# Lecture 14
# The Fundamental Theorem of Surface Theory

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**Review of Notation.**

- In what follows, $(t_1, t_2) \mapsto \mathcal{F}(t_1, t_2)$ is a parametric surface in $\mathbf{R}^3$, $\mathcal{F} : \mathcal{O} \to \mathbf{R}^3$.
- Partial derivatives with respect to $t_1$ and $t_2$ are denoted by subscripts: $\mathcal{F}_{t_i} := \frac{\partial \mathcal{F}}{\partial t_i}$, $\mathcal{F}_{t_i t_j} := \frac{\partial^2 \mathcal{F}}{\partial t_i t_j}$, etc.
- The standard basis for $T_p\mathcal{F}$—the tangent space to $\mathcal{F}$ at a point $p \in \mathcal{O}$—is $\mathcal{F}_{t_1}, \mathcal{F}_{t_2}$.
- The unit normal to $\mathcal{F}$ at $p$ is $\nu(p) = \frac{\mathcal{F}_{t_1}(p) \times \mathcal{F}_{t_2}(p)}{\left\| \mathcal{F}_{t_1}(p) \times \mathcal{F}_{t_2}(p)) \right\|}$.
- The matrix $g$ of the First Fundamental Form with respect to the standard basis is the $2 \times 2$ matrix $g_{ij} = \mathcal{F}_{t_i}(p) \cdot \mathcal{F}_{t_j}(p)$.
- The Shape operator at $p$ is the self-adjoint operator $-D\nu_p : T_p\mathcal{F} \to T_p\mathcal{F}$.
- The Shape operator defines the Second Fundamental Form which has the $2 \times 2$ matrix of coefficients $\ell$ given by $\ell_{ij} = -\nu_{t_i} \cdot \mathcal{F}_{t_j} = \nu \cdot \mathcal{F}_{t_i t_j}$.
- The matrix of the Shape operator in the standard basis is $g^{-1}\ell$.

## 14.1 The Frame Equations.

At each point $p \in \mathcal{O}$ we define the *standard frame* at $p$, $\mathfrak{f}(p) = (\mathfrak{f}_1(p), \mathfrak{f}_2(p), \mathfrak{f}_3(p))$ to be the basis of $\mathbf{R}^3$ given by $\mathfrak{f}_1(p) := \mathcal{F}_{t_1}(p), \mathfrak{f}_2(p) := \mathcal{F}_{t_2}(p), \mathfrak{f}_3(p) := \nu(p)$. Note that $\mathfrak{f}_1(p), \mathfrak{f}_2(p)$ is just the standard basis for $T_p\mathcal{F}$. We will regard $\mathfrak{f}$ as a map from $\mathcal{O}$ into $3 \times 3$ matrices, with the rows being the three basis elements. Since $\mathfrak{f}(p)$ is a basis for $\mathbf{R}^3$, any $v \in \mathbf{R}^3$ can be written uniquely as a linear combination of the $\mathfrak{f}_i(p)$: $v = \sum_i c_i \mathfrak{f}_i(p)$. In particular, we can take in turn for $v$ each of $(\mathfrak{f}_j)_{t_k}(p)$ and this defines uniquely a $3 \times 3$ matrix $P^k_{ji}(p)$ such that $(\mathfrak{f}_j)_{t_k}(p) = \sum_j P^k_{ji}(p)\mathfrak{f}_i(p)$. We can write these equations as a pair of equations between five matrix-valued functions $\mathfrak{f}$, $\mathfrak{f}_{t_1}$, $\mathfrak{f}_{t_2}$, $P^1$, and $P^2$, defined on $\mathcal{O}$, namely:

$$\mathfrak{f}_{t_1} = \mathfrak{f}\, P^1$$
$$\mathfrak{f}_{t_2} = \mathfrak{f}\, P^2$$

and we call these equations the *frame equations* for the surface $\mathcal{F}$.

What makes these equations so interesting and important is that, as we will see below, the matrix-valued functions $P^1$ and $P^2$ can be calculated explicitly from formulas that display them as fixed expressions in the coeffients $g_{ij}$ and $\ell_{ij}$ of the First and Second Fundamental Forms and their partial derivatives. Thus, **if the First and Second Fundamental Forms are known, we can consider the Frame Equations as a coupled pair of first order PDE for the frame field** $\mathfrak{f}$, and it follows from the Frobenius Theorem that we can solve these equations and find the frame field, and then with another integration we can recover the surface $\mathcal{F}$. Thus the frame equations are analogous to the Frenet equations of curve theory, and lead in the same way to a Fundamental Theorem. Now the fun begins!

**14.1.1 Lemma.** *Let $V$ be an inner-product space, $\mathfrak{f} = (\mathfrak{f}_1, \ldots, \mathfrak{f}_n)$ a basis for $V$, and $G$ the matrix of inner-products $G_{ij} = \langle \mathfrak{f}_i, \mathfrak{f}_j \rangle$. Given $x = \sum_{i=1}^n x_i \mathfrak{f}_i$ in $V$, let $\xi_i := \langle x, \mathfrak{f}_i \rangle$. Then $\xi_i = \sum_{j=1}^n G_{ji} x_j = \sum_{j=1}^n G^t_{ij} x_j$, or in matrix notation, $(\xi_1, \ldots, \xi_n) = (x_1, \ldots, x_n) G^t$, so $(x_1, \ldots, x_n)^t = G^{-1}(\xi_1, \ldots, \xi_n)^t$.*

PROOF. $\xi_i = \left\langle \sum_{j=1}^n x_j \mathfrak{f}_j, \mathfrak{f}_i \right\rangle = \sum_{j=1}^n x_j \langle \mathfrak{f}_j, \mathfrak{f}_i \rangle$. ∎

**14.1.2 Remark.** Here is another way of phrasing this result. The basis $\mathfrak{f}_1, \ldots, \mathfrak{f}_n$ for $V$ determines two bases for the dual space $V^*$, the dual basis $\ell_i$ defined by $\ell_i(\mathfrak{f}_j) = \delta_{ij}$ and the basis $\mathfrak{f}_i^*$, defined by $\mathfrak{f}_i^*(v) := \langle v, \mathfrak{f}_i \rangle$, and these two bases are related by $\mathfrak{f}_i^* = \sum_{j=1}^n G_{ij} \ell_j$.

**14.1.3 Theorem.**

$$P^1 = G^{-1}A^1 = \begin{pmatrix} g^{11} & g^{12} & 0 \\ g^{12} & g^{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}(g_{11})_{t_1} & \frac{1}{2}(g_{11})_{t_2} & -\ell_{11} \\ (g_{12})_{t_1} - \frac{1}{2}(g_{11})_{t_2} & \frac{1}{2}(g_{22})_{t_1} & -\ell_{12} \\ \ell_{11} & \ell_{12} & 0 \end{pmatrix}$$

$$P^2 = G^{-1}A^2 = \begin{pmatrix} g^{11} & g^{12} & 0 \\ g^{12} & g^{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}(g_{11})_{t_2} & (g_{12})_{t_2} - \frac{1}{2}(g_{22})_{t_1} & -\ell_{12} \\ \frac{1}{2}(g_{22})_{t_1} & \frac{1}{2}(g_{22})_{t_2} & -\ell_{22} \\ \ell_{12} & \ell_{22} & 0 \end{pmatrix}$$

PROOF. In the lemma (with $n = 3$) take $\mathfrak{f} = \mathfrak{f}(p)$, the standard frame for $\mathcal{F}$ at $p$, so that

$$G = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and hence

$$G^{-1} = \begin{pmatrix} g^{11} & g^{12} & 0 \\ g^{12} & g^{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(recall that the $g^{-1} = g^{ij}$ is the matrix inverse to $g = g_{ij}$), and take $x = (\mathfrak{f}_j)_{t_k}$, so by the conclusion of the Lemma, $A^k_{ij} = \xi_i = (\mathfrak{f}_j)_{t_k} \cdot \mathfrak{f}_i$. Thus, for example, for $i = 1, 2$:

$$A^k_{ii} = (\mathfrak{f}_i)_{t_k} \cdot \mathfrak{f}_i = \mathcal{F}_{t_i t_k} \cdot \mathcal{F}_{t_i} = \frac{1}{2}(\mathcal{F}_{t_i} \cdot \mathcal{F}_{t_i})_{t_k} = \frac{1}{2}(g_{ii})_{t_k}.$$

Next note that $(g_{12})_{t_1} = (\mathcal{F}_{t_1} \cdot \mathcal{F}_{t_2})_{t_1} = \mathcal{F}_{t_1 t_1} \cdot \mathcal{F}_{t_2} + \mathcal{F}_{t_1} \cdot \mathcal{F}_{t_1 t_2} = \mathcal{F}_{t_1 t_1} \cdot \mathcal{F}_{t_2} + \frac{1}{2}(g_{11})_{t_2}$, so:

$$A^1_{21} = (\mathfrak{f}_1)_{t_1} \cdot \mathfrak{f}_2 = \mathcal{F}_{t_1 t_1} \cdot \mathcal{F}_{t_2} = (g_{12})_{t_1} - \frac{1}{2}(g_{11})_{t_2}.$$

and interchanging the roles of $t_1$ and $t_2$ gives

$$A^2_{12} = (\mathfrak{f}_2)_{t_2} \cdot \mathfrak{f}_1 = \mathcal{F}_{t_2 t_2} \cdot \mathcal{F}_{t_1} = (g_{12})_{t_2} - \frac{1}{2}(g_{22})_{t_1}. \text{ Also}$$

$$A^1_{12} = (\mathfrak{f}_2)_{t_1} \cdot \mathfrak{f}_1 = \mathcal{F}_{t_1 t_2} \cdot \mathcal{F}_{t_1} = \frac{1}{2}(\mathcal{F}_{t_1} \cdot \mathcal{F}_{t_1})_{t_2} = \frac{1}{2}(g_{11})_{t_2} \text{ and}$$

$$A^2_{21} = (\mathfrak{f}_1)_{t_2} \cdot \mathfrak{f}_2 = \mathcal{F}_{t_1 t_2} \cdot \mathcal{F}_{t_2} = \frac{1}{2}(\mathcal{F}_{t_1} \cdot \mathcal{F}_{t_1})_{t_2} = \frac{1}{2}(g_{11})_{t_2}.$$

For $i = 1, 2$,

$$A^k_{i3} = (\mathfrak{f}_3)_{t_k} \cdot \mathfrak{f}_i = \nu_{t_k} \cdot \mathcal{F}_{t_i} = D\nu(\mathcal{F}_{t_k}) \cdot \mathcal{F}_{t_i} = -\ell_{ki},$$

and since $\mathfrak{f}_3$ is orthogonal to $\mathfrak{f}_i$, $0 = (\mathfrak{f}_3 \cdot \mathfrak{f}_i)_{t_k} = (\mathfrak{f}_3)_{t_k} \cdot \mathfrak{f}_i + \mathfrak{f}_3 \cdot (\mathfrak{f}_i)_{t_k}$, hence

$$A^k_{3i} = (\mathfrak{f}_i)_{t_k} \cdot \mathfrak{f}_3 = -(\mathfrak{f}_3)_{t_k} \cdot \mathfrak{f}_i = -A^k_{i3}.$$

Finally, since $\mathfrak{f}_3 \cdot \mathfrak{f}_3 = \|\nu\|^2 = 1$, $(\mathfrak{f}_3 \cdot \mathfrak{f}_3)_{t_k} = 0$, so

$$A^k_{33} = (\mathfrak{f}_3)_{t_k} \cdot \mathfrak{f}_3 = \frac{1}{2}(\mathfrak{f}_3 \cdot \mathfrak{f}_3)_{t_k} = 0. \quad \blacksquare$$

Henceforth we regard the $3 \times 3$ matrix-valued functions $G$, $G^{-1}$, $A^k$ and $P^k$ in $\mathcal{O}$ as being **defined** by the formulas in the statement of the above theorem.

**14.1.4 Corollary (Gauss-Codazzi Equations).** *If $g_{ij}$ and $\ell_{ij}$ are the coefficients of the First and Second Fundamental Forms of a surface $\mathcal{F} : \mathcal{O} \to \mathbf{R}^3$, then the matrix-valued functions $P^1$ and $P^2$ defined in $\mathcal{O}$ by the above Theorem satisfy the matrix identity*

$$P^1_{t_2} - P^2_{t_1} = P^1 P^2 - P^2 P^1$$

*called the Gauss-Codazzi Equations.*

PROOF. Differentiate the first of the Frame Equations with respect to $t_2$ and the second with respect to $t_1$ and set $\mathfrak{f}_{t_1 t_2} = \mathfrak{f}_{t_2 t_1}$. This gives $\mathfrak{f}_{t_2} P^1 + \mathfrak{f} P^1_{t_2} = \mathfrak{f}_{t_1} P^2 + \mathfrak{f} P^2_{t_1}$. Substituting for $\mathfrak{f}_{t_1}$ and $\mathfrak{f}_{t_2}$ their values from the Frame Equations, gives

$$\mathfrak{f}\,(P^1_{t_2} - P^2_{t_1} - (P^1 P^2 - P^2 P^1)) = 0,$$

and since $\mathfrak{f}$ is a non-singular matrix, the corollary follows. $\blacksquare$

## 14.2 Gauss's "Theorema Egregium" (Remarkable Theorem).

Karl Friedrich Gauss was one of the great mathematicians of all time, and it was he who develped the deeper aspects of surface theory in the first half of the nineteenth century. There is one theorem that he proved that is highly surprising, namely that $K$, the Gauss Curvature of a surface (the determinant of the shape operator), is an **intrinsic** quantity that is it can be computed from a knowledge of only the First Fundamental Form, and so it can be found by doing measurements within the surface without reference to how the surface is embedded in space. Gauss thought so highly of this result that in his notes he called it the "Theorema Egregium", or the remarkable (or outstanding) theorem. Notice that by Theorem 14.1.3, the matrix entries of $P^k_{ij}$ with $1 \leq i, j \leq 2$ depend only on the $g_{ij}$ and their partial derivatives, so to prove the intrinsic nature of $K$ it will suffice to get a formula for it involving only these quantities and the $g^{ij}$.

**14.2.1 Theorema Egregium.**

$$K = -\frac{(P^1_{12})_{t_2} - (P^2_{12})_{t_1} - \sum_{j=1}^{2}(P^1_{1j}P^2_{j2} - P^2_{1j}P^1_{j2})}{g^{11}(g_{11}g_{22} - g_{12}^2)}.$$

PROOF. In the matrix Gauss-Codazzi Equation, consider the equation for the first row and second column,

$$(P^1_{12})_{t_2} - (P^2_{12})_{t_1} = \sum_{j=1}^{3}(P^1_{1j}P^2_{j2} - P^2_{1j}P^1_{j2}).$$

If we move all terms involving $P^k_{i,j}$ with $i, j < 3$ to the left hand side of the equation, the result is:

$$(P^1_{12})_{t_2} - (P^2_{12})_{t_1} - \sum_{j=1}^{2}(P^1_{1j}P^2_{j2} - P^2_{1j}P^1_{j2}) = P^1_{13}P^2_{32} - P^2_{13}P^1_{32}.$$

Now use Theorem 14.1.3 to find the matrix elements on the right hand side of this equation: $P^1_{13} = -(g^{11}\ell_{11} + g^{12}\ell_{12})$, $P^2_{13} = -(g^{11}\ell_{12} + g^{12}\ell_{22})$, $P^1_{32} = \ell_{12}$, $P^2_{32} = \ell_{22}$. Thus:

$$P^1_{13}P^2_{32} - P^2_{13}P^1_{32} = -(g^{11}\ell_{11} + g^{12}\ell_{12})\ell_{22} + (g^{11}\ell_{12} + g^{12}\ell_{22})\ell_{12} = -g^{11}(\ell_{11}\ell_{22} - \ell_{12}^2).$$

Since by an earlier remark (13.3.8),

$$K = \frac{\det(\ell)}{\det(g)} = \frac{\ell_{11}\ell_{22} - \ell_{12}^2}{g_{11}g_{22} - g_{12}^2},$$

$P^1_{13}P^2_{32} - P^2_{13}P^1_{32} = -g^{11}(\ell_{11}\ell_{22} - \ell_{12}^2) = -g^{11}(g_{11}g_{22} - g_{12}^2)K$
and the claimed formula for $K$ follows. ∎

**14.2.2 The Fundamental Theorem of Surfaces.** *Congruent parametric surfaces in $\mathbf{R}^3$ have the same First and Second Fundamental Forms and conversely two parametric surfaces in $\mathbf{R}^3$ with the same First and Second Fundamental Forms are congruent. Moreover, if $\mathcal{O}$ is a domain in $\mathbf{R}^2$ and $I = \sum_{ij=1}^2 g_{ij}\,dt_i\,dt_j$, $II = \sum_{ij=1}^2 \ell_{ij}\,dt_i\,dt_j$ are $C^2$ quadratic forms in $\mathcal{O}$ with $I$ positive definite, then there exists a parametric surface $\mathcal{F} : \mathcal{O} \to \mathbf{R}^3$ with $I^{\mathcal{F}} = I$ and $II^{\mathcal{F}} = II$ provided the Gauss-Codazzi equations are satisfied.*

PROOF. We have already discussed the first statement. If $\mathcal{F}^i : \mathcal{O} \to \mathbf{R}^3$, $i = 1, 2$ have the same First and Second Fundamental forms, then after translations we can assume that they both map some point $p \in \mathcal{O}$ to the origin. Then, since $\mathcal{F}^1_{t_i}(p) \cdot \mathcal{F}^1_{t_j}(p) = g_{ij}(p) = \mathcal{F}^2_{t_i}(p) \cdot \mathcal{F}^2_{t_j}(p)$, it follows that after transforming one of the surfaces by an orthogonal transformation we can assume that the standard frame $\mathfrak{f}^1$ for $\mathcal{F}^1$ and the standard frame $\mathfrak{f}^2$ for $\mathcal{F}^2$ agree at $p$. But then, since $\mathcal{F}^1$ and $\mathcal{F}^2$ have identical frame equations, by the uniqueness part of the Frobenius Theorem it follows that $\mathfrak{f}^1$ and $\mathfrak{f}^2$ agree in all of $\mathcal{O}$, so in particular $\mathcal{F}^1_{t_i} = \mathcal{F}^2_{t_i}$. But then $\mathcal{F}^1$ and $\mathcal{F}^2$ differ by a constant, and since they agree at $p$, they are identical, proving the second statement.

To prove the third and final statement of the theorem, we first note that since $g_{ij}$ is positive definite, it is in particular invertible, so the matrix $G^{-1}$ and the matrices $P^k$ of Theorem 14.1.3, are well-defined. Moreover, since the Gauss-Codazzi equations are just the compatibility conditions of Frobenius Theorem, it follows that we can solve the "frame equations" $\mathfrak{f}_{t_i} = \mathfrak{f}\, P^k$, $k = 1, 2$ uniquely given an arbitrary initial value for $\mathfrak{f}$ at some point $p \in \mathcal{O}$, and for this initial frame we choose a basis $\mathfrak{f}(p)$ for $\mathbf{R}^3$ such that $\mathfrak{f}_i(p) \cdot \mathfrak{f}_j(p) = G_{ij}(p)$ (which is possible since $g_{ij}$ and hence $G_{ij}$ is positive definite).

Having solved for the frame field $\mathfrak{f}$, we now need to solve the system $\mathcal{F}_{t_i} = \mathfrak{f}_i$, $i = 1, 2$ to get the surface $\mathcal{F} : \mathcal{O} \to \mathbf{R}^3$. This is another Frobenius problem, and now the compatibility condition is $(\mathfrak{f}_1)_{t_2} = (\mathfrak{f}_2)_{t_1}$ or by the frame equation, $\sum_{j=1}^3 P^2_{j1}\mathfrak{f}_j = \sum_{j=1}^3 P^1_{j2}\mathfrak{f}_j$. But by inspection, the second column of $P^1$ is indeed equal to the first column of $P^2$, so the compatibility equations are satisfied and we can find a unique $\mathcal{F}$ with $\mathcal{F}(p) = 0$ and $\mathcal{F}_{t_i} = \mathfrak{f}_i$, for $i = 1, 2$.

It remains to show that $\mathcal{F}$ is a surface in $\mathbf{R}^3$ with $\sum_{ij} g_{ij} dt_i\, dx_j$ and $\sum_{ij} \ell_{ij} dt_i\, dt_j$ as its first and second fundamental forms, i.e.,

- $\mathcal{F}_{t_1}$ and $\mathcal{F}_{t_2}$ are independent,
- $\mathfrak{f}_3$ is orthogonal to $\mathcal{F}_{t_1}$ and $\mathcal{F}_{t_2}$,
- $\|\mathfrak{f}_3\| = 1$,
- $\mathcal{F}_{t_i} \cdot \mathcal{F}_{t_j} = g_{ij}$, and
- $(\mathfrak{f}_3)_{x_i} \cdot f_{x_j} = -\ell_{ij}$.

The first step is to prove that the $3 \times 3$ matrix function $\Phi = (\mathfrak{f}_i \cdot \mathfrak{f}_j)$ is equal to $G$. We compute the partial derivatives of $\Phi$. Since $\mathfrak{f}$ satisfy the frame equations,

$$\begin{aligned}
(\mathfrak{f}_i \cdot \mathfrak{f}_j)_{t_1} &= (\mathfrak{f}_i)_{t_1} \cdot \mathfrak{f}_j + \mathfrak{f}_i \cdot (\mathfrak{f}_j)_{t_1} \\
&= \sum_k P^1_{ki}\mathfrak{f}_k \cdot \mathfrak{f}_j + P^1_{kj}\mathfrak{f}_k \cdot \mathfrak{f}_i = \sum_k P^1_{ki} g_{jk} + g_{ik} P^1_{kj} \\
&= (GP^1)_{ji} + (GP^1)_{ij} = (GP^1 + (GP^1)^t)_{ij}.
\end{aligned}$$

But $GP^1 = G(G^{-1}A^1) = A^1$, so $\Phi_{t_1} = G_{t_1}$ and a similar computation gives $\Phi_{t_2} = G_{t_2}$. Thus $\Phi$ and $G$ differ by a constant, and since they agree at $p$ they are identical. Thus $\mathfrak{f}_i \cdot \mathfrak{f}_j = G_{ij}$, which proves all of the above list of bulleted items but the last.

To compute the Second Fundamental Form of $\mathcal{F}$, we again use the frame equations:

$$\begin{aligned}
-(\mathfrak{f}_3)_{t_1} \cdot \mathfrak{f}_j &= (g^{11}\ell_{11} + g^{12}\ell_{12})\mathfrak{f}_1 \cdot \mathfrak{f}_j + (g^{12}\ell_{11} + g^{22}\ell_{12})\mathfrak{f}_2 \cdot \mathfrak{f}_j \\
&= (g^{11}\ell_{11} + g^{12}\ell_{12}) g_{1j} + (g^{12}\ell_{11} + g^{22}\ell_{12}) g_{2j} \\
&= \ell_{11}(g^{11}g_{1j} + g^{12}g_{2j}) + \ell_{12}(g^{21}g_{1j} + g^{22}g_{2j}) \\
&= \ell_{11}\delta_{1j} + \ell_{12}\delta_{2j}.
\end{aligned}$$

So $-(\mathfrak{f}_3)_{t_1} \cdot \mathfrak{f}_1 = \ell_{11}$, $\quad -(\mathfrak{f}_3)_{t_1} \cdot \mathfrak{f}_2 = \ell_{12}$, and similar computations show that $-(\mathfrak{f}_2)_{t_2} \cdot \mathfrak{f}_j = \ell_{2j}$, proving that $\sum_{ij} \ell_{ij} dt_i\, dt_j$ is the Second Fundamental Form of $\mathcal{F}$. ∎

**The Eighth and Final Matlab Project.**

The primary Matlab M-File should be called SurfaceFT.m and should start out:

```
function   F = SurfaceFT(g11,g12,g22,l11,l12,l22,a1,b1,a2,b2,T1Res,T2Res)
```

where $I := \sum_{ij}$ `gij(t1,t2) dti dtj` and $II := \sum_{ij}$ `lij(t1,t2) dti dtj` are quadratic forms in $\mathcal{O}$, and the function $F : \mathcal{O} \to \mathbf{R}^3$ returned is supposed to be the surface having $I$ and $II$ as its First and Second Fundamental Forms. For this surface to exist, we know from the Fundamental Theorem that it is necessary and sufficient that $I$ be positive definite and that the Gauss-Codazzi equations be satisfied.

Of course the heavy lifting of the SurfaceFT will be done by AlgorithmF (i.e., the solution of the Frobenius Problem) which you will apply to integrate the frame equations in order to get the frame field $\mathfrak{f}$—after which you must apply AlgorithmF a second time to get the surface $\mathcal{F}$ from $\mathfrak{f}_1$ and $\mathfrak{f}_2$.

But to carry out the first application of AlgorithmF, you must first compute the matrices $P^1 = G^{-1}A^1$ and $P^2 = G^{-1}A^2$ that define the right hand sides of the two frame equations. Recall that $G^{-1}$ is the inverse of the $3 \times 3$ matrix

$$G = \begin{pmatrix} \texttt{g11(t1,t2)} & \texttt{g12(t1,t2)} & 0 \\ \texttt{g12(t1,t2)} & \texttt{g22(t1,t2)} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and so it can be easily computed from the `gij(t1,t2)` by using Cramer's Rule, while the two $3 \times 3$ matrices $A^1$ and $A^2$ are given explicitly (see 14.1.3 above) in terms of the `g11(t1,t2)` **and their partial derivatives with respect to the** $t_i$ . (Of course, once you have both $G^{-1}$ and $A^i$, you get $P^i$ as their matrix product.)

In order to be able to compute the $A^i$, you will first need to define some auxilliary functions. Most of them can probably be subfunctions defined in the same file, though some could be separate M-Files. For example you will want to have a function called `g(t1,t2)` that returns a $2 \times 2$ matrix $\begin{pmatrix} \texttt{g11(t1,t2)} & \texttt{g12(t1,t2)} \\ \texttt{g12(t1,t2)} & \texttt{g22(t1,t2)} \end{pmatrix}$ and another called `l(t1,t2)` that returns a $2 \times 2$ matrix $\begin{pmatrix} \texttt{l11(t1,t2)} & \texttt{l12(t1,t2)} \\ \texttt{l12(t1,t2)} & \texttt{l22(t1,t2)} \end{pmatrix}$. You will then want to create the functions `G(t1,t2)` and `invG(t2,t2)` that return the $3 \times 3$ matrices $G$ and $G^{-1}$.

There is another complication before you can define the Matlab functions `A1` and `A2` that represent $A^1$ and $A^2$. You not only need the functions `gij(t1,t2)` but also their first partial derivatives with respect to the variables `t1` and `t2`. I recommend that along with the function `g(t1,t2)` you also define two more functions `g_t1(t1,t2)` and `g_t2(t1,t2)` that return $2 \times 2$ matrices whose entries are the partial derivatives of the `gij(t1,t2)` with respect to `t1` and `t2` respectively. As usual you can compute these partial derivatives using symmetric differencing—you don't need to do it symbolically.

You should define a Matlab function GaussCodazziCheck that will check whether or not the Gauss-Codazzi Equations are satisfied. Once you have defined the two $3 \times 3$ matrix-valued functions $P^1$ and $P^2$, it will be easy to write GaussCodazziCheck, since the Gauss-Codazzi equations are just $P^1_{t_2} - P^2_{t_1} = P^1P^2 - P^2P^1$. The idea is to check the identities numerically, matrix element by matrix element, at a sufficiently dense set of points, again using symmetric differencing to compute the derivatives.

Of course, when you are all done you will need some good test cases on which to try out you algorithm. We will discuss this elsewhere.

**GOOD LUCK, AND HAVE FUN!**