bounded.*

Corollary 92. $M \setminus C_m(p)$ *is a normal neighborhood of p that is homeomorphic to a ball, open, dense and star-shaped. In particular,* $d^2(p, \cdot) = \| \exp_p^{-1}(\cdot) \|^2$ *is smooth in* $M \setminus C_m(p)$.

Exercise. Show that $C_m(p)$ has measure 0 (Sug.: show that $C_m(p) \cap B_r(p)$ has measure 0). In fact, $C(p)$ and $C_m(p)$, and even $C(N)$ and $C_m(N)$, are Lipschitz submanifolds; see [**IT**].

§29. Bishop-Gromov volume comparison, I ([Pe])

Consider a normal ball $B_{r_0}(p) \subset M^n$, $r < r_0$ (but the same computation works for normal neighborhoods) and set $\mathbb{S} = \mathbb{S}^{n-1} = \mathbb{S}_1^{n-1}(0_p) \subset T_pM$. Let $v \in \mathbb{S}$, $\gamma = \gamma_v$, $\{e_i\}$ an o.n. basis of $v^\perp \subset T_pM$ and $J_i(t) = t(d\exp_p)_{tv}(e_i) \in \mathfrak{X}_\gamma^J$. Then,

$$\mathrm{Vol}(\mathbb{S}_r^{n-1}(p)) = \int_{\mathbb{S}} \det\left((d\exp_p)_{rv}\right) r^{n-1} dv = \int_{\mathbb{S}} j_v(r)^{n-1} dv,$$

where $j_v^{n-1} = \|J_1 \wedge \cdots \wedge J_{n-1}\|$ is the volume in $\gamma'^\perp$ of the parallelepiped spanned by $\{J_i\}$. Therefore, $j_v' = h_v j_v$, where $h_v(r) = \frac{1}{n-1}\mathrm{trace}(A(r))$ is the mean curvature and $A(r)$ the sec.fund.form of $\mathbb{S}_r^{n-1}(p)$ at $\gamma_v(r)$ as seen in Section 14. Writing $A = h_v Id + A_0$ with A_0 symmetric and traceless, by (2),

$$h_v' + h_v^2 + \mathcal{R}_v = 0, \quad \text{with} \;\; \mathcal{R}_v := Ric(\gamma') + \frac{\|A_0\|^2}{n-1} \geq Ric(\gamma').$$

So, $j_v' = h_v j_v \Rightarrow j_v'' + \mathcal{R}_v j_v = 0$, with $j_v(0) = 0$ and $j_v'(0) = 1$. In particular, for $M^n = \mathbb{Q}_k^n$, we have $\bar{j}'' + k\bar{j} = 0$ (indep. of v !!).

Now assume that $Ric \geq k \Rightarrow$ by Sturm Theorem 68, $j_v/\bar{j}$ is decreasing $\Rightarrow q_v := (j_v/\bar{j})^{n-1}$ is decreasing $\Rightarrow$

$$\textit{the map} \;\; r \mapsto \mathrm{Vol}(\mathbb{S}_r^{n-1}(p))/\mathrm{Vol}(\mathbb{S}_{r,k}^{n-1}) \;\; \textit{is decreasing} \; !!$$

where $B_{r,k}^n$ is a ball of radius r in $\mathbb{Q}_k^n$ and $\mathbb{S}_{r,k}^{n-1}$ its geodesic sphere. Moreover, setting $V_r(p) := \mathrm{Vol}(B_r(p))$ and $V_r^k := \mathrm{Vol}(B_{r,k}^n)$, by Gauss Lemma $V_r(p)/V_r^k = \mathrm{Vol}(\mathbb{S})^{-1} \int_{\mathbb{S}} m_v(r) dv$, where $m_v(r) := \int_0^r q_v \bar{j}^{n-1} / \int_0^r \bar{j}^{n-1}$ is the ($\bar{j}^{n-1}$-weighted) average of q_v. Since q_v is decreasing, so is m_v, and we conclude:

Theorem 93. *(Bishop–Gromov, local: for normal balls). If* $Ric_M \geq k$*, the function* $r \mapsto V_r(p)/V_r^k$ *is non-increasing,* $0 \leq r \leq i(p)$*. If, in addition,* $V_s(p)/V_s^k = V_r(p)/V_r^k$ *for some* $0 < s < r \leq diam(M)$*, then* $B_r(p)$ *is isometric to* $B_{r,k}^n$*.*

Proof: We already proved the first part, so we only need to check the equality case. But in this case by monotonicity of m_v we get $m_v(s) = m_v(r)$ $\forall v \in \mathbb{S}$. By monotonicity of q_v this implies that $q_v \equiv 1$ on $[0, r]$ $\forall v$. By the equality in Sturm Theorem 68, $\mathcal{R}_v \equiv k \Rightarrow Ric(\gamma') \equiv k$ and $A_0 \equiv 0 \Rightarrow A$ agrees to that for $\mathbb{Q}_k^n$ $\Rightarrow$ the Jacobi fields along γ are $sn_k(t)e(t)$ with $e(t)$ parallel (as for $\mathbb{Q}_k^n$) $\Rightarrow f$ in Proposition 74 is an isometry. ∎

Remark 94. B-G Theorem 93 does not hold for $Ric \leq k$ because of A_0, but the non-increasing statement works for $K \leq k$ using the same idea as in the proof of Rauch Theorem 69. (exercise)

§30. Bishop-Gromov volume comparison, II ([Pe])

Theorem 95. *(Bishop-Gromov) If* M *is complete, Theorem 93 holds for all* $r \geq 0$ *(i.e., no restriction* $r \leq i(p)$*).*

Proof: Since all the arguments in Section 29 need only for $\exp_p$ to be a chart, we can repeat everything on $M \setminus C_m(p)$ using Corollary 92. Hence, $\mathrm{Vol}(B_p(r)) = \int_{\mathbb{S}} \int_0^r j_v(t)^{n-1} dt dv$ still holds once we extend $j_v(t)$ as 0 for $t > \rho(v)$. Indeed, all that is needed is that the functions $q_v = j_v/\bar{j}$ are still decreasing. ∎

Corollary 96. *(Cheng) If diam$(M^n) = \pi k$ in Bonnet-Myers Theorem 59, then M^n is isometric to $\mathbb{S}^n(k) = \mathbb{Q}^n_{1/k^2}$.*

Proof: WLG $k = 1$, and take $p_1, p_2 \in M^n$ with $d(p_1, p_2) = \pi$. Then, we have that $M^n = \overline{B_\pi(p_i)}$, and $B_{\frac{\pi}{2}}(p_1) \cap B_{\frac{\pi}{2}}(p_2) = \emptyset$. But $\mathrm{Vol}(M^n)/\mathrm{Vol}(B_{\frac{\pi}{2}}(p_i)) = V_\pi(p_i)/V_{\frac{\pi}{2}}(p_i) \leq V^1_\pi/V^1_{\frac{\pi}{2}} = 2$. So, $\mathrm{Vol}(M^n) \leq \mathrm{Vol}(B_{\frac{\pi}{2}}(p_1) \cup B_{\frac{\pi}{2}}(p_2)) \leq \mathrm{Vol}(M^n) \Rightarrow V_\pi(p_i)/V_{\frac{\pi}{2}}(p_i) = 2$ $\Rightarrow$ by the equality case in Theorem 95 $B_\pi(p_i)$ and $B^n_{\pi,1} = \mathbb{S}^n \setminus \{N\}$ are isometric $\Rightarrow B_\pi(p_i) = M^n \setminus \{p_{i+1}\} \Rightarrow M^n = \mathbb{S}^n$. ∎

Corollary 97. *(Calabi-Yau) M^n complete noncompact with $Ric \geq 0 \Rightarrow \mathrm{Vol}(B_r(p)) \geq r\frac{\mathrm{Vol}(B_{r_0}(p))}{2^{n+3}r_0}$ if $r \geq 6r_0$, i.e., it grows at least linearly in r (notice that it grows linearly in $\mathbb{S}^n \times \mathbb{R}$).*

Proof: $V_t = V_t(p) = \mathrm{Vol}(B_t(p))$, $\hat{V}_t = t^n w_{n-1}$ in $\mathbb{R}^n$. For a ray γ at p, $t \geq 2r_0$, and $q = \gamma(t + r_0)$ we have $V_{3t} \geq V_t(q) \geq \frac{V_{t+2r_0}(q) - V_t(q)}{\hat{V}_{t+2r_0} - \hat{V}_t}\hat{V}_t \geq \frac{V_{r_0}\ t^n}{(t+2r_0)^n - t^n} = \frac{V_{r_0}\ t}{2r_0 \sum_{i=1}^n \binom{n}{i}(2r_0/t)^{i-1}} \geq t\frac{V_{r_0}}{2r_0(2^n-1)}$. ∎

Corollary 98. *If M is complete with finite volume and $Ric \geq 0$ (in particular, if M is flat), then M is compact.*

§31. The Toponogov Theorem ([Me])

A global generalization of Rauch Theorem 69 is the following.

Theorem 99. *(Toponogov, hinge version) M complete with $K \geq k$, and γ_1, γ_2 normalized geodesics arcs with $\gamma_1(0) = \gamma_2(0)$. Assume γ_1 is minimizing and, if $k > 0$, that $L(\gamma_2) \leq \pi/\sqrt{k}$. Let $\hat{\gamma}_1, \hat{\gamma}_2$ be the corresponding hinge in $\mathbb{Q}^2_k$, that is, $L(\hat{\gamma}_i) = L(\gamma_i)$ and $\angle(\hat{\gamma}_1'(0), \hat{\gamma}_2'(0)) = \angle(\gamma_1'(0), \gamma_2'(0))$. Then, $d(\gamma_1(\ell_1), \gamma_2(\ell_2)) \leq \hat{d}(\hat{\gamma}_1(\ell_1), \hat{\gamma}_2(\ell_2))$.*

Remark 100. Theorem 99 is immediate from Proposition 74 when γ_1 and γ_2 are contained in a metric ball centered at p onto which $\exp_p$ is nonsingular, and $L(\gamma_i) \leq \pi/\sqrt{4k}$, $i = 1, 2$, when $k > 0$.

There are several versions of Toponogov Theorem 99, some of which do not need anything but distances. For example:

Theorem 101. *Let M be complete with $K \geq k$. If $\{\gamma_j\}$ is a minimizing geodesic triangle in M, then there is a unique minimizing geodesic triangle $\{\hat{\gamma}_j\}$ in $\mathbb{Q}^2_k$ with $L(\hat{\gamma}_j) = L(\gamma_j)$, $j = 0, 1, 2$, and satisfies $d(o, \gamma_0(t)) \geq \hat{d}(\hat{o}, \hat{\gamma}_0(t))$ $\forall t \in [0, L(\gamma_0)]$.*

Theorem 99 follows easily from Theorem 102 below (which in turn is slightly more general than Theorem 101) using the Exercise in Section 11 and the fact that in $\mathbb{Q}^2_k$ the length of a closing edge in a hinge with minimal geodesics and the hinge angle are in a monotone relation; see [**Me**], page 16 Remarks 3 and 5. However, they are actually equivalent. Hence, we will prove:

Theorem 102. *(Toponogov, metric version) M complete, $p_1 \neq o \neq p_2 \in M$, γ_i a minimizing geodesic between o and p_i, $i = 1, 2$, and γ_0 a non-constant geodesic between p_1 and p_2 satisfying $L(\gamma_0) \leq L(\gamma_1) + L(\gamma_2)$, all p.b.a.l.. If $K \geq k$, and $L(\gamma_0) \leq \pi/\sqrt{k}$ when $k > 0$, then there is a minimizing geodesic triangle $\{\hat{\gamma}_j\}$ in $\mathbb{Q}^2_k$ with $L(\hat{\gamma}_j) = L(\gamma_j)$, $j = 0, 1, 2$, and it satisfies that $d(o, \gamma_0(t)) \geq \hat{d}(\hat{o}, \hat{\gamma}_0(t))$ $\forall t \in [0, L(\gamma_0)]$.*

Proof: Let $\rho = d(o, \cdot)$, $\hat{\rho} = \hat{d}(\hat{o}, \cdot)$. If $A = \mathrm{Hess}_\rho|_{\nabla\rho^\perp}$ is the second fundamental form of (pieces of) geodesic spheres centered at o, Rauch says that $A \leq \hat{A} = \frac{s'}{s}I$, where s is the solution of $s''+ks = 0, s(0) = 0, s'(0) = 1$ (see Remark 71). To get a uniform Hessian estimate (not just on $\nabla\rho^\perp$), take f such that $f' = s$. Then, $f'' + kf = C =$ constant. So, if $\sigma := f \circ \rho$ and $\hat{\sigma} := f \circ \hat{\rho}$ we have $\text{Hess}_\sigma = (f'' \circ \rho)d\rho \otimes d\rho + (f' \circ \rho)\text{Hess}_\rho$ and therefore $\text{Hess}_\sigma \leq (-k\sigma + C)I$ on $M \setminus C_m(o)$ and $\text{Hess}_{\hat{\sigma}} = (-k\hat{\sigma} + C)I$.

If $k > 0$, assume first that $L(\gamma_0)+L(\gamma_1)+L(\gamma_2) < 2\pi/\sqrt{k}$, so the corresponding minimizing geodesic triangle exists in $\mathbb{Q}^2_k$ and it is not a great circle. In particular, $\ell := L(\gamma_0) < \pi/\sqrt{k}$.

Consider now $\delta := \sigma \circ \gamma_0 - \hat{\sigma} \circ \hat{\gamma}_0$ on $[0, \ell]$. Since $\text{diam}(M) \leq \pi/\sqrt{k}$ if $k > 0$ by Bonnet-Myers Theorem 59, in any case f is monotonous increasing and we only have to see that $\delta \geq 0$. Observing that $\delta(0) = \delta(\ell) = 0$, assume that $m := \min \delta < 0$. If $k > 0$, comparing with a sphere of curvature $k - \epsilon$ for $\epsilon \to 0$, we may assume that $\text{diam}(M) < \pi/\sqrt{k}$ (or use Theorem 96!). Hence, there exist $k' > k$ and $\tau > 0$ such that $\ell < \pi/\sqrt{k'} - \tau$. In any case, it is easy to find a function a_0 such that $a_0''+k'a_0 = 0$, $a_0(-\tau) = 0$ and $a_0|_{[0,\ell]} \leq m$. Thus, there is $\lambda > 0$ such that the function $a = \lambda a_0$ satisfies $a'' + k'a = 0$, $a \leq \delta$, and $a(t_0) = \delta(t_0) < 0$ for some $t_0 \in (0, \ell)$. (make a picture!)

<u>*Case 1*</u>. $x := \gamma_0(t_0) \notin C_m(o)$. Then δ is smooth in a neighborhood of t_0, and $\delta'' = \langle \text{Hess}_\sigma \gamma_0', \gamma_0' \rangle - \langle \text{Hess}_{\hat{\sigma}} \hat{\gamma}_0', \hat{\gamma}_0' \rangle \leq -k\delta$. Hence, $(\delta - a)''(t_0) \leq (k' - k)\delta(t_0) < 0$, which contradicts the fact that t_0 is a minimum of $\delta - a$.

<u>*Case 2*</u>. $x \in C_m(o)$. Let β be a minimizing geodesic from o to x, $o_\epsilon := \beta(\epsilon)$, and replace ρ by $\rho_\epsilon = d(o, o_\epsilon) + d(o_\epsilon, \cdot)$. By the

triangle inequality, $\rho_\epsilon \geq \rho$ with equality at x, i.e., ρ_ϵ is an *upper support function (USF) of ρ at x*. Moreover, $x \notin C_m(o_\epsilon)$, and so ρ_ϵ is smooth at x. Since f is monotonously increasing, $\sigma_\epsilon := f \circ \rho_\epsilon$ is then an USF of σ at x. Thus $\delta_\epsilon - a$ is also an USF of $\delta - a$ at t_0, and therefore it also attains its minimum at t_0. Since we get the same estimates as in Case 1 up to a small error, $\delta_\epsilon'' \leq -k\delta_\epsilon + O(\epsilon)$ (exercise), we have $(\delta_\epsilon - a)''(t_0) \leq (k' - k)\delta(t_0) + O(\epsilon) < 0$ for ϵ small enough, again a contradiction.

Finally, we need to argue for $L(\gamma_0) + L(\gamma_1) + L(\gamma_2) \geq 2\pi/\sqrt{k}$ if $k > 0$. The "=" case follows from the "<" case with a limit argument in $k - \epsilon$ as we did with the diameter. For the ">" case, take $r < k$ given by $L(\gamma_0) + L(\gamma_1) + L(\gamma_2) = 2\pi/\sqrt{r}$ and use the "=" case comparing with $\mathbb{Q}^2_r$: the comparison triangle in $\mathbb{Q}^2_r$ has to be a great circle, so $-\hat{o} = \hat{\gamma}_0(s_0)$ and therefore $\pi/\sqrt{r} = \hat{d}(\hat{o}, -\hat{o}) \leq d(o, \gamma_0(s_0)) \leq \pi/\sqrt{k} < \pi/\sqrt{r}$, a contradiction. ∎

Application. For noncompact M, $\pi_1(M)$ may not be finitely generated (exercise). However, this does not happen if $K \geq 0$; in fact, there is an *a-priori* bound on the number of generators:

Theorem 103. *(Gromov) M^n complete with $K \geq 0 \Rightarrow \pi_1(M^n)$ can be generated by less than 3^n elements.*

Proof: Fix $x \in \tilde{M}$, and for $f \in \Gamma = \mathrm{Deck}(\pi)$ define $\|f\| = d(x, f(x))$. Notice that $\{g \in \Gamma\colon \|g\| \leq r\}$ is finite for all $r > 0$. So choose $f_1 \in \Gamma$ such that $\|f_1\| = \min\{\|f\|\colon f \in \Gamma\}$, and $f_k \in \Gamma$ with $\|f_k\| = \min\{\|f\|\colon f \in \Gamma \backslash \langle f_1, \ldots, f_{k-1} \rangle\}$. Setting $l_i := \|f_i\|$ and $l_{ij} := d(f_i(x), f_j(x))$, we have for $i < j$ that $l_{ij} = d(x, f_i^{-1} f_j(x)) \geq l_j \geq l_i$ since $f_i^{-1} f_j \notin \langle f_1, \ldots, f_{j-1} \rangle$.

Now choose a minimizing geodesic γ_i from x to $f_i(x)$ of length l_i, and for $i < j$ a minimizing geodesic γ_{ij} from $f_i(x)$ to $f_j(x)$ of length l_{ij}. Take $\alpha_{ij} = \langle \gamma_i'(0), \gamma_j'(0) \rangle$ that is bounded from below by the angle $\tilde{\alpha}_{ij}$ of the corresponding minimizing triangle in $\mathbb{R}^2$ by Toponogov's Theorem 99. The cosine law says that $\cos \tilde{\alpha}_{ij} = (l_i^2 + l_j^2 - l_{ij}^2)/2l_i l_j \leq (l_i^2 + l_j^2 - l_j^2)/2l_i^2 = 1/2$. Hence, $\alpha_{ij} \geq \pi/3$, and so the balls $B^n_{1/2,0}(\gamma_i'(0))$ are disjoint in $B^n_{3/2,0}(0) \subset T_x\tilde{M}$. The estimate follows easily comparing volumes. ∎

Remark 104. Essentially the same proof shows that if M^n is complete with K bounded from below, $K \geq -\lambda^2$, and bounded diameter, $\operatorname{diam}(M^n) \leq D$, then $\pi_1(M^n)$ is generated by less than $\sqrt{n\pi/2}\,(2+2\cosh(2\lambda D))^{\frac{n-1}{2}}$ elements (see Theorem 3.1 in [**Me**]).

To estimate the maximum number of balls of a fixed radius r that fit in the unit n-sphere is an old subject. For $\pi/6$ an exponential known bound is 1.321^n ([**CZ**]). But we have a natural:

Open problem: Is there a linear (or polynomial, or even subexponential) bound in n for Theorem 103?

§32. On Alexandrov Spaces ([BBI])

Toponogov's Theorem 102 (or even Proposition 74) gives rise to curvature notions for metric (length) spaces(!):

Def.: (E, d) a metric space $\Rightarrow d_i = \inf\{L(c)\}$ (may be $+\infty$) is called the *interior distance.* If $d_i = d$, (E, d) is called a *length space* (actually, $d_{ii} = d_i$).

Hopf-Rinow Theorem 32 holds for locally compact length spaces: *If a locally compact length space (E, d) is complete, then any* *two points in E can be connected by a minimizing geodesic, and any bounded closed set of E is compact.*

Def.: A length space (E, d) is called an *Alexandrov space with curvature* $\geq c$ if for all $x \in E$ there exists a neighborhood U_x of x such that, for every triangle pqr in U_x, $q' \in \overline{pr}$ and $p' \in \overline{qr}$, it holds that $d(p', q') \geq \hat{d}(\hat{p}', \hat{q}')$, where $\hat{p}'$ and $\hat{q}'$ are the corresponding points on the comparison triangle $\hat{p}\hat{q}\hat{r}$ in $\mathbb{Q}_c^2$.

Remark 105. In the same way that the local Proposition 74 gives rise to its global version Toponogov Theorem 102 for complete manifolds, the previous local definition implies the corresponding global theorem for complete Alexandrov spaces, a result due to Burago, Gromov and Perelman (for a proof, see [**LS**]).

Alexandrov spaces appear as <u>*limits of manifolds:*</u>

Given two compact metric spaces X, Y we define the *Gromov-Hausdorff distance* $d_{GH}(X, Y) = \inf\{d_H(f(X), g(Y))\}$ where the infimum is taken over all metric spaces Z and all distance preserving maps $f\colon X \to Z$, $g\colon Y \to Z$, and d_H is the *Hausdorff distance* given by $d_H(R, S) = \inf\{\epsilon \geq 0\colon R \subseteq B_\epsilon(S), S \subseteq B_\epsilon(R)\}$. With d_{GH} the isometry classes of compact metric spaces $\mathcal{C}$ is itself a metric space(!) and we can talk about convergence of compact metric spaces(!!). A celebrated result by M. Gromov states that

$$\mathcal{M}(n, c, D) = \{M^n \text{ compact} : Ric \geq c,\ \text{diam}(M) \leq D\}$$

is precompact in $\mathcal{C}$. Limits of converging sequences with bounded K are Alexandrov spaces that are not in general manifolds.

§33. The Preissman Theorem

M^n complete, $K < 0 \Rightarrow \tilde{M}^n \cong \mathbb{R}^n \Rightarrow \pi_k(M^n) = 0 \ \ \forall\, k \geq 2$. But how is $\pi_1(M^n)$ when M^n is compact?

Def.: *Free homotopy classes*: $\hat{\pi}_1(M)$.

Def.: *Closed geodesics* and *geodesic loops*.

Theorem 106. *(Cartan) M^n compact $\Rightarrow \exists$ a closed geodesic in each free homotopy class.*

Proof: Fix $w \in \hat{\pi}_1(M)$ nontrivial, and take a sequence of closed piecewise geodesics $\gamma_n : \mathbb{S}^1 \to M$ such that $L(\gamma_n) \to \ell := \inf\{L(c) \colon c \in w\}$. $\{\gamma_n\}$ is equicontinuous $\Rightarrow \gamma_n \to \sigma \in C^0$ uniformly. Define γ as the closed broken geodesic joining $\sigma(t_i)$ to $\sigma(t_{i+1})$, where $\sigma([t_i, t_{i+1}])$ is inside a convex ball $\Rightarrow \gamma \in w \Rightarrow L(\gamma) \geq \ell$. But $L(\gamma) \leq \ell \Rightarrow \gamma$ is not broken. ∎

Remark 107. Compactness is necessary. Yet, every compact Riemannian manifold has a closed geodesic (Lyusternik-Fet '51).

Def.: $g \in \mathrm{Iso}(N)$ without fixed points is a *translation along* γ if $g(\gamma) = \gamma$ (the images as sets), for some geodesic γ of N.

Lemma 108. *M compact, $\pi \colon \tilde{M} \to M$ its universal cover with the covering metric. Then, every $f \in \mathrm{Deck}(\pi) \subset \mathrm{Iso}(\tilde{M})$ is a translation.*

Proof: Let j be the isomorphism in Corollary 37 and $\gamma \in j^{-1}(f)$ as in Cartan's Theorem 106 (as a free homotopy class) with lift $\tilde{\gamma}$. Then, $f(\tilde{\gamma}(s)) = \tilde{\gamma}(s + r)$, where r is the period of γ (it is s and not $-s$ since otherwise $\tilde{\gamma}(r/2)$ would be a fixed point of f). ∎

Lemma 109. *If $H \neq 1$ is a subgroup of* $\mathrm{Deck}(\pi)$ *all whose elements leave invariant the same geodesic γ, then $H \cong \mathbb{Z}$.*

Proof: $h(\gamma(0)) = \gamma(\tau(h))$, with $\tau\colon H \to (\mathbb{R}, +)$ an injective group homomorphism. H acts discontinuously $\Rightarrow \tau(H) \cong \mathbb{Z}$. ∎

Lemma 110. *A, B, C a geodesic triangle in a Hadamard manifold $\Rightarrow$ i) $A^2 + B^2 - 2AB\cos(\gamma) \leq C^2$ ($<$ if $K < 0$), ii) $\alpha + \beta + \gamma \leq \pi$ ($<$ if $K < 0$).*

Proof: Consequence of Proposition 74 ($\exp_p$ is an expansion). ∎

Proposition 111. *Let $\tilde{M}$ be a Hadamard manifold with $K < 0$, and $f \neq Id$ a translation along $\gamma \Rightarrow \gamma$ is unique.*

Proof: Suppose there are two, $\gamma_1, \gamma_2 \Rightarrow \gamma_1 \cap \gamma_2 = \emptyset \Rightarrow$ there is a geodesic quadrilateral which contradicts Lemma 110. ∎

Corollary 112. *If $g \in$* $\mathrm{Iso}(\tilde{M})$ *commutes with an f as in Proposition 111 $\Rightarrow g$ is also a translation along γ.*

Theorem 113. *(Preissman) M compact with $K < 0 \Rightarrow$ any nontrivial abelian subgroup of $\pi_1(M)$ is infinite cyclic.*

Proof: Lemma 108 + Corollary 112 + Lemma 109. ∎

Corollary 114. *Many compact manifolds that admit metric with $K \leq 0$ admit no metric with $K < 0$: T^n, $N^2 \times \mathbb{S}^1$ for a compact N^2. Nor $M \times N$ for compact M and N. Etc...*

Lemma 115. *If M complete with $K \leq 0$ and* $\mathrm{Deck}(\pi)$ *fixes the same geodesic $\tilde{\gamma}$, then M is not compact (in fact, every geodesic orthogonal to $\pi(\tilde{\gamma})$ is a ray).*

Proof: Take β a unit orthogonal geodesic to γ at $p = \gamma(0)$, α_t a minimizing geodesic joining p to $\beta(t)$, and lift β and α_t to $\tilde{M}$. By Lemma 110 (i), $t \leq L(\tilde{\alpha}_t) = L(\alpha_t) = d(p, \beta(t)) \leq t$. ∎

Corollary 116. *(Preissman) If M is compact with $K < 0$, then $\pi_1(M)$ is not abelian.*

Theorem 117. *(Byers) If M is compact with $K < 0$ and $1 \neq H \subset \pi_1(M)$ is solvable, then $H \cong \mathbb{Z}$. Moreover, any such subgroup has infinite index.*

Proof: $H = H_0 \supset H_1 \supset \cdots \supset H_{k-1} \supset H_k = 1$ with H_i normal in H_{i+1} and abelian quotients $\Rightarrow H_{k-1} =< g >\cong \mathbb{Z}$ with g fixing γ. If $h \in H_{k-2}$, $[h, g] = g^m$ for some $m \Rightarrow h$ also leaves γ invariant $\Rightarrow H_{k-2} \cong \mathbb{Z}$, and so on $\Rightarrow H \cong \mathbb{Z}$ (abelian quotients only needed for H_{k-1}).
For the second part, suppose $H =< g >\cong \mathbb{Z} \subset \pi_1(M)$ has finite index, and take $h \in \pi_1(M) \Rightarrow$ for some n, m, $h^n = g^m \Rightarrow h^n$ fixes γ. By Proposition 111 h also fixes $\gamma \Rightarrow \pi_1(M)$ fixes γ. This contradicts Corollary 116 by Lemma 109. ∎

Remark 118. For (much) more about manifolds with non-negative curvature, see [**BGS**].

§34. On the differentiable sphere Theorem

Let M^n be a compact manifold with positive sectional curvature. Then, $K_{min} \leq K \leq K_{max}$ (i.e., $K_{min}(p) \leq K(\sigma_p) \leq K_{max}(p)$).

Def.: The function K_{min}/K_{max} is called the *pinching function* of M. We say that M is *δ-pinched*, or that $\delta \in \mathbb{R}$ is a *pinching* *of* M, if $\delta < K_{min}/K_{max}$, i.e.,

$$\delta K_{max}(p) < K(\sigma_p) \leq K_{max}(p), \ \ \forall \sigma_p \subset T_pM, \ \forall p \in M.$$

The old question: $\delta \sim 1 \Rightarrow M^n \cong \mathbb{S}^n/\Gamma$?

The answer was **yes**, but how close δ has to be from 1, and what does "$\cong$" mean? *Lots* of development and people involved.
At least for n even, $\delta \geq 1/4 : \mathbb{CP}^n$.
Extrinsic geometric flows: Curvature flow for closed embedded curves in compact and complete surfaces. Watch this and this youtube videos to get an intuition.
Very global in nature: smooth a triangle at its vertices.
Mean curvature flow (MCF): $f' = -HN$; inverse MCF, etc...: $f' = -\nabla E(f)$ for some *energy functional* E ($E =$ vol for MCF).

Def.: *Hamilton's Ricci flow*: $g_t' = -Ric_{g_t}$.

Def.: *Normalized Ricci flow*: $g_t' = -Ric_{g_t} + \frac{1}{n}(\int_M scal_{g_t})g_t$.

These are diffusion equations that tend to 'distribute' the curvature uniformly over the manifold (preserving the volume for the normalized flow). So they should somehow make the metric more 'symmetric'. In general, although we always have existence of flux for small time (Hamilton), singularities (where $K \to \infty$) appear.

Remark 119. Perelman's proof of Thurston's geometrization (and hence Poincaré's) conjecture is based on the classification of the singularity types of the Ricci flow, and their desingularization using (discrete!) surgeries. The number of surgeries is finite for compact simply connected 3-dimensional manifolds, proving Poincaré's conjecture. Apart from the beautiful and tough math, the story behind this is well known (and quite sad... to say the least: see [**NG**]).

The two important questions for us are:

1. *Which are <u>invariant conditions</u> under the Ricci flow?*
2. *Does the metric converge under an invariant condition?*

Under some invariant conditions the Ricci flow develops no singularities, like it was shown in the seminal work [**BW**]:

Theorem 120. *(Böhm-Wilking) Positive and 2-positive curvature operator are invariant conditions, and the metrics converge to a metric with constant sectional curvature. In particular, M is diffeomorphic to a spherical space form, $\mathbb{S}^n/\Gamma$.*

The key main technique behind this beautiful result is the use of *pinching-families*, that are barriers in the sense of PDEs.

Theorem 121. *(Yau-Zheng) If M is $1/4$-pinched $\Rightarrow K_{\mathbb{C}} > 0$.*

Theorem 122. *(Ni-Wolfson, [**NW**]) Both $K_{\mathbb{C}} \geq 0$ and $K_{\mathbb{C}} > 0$ are invariant conditions under the Ricci flow.*

These three results, together with a pinching-family construction as [**BW**], immediately give the *differentiable sphere theorem*:

Corollary 123. *(Brendle-Schoen) If M is (pointwise) 1/4-pinched, then M is diffeomorphic to a spherical space form.*

Actually, Ni and Wolfson in their beautiful and short work [**NW**] proved a stronger version of the differentiable sphere theorem Corollary 123, where even zero curvatures are allowed:

Theorem 124. *(Ni-Wolfson) Assume there exist continuous functions* $k(p), \delta(p) \geq 0$*, such that* $\mathcal{P} := \{p \in M : k(p) > 0\}$ *is dense and* $\delta \not\equiv 0$*, satisfying that, for all* $p \in M$*,* $\sigma \subset T_pM$*,*

$$\frac{1}{4}(1 + \delta(p))k(p) \leq K(\sigma) \leq (1 - \delta(p))k(p).$$

Then, the normalized Ricci flow deforms g *into a metric of constant sectional curvature. In particular,* $M^n \cong \mathbb{S}^n/\Gamma$*.*

Remark 125. It is a pity that the paper [**NW**] by Ni and Wolfson was never published in print (as neither were the three papers where Perelman proves Thurston's geometrization conjecture). But the really interesting question is: *why?*

For details about the Ricci flow, Böhm-Wilking superb work [**BW**] and the differentiable sphere theorem, see the survey [**Ri**].

§35. Busemann functions

These functions are one of the main tools to study the behavior "at infinity" of complete noncompact manifolds.

First, recall: Integration by parts $\Rightarrow$ *weak solutions of PDEs* = good spaces where things converge nicely, as opposed to $C^k(M,\mathbb{R})$. *Regularity theory of elliptic PDEs*: weak solutions are strong. *Max. pple*: $f \in C^2(M, \mathbb{R}), f \geq 0$, $f(p_0) = 0$, $\Delta f \leq 0 \Rightarrow f \equiv 0$. *Support functions and the strong maximum principle*: Let $f \in C^0(M, \mathbb{R})$, $f \geq 0$, $f(p_0) = 0$. Suppose that $\forall x \in M$ and $\forall \epsilon > 0$, $\exists\ g_\epsilon^x \in C^2(U_x)$ with $g_\epsilon^x \geq f$, $g_\epsilon^x(x) = f(x)$ and $\Delta g_\epsilon^x(x) \leq \epsilon$. Then, $f \equiv 0$.

Def.: A *ray* $\gamma\colon [0,+\infty) \to M$ is a (normalized) geodesic such that $d(p,\gamma(t)) = t, \forall t > 0$, while a *line* is a (normalized) geodesic $\gamma\colon \mathbb{R} \to M$ with $d(\gamma(t),\gamma(s)) = |t-s|,\ \forall t,s \in \mathbb{R}$.

For a ray γ and $t \geq 0$, set $b_t = b_t^\gamma := d(\gamma(t),\cdot) - t\colon M \to \mathbb{R}$. If $p := \gamma(0)$, triangle inequality $\Rightarrow$ $b_t \leq b_s$ if $t \geq s$, $b_t \geq -d(p,\cdot)$, and $|b_t(x) - b_t(y)| \leq d(x,y)\ \forall x,y \in M \Rightarrow$ the *Busemann function of* γ given by $b^\gamma := \lim_{t\to+\infty} b_t^\gamma$ is well defined and Lipschitz.

Lemma 126. *If* $f\colon M \to \mathbb{R}$ *is* C^2 *with* $\|\nabla f\| \equiv 1$*, then*

$$-(n-1)Ric(\nabla f) \geq \nabla f(\Delta f) + \|\mathrm{Hess}_f\|^2 \geq \nabla f(\Delta f) + \frac{(\Delta f)^2}{n-1}.$$

Proof: The first inequality follows taking an o.n.b. diagonalizing Hess_f, while the second one is Cauchy-Schwarz on $(\nabla f)^\perp$. ∎

Corollary 127. *(Calabi) If* $Ric \geq 0$*, then for* $\rho := d(p,\cdot)$ *it holds that* $\Delta\rho \leq (n-1)/\rho$ *on* $M \setminus C_m(p) \cup \{p\}$*.*

Proof: If γ is a minimizing geodesic starting at p, and $\lambda := \frac{1}{n-1}\Delta\rho \circ \gamma$, then $\lim_{t\to 0} \frac{1}{\lambda(t)} = \lim_{t\to 0} t = 0$, and $\lambda' + \lambda^2 \leq 0$ by Lemma 126. Since $\mu(t) = 1/t$ satisfies that $\mu' + \mu^2 = 0$, we conclude that $\lambda(t) \leq 1/t = 1/\rho(\gamma(t))$. ∎

Corollary 128. $Ric \geq 0 \Rightarrow$ *a.e.* $\Delta b_t^\gamma \leq \frac{n-1}{t-d(p,\cdot)} \to 0$ *on compacts as* $t \to +\infty$*. In particular,* b^γ *is weakly subharmonic.*

§36. The Cheeger-Gromoll splitting Theorem

While any complete noncompact manifold has a ray, lines only appear in products under nonnegative Ricci curvature:

Theorem 129. *(Cheeger-Gromoll) Let M be complete with $Ric \geq 0$. If M has a line, then M is isometric to $N \times \mathbb{R}$.*

Proof: Take γ a line, $x \in M$, and $\mu_+ = \lim_s \mu_s \colon [0, +\infty) \to M$ a future asymptote to γ with $\mu_+(0) = x$. Since μ_+ is a ray starting at x, $g_t^x := b_t^{\mu_+} + b^\gamma(x)$ is smooth at x. In fact, $g_t^x(x) = b^\gamma(x)$ and, since $d(\gamma(s), x) - t \geq d(\gamma(s), \mu_+(t)) - d(\mu_s(t), \mu_+(t))$,

$$g_t^x = \lim_{s \to +\infty} \left(d(\mu_+(t), \cdot) + d(\gamma(s), x) - t - s \right) \geq b^\gamma.$$

That is, g_t^x is an upper support function for b^γ at x.

Now repeat the same for the past of γ: $b^{-\gamma}$, μ_-, $\tilde{g}_t^x$. The function $b := b^\gamma + b^{-\gamma}$, satisfies $b \geq 0$ and $b = 0$ over γ. But $h_t^x := g_t^x + \tilde{g}_t^x$ is an upper support function for b at x and, by Corollary 127, $\Delta h_t^x(x) \leq 2(n-1)/t$. By the strong maximum principle, $b \equiv 0$, and by Corollary 128 both $b^{\pm\gamma}$ are harmonic, hence smooth. By Lemma 126, $\mathrm{Hess}_{b^\gamma} \equiv 0$, ∇b^γ is parallel ($\Rightarrow$ Killing), the level sets $N_t = (b^\gamma)^{-1}(t)$ of b^γ are smooth embedded totally geodesic isometric hypersurfaces, and the (global!) flux of ∇b^γ restricted to $N_0 \times \mathbb{R}$ is a bijective local isometry, hence an isometry. ∎

Exercise. If M is compact with $Ric \geq 0$, then its universal cover splits isometrically as $N \times \mathbb{R}^k$, with N compact and simply connected.

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