ON TOPONOGOV'S COMPARISON THEOREM FOR ALEXANDROV SPACES

URS LANG AND VIKTOR SCHROEDER

Introduction

In this expository note, we present a transparent proof of Toponogov's theorem for Alexandrov spaces in the general case, not assuming local compactness of the underlying metric space. More precisely, we show that if M is a complete geodesic metric space such that the Alexandrov triangle comparisons for curvature greater than or equal to $\kappa \in \mathbb{R}$ are satisfied locally, then these comparisons also hold in the large; see Theorem 2.3. The core of the proof is Proposition 2.2. It states that a hinge $H = px \cup py$ in M has the desired comparison property if every hinge $H' = p'x' \cup py'$ with an endpoint on H and perimeter $|p'x'| + |p'y'| + |x'y'|$ less than some fixed fraction of the perimeter of H has this property. The argument involves simple inductive constructions in M and the model space $\mathbb{M}^2_\kappa$ of constant curvature, leading to two monotonic quantities (see (5) and (6)), whose limits agree. This immediately gives the required inequality.

The history of Toponogov's theorem starts with the work of Alexandrov [3], who proved it for convex surfaces. Toponogov [10, 11, 12] established the result for Riemannian manifolds, in which case the local comparison inequalities are equivalently expressed as a respective lower bound on the sectional curvature. A first purely metric local-to-global argument was given in [8] for geodesic metric spaces with extendable geodesics. In its most general form, without the assumption of local compactness, the theorem was proved in [6] (in [5] the result is attributed to Perelman). An independent approach, building on [8], was then described by Plaut [9]. The present note further simplifies his argument.

In fact, the statements in both [6] and [9] differ from what is shown here in that the metric of M is merely assumed to be intrinsic (that is, $d(p, q)$ equals the infimum of the lengths of all curves connecting p and q, but it is not required that the infimum is attained); correspondingly, the Alexandrov comparisons are formulated without reference to shortest curves in M. However, assuming M to be geodesic is not a severe restriction. By [9, Theorem 1.4], for every point p in a complete, intrinsic metric space M of curvature locally bounded below there is a dense G_δ subset J_p of M such that for all $q \in J_p$ there exists a shortest curve from p to q. A proof of Toponogov's theorem for intrinsic spaces via essentially the same construction as here, which was obtained independently by Petrunin, is contained in the preliminary version of the forthcoming book [1]. Nevertheless, we felt that it would be worthwhile to make the argument in the present sleek form for geodesic spaces (such as complete Riemannian manifolds) available in the literature.

Date: 25 July 2012.

1. Preliminaries

In this section we fix the notation and recall some basic definitions and facts from metric geometry.

Let M be a metric space with metric d. By a *segment* connecting two points p, q in M we mean the image of an isometric embedding $[0, d(p,q)] \to M$ that maps 0 to p and $d(p,q)$ to q. We will write pq for some such segment (assuming there is one), despite the fact that it need not be uniquely determined by p and q. We will use the symbol $|pq|$ as a shorthand for $d(p,q)$, regardless of the existence of a segment pq. By a *hinge* $H = H_p(x,y)$ in M we mean a collection of three points p, x, y and two nondegenerate segments px, py in M; thus $p \notin \{x, y\}$ (but possibly $x = y$). We call p the *vertex*, x, y the *endpoints*, and px, py the *sides* of H. The *perimeter* of a triple (p, x, y) of points in M is the number

$$\operatorname{per}(p,x,y) := |px| + |py| + |xy|.$$

By the perimeter $\operatorname{per}(H)$ of a hinge $H = H_p(x,y)$ we mean the perimeter of the triple (p,x,y).

We denote by $\mathbb{M}^m_\kappa$ the m-dimensional, complete and simply connected model space of constant sectional curvature $\kappa \in \mathbb{R}$. We write

$$D_\kappa := \operatorname{diam}(\mathbb{M}^m_\kappa) = \begin{cases} \pi/\sqrt{\kappa} & \text{if } \kappa > 0, \\ \infty & \text{if } \kappa \le 0 \end{cases}$$

for the diameter of $\mathbb{M}^m_\kappa$. Some trigonometric formulae for the model spaces are collected in the appendix. The following basic monotonicity property follows readily from the law of cosines, equation (18).

Lemma 1.1. *Let $\kappa \in \mathbb{R}$, and let $a, b \in (0, D_\kappa)$ be fixed. For $\gamma \in [0, \pi]$, let $H_p(x,y)$ be a hinge in $\mathbb{M}^2_\kappa$ with $|px| = b$ and $|py| = a$ such that the hinge angle $\angle_p(x,y)$ (between px and py) equals γ, and put $c_{a,b}(\gamma) := |xy|$. The function $c_{a,b}$ so defined is continuous and strictly increasing on $[0,\pi]$.*

The next lemma goes back to Alexandrov [3], compare [5, Lemma 4.3.3].

Lemma 1.2. *Suppose that $H_p(q,y)$ and $H_q(x,y)$ are two hinges in $\mathbb{M}^2_\kappa$ with $|py|, |qy|, |pq| + |qx| < D_\kappa$, and $H_{\overline{p}}(\overline{x}, \overline{y})$ is a hinge in $\mathbb{M}^2_\kappa$ such that $|\overline{p}\,\overline{x}| = |pq| + |qx|$, $|\overline{p}\,\overline{y}| = |py|$, and $|\overline{x}\,\overline{y}| = |xy|$. Then $\angle_q(p,y) + \angle_q(x,y) \le \pi$ if and only if $\angle_p(q,y) \ge \angle_{\overline{p}}(\overline{x}, \overline{y})$, and $\angle_q(p,y) + \angle_q(x,y) \ge \pi$ if and only if $\angle_p(q,y) \le \angle_{\overline{p}}(\overline{x}, \overline{y})$.*

Proof. Prolongate pq to a segment px' of length $|px'| = |pq| + |qx|$; see Figure 1. Consider the following obvious identities:

$$\pi - \angle_q(p,y) - \angle_q(x,y) = \angle_q(x',y) - \angle_q(x,y), \tag{1}$$

$$|x'y| - |xy| = |x'y| - |\overline{x}\,\overline{y}|, \tag{2}$$

$$\angle_p(x',y) - \angle_{\overline{p}}(\overline{x}, \overline{y}) = \angle_p(q,y) - \angle_{\overline{p}}(\overline{x}, \overline{y}). \tag{3}$$

By Lemma 1.1, the right side of (1) and the left side of (2) have the same sign, and also the right side of (2) and the left side of (3) have equal sign. Hence, the same holds for the left side of (1) and the right side of (3). □

Let again M be a metric space, and let $\kappa \in \mathbb{R}$. Given $p, x, y \in M$, a triple $(\overline{p}, \overline{x}, \overline{y})$ of points in $\mathbb{M}^2_\kappa$ is called a *comparison triple* for (p,x,y) if $|\overline{p}\,\overline{x}| = |px|$, $|\overline{p}\,\overline{y}| = |py|$, and $|\overline{x}\,\overline{y}| = |xy|$. If $\kappa \le 0$, such a comparison triple always exists,

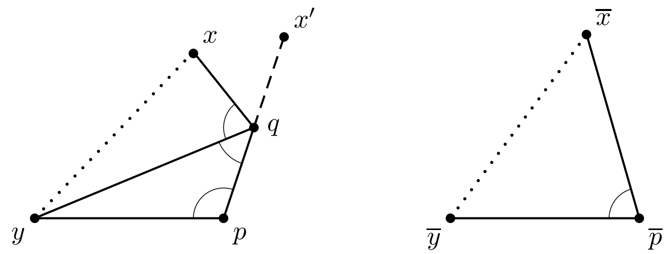

FIGURE 1. Proof of Lemma 1.2

and if $\kappa > 0$, a comparison triple exists if and only if $\mathrm{per}(p,x,y) \le 2D_\kappa$. This is obvious if one of the distances $a := |py|$, $b := |px|$, and $c := |xy|$ is zero or equal to D_κ. Otherwise, when $a,b,c \in (0, D_\kappa)$, the assertion follows from Lemma 1.1: Depending on whether $a+b < D_\kappa$ or $a+b \ge D_\kappa$, the function $c_{a,b}$ maps $[0,\pi]$ bijectively onto $[|a-b|, a+b]$ or $[|a-b|, 2D_\kappa - a - b]$. In either case, the given number c is contained in the image of $c_{a,b}$, so there exists a unique $\gamma \in [0,\pi]$ such that $c_{a,b}(\gamma) = c$.

Now consider a triple (p,x,y) of points in M such that $p \notin \{x,y\}$. In case $\kappa > 0$, suppose that $|px|, |py| < D_\kappa$ and $\mathrm{per}(p,x,y) \le 2D_\kappa$. Then any comparison triple $(\overline{p}, \overline{x}, \overline{y})$ in $\mathbb{M}^2_\kappa$ uniquely determines a hinge $H_{\overline{p}}(\overline{x}, \overline{y})$ and one defines the *comparison angle* $\angle^\kappa_p(x,y) \in [0,\pi]$ as the hinge angle, thus

$$\angle^\kappa_p(x,y) := \angle_{\overline{p}}(\overline{x}, \overline{y}).$$

For an arbitrary hinge $H_p(x,y)$ in M, the (Alexandrov) *angle* or *upper angle* of $H_p(x,y)$ is then defined by

$$\angle_p(x,y) := \limsup_{\substack{u \in px,\, v \in py \\ u,v \to p}} \angle^\kappa_p(u,v).$$

The number $\angle_p(x,y)$ is clearly independent of $\kappa \in \mathbb{R}$. Furthermore, if px, py, pz are three nondegenerate segments, the triangle inequality

$$\angle_p(x,y) + \angle_p(y,z) \ge \angle_p(x,z) \tag{4}$$

holds, see [2] or [4, Part I, Proposition 1.14].

Let again $H = H_p(x,y)$ be a hinge in M, and suppose that $\mathrm{per}(H) < 2D_\kappa$. Let $(\overline{p}, \overline{x}, \overline{y})$ be a comparison triple in $\mathbb{M}^2_\kappa$ for (p,x,y), and let $H_{\hat{p}}(\hat{x}, \hat{y})$ be a *comparison hinge* in $\mathbb{M}^2_\kappa$ for H, that is, $|\hat{p}\hat{x}| = |px|$, $|\hat{p}\hat{y}| = |py|$, and $\angle_{\hat{p}}(\hat{x}, \hat{y}) = \angle_p(x,y)$. We are interested in the following comparison properties that H may or may not have:

- (A_κ) (Angle comparison) $\angle_p(x,y) \ge \angle^\kappa_p(x,y)$ $(= \angle_{\overline{p}}(\overline{x}, \overline{y}))$;
- (H_κ) (Hinge comparison) $|xy| \le |\hat{x}\hat{y}|$;
- (D_κ) (Distance comparison) $|uv| \ge |\overline{u}\,\overline{v}|$ whenever $u \in px$, $v \in py$, $\overline{u} \in \overline{p}\,\overline{x}$, $\overline{v} \in \overline{p}\,\overline{y}$, and $|pu| = |\overline{p}\,\overline{u}|$, $|pv| = |\overline{p}\,\overline{v}|$.

It follows easily from Lemma 1.1 that, for an individual hinge H as above,

$$(\mathrm{D}_\kappa) \Rightarrow (\mathrm{A}_\kappa) \Leftrightarrow (\mathrm{H}_\kappa).$$

For the implication $(\mathrm{A}_\kappa) \Rightarrow (\mathrm{D}_\kappa)$, see Lemma 1.3 below. The metric space M is called a *space of curvature* $\ge \kappa$ *in the sense of Alexandrov* if every point q has a neighborhood U_q such that any two points in U_q are connected by a segment in M and every hinge $H = H_p(x,y)$ with $p,x,y \in U_q$ (and $\mathrm{per}(H) < 2D_\kappa$) satisfies

(D_κ). Again due to Lemma 1.1, the upper angle between two segments in such a space M always exists as a limit, by monotonicity. We call a segment px in a metric space *balanced* if, for every nondegenerate segment qy with $q \in px \setminus \{p, x\}$, the angles formed by qy and the subsegments qp, qx of px satisfy $\angle_q(p, y) + \angle_q(x, y) = \pi$. Note that, by (4), the inequality $\angle_q(p, y) + \angle_q(x, y) \geq \pi$ always holds, since $\angle_q(p, x) = \pi$. Of course, in a Riemannian manifold every segment is balanced.

Lemma 1.3. *Let $\kappa \in \mathbb{R}$, and let M be a metric space. Then:*

(i) *If M is a space of curvature $\geq \kappa$ in the sense of Alexandrov, then all segments in M are balanced.*

(ii) *Let $H = H_p(x, y)$ be a hinge in M with balanced sides and* $\mathrm{per}(H) < 2D_\kappa$*. Suppose that every pair of points in $px \cup py$ is connected by a segment in M and every hinge with one side contained in px or py and the opposite endpoint on the other side of H satisfies* (A_κ)*. Then H satisfies* (D_κ).

Proof. For (i), let px, qy be two nondegenerate segments in M such that $q \in px \setminus \{p, x\}$. Let $u \in qp$, $v \in qx$, $w \in qy$ be points distinct from q, and assume that $u \neq w$. If u, v, w are sufficiently close to q, then there is a segment uw such that the hinge $H_u(v, w)$ with $uv \subset px$ satisfies (D_κ). Let $(\overline{u}, \overline{v}, \overline{w})$ be a comparison triple in $\mathbb{M}^2_\kappa$ for (u, v, w), and let $\overline{q} \in \overline{u}\,\overline{v}$ be the point with $|\overline{q}\,\overline{u}| = |qu|$. Then $|qw| \geq |\overline{q}\,\overline{w}|$ and so $\angle^\kappa_u(q, w) \geq \angle_{\overline{u}}(\overline{q}, \overline{w}) = \angle_{\overline{u}}(\overline{v}, \overline{w})$ by Lemma 1.1. Now Lemma 1.2 shows that $\angle^\kappa_q(u, w) + \angle^\kappa_q(v, w) \leq \pi$. Passing to the limit for $u, v, w \to q$ we get $\angle_q(p, y) + \angle_q(x, y) \leq \pi$.

We prove (ii). Let $(\overline{p}, \overline{x}, \overline{y})$ be a comparison triple in $\mathbb{M}^2_\kappa$ for (p, x, y), and let u, v and $\overline{u}, \overline{v}$ be given as in (D_κ). We first show that $|uy| \geq |\overline{u}\,\overline{y}|$. Omitting some trivial cases, we assume $u \notin \{p, x, y\}$. Choose a segment uy. Then $\angle^\kappa_u(p, y) + \angle^\kappa_u(x, y) \leq \angle_u(p, y) + \angle_u(x, y) = \pi$ by the assumptions and so Lemma 1.2 yields $\angle^\kappa_p(u, y) \geq \angle_{\overline{p}}(\overline{x}, \overline{y}) = \angle_{\overline{p}}(\overline{u}, \overline{y})$. By Lemma 1.1, $|uy| \geq |\overline{u}\,\overline{y}|$. An analogous argument shows that $|uv| \geq |\tilde{u}\,\tilde{v}|$ if $(\tilde{p}, \tilde{u}, \tilde{y})$ is a comparison triple for (p, u, y) and $\tilde{v} \in \tilde{p}\,\tilde{y}$ is such that $|pv| = |\tilde{p}\,\tilde{v}|$. Since $|\tilde{u}\,\tilde{y}| = |uy| \geq |\overline{u}\,\overline{y}|$, we have $\angle_{\tilde{p}}(\tilde{u}, \tilde{v}) = \angle_{\tilde{p}}(\tilde{u}, \tilde{y}) \geq \angle_{\overline{p}}(\overline{u}, \overline{y}) = \angle_{\overline{p}}(\overline{u}, \overline{v})$ (assuming $p \notin \{u, v\}$) and hence $|\tilde{u}\,\tilde{v}| \geq |\overline{u}\,\overline{v}|$ by Lemma 1.1. So $|uv| \geq |\overline{u}\,\overline{v}|$. ☐

2. The globalization theorem

Now we prove Toponogov's theorem, in the form stated in Theorem 2.3 below. The central piece of the argument is Proposition 2.2, the following lemma and the concluding part of the proof are standard techniques.

Lemma 2.1. *Let $\kappa \in \mathbb{R}$, let M be a metric space, and let $H = H_p(x, y)$ be a hinge in M with* $\mathrm{per}(H) < 2D_\kappa$*. Suppose that there exist a point q on px, distinct from p, x, y, and a segment qy such that each of the three hinges $H_p(q, y), H_q(p, y), H_q(x, y)$ with sides in $px \cup py \cup qy$ satisfies* (A_κ)*, and $\angle_q(p, y) + \angle_q(x, y) = \pi$. Then H satisfies* (A_κ) *as well.*

Proof. Note that $\mathrm{per}(p, q, y), \mathrm{per}(q, x, y) \leq \mathrm{per}(H) < 2D_\kappa$. Since $H_p(q, y)$ satisfies (A_κ), we have $\angle_p(x, y) = \angle_p(q, y) \geq \angle^\kappa_p(q, y)$. By the remaining assumptions, $\angle^\kappa_q(p, y) + \angle^\kappa_q(x, y) \leq \angle_q(p, y) + \angle_q(x, y) = \pi$ and so Lemma 1.2 gives $\angle^\kappa_p(q, y) \geq \angle^\kappa_p(x, y)$. Thus $\angle_p(x, y) \geq \angle^\kappa_p(x, y)$. ☐

Proposition 2.2. *Let $\kappa \in \mathbb{R}$, and let M be a metric space such that every pair of points in M at distance $< D_\kappa$ is connected by a balanced segment. Let $H_p(x, y)$ be*

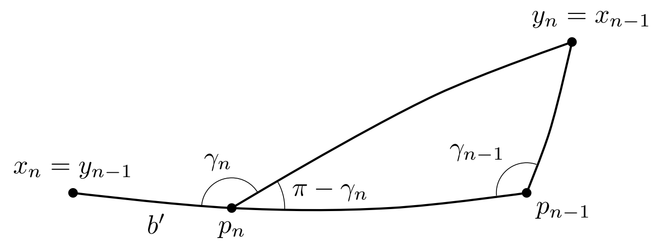

FIGURE 2. Constructing H_n from H_{n-1}

a hinge in M with balanced sides and $\mathrm{per}(p,x,y) < 2D_\kappa$. *If every hinge* $H_{p'}(x',y')$ *in M with balanced sides,* $\mathrm{per}(p',x',y') < \frac{4}{5}\,\mathrm{per}(p,x,y)$*, and* $\{x',y'\} \cap (px \cup py) \neq \emptyset$ *satisfies* (A_κ)*, then* $H_p(x,y)$ *satisfies* (A_κ) *as well.*

Proof. We prove the following assertion, from which the general result follows easily by a repeated application of Lemma 2.1: *Let* $H_0 = H_{p_0}(x_0,y_0)$ *be a hinge in M with balanced sides and* $|p_0x_0| < \min\{\frac{1}{5}|p_0y_0|, D_\kappa - |p_0y_0|\}$. *If every hinge* $H_{p'}(x',y')$ *in M with balanced sides,* $\mathrm{per}(p',x',y') < \frac{4}{5}\,\mathrm{per}(H_0)$*, and* $\{x',y'\} \cap \{x_0,y_0\} \neq \emptyset$ *satisfies* (A_κ)*, then H_0 satisfies* (A_κ) *as well.* We put $a := |p_0y_0|$ and $b := |p_0x_0|$, so $b < \frac{1}{5}a$ and $a + b < D_\kappa$.

First, starting from H_0, we will inductively construct a particular sequence of hinges $H_n = H_{p_n}(x_n,y_n)$ in M with balanced sides such that $\{x_n,y_n\} = \{x_0,y_0\}$ and the numbers $l_n := |p_nx_n| + |p_ny_n|$ satisfy

$$a + b = l_0 \geq l_1 \geq l_2 \geq \ldots \geq |x_0y_0|; \tag{5}$$

furthermore, for $n \geq 1$, $|p_nx_n| = b' := \frac{2}{5}a$ and hence

$$|p_ny_n| \geq |x_ny_n| - |p_nx_n| = |x_0y_0| - b' \geq a - b - b' > b'.$$

The hinge H_0 is already given. For $n \geq 1$, if H_{n-1} is constructed, let $p_n \in p_{n-1}y_{n-1}$ be the point at distance b' from y_{n-1}, and put $x_n := y_{n-1}$ and $y_n := x_{n-1}$. Note that

$$|p_ny_n| \leq |p_{n-1}p_n| + |p_{n-1}y_n| = l_{n-1} - b' \leq a + b - b' < \tfrac{4}{5}a$$

and hence $\mathrm{per}(p_{n-1},p_n,y_n) < \frac{8}{5}a \leq \frac{4}{5}\,\mathrm{per}(H_0)$. The sides of H_n are the subsegment p_nx_n of $p_{n-1}y_{n-1}$ and an arbitrarily chosen balanced segment p_ny_n. Denote the angle of H_n by γ_n, and note that since $p_{n-1}y_{n-1}$ is balanced, the adjacent angle between p_ny_n and the subsegment p_np_{n-1} of $p_{n-1}y_{n-1}$ equals $\pi - \gamma_n$. See Figure 2. Clearly $l_n \leq l_{n-1}$.

Now we will construct a sequence of hinges $\overline{H}_n := H_{\overline{p}_n}(\overline{x}_n,\overline{y}_n)$ in $\mathbb{M}^2_\kappa$ such that $|\overline{p}_n\overline{x}_n| = |p_nx_n|$, $|\overline{p}_n\overline{y}_n| = |p_ny_n|$,

$$|\overline{x}_0\overline{y}_0| \geq |\overline{x}_1\overline{y}_1| \geq |\overline{x}_2\overline{y}_2| \geq \ldots, \tag{6}$$

and such that the angle $\bar{\gamma}_n$ of $\overline{H}_n$ is greater than or equal to γ_n. Let $\overline{H}_0$ be a comparison hinge for H_0, thus $|\overline{p}_0\overline{x}_0| = b$, $|\overline{p}_0\overline{y}_0| = a$, and $\bar{\gamma}_0 = \gamma_0$. For $n \geq 1$, given $\overline{H}_{n-1}$, let $\overline{p}_n \in \overline{p}_{n-1}\overline{y}_{n-1}$ be the point at distance b' from $\overline{y}_{n-1}$, put $\overline{x}_n := \overline{y}_{n-1}$, and choose $\overline{y}_n$ such that $(\overline{p}_{n-1},\overline{p}_n,\overline{y}_n)$ is a comparison triple for (p_{n-1},p_n,y_n). This determines $\overline{H}_n$. Put $\bar{\omega}_n := \angle_{\overline{p}_{n-1}}(\overline{p}_n,\overline{y}_n) = \angle_{\overline{p}_{n-1}}(\overline{x}_n,\overline{y}_n)$. See Figure 3. Since $\mathrm{per}(p_{n-1},p_n,y_n) < \frac{4}{5}\,\mathrm{per}(H_0)$ and $y_n \in \{x_0,y_0\}$, the inequalities $\gamma_{n-1} \geq \bar{\omega}_n$

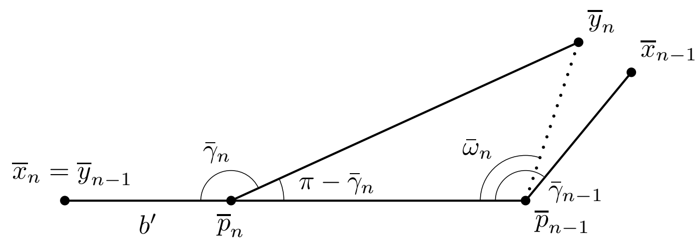

FIGURE 3. Constructing $\overline{H}_n$ from $\overline{H}_{n-1}$

and $\pi - \gamma_n \geq \pi - \bar{\gamma}_n$ hold by assumption. Hence, $\bar{\gamma}_{n-1} \geq \gamma_{n-1} \geq \bar{\omega}_n$ and so $|\overline{x}_{n-1}\overline{y}_{n-1}| \geq |\overline{x}_n\overline{y}_n|$ by Lemma 1.1.

Now we can easily conclude the proof. For $n \to \infty$, we have

$$|\overline{p}_{n-1}\overline{p}_n| + |\overline{p}_{n-1}\overline{y}_n| - |\overline{p}_n\overline{y}_n| = l_{n-1} - l_n \to 0$$

by (5), consequently $\bar{\omega}_n \to \pi$ and $\bar{\gamma}_n \to \pi$ (note that $|\overline{p}_{n-1}\overline{p}_n| \geq a - b - 2b' > 0$ and $|\overline{p}_{n-1}\overline{y}_n| = b' > 0$ for $n \geq 2$). This implies in turn that

$$l_n - |\overline{x}_n\overline{y}_n| = |\overline{p}_n\overline{x}_n| + |\overline{p}_n\overline{y}_n| - |\overline{x}_n\overline{y}_n| \to 0$$

as $n \to \infty$ (recall that $l_n \leq a + b < D_\kappa$). In view of (6) and (5), this gives $|\overline{x}_0\overline{y}_0| \geq |x_0y_0|$, so H_0 satisfies (H_κ) and hence also (A_κ). □

Theorem 2.3. *Let $\kappa \in \mathbb{R}$, and let M be a complete metric space of curvature $\geq \kappa$ in the sense of Alexandrov. Suppose that every pair of points in M at distance $< D_\kappa$ is connected by a segment. Then every hinge $H_p(x,y)$ in M with* $\mathrm{per}(p,x,y) < 2D_\kappa$ *satisfies* (A_κ), (H_κ), *and* (D_κ).

Proof. Recall that by Lemma 1.3 all segments in M are balanced; furthermore, it suffices to prove that every hinge in M with perimeter less than $2D_\kappa$ satisfies (A_κ). Suppose to the contrary that there exists a hinge H in M with $\mathrm{per}(H) < 2D_\kappa$ that does not satisfy (A_κ). Then, by Proposition 2.2, there exists a hinge H_1 with $\mathrm{per}(H_1) < \frac{4}{5}\,\mathrm{per}(H)$ and an endpoint on the union of the sides of H such that H_1 does not satisfy (A_κ) either. Inductively, for $n = 2, 3, \ldots$, there exist hinges H_n such that $\mathrm{per}(H_n) < \frac{4}{5}\,\mathrm{per}(H_{n-1}) < \left(\frac{4}{5}\right)^n \mathrm{per}(H)$, some endpoint of H_n lies on the union of the sides of H_{n-1}, and H_n does not satisfy (A_κ). Let p_n denote the vertex of H_n. Clearly the sequence (p_n) is Cauchy and thus converges to a point $q \in M$. However, since M has curvature $\geq \kappa$, all hinges with vertex and endpoints in an appropriate neighborhood of q satisfy (A_κ). This gives a contradiction, as $p_n \to q$ and $\mathrm{per}(H_n) \to 0$. □

APPENDIX: TRIGONOMETRY OF MODEL SPACES

In this appendix, we collect some trigonometric formulae for the model spaces $\mathbb{M}^2_\kappa$, stated in a unified way for all $\kappa \in \mathbb{R}$ in terms of the generalized sine and cosine functions.

For $\kappa \in \mathbb{R}$ we denote by $\mathrm{sn}_\kappa \colon \mathbb{R} \to \mathbb{R}$ and $\mathrm{cs}_\kappa \colon \mathbb{R} \to \mathbb{R}$ the solutions of the second order differential equation $f'' + \kappa f = 0$ satisfying the initial conditions

$$\mathrm{sn}_\kappa(0) = 0, \quad \mathrm{sn}'_\kappa(0) = 1, \qquad \mathrm{cs}_\kappa(0) = 1, \quad \mathrm{cs}'_\kappa(0) = 0.$$

Explicitly,

$$\operatorname{sn}_\kappa(x) = \sum_{n=0}^{\infty} \frac{(-\kappa)^n}{(2n+1)!} x^{2n+1} = \begin{cases} \sin(\sqrt{\kappa}x)/\sqrt{\kappa} & \text{if } \kappa > 0, \\ x & \text{if } \kappa = 0, \\ \sinh(\sqrt{-\kappa}x)/\sqrt{-\kappa} & \text{if } \kappa < 0, \end{cases}$$

$$\operatorname{cs}_\kappa(x) = \sum_{n=0}^{\infty} \frac{(-\kappa)^n}{(2n)!} x^{2n} = \begin{cases} \cos(\sqrt{\kappa}x) & \text{if } \kappa > 0, \\ 1 & \text{if } \kappa = 0, \\ \cosh(\sqrt{-\kappa}x) & \text{if } \kappa < 0. \end{cases}$$

Note that

$$\operatorname{sn}_\kappa' = \operatorname{cs}_\kappa, \quad \operatorname{cs}_\kappa' = -\kappa \operatorname{sn}_\kappa,$$

and

$$\operatorname{cs}_\kappa^2 + \kappa \operatorname{sn}_\kappa^2 = 1. \tag{7}$$

The following functional equations hold. For $x, y \in \mathbb{R}$,

$$\operatorname{sn}_\kappa(x+y) = \operatorname{sn}_\kappa(x)\operatorname{cs}_\kappa(y) + \operatorname{cs}_\kappa(x)\operatorname{sn}_\kappa(y), \tag{8}$$

$$\operatorname{cs}_\kappa(x+y) = \operatorname{cs}_\kappa(x)\operatorname{cs}_\kappa(y) - \kappa\operatorname{sn}_\kappa(x)\operatorname{sn}_\kappa(y); \tag{9}$$

in particular,

$$\operatorname{sn}_\kappa(2x) = 2\operatorname{sn}_\kappa(x)\operatorname{cs}_\kappa(x), \tag{10}$$

$$\begin{aligned} \operatorname{cs}_\kappa(2x) &= \operatorname{cs}_\kappa^2(x) - \kappa\operatorname{sn}_\kappa^2(x) \\ &= 2\operatorname{cs}_\kappa^2(x) - 1 \\ &= 1 - 2\kappa\operatorname{sn}_\kappa^2(x). \end{aligned} \tag{11}$$

Replacing x by $x/2$ in the last three lines one gets

$$\kappa\operatorname{sn}_\kappa^2\left(\frac{x}{2}\right) = \frac{1-\operatorname{cs}_\kappa(x)}{2}, \tag{12}$$

$$\operatorname{cs}_\kappa^2\left(\frac{x}{2}\right) = \frac{1+\operatorname{cs}_\kappa(x)}{2}. \tag{13}$$

Karcher [7] defined a "modified distance function" $\operatorname{md}_\kappa \colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\operatorname{md}_\kappa(x) := \int_0^x \operatorname{sn}_\kappa(t)\,dt = \begin{cases} (1-\operatorname{cs}_\kappa(x))/\kappa & \text{if } \kappa \neq 0, \\ x^2/2 & \text{if } \kappa = 0. \end{cases}$$

In view of (12), this can be written as

$$\operatorname{md}_\kappa(x) = 2\operatorname{sn}_\kappa^2\left(\frac{x}{2}\right).$$

It is easy to check that

$$\operatorname{cs}_\kappa + \kappa\operatorname{md}_\kappa = 1, \tag{14}$$

$$\begin{aligned} \operatorname{md}_\kappa(x+y) &= \operatorname{md}_\kappa(x-y) + 2\operatorname{sn}_\kappa(x)\operatorname{sn}_\kappa(y) \\ &= \operatorname{md}_\kappa(x) + \operatorname{cs}_\kappa(x)\operatorname{md}_\kappa(y) + \operatorname{sn}_\kappa(x)\operatorname{sn}_\kappa(y) \\ &= \operatorname{md}_\kappa(x)\operatorname{cs}_\kappa(y) + \operatorname{md}_\kappa(y) + \operatorname{sn}_\kappa(x)\operatorname{sn}_\kappa(y), \end{aligned} \tag{15}$$

$$\begin{aligned} \operatorname{md}_\kappa(2x) &= 2\operatorname{sn}_\kappa^2(x) \\ &= 2(1+\operatorname{cs}_\kappa(x))\operatorname{md}_\kappa(x). \end{aligned} \tag{16}$$

We turn to trigonometry. Consider a triangle in $\mathbb{M}^2_\kappa$ with vertices x, y, z and (possibly degenerate) sides of length $a, b, c \geq 0$, where $a = |yz|$, $b = |zx|$, and $c = |xy|$, and let $\alpha, \beta, \gamma \in [0, \pi]$ denote the angles at x, y, z, respectively, whenever they are defined. The law of cosines can be stated in a unified way as

$$\begin{aligned} \mathrm{md}_\kappa(c) &= \mathrm{md}_\kappa(a+b) - \mathrm{sn}_\kappa(a)\,\mathrm{sn}_\kappa(b)(1+\cos(\gamma)) \\ &= \mathrm{md}_\kappa(a-b) + \mathrm{sn}_\kappa(a)\,\mathrm{sn}_\kappa(b)(1-\cos(\gamma)) \\ &= \mathrm{md}_\kappa(a) + \mathrm{cs}_\kappa(a)\,\mathrm{md}_\kappa(b) - \mathrm{sn}_\kappa(a)\,\mathrm{sn}_\kappa(b)\cos(\gamma) \\ &= \mathrm{md}_\kappa(a)\,\mathrm{cs}_\kappa(b) + \mathrm{md}_\kappa(b) - \mathrm{sn}_\kappa(a)\,\mathrm{sn}_\kappa(b)\cos(\gamma) \end{aligned} \tag{17}$$

(compare (15)), or, in terms of sn_κ, as

$$\begin{aligned} \mathrm{sn}^2_\kappa\Big(\frac{c}{2}\Big) &= \mathrm{sn}^2_\kappa\Big(\frac{a+b}{2}\Big) - \mathrm{sn}_\kappa(a)\,\mathrm{sn}_\kappa(b)\cos^2\Big(\frac{\gamma}{2}\Big) \\ &= \mathrm{sn}^2_\kappa\Big(\frac{a-b}{2}\Big) + \mathrm{sn}_\kappa(a)\,\mathrm{sn}_\kappa(b)\sin^2\Big(\frac{\gamma}{2}\Big). \end{aligned} \tag{18}$$

Multiplying any of these equations by κ one obtains the more familiar formula

$$\mathrm{cs}_\kappa(c) = \mathrm{cs}_\kappa(a)\,\mathrm{cs}_\kappa(b) + \kappa\,\mathrm{sn}_\kappa(a)\,\mathrm{sn}_\kappa(b)\cos(\gamma) \tag{19}$$

for the hyperbolic and spherical geometries. The "dual law of cosines" or "law of cosines for angles" is the identity

$$\cos(\gamma) = \sin(\alpha)\sin(\beta)\,\mathrm{cs}_\kappa(c) - \cos(\alpha)\cos(\beta); \tag{20}$$

in the Euclidean case it represents the fact that $\alpha + \beta + \gamma = \pi$. The law of sines is given by

$$\mathrm{sn}_\kappa(a)\sin(\beta) = \mathrm{sn}_\kappa(b)\sin(\alpha). \tag{21}$$

Let l denote the distance from the midpoint of the side xy of the triangle to the vertex z. Then

$$2\,\mathrm{cs}_\kappa\Big(\frac{c}{2}\Big)\,\mathrm{md}_\kappa\, l = \mathrm{md}_\kappa(a) + \mathrm{md}_\kappa(b) - 2\,\mathrm{md}_\kappa\Big(\frac{c}{2}\Big); \tag{22}$$

equivalently,

$$2\,\mathrm{cs}_\kappa\Big(\frac{c}{2}\Big)\,\mathrm{sn}^2_\kappa\Big(\frac{l}{2}\Big) = \mathrm{sn}^2_\kappa\Big(\frac{a}{2}\Big) + \mathrm{sn}^2_\kappa\Big(\frac{b}{2}\Big) - 2\,\mathrm{sn}^2_\kappa\Big(\frac{c}{4}\Big). \tag{23}$$

(This equation may be used to define spaces of curvature $\geq \kappa$ or $\leq \kappa$.) Multiplying by κ one obtains the simple formula

$$2\,\mathrm{cs}_\kappa\Big(\frac{c}{2}\Big)\,\mathrm{cs}_\kappa(l) = \mathrm{cs}_\kappa(a) + \mathrm{cs}_\kappa(b) \tag{24}$$

for the hyperbolic and spherical geometries.

Proof of (22). (We omit all subscripts κ.) By (17),

$$\begin{aligned} \mathrm{md}(l) &= \mathrm{md}(b)\,\mathrm{cs}\Big(\frac{c}{2}\Big) + \mathrm{md}\Big(\frac{c}{2}\Big) - \mathrm{sn}(b)\,\mathrm{sn}\Big(\frac{c}{2}\Big)\cos(\alpha), \\ \mathrm{md}(a) &= \mathrm{md}(b)\,\mathrm{cs}(c) + \mathrm{md}(c) - \mathrm{sn}(b)\,\mathrm{sn}(c)\cos(\alpha). \end{aligned}$$

Using (13) and (10) we get

$$2\,\mathrm{cs}\Big(\frac{c}{2}\Big)\,\mathrm{md}(l) - \mathrm{md}(a) = \mathrm{md}(b) + 2\,\mathrm{cs}\Big(\frac{c}{2}\Big)\,\mathrm{md}\Big(\frac{c}{2}\Big) - \mathrm{md}(c).$$

Now the formula follows from (16). □

References

[1] Alexander, S., V. Kapovitch, and A. Petrunin. *Alexandrov Geometry.* In preparation.

[2] Alexandrov, A. D. A theorem on triangles in a metric space and some of its applications. (Russian) *Trudy Mat. Inst. Steklov 38* (1951), 5–23.

[3] ——— *Die innere Geometrie der konvexen Flächen.* Akademie-Verlag 1955.

[4] Bridson, M. R. and A. Haefliger. *Metric Spaces of Non-Positive Curvature.* Springer 1999.

[5] Burago, D., Y. Burago, and S. Ivanov. *A Course in Metric Geometry.* Amer. Math. Soc. 2001.

[6] Burago, Y. D., M. Gromov, and G. Perelman. A. D. Alexandrov spaces with curvature bounded below. *Russian Math. Surveys 47* (1992), 1–58.

[7] Karcher, H. Riemannian comparison constructions. In *Global differential geometry* (S. S. Chern, ed.), Math. Assoc. America 1989, pp. 170–222.

[8] Plaut, C. Almost Riemannian spaces. *J. Differential Geom. 34* (1991), 515–537.

[9] ——— Spaces of Wald–Berestovskii curvature bounded below. *J. Geom. Anal. 6* (1996), 113–134.

[10] Toponogov, V. A. On convexity of Riemannian spaces of positive curvature. (Russian) *Dokl. Akad. Nauk SSSR 115* (1957), 674–676.

[11] ——— Riemannian spaces having their curvature bounded below by a positive number. (Russian) *Dokl. Akad. Nauk SSSR 120* (1958), 719–721.

[12] ——— Riemann spaces with curvature bounded below. (Russian) *Uspehi Mat. Nauk 14* (1959), 87–130.

Departement Mathematik, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland
E-mail address: lang@math.ethz.ch

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, 8057 Zürich, Switzerland
E-mail address: viktor.schroeder@math.uzh.ch