Notes on Fibre Bundles

The aim of these notes is to describe enough of the theory of fibre bundles, principal G-bundles and associated fibre bundles to prove that, whenever we have groups $G \leq H$ admitting a principal G-bundle $p\colon H \to H/G$, there is an equivalence of categories between the category of G-spaces with G-equivariant maps and the category of H-bundles over H/G with H-equivariant gauge morphisms.

1 Spaces and Morphisms

We will work in a fairly general setting, so when we write 'space' and 'morphism', we shall mean for example one of the following

1. topological spaces and continuous maps.
2. differentiable manifolds and differentiable maps.
3. algebraic varieties and regular maps.
4. separated schemes and regular maps.

We shall write **Space** for whichever category we are interested in.

Note that **Space** has finite products and a terminal object 1 (for example, a singleton in the category of topological spaces, or $\operatorname{Spec} k$ in the category of k-schemes). Also, most proofs will proceed by considering points in our spaces; for schemes one needs to be a little more careful, but one can take the functorial point-of-view and consider R-valued points for all rings R.

1.1 Groups and Group Actions

Suppose now that we have a 'group object' in our category; that is, we have a space G together with a group structure on the points of G such that the multiplication, inversion, and unit are all given by morphisms

$$G \times G \to G, \quad (g,h) \mapsto gh, \quad G \to G, \quad g \mapsto g^{-1}, \quad \text{and} \quad 1 \to G.$$

A (left) G-space is a space X together with a morphism

$$G \times X \to X, \quad (g,x) \mapsto gx$$

giving a (left) group action on X. If X and Y are G-spaces, then a G-equivariant morphism $X \to Y$ is one which commutes with the G-actions. Categorically, this says that we have a commutative square

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d] & X \arrow[d] \\ G \times Y \arrow[r] & Y \end{tikzcd}$$

We shall write $G-\textbf{Space}$ for the category of G-spaces and G-equivariant morphisms.

1.2 Gluing

There is one construction that we will need in the category **Space** — that of gluing. We recall the general procedure.

Let X be a space having an open covering $\mathcal{U} = \{U_i\}$, and let $\phi_i \colon X_i \to U_i$ be isomorphisms. Define $U_{ij} := \phi_i^{-1}(U_i \cap U_j)$, an open subspace of X_i, and $T_{ij} := \phi_j^{-1}\phi_i$, an isomorphism $U_{ij} \to U_{ji}$. The T_{ij} are called transition functions, and clearly satisfy

(a) $T_{ji} = T_{ij}^{-1}$ for all i, j.

(b) $T_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all i, j, k, and $T_{jk}T_{ij} = T_{ik}$ on $U_{ij} \cap U_{ik}$.

Gluing is the reverse procedure: we start with spaces X_i, containing open subspaces $U_{ij} \subset X_i$ for $i \neq j$, and with isomorphisms $T_{ij} \colon U_{ij} \to U_{ji}$ satisfying (a) and (b) above. We wish to construct a space X together with an open covering U_i and isomorphisms $\phi_i \colon X_i \to U_i$ such that $\phi_i(U_{ij}) = U_i \cap U_j$ and $\phi_i = \phi_j T_{ij}$ on U_{ij}.

Unfortunately, this is usually not quite enough, since the resulting topological space may not have the required separation property: for example, manifolds are generally required to be Hausdorff, and schemes are often required to be separated.

2 Fibre Bundles

A fibre bundle with total space E, base B, fibre F and structure group G is given by the following data:

1. a morphism $p \colon E \to B$
2. a left G-action on F, written $G \times F \to F$, $(g, f) \mapsto gf$
3. an open covering $\mathcal{U} = \{U_i\}$ of B and isomorphisms $\phi_i \colon U_i \times F \to p^{-1}(U_i)$, called co-ordinate charts, such that $p\phi_i = \mathrm{pr}_1$, i.e.
$$p\phi_i(u, f) = u$$
4. morphisms $t_{ij} \colon U_{ij} \to G$, where $U_{ij} := U_i \cap U_j$, giving transfer functions T_{ij}
$$T_{ij} \colon U_{ij} \times F \to U_{ij} \times F, \quad (u, f) \mapsto (u, t_{ij}(u)f),$$
so $\phi_i = \phi_j T_{ij}$.

By abuse of terminology we also call the t_{ij} transition functions.

It follows that E is formed by gluing together the spaces $U_i \times F$, using the open subspaces $U_{ij} \times F$ and transition functions T_{ij}.

Condition 3 is often referred to by saying that p is locally trivial (it would be trivial if $E = B \times F$ and $p = \mathrm{pr}_1$). Condition 4 ties together the group action on the fibres and the transition functions, which in turn affects the global structure of the space E.

We can rewrite the compatibility conditions (a) and (b) for the T_{ij} in terms of the t_{ij}, getting (a') $t_{ji}(u) = t_{ij}(u)^{-1}$ for all $u \in U_{ij}$.

(b') $t_{ik}(u) = t_{jk}(u)t_{ij}(u)$ for all $u \in U_{ijk} := U_i \cap U_j \cap U_k$.

Condition (a') states that the transfer functions form a Čech 1-cochain, relative to $\mathcal{U}$ and with values in G, whereas (b') states that in fact they form a Čech 1-cocycle.

2.1 Local Construction Fibre Bundles

We noted earlier that we cannot glue spaces together arbitrarily. For fibre bundles, however, this is not a problem.

Theorem 2.1. *Suppose we have a left G-action on F, an open covering $\mathcal{U}$ of B, and functions $t_{ij}\colon U_{ij} \to G$ satisfying the Čech 1-cocycle condition. Then there is a fibre bundle E over B with fibre F and structure group G having transition functions t_{ij}.*

Proof. We want to glue the spaces $U_i \times F$ together using the transition functions t_{ij} to form a space E. For this, we just need to check that the gluing procedure preserves the relevant separation property (for example, Hausdorff topological spaces or separated schemes).

Both the Hausdorff axiom for topological spaces and the separated axiom for schemes can be given by requiring that the diagonal morphism $E \to E \times E$ has closed image Δ_E. (The difference in the two situations lies in the different topologies used on the product.)

Since the $V_i := p^{-1}(U_i)$ form an open covering of E, the sets $V_i \times V_j$ form an open covering of $E \times E$. It is then sufficient to check that $\Delta \cap (V_i \times V_j)$ is closed for all i, j. Using the isomorphisms ϕ_i and ϕ_j, this is just the set

$$\{(u, u', f, f') \in U_i \times U_j \times F \times F : \phi_i(u, f) = \phi_j(u', f')\},$$

which we can view as the image Δ_{ij} of the map

$$U_{ij} \times F \to U_i \times U_j \times F \times F, \quad (u, f) \mapsto (u, u, f, t_{ij}(u)f).$$

Since B is separated, we know that the morphism $U_{ij} \to U_i \times U_j$ has closed image X_{ij} for all i, j. Also, since F is separated, we know that $\Delta_F \subset F \times F$ is closed. Thus $X_{ij} \times \Delta_F$ is closed in $U_i \times U_j \times F \times F$.

Finally, consider the morphism

$$\theta\colon X_{ij} \times F \times F \to X_{ij} \times F \times F, \quad (u, u, f, f') \mapsto (u, u, t_{ij}(u)f, f').$$

Then $\Delta_{ij} = \theta^{-1}(X_{ij} \times \Delta_F)$, so Δ_{ij} is closed in $X_{ij} \times F \times F$, and hence is closed in $U_i \times U_j \times F \times F$. This proves that E is separated. $\square$

More generally, the same ideas can be used to prove that a morphism $p\colon E \to B$ is separated (that is the diagonal $E \to E \times_B E$ has closed image) if and only if B has an open covering U_i such that each $p^{-1}(U_i) \to U_i$ is separated. For fibre bundles, this is equivalent to the fibre F being separated. If moreover B is separated, then E itself is separated. Compare Hartshorne, II, Corollary 4.6.

3 Morphisms of Fibre Bundles

There are various notions of morphisms between fibre bundles, depending on the degree of generality one wishes to work in. We begin with the most restrictive.

3.1 Equivalence Classes of Fibre Bundles

Let E and E' be fibre bundles having the same base B, fibre F and structure group G. By passing to a common refinement, we may assume that both bundles are given with respect to the same open covering.

The fibre bundles $E' \to E$ are said to be equivalent provided there exist morphisms $\theta_i \colon U_i \to G$ such that

$$(u, \theta_j(u)t'_{ij}(u)f) = (u, t_{ij}(u)\theta_i(u)f) \quad \text{on} \quad U_{ij} \times F.$$

It follows that the θ_i glue together to give a morphism $\theta \colon E' \to E$ satisfying

$$p\theta = p' \quad \text{and} \quad \theta\phi'_i(u, f) = \phi_i(u, \theta_i(u)f).$$

This notion defines an equivalence relation on the set of fibre bundles having the same base, fibre and structure group. For the symmetry relation, we use the morphism θ^{-1} defined locally by $\theta_i^{-1}(u) := \theta_i(u)^{-1}$.

3.2 Gauge Morphisms of Fibre Bundles

The next level of generality is provided by the notion of gauge morphisms.

Let E and E' be two fibre bundles having the same base B, but with respective fibres F and F', and structure groups G and G'. Again we may pass to a common refinement, and so assume that both bundles are given with respect to the same open covering.

A gauge morphism $E' \to E$ is given by a morphism $\theta \colon E' \to E$ such that $p\theta = p'$, or equivalently it is given locally by morphisms $\theta_i \colon U_i \to \mathrm{Hom}(F', F)$ such that

$$(u, t_{ij}(u)\theta_i(u)(f')) = (u, \theta_j(u)(t'_{ij}(u)f')) \quad \text{on} \quad U_{ij} \times F'.$$

Observe that every equivalence between fibre bundles having the same base, fibre and structure group is necessarily a gauge morphism, but the converse is not true.

Also, one should be clear that when the fibre bundles have the same structure group G, the θ_i are not G-equivariant: this does not even hold for equivalences (provided the group G is not abelian).

3.3 Morphisms of Fibre Bundles

Finally, we have the most general notion of a morphism.

Let E and E' be fibre bundles, with respective bases B and B', fibres F and F', and structure groups G and G'.

A morphism $E' \to E$ is given by morphisms $\psi\colon E' \to E$ and $\theta\colon B' \to B$ such that $p\psi = \theta p'$, so giving a commutative square

$$\begin{array}{ccc} E' & \xrightarrow{\psi} & E \\ \big\downarrow{\scriptstyle p'} & & \big\downarrow{\scriptstyle p} \\ B' & \xrightarrow{\theta} & B \end{array}$$

Note that a gauge morphism between fibre bundles having the same base is the same as a morphism with $\bar{\theta} = \mathrm{id}$.

4 Pull-Backs of Fibre Bundles

Let $p\colon E \to B$ be a fibre bundle with fibre F and structure group G, and let $\theta\colon B' \to B$ be a morphism. Then we can take the fibre product, or pull-back, of p along θ, denoted $B' \times_B E$ or $\theta^* E$, yielding the commutative diagram

$$\begin{array}{ccc} \theta^* E & \xrightarrow{\bar{\theta}} & E \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle p} \\ B' & \xrightarrow{\theta} & B \end{array}$$

Proposition 4.1. *With the notation as above, $q\colon \theta^* E \to B'$ is a fibre bundle with the same fibre and structure group as $p\colon E \to B$. Moreover, if $p\colon E \to B$ has co-ordinate charts using the open covering U_i and transition functions $t_{ij}\colon U_{ij} \to G$, then q has co-ordinate charts using the open covering $U_i' := \theta^{-1}(U_i)$ of B' and transition functions*

$$s_{ij} := t_{ij}\theta\colon U_{ij}' \to G.$$

Proof. The construction of the fibre product is local, so determined by

$$\begin{array}{ccc} U_i' \times F & \xrightarrow{\theta \times \mathrm{id}_F} & U_i \times F \\ \big\downarrow{\scriptstyle \mathrm{pr}_1} & & \big\downarrow{\scriptstyle \mathrm{pr}_1} \\ U_i' & \xrightarrow{\theta} & U_i \end{array}$$

Let $\phi_i'\colon U_i' \times F \to B' \times_B E$ be the inclusion map. This satisfies

$$\bar{\theta}\phi_i' = \phi_i(\theta \times \mathrm{id}_F) \quad \text{and} \quad q\phi_i' = \mathrm{pr}_1.$$

In particular, we see that q is locally trivial with fibre F, using the co-ordinate charts ϕ_i' over U_i'.

To compute the transition functions, take $u' \in U_{ij}'$. Then

$$\bar{\theta}\phi_i'(u', f) = \phi_i(\theta(u'), f) = \phi_j(\theta(u'), t_{ij}(\theta(u'))f) = \bar{\theta}\phi_j'(u', t_{ij}(\theta(u'))f)$$

and clearly

$$q\phi_i'(u', f) = u' = q\phi_j'(u', t_{ij}(\theta(u'))f).$$

It follows that

$$\phi_i'(u', f) = \phi_j'(u', t_{ij}(\theta(u'))f),$$

and so the transition functions are given by

$$s_{ij} := t_{ij}\theta\colon U'_{ij} \to G. \qquad \square$$

As a special case, we may construct the pull-back of a fibre bundle $E \to B$ along another fibre bundle $E' \to B$. Then this becomes a fibre bundle over B, called the bundle product of E and E'. In the case of vector bundles (see the next section), this construction is called the Whitney sum.

Proposition 4.2. *Let $p\colon E \to B$ and $p'\colon E' \to B$ be fibre bundles, with respective fibres F and F', and structure groups G and G'. Then their fibre product $E' \times_B E$ is a fibre bundle over B with fibre $F' \times F$ and structure group $G' \times G$.*

Proof. After passing to a common refinement, we may assume that p and p' are both defined with respect to the same open covering U_i of B. We therefore have co-ordinate charts

$$\phi_i\colon U_i \times F \to p^{-1}(U_i), \quad \phi'_i\colon U_i \times F' \to {p'}^{-1}(U_i),$$

and transition functions

$$t_{ij}\colon U_i \to G, \quad t'_{ij}\colon U_{ij} \to G'.$$

Denote the projection maps by $q'\colon E' \times_B E \to E'$ and $q\colon E' \times_B E \to E$, and set $P\colon E' \times_B E \to B$ to be $P := pq = p'q'$. Define

$$\Phi_i\colon U_i \times F' \times F \to P^{-1}(U_i)$$

via

$$q'\Phi_i = \phi'_i\mathrm{pr}_{12} \quad \text{and} \quad q\Phi_i = \phi_i\mathrm{pr}_{13}.$$

Then the Φ_i are isomorphisms, and $P\Phi_i = \mathrm{pr}_1\colon U_i \times F' \times F \to U_i$, so the Φ_i determine co-ordinate charts for P and P is locally trivial with fibre $F' \times F$.

To compute the transition functions, take $u \in U_{ij}$. Then

$$q'\Phi_i(u, f', f) = \phi'_i(u, f') = \phi'_j(u, t'_{ij}(u)f')$$

and

$$q\Phi_i(u, f', f) = \phi_i(u, f) = \phi_j(u, t_{ij}(u)f).$$

Thus

$$\Phi_i(u, f', f) = \Phi_j(u, t'_{ij}(u)f', t_{ij}(u)f),$$

so the transition functions are given by

$$t'_{ij} \times t_{ij}\colon U_{ij} \to G' \times G. \qquad \square$$

Coming back to a general pull-back, we see that $\bar{\theta}\colon \theta^*E \to B'$ and $\theta\colon B' \to B$ yield a morphism of bundles having the same fibre and structure group. In fact, θ is given locally by the morphism $\theta \times \mathrm{id}_F\colon V_i \times F \to U_i \times F$, and so has a particularly simple form. The next result shows that in general, every bundle morphism factorises as a composition of a gauge morphism followed by such a pull-back morphism.

Lemma 4.3. *Let $p\colon E \to B$ and $p'\colon E' \to B'$ be fibre bundles, and let $\psi\colon E' \to E$ and $\theta\colon B' \to B$ determine a morphism between them. Then we can factorise this morphism as a gauge morphism $E' \to \theta^*E$ followed by the pull-back morphism $\theta^*E \to E$.*

Proof. By virtue of the universal property of pull-backs, we obtain a morphism $\bar{\psi}\colon E' \to \theta^*E$ such that $q\bar{\psi} = p'$ and $\bar{\theta}\bar{\psi} = \psi$. It follows that $\bar{\psi}$ is a gauge morphism. $\square$

5 Special Types of Fibre Bundles

There are many special types of fibre bundles which one may be interested in, where one fixes the type of fibre and/or the type of structure group. One well-known class is that of vector bundles, where one takes as fibres vector spaces, and as structure group the corresponding general linear group with its natural action. In this case, one usually restricts to those morphisms which act linearly on fibres; that is, locally $\theta_i(u)$ is linear for all $u \in U_i$.

One may however also be interested in taking the fibre to be a sphere, or a projective space.

One may also consider H-bundles for a group H, which are fibre bundles in the category $H - \mathbf{Space}$. In other words, we require all spaces to be H-spaces, and all morphisms are required to be H-equivariant. In the following sections we develop some of the theory of H-bundles.

6 Principal G-bundles

A principal G-bundle over B is given by a fibre bundle with fibre G and structure group G acting by left translations. Thus locally we have charts $\phi_i\colon U_i \times G \to p^{-1}(U_i)$ and transition functions $t_{ij}\colon U_{ij} \to G$ such that $\phi_i(u, g) = \phi_j(u, t_{ij}(u)g)$, where $t_{ij}(u)g$ is just the group multiplication.

It follows that we have a right G-action on the total space. For, locally we have the right G-action given by right translations

$$U_i \times G \times G \to U_i \times G, \quad (u, g, h) \mapsto (u, gh).$$

Using the charts ϕ_i, this yields a right G-action on $p^{-1}(U_i)$. This action is independent of the chart, since locally the right G-action commutes with the transition functions. (This is just the associativity of the group multiplication.) Hence we obtain a right G-action on the total space.

The following result shows that this characterises principal G-bundles. Observe that we do not *a priori* assume anything about that the fibre and structure group of the fibre bundle $p\colon E \to B$. In particular, a principal G-bundle is a (right) G-bundle in the sense of the previous section, where G acts trivially on the base space B.

Theorem 6.1. *Let $p\colon E \to B$ be a fibre bundle and suppose that there is a right G-action on E inducing a free and transitive action on each fibre $p^{-1}(b)$. Then there exists a principal G-bundle E' and a right G-equivariant gauge isomorphism $E' \to E$.*

Proof. Let E have fibre F and structure group H.

Locally we have the charts $\phi_i\colon U_i \times F \to p^{-1}(U_i)$. Since the G-action preserves the fibres of p, it restricts to an action on each $p^{-1}(U_i)$, and hence determines an action on $U_i \times F$, say

$$U_i \times F \times G \to U_i \times F, \quad (u, f, g) \mapsto (u, f \cdot g) := \phi_i^{-1}(\phi_i(u, f) \cdot g).$$

Note however that the action $f \cdot g$ may depend on u.

Fixing a point $x \in F$, we obtain a morphism

$$\psi_i\colon U_i \times G \to U_i \times F, \quad (u, g) \mapsto (u, x \cdot g).$$

Since the G-action is free and transitive on each fibre, we deduce that ψ_i is an isomorphism.

We define transition functions S_{ij} on the co-ordinate charts $U_i \times G$ by transfer of structure, so

$$S_{ij}(u, g) = \psi_j^{-1} T_{ij} \psi_i(u, g) = \psi_j^{-1}(u, t_{ij}(u)(x \cdot g)).$$

Since $\phi_i(u, x \cdot g) = \phi_i(u, x) \cdot g$, we see that

$$\begin{aligned}\phi_j(u, t_{ij}(u)(x \cdot g)) &= \phi_i(u, x \cdot g) = \phi_i(u, x) \cdot g \\ &= \phi_j(u, t_{ij}(u)x) \cdot g = \phi_j(u, (t_{ij}(u)x) \cdot g).\end{aligned}$$

Also, since the G-action is free and transitive, there exists a unique $s_{ij}(u) \in G$ such that $(u, t_{ij}(u)x) = (u, x \cdot s_{ij}(u))$. Thus

$$(u, t_{ij}(u)(x \cdot g) = (u, (t_{ij}(u)x) \cdot g) = (u, (x \cdot s_{ij}(u)) \cdot g) = (u, x \cdot (s_{ij}(u)g)).$$

Hence the transition functions S_{ij} are given by

$$S_{ij}(u, g) = (u, s_{ij}(u)g) \quad \text{for maps} \quad s_{ij}\colon U_{ij} \to G.$$

Since the S_{ij} are morphisms, so too are the s_{ij}.

The S_{ij} or s_{ij} clearly satisfy the compatibility properties required for a fibre bundle, and hence glue together to form a principal G-bundle E'. The ψ_i then glue to give a gauge isomorphism $\psi\colon E' \to E$. Since the ψ_i are right G-equivariant, so too is ψ. $\square$

Using the right G-action, it is easy to determine equivalences between principal G-bundles.

Proposition 6.2. *An equivalence between principal G-bundles over B is the same as a right G-equivariant gauge morphism.*

Proof. Every equivalence is given locally by left translation using $\theta_i\colon U_i \to G$, and hence is right G-equivariant. On the other hand, suppose we have a right G-equivariant gauge morphism. Then locally we have that each θ_i is right G-equivariant, so $(u, \theta_i(u)(g)) = (u, \theta_i(u)(1)g)$. We therefore define $s_i\colon U_i \to G$ by $s_i(u) := \theta_i(u)(1)$, and see that $\theta\phi_i'(u, g) = \phi_i(u, s_i(u)g)$, so that θ is an equivalence. $\square$

As a corollary we note that if we had chosen a different base point $y \in F$ in the proof of Theorem 6.1, then we would obtain a right G-equivariant gauge isomorphism between the two principal G-bundles, and hence an equivalence between them.

6.1 Classifying Principal G-bundles

Recall that a Čech 1-cocycle, relative to an open covering $\mathcal{U}$ and with values in G is given by morphisms $t_{ij}\colon U_{ij} \to G$ for $i \neq j$ such that

$$t_{ji}(u) = t_{ij}(u)^{-1} \quad \text{and} \quad t_{ik}(u) = t_{jk}(u)t_{ij}(u)$$

over the appropriate domains.

As in the definition of equivalences of fibre bundles, we define an equivalence relation on the set of all Čech 1-cocycles by saying that t_{ij} and t'_{ij} are equivalent provided there exist $s_i\colon U_i \to G$ such that $s_j(u)t'_{ij}(u) = t_{ij}(u)s_i(u)$. We define the first Čech cohomology $H^1(\mathcal{U}; G)$ to be the set of equivalence classes.

The set of all open coverings of B form a directed system with respect to refinement, so $\mathcal{U} \leq \mathcal{U}'$ if each U_i is a union of U'_j. Restriction yields a map $H^1(\mathcal{U}; G) \to H^1(\mathcal{U}'; G)$, so we define

$$H^1(B; G) := \varinjlim H^1(\mathcal{U}; G),$$

the first Čech cohomology of B with coefficients in G. Note that $H^1(B; G)$ is not a group when G is not abelian.

We have seen that every principal G-bundle determines a Čech 1-cocyle with respect to some open covering, that every Čech 1-cocycle determines a principal G-bundle over B, and that two such principal G-bundles are equivalent if and only if their corresponding Čech 1-cocycles are equivalent (after passing to a common refinement). This proves the following theorem.

Theorem 6.3. *The isomorphism classes of principal G-bundles over B are in bijection with the elements in the first Čech cohomology $H^1(B; G)$.*

6.2 Associated Bundles

Let $p\colon E \to B$ be a principal G-bundle, say given by charts $\phi_i\colon U_i \times G \to p^{-1}(U_i)$ and transition functions $t_{ij}\colon U_{ij} \to G$ such that $\phi_i(u, g) = \phi_j(u, t_{ij}(u)g)$.

If F is a left G-space, then we can form an associated bundle $E \times^G F \to B$ with fibre F and structure group G by taking charts $U_i \times F$, together with the same transition functions $t_{ij}\colon U_i \to G$, and gluing. The resulting fibre bundle is denoted $p^F\colon E \times^G F \to B$, and has fibre F and structure group G.

Proposition 6.4. *Let $E \to B$ be a principal G-bundle, and let F be a left G-space. Then there is a principal G-bundle*

$$E \times F \to E \times^G F,$$

where the right G-action on $E \times F$ is given by

$$E \times F \times G \to E \times F, \quad (e, f, g) \mapsto (eg, g^{-1}f).$$

In particular, $E \times^G F \cong (E \times F)/G$.

Proof. Since both $p\colon E \to B$ and $p^F\colon E \times^G F \to B$ are fibre bundles, we know that the pull-back $p^*(E \times^G F)$ is a fibre bundle over B with fibre $G \times F$. Furthermore, since both bundles p and p^F have the same transition functions t_{ij}, we see that they are also the transition functions for the pull-back, where they now act diagonally; that is, we have co-ordinate charts $U_i \times G \times F$ and transition functions

$$U_{ij} \times G \times F \to U_{ij} \times G \times F, \quad (u, g, f) \mapsto (u, t_{ij}(u)g, t_{ij}(u)f).$$

In this case, we can describe the pull-back even more explicitly by constructing an isomorphism $\theta \colon E \times F \to p^*(E \times^G F)$. Observe that $E \times F$ is built by gluing the charts $U_i \times G \times F$ using the transition functions

$$U_{ij} \times G \times F \to U_{ij} \times G \times F, \quad (u, g, f) \mapsto (u, t_{ij}(u)g, f).$$

The isomorphism θ is thus given locally by

$$\theta_i \colon U_i \times G \times F \to U_i \times G \times F, \quad (u, g, f) \mapsto (u, g, gf).$$

Note that the composition of θ with the morphism $p^*(E \times^G F) \to E$ is just the projection pr_1, whereas its composition with $p^*(E \times^G F) \to E \times^G F$ is given locally by

$$U_i \times G \times F \to U_i \times F, \quad (u, g, f) \mapsto (u, gf).$$

Now, since $p \colon E \to B$ is a principal G-bundle, so too is $p^*(E \times^G F) \to E \times^G F$. The right G-action is given locally by

$$U_i \times G \times F \times G \to U_i \times G \times F, \quad (u, g, f, h) \mapsto (u, gh, f).$$

Composing this with θ^{-1} we get the right G-action on $E \times F$ given locally by

$$U_i \times G \times F \times G \to U_i \times G \times F, \quad (u, g, f, h) \mapsto (u, gh, h^{-1}f).$$

We can therefore describe this globally, using the right G-action on E, as

$$E \times F \times G \to E \times F, \quad (e, f, g) \mapsto (eg, g^{-1}f). \qquad \square$$

As a corollary, note that $E \times^G G \cong E$, and so if we take the pull-back of a principal G-bundle along itself, then the resulting bundle $E \times_B E \to E$ is trivial.

Conversely, suppose we have a fibre bundle $E' \to B$ with fibre F and structure group G, say with charts $U_i \times F$ and transition functions $t_{ij} \colon U_i \to G$. Then we can define a principal G-bundle E over B by taking the charts $U_i \times G$ and with the same transition functions $t_{ij} \colon U_i \to G$, acting via left translation.

Proposition 6.5. *Let $E' \to B$ be a fibre bundle with fibre F and structure group G. Let $E \to B$ be the corresponding principal G-bundle. Then there is a gauge isomorphism $E' \cong E \times^G F$.*

Proof. Locally we just take the identity map $U_i \times F \to U_i \times F$. $\square$

7 An Equivalence of Categories

Let $p \colon E \to B$ be a principal G-bundle. Then the associated bundle construction determines a functor

$$\begin{array}{ccc} G - \mathbf{Space} & & F \\ \downarrow{\scriptstyle \mathcal{F}} & & \downarrow \\ \{\text{bundles over } B \text{ with gauge morphisms}\} & & E \times^G F \end{array}$$

This sends a G-equivariant morphism $\theta\colon F' \to F$ to the gauge morphism

$$\Theta\colon E \times^G F' \to E \times^G F$$

given locally by

$$U_i \times F' \to U_i \times F, \quad (u, f') \mapsto (u, \theta(f')).$$

Since θ is G-equivariant, we see that the condition on $U_{ij} \times F$ is satisfied

$$(u, t_{ij}(u)\theta(f')) = (u, \theta(t_{ij}(u)f')).$$

We would like to describe the image of this functor, but this seems to be difficult. We therefore restrict ourselves to the special case when we have groups $G \leq H$ and a principal G-bundle $p\colon H \to H/G$, $h \mapsto hG$. We can then use the extra structure given by the group H to determine the image of $\mathcal{F}$.

Note that we do not always have such a principal G-bundle $H \to H/G$. For example, if H is an algebraic group, then for the quotient H/G to exist, G needs to be a closed subgroup; however, the natural morphism $H \to H/G$ may still not be locally trivial.

We begin with a useful lemma.

Lemma 7.1. *Let $p\colon E \to B$ be a principal G-bundle. If there is a left H-action on E commuting with the right G-action, then it descends to a left H-action on B such that p is H-equivariant.*

Proof. Locally, we have a morphism

$$H \times U_i \to H \times U_i \times G \to H \times E \to E \to B, \quad (h, u) \mapsto p(h \cdot \phi_i(u, 1)).$$

To check that these maps glue together, take $u \in U_{ij}$. Then, since the left H-action on E commutes with the right G-action, we have

$$h \cdot \phi_i(u, 1) = h \cdot \phi_j(u, t_{ij}(u)) = h \cdot (\phi_j(u, 1)t_{ij}(u)) = (h \cdot \phi_j(u, 1))t_{ij}(u).$$

Since the right G-action preserves the fibres, we deduce that

$$p(h \cdot \phi_i(u, 1)) = p(h \cdot \phi_j(u, t_{ij}(u)) = p(h \cdot \phi_j(u, 1))$$

as required.

Our construction of the left H-action on B automatically means that p is H-equivariant. $\square$

Corollary 7.2.
1. *There is a left H-action on H/G given by $(h, h'G) \mapsto (hh')G$, and the canonical map $H \to H/G$ is H-equivariant.*
2. *Let F have a left G-action. Then there is a left H-action on the associated fibre bundle $H \times^G F$, and the morphism $p^F\colon H \times^G F \to H/G$ is H-equivariant.*

Proof. For the first, we just apply the lemma to the principal G-bundle $H \to H/G$, using the left H-action on H given by left translation.

For the second, we first apply the lemma to the principal G-bundle $q\colon H \times F \to H \times^G F$, using the left H-action on $H \times F$ given by left translation on the first component. This clearly commutes with the right G-action on $H \times F$, which we recall is given by $(h, f, g) \mapsto (hg, g^{-1}f)$.

Now, since the composition $H \times F \to H \times^G F \to H/G$ equals the composition $H \times F \to H \to H/G$, it is H-equivariant, and hence p^F is also H-equivariant. $\square$

It follows that the image of our functor $\mathcal{F}$ is contained in the category of bundles $q\colon E \to H/G$ such that there is a left H-action on E for which q is H-equivariant. Moreover, if $\theta\colon F' \to F$ is G-equivariant, then the morphism $\Theta\colon H \times^G F' \to H \times^G F$ is H-equivariant, since it is induced by the morphism $\mathrm{id}_H \times \theta\colon H \times F' \to H \times F$, which is clearly H-equivariant.

So, we are interested in the category of bundles over H/G having a left H-action together with H-equivariant gauge morphisms. We call this the category of H-bundles over H/G, and observe that we can consider this as the category of bundles over H/G inside the category $H - \mathbf{Space}$.

Theorem 7.3. *Suppose that we have a principal G-bundle $p\colon H \to H/G$ for groups $G \leq H$. Then there is an equivalence of categories between the category of G-spaces and the category of H-bundles over H/G. This is given by sending a G-space F to the associated bundle $p^F\colon H \times^G F \to H/G$, and by sending an H-bundle $q\colon E \to H/G$ to the G-space $q^{-1}(G)$.*

Proof. We have already seen the functor $\mathcal{F}$ sending a G-space F to the associated H-bundle $p^F\colon H \times^G F \to H/G$, and a G-equivariant morphism $\theta\colon F' \to F$ to the H-equivariant morphism $\Theta\colon H \times^G F' \to H \times^G F$.

Conversely, suppose that we have an H-bundle $q\colon E \to H/G$. Then the left H-action on E restricts to a left G-action on E, and since q is equivariant with respect to this action, we obtain a left G-action on the fibre $q^{-1}(G)$. Thus $q^{-1}(G)$ is naturally a G-space.

Now let $p\colon E \to H/G$ and $p'\colon E' \to H/G$ be H-bundles, and let $\theta\colon E' \to E$ be an H-equivariant gauge morphism. Then θ restricts to a G-equivariant morphism ${p'}^{-1}(G) \to p^{-1}(G)$. We have therefore constructed a functor $\mathcal{G}$ from the category of H-bundles over H/G to the category of G-spaces.

We next construct a natural isomorphism $\mathcal{G}\mathcal{F} \cong \mathrm{id}$. To do this, observe that $\mathcal{F}(F)$ is the H-bundle $p^F\colon H \times^G F \to H/G$. Using the principal G-bundle $H \times F \to H \times^G F$, we can write elements in $H \times^G F$ as equivalence classes $[h, f]$. By the corollary, this has left H-action given by $h \cdot [h', f] = [hh', f]$. In particular, we have an isomorphism $F \to (p^F)^{-1}(G)$ given by $f \mapsto [1, f]$, and since $g \cdot [1, f] = [g, f] = [1, gf]$, the induced left G-action on F is simply the original G-action. This proves that we have an isomorphism $\mathcal{G}\mathcal{F}(F) \cong F$.

If $\theta\colon F' \to F$ is G-equivariant, then

$$\Theta\colon H \times^G F' \to H \times^G F, \quad [h, f'] \mapsto [h, \theta(f')]$$

is H-equivariant. We therefore get the commutative square

$$\begin{array}{ccccccc}
F' & \longrightarrow & (p^{F'})^{-1}(G) & \qquad & f' & \longrightarrow & [1, f'] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F & \longrightarrow & (p^F)^{-1}(G) & & \theta(f) & \longrightarrow & [1, \theta(f)] = \Theta([1, f'])
\end{array}$$

and hence a natural isomorphism $\mathcal{G}\mathcal{F} \cong \mathrm{id}$.

Finally, we construct a natural isomorphism $\mathcal{F}\mathcal{G} \cong \mathrm{id}$. Let $q\colon E \to H/G$ be an H-bundle. We first construct an H-equivariant morphism

$$\tilde{\psi}\colon H \times q^{-1}(G) \to E, \quad (h, e) \mapsto h \cdot e,$$

which is just the restriction to $q^{-1}(G)$ of the left H-action on E. Now, this morphism is G-invariant under the right G-action

$$H \times q^{-1}(G) \times G \to H \times q^{-1}(G), \quad (h, e, g) \mapsto (hg, g^{-1} \cdot e),$$

and so restricts to an H-equivariant morphism

$$\psi \colon H \times^G q^{-1}(G) \to E, \quad [h, e] \mapsto h \cdot e.$$

Moreover, this is a gauge morphism over H/G

$$q\psi([h, e]) = q(h \cdot e) = hq(e) = hG.$$

We can construct an inverse map as follows. Locally we have a morphism

$$\eta_i \colon q^{-1}(U_i) \to H, \quad e \mapsto \phi_i(q(e), 1),$$

so we can define a morphism

$$q^{-1}(U_i) \to H \times^G q^{-1}(G), \quad e \mapsto [\eta_i(e), \eta_i(e)^{-1} \cdot e].$$

These can be glued together, since if $q(e) = u \in q^{-1}(U_{ij})$, then

$$\eta_i(e) = \phi_i(u, 1) = \phi_j(u, t_{ij}(u)) = \eta_j(e)t_{ij}(u),$$

and so

$$[\eta_j(e), \eta_j(e)^{-1} \cdot e] = [\eta_i(e)t_{ij}(u), t_{ij}(u)^{-1}\eta_i(e)^{-1} \cdot e] = [\eta_i(e), \eta_i^{-1}(e) \cdot e].$$

This gives a morphism $\eta \colon E \to H \times^G q^{-1}(G)$.

Clearly $\psi\eta = \mathrm{id}$, whereas if $(h, e) \in H \times q^{-1}(G)$ with $q(h \cdot e) = hG \in U_i$, then

$$\eta\psi[h, e] = \eta(h \cdot e) = [h', h'^{-1}h \cdot e], \quad \text{where} \quad h' = \eta_i(h \cdot e) = \phi_i(hG, 1).$$

Now, $p(h') = p\phi_i(hG, 1) = hG$, so $h' = hg$ for some $g \in G$. Thus

$$\eta\psi[h, e] = [hg, (hg)^{-1}h \cdot e] = [hg, g^{-1} \cdot e] = [h, e],$$

so that $\eta\psi = \mathrm{id}$.

Let $q' \colon E' \to H/G$ be another H-bundle, and let $\Theta \colon E' \to E$ be an H-equivariant gauge morphism. We then have a commutative square

$$\begin{array}{ccccc}
H \times^G q'^{-1}(G) & \longrightarrow & H \times^G q^{-1}(G) & [h, e] \longrightarrow & [h, \Theta(e')] \\
\downarrow & & \downarrow & \downarrow & \downarrow \\
E' & \longrightarrow & E & h \cdot e' \longrightarrow & h \cdot \Theta(e') = \Theta(h \cdot e')
\end{array}$$

proving that we have a natural isomorphism $\mathcal{FG} \to \mathrm{id}$. $\square$