The Immersion Conjecture for Differentiable Manifolds
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The immersion conjecture for differentiable manifolds

By Ralph L. Cohen*

To Fran

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Introduction

An old problem in differential topology is to find the smallest integer k_n with the property that every compact, C^∞ n-dimensional manifold M^n immerses

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in $\mathbf{R}^{n+k_n}$. A well known result of H. Whitney [26] states that for $n > 1$, $k_n \leq n - 1$.

The first obstructions to finding an immersion of M^n in $\mathbf{R}^{n+k}$ are the Stiefel-Whitney characteristic classes of the stable normal bundle of M^n, $w_i(M^n)$, for $i > k$. In 1960, W. Massey [20] proved that $w_i(M^n) = 0$ for $i > n - \alpha(n)$, where $\alpha(n)$ is the number of ones in the dyadic expansion of n. That is, if one writes

$$n = 2^{i_1} + \cdots + 2^{i_l} \quad \text{where } i_1 < i_2 < \cdots < i_l,$$

then $\alpha(n) = l$. Massey's result is best possible because a standard computation shows that if

$$M^n = RP^{2^{i_1}} \times \cdots \times RP^{2^{i_l}},$$

then $w_{n-\alpha(n)}(M^n) \neq 0$. This in particular implies that $k_n \geq n - \alpha(n)$.

The classical immersion conjecture is that $k_n = n - \alpha(n)$. The object of this paper is to give a proof of this conjecture. That is, we shall prove the following:

THEOREM. *If M^n is a compact, C^∞, n-dimensional manifold, $n > 1$, then there exists a differentiable immersion of M^n into* $\mathbf{R}^{2n-\alpha(n)}$

A scheme for proving this theorem has been developed and partially carried out by E. H. Brown and F. P. Peterson [4], [6], [8]. This paper can be viewed as a completion of their program.

Brown and Peterson's program began in 1963 [4] when they strengthened Massey's algebraic results by computing the ideal of all relations among the Stiefel-Whitney classes of stable normal bundles of n-manifolds. That is, if

$$\nu^*_{M^n}\colon H^*(BO;\mathbf{Z}_2) \to H^*(M^n;\mathbf{Z}_2)$$

is the homomorphism induced by the classifying map of the stable normal bundle of an n-manifold M^n, let $I_M = \ker \nu^*_M$, and let

$$I_n = \bigcap I_{M^n}$$

where the intersection is taken over all n-manifolds M. In [4], Brown and Peterson computed the ideal I_n explicitly.

By definition, the stable normal bundle homomorphism

$$\nu^*_M\colon H^*(BO;\mathbf{Z}_2) \to H^*(M^*;\mathbf{Z}_2)$$

factors through a homomorphism $H^*(BO)/I_n \xrightarrow[\bar\nu^*_M]{} H^*(M)$. Combining this with Massey's result, we have the following commutative diagram of groups and homomorphisms:

$$\begin{tikzcd} H^*(BO;\mathbf{Z}_2)/I_n \arrow[r, "\tilde{\nu}_M^*"] & H^*(M;\mathbf{Z}_2) \\ H^*(BO(n-\alpha(n));\mathbf{Z}_2) \arrow[u, "\rho_n^*"] & H^*(BO;\mathbf{Z}_2) \arrow[l] \arrow[ul, "\rho^*"'] \arrow[u, "\nu_M^*"'] \end{tikzcd}$$

where ρ^* is the natural projection, and ρ_n^* is the induced factorization through $H^*(BO(n-\alpha(n)))$.

The goal of Brown and Peterson's program is to realize this diagram by spaces and maps of spaces. That is, the idea is to complete the following two steps.

Step 1. Construct a space BO/I_n together with a map $\rho\colon BO/I_n \to BO$ satisfying the following properties:

a. $H^*(BO/I_n;\mathbf{Z}_2) = H^*(BO;\mathbf{Z}_2)/I_n$ and ρ induces the natural projection in cohomology.

b. The stable normal map $\nu_M\colon M \to BO$ of any n-manifold M can be factored up to homotopy as a composition $\nu_M\colon M \underset{\tilde{\nu}_M}{\to} BO/I_n \underset{\rho}{\to} BO$.

Step 2. Show that the map $\rho\colon BO/I_n \to BO$ factors through a map $\rho_n\colon BO/I_n \to BO(n-\alpha(n))$.

In [8], Brown and Peterson carried out step 1 and conjectured that step 2 could be carried out. The main goal of this paper is to prove their conjecture by constructing the requisite map $\rho_n\colon BO/I_n \to BO(n-\alpha(n))$.

We remark that this is sufficient for proving the immersion conjecture since once carried out, we will have constructed a lifting of the stable normal bundle map $\nu_M\colon M \to BO$ of any n-manifold M, to $BO(n-\alpha(n))$; namely,

$$\rho_n \circ \tilde{\nu}_M\colon M \to BO/I_n \to BO(n-\alpha(n)).$$

By M. Hirsch's immersion theorem [15], this guarantees the existence of an immersion of M^n into $\mathbf{R}^{2n-\alpha(n)}$.

The construction of the lifting $\rho_n\colon BO/I_n$—$BO(n-\alpha(n))$ will be spread over the following five sections. Section 0 is a collection of relatively elementary results about fibrations, cofibrations, and spectra. Besides collecting these results this section will give us an opportunity to establish certain conventions, both in terminology and in notation. In Section 1 we describe a crucial lemma and show that it implies the existence of the map ρ_n. In Sections 2 and 3 we prove this lemma modulo a technical result concerning pairings of the spaces BO/I_n. This result is proved in Section 4.

The influence of the work of Brown and Peterson on this paper is obvious even to the most casual reader. The author is indebted to them not only for all

their preliminary work toward the solution of the immersion conjecture, but also for the many helpful conversations and constant encouragement they have shown during the preparation of this work. The author is particularly grateful to Ed Brown for going through preliminary versions of this paper carefully. The author is also grateful to G. Brumfiel, S. Gitler, M. Mahowald, R. J. Milgram, P. Selick, and V. Snaith for many helpful conversations.

Throughout the rest of this paper, all (co)homology will be taken with $\mathbf{Z}/2$ coefficients, and by the term "n-manifold" we shall mean a compact, C^∞, n-dimensional manifold.

Part I

0. Fibrations, cofibrations, and spectra

In this section we will discuss relatively straightforward properties of maps between homotopy fibration and cofibration sequences of spaces and spectra. The results of this section are elementary and well-known. The primary purpose of this section is simply to collect some of these results, and to establish some terminology and notation that will be used throughout the paper.

All spaces will be of the homotopy type of based CW complexes. Let $f\colon X \to Y$ be a map of based spaces. By the *homotopy fiber* of f we mean the space $F(f) = \{(x, \alpha) \in X \times Y^I\colon \alpha(0) = f(x) \text{ and } \alpha(1) = *\}$, where Y^I denotes the space of paths in Y and where $* \in Y$ is a distinguished basepoint. If $i\colon F(f) \to X$ is defined by $i(x, \alpha) = x$ then it is well known that the sequence $F(f) \xrightarrow{i} X \xrightarrow{f} Y$ is homotopic to a fibration sequence.

Similarly, we define the *homotopy cofiber* of f to be the mapping cone $C(f) = Y \cup_f \hat{X} = Y \coprod X \times I/\sim$ where $(x, 0) \sim f(x) \in Y$, $(x, 1) \sim * \in Y$ and $(*, t) \sim *$ for all t. If $j\colon Y \to Y \cup_f \hat{X} = C(f)$ is the inclusion, then the sequence $X \underset{f}{\to} Y \underset{j}{\to} M(f)$ is homotopic to a cofibration sequence.

Suppose we are given a square diagram where all maps are basepoint preserving.

(0.1)
$$\begin{array}{ccc} X & \xrightarrow{h_1} & W \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[h_2]{} & Z. \end{array}$$

If this diagram strictly commutes (i.e., $g \circ h_1$ and $h_2 \circ f$ are equal maps) then by the *induced map of homotopy fibers* $h_F\colon F(f) \to F(g)$ we mean the map defined by the formula

$$h_F(x, \alpha) = \big(h_1(x), h_2 \circ \alpha\big) \in F(g)$$

where $(x, \alpha) \in F(f)$. The fact that $h_2 \circ \alpha(0) = g(h_1(x))$ and $h_2 \circ \alpha(1) = *$ follows from the commutativity of this square. By the *induced map of homotopy cofibers* h_C: $C(f) \to C(g)$ we mean the map

$$h_C\colon Y \cup_f \hat{X} \to Z \cup_g \hat{W}$$

defined by

$$h_C(y) = h_2(y) \qquad \text{if } y \in Y \subset Y \cup_f \hat{X}$$

and
$$h_C(x, t) = \big(h_1(x), t\big) \quad \text{if } (x, t) \in \hat{X} \subset Y \cup_f \hat{X}.$$

The fact that h_C is well-defined again follows from the commutativity of the diagram.

Finally we will say that a strictly commutative diagram as above *induces a map of pairs* h_P by which we will mean the map of pairs

$$h_2 \cup h_1 \times 1\colon \big(Y \cup_f X \times I, X \times \{1\}\big) \to \big(Z \cup_g W \times I, W \times \{1\}\big)$$

where $Y \cup_f X \times I$ and $Z \cup_g W \times I$ denote the mapping cylinders of f and g respectively.

Now suppose that square (0.1) is not strictly commutative, but only homotopy commutative. Say H: $X \times I \to \mathbf{Z}$ is a homotopy between the based maps $g \circ h_1$ and $h_2 \circ f$. That is, $H(x, 0) = g \circ h_1(x)$, $H(x, 1) = h_2 \circ f(x)$ and $H(*, t) = *$ for all t. The homotopy H induces a map of homotopy fibers, cofibers, and a map of pairs as follows:

$$h_F(H)\colon F(f) \to F(g)$$

is defined by $h_F(H)(x, \alpha) = (h_1(x), (h_2 \circ \alpha)^* H_x)$ where H_x: $I \to \mathbf{Z}$ is the path given by $H_x(t) = H(x, t)$ and where $(h_2 \circ \alpha)^* H_x$ is the path sum of $h_2 \circ \alpha$ and H_x. That is,

$$(h_2 \circ \alpha)^* H_x(t) = \begin{cases} H_x(2t), & t \le \frac{1}{2} \\ (h_2 \circ \alpha)(2t - 1), & \frac{1}{2} \le t \le 1. \end{cases}$$

On the cofiber level we have

$$h_C(H)\colon Y \cup_f \hat{X} \to Z \cup_g \hat{W}$$

defined by

$$h_C(H)(y) = h_2(y) \in Z \subset Z \cup_g \hat{W}, \quad \text{for } y \in Y$$

and

$$h_C(H)(x,t) = \begin{cases} H(x, 1-2t) \in Z \subset Z \cup_g \hat{W}, & \text{for } (x,t) \in \hat{X} \text{ and } t \le \frac{1}{2} \\ (h_1(x), 2t-1) \in \hat{W} \subset Z \cup_g \hat{W}, & \text{for } (x,t) \in \hat{X} \text{ and } \frac{1}{2} \le t \le 1. \end{cases}$$

Similarly, we get a map of pairs

$$h_P(H)\colon (Y \cup_f X \times I,\ X \times \{1\}) \to (Z \cup_g W \times I,\ W \times \{1\})$$

defined by

$$h_P(H)(y) = h_2(y) \in Z \subset Z \cup_g W \times I$$

and

$$h_P(H)(x,t) = \begin{cases} H(x, 1-2t) \in Z \subset Z \cup_g W \times I & \text{for } (x,t) \in X \times I,\ t \le \frac{1}{2} \\ (h_1(x), 2t-1) \in W \times I & \text{for } (x,t) \in X \times I,\ \frac{1}{2} \le t \le 1. \end{cases}$$

Notice that the homotopy type of the maps $h_F(H)$, $h_C(H)$ and $h_P(H)$ depend on the choice of homotopy H. Notice furthermore that in the case when diagram (0.1) strictly commutes, the maps h_F, h_C, and h_P described above are the maps induced by the constant homotopy H: $X \times I \to Z$ given by $H(x,t) = h_2 f(x) = g \circ h_1(x)$.

The following is an easy and standard exercise which describes the relation between the maps $h_F(H)$, $h_C(H)$ and $h_p(H)$.

PROPOSITION 0.2. *The following diagrams commute.*

$$\begin{tikzcd} \pi_q(Y \cup_f X \times I, X \times \{1\}) \arrow[r, "h_P(H)_*"] \arrow[d, "\phi", "\cong"'] & \pi_q(Z \cup_g W \times I, W \times \{1\}) \arrow[d, "\phi", "\cong"'] \\ \pi_{q-1}(F(f)) \arrow[r, "h_F(H)_*"'] & \pi_{q-1}(F(g)) \end{tikzcd}$$

$$\begin{tikzcd} H_q(Y \cup_f X \times I, X \times \{1\}) \arrow[r, "h_P(H)_*"] \arrow[d, "\phi", "\cong"'] & H_q(Z \cup_g W \times I, W \times \{1\}) \arrow[d, "\phi", "\cong"'] \\ H_q(C(f)) \arrow[r, "h_C(H)_*"'] & H_q(C(g)) \end{tikzcd}$$

where the vertical isomorphisms ϕ are the natural ones obtained by comparing the appropriate long exact sequences and using the five lemma.

Now we would like to do similar constructions with spectra. That is we would like to take a homotopy commutative diagram of spectra, together with a homotopy, and produce an induced map of homotopy cofibers. To do this we need to make precise our notation of spectra, and maps between them. For this we follow May [28], [29]. That is, by a spectrum E we will mean a strict Ω-spectrum; $\{E_n, e_n\}$ where $e_n\colon E_n \to \Omega E_{n+1}$ is a *homeomorphism*. A map $g\colon E \to F$ between spectra $E = \{E_n, e_n\}$ and $F = \{F_n, f_n\}$ is a sequence of maps $g_n\colon E_n \to F_n$ so that $\Omega g_n \circ e_n = f_n \circ g_{n-1}\colon E_{n-1} \to \Omega F_n$. In [28], [29] May showed that this category of spectra is equivalent to other standard categories of spectra. In particular a classical spectrum $X = \{X_n, \varepsilon_n\}$ where $\varepsilon_n\colon \Sigma X_n \to X_{n+1}$ (or equivalently $\varepsilon_n\colon X_n \to \Omega X_{n+1}$) are arbitrary maps, May calls a *prespectrum*. In particular a spectrum is a prespectrum, and there is a standard procedure for producing a spectrum out of a prespectrum. Namely if $X = \{X_n, \varepsilon_n\}$, is a prespectrum we let $\overline{X} = \{\overline{X}_n, \bar{\varepsilon}_n\}$ be the spectrum defined by $\overline{X}_n = \varinjlim_k \Omega^k X_{n+k}$ where the limit is taken with respect to the maps $\Omega^k \varepsilon_{n+k}\colon \Omega^k X_{n+k} \to \Omega^{k+1} X_{n+k+1}$. Also $\bar{\varepsilon}_n\colon \overline{X}_n \to \Omega \overline{X}_{n+1}$ is the homeomorphism

$$\varinjlim_k \Omega^k X_{n+k} \cong \Omega \varinjlim_k \Omega^{k-1} X_{(n+1)+(k-1)} = \Omega \overline{X}_{n+1}.$$

Because of this correspondence between spectra and prespectra we will not distinguish between them in the text of the paper. This sloppiness in notation should not, however, cause confusion since the reader can always take any "spectrum" referred to in the paper and convert it to a strict Ω-spectrum by the above procedure.

The advantage of using May's category of spectra is that dealing with connective spectra (which is all we will work with in this paper) is equivalent to working with infinite loop spaces (namely the zero$^{\text{th}}$ spaces) and infinite loop maps. So if readers prefer, they can translate every statement about spectra made in this paper to a statement about their corresponding zero$^{\text{th}}$ spaces. In particular we will use the following definition.

Definition 0.3. A diagram of maps of spectra

$$\begin{array}{ccc} E & \longrightarrow & F \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

is said to commute strictly if the corresponding diagram of zero$^{\text{th}}$ spaces strictly commutes.

Now in May's category one can construct mapping cylinders, mapping cones and the like (these are most conceptually done on the prespectrum level and then converted to spectra), and so if given a homotopy commutative diagram of spectra

$$\begin{array}{ccc} E & \xrightarrow{h_1} & F \\ \downarrow f & & \downarrow g \\ X & \xrightarrow{h_2} & Y \end{array}$$

together with a homotopy H: $E \wedge I^+ \to Y$ (where I^+ denotes the suspension spectrum of I together with a disjoint basepoint) we can then define the induced map of homotopy mapping cones of spectra

$$h_C(H): X \cup_f \hat{E} \to Y \cup_g \hat{F}$$

in a manner analogous to what was done with spaces.

The following result, whose proof is standard obstruction theory, describes how certain changes in the choice of homotopies affect the induced maps of homotopy cofibers of spectra.

Consider a strictly commutative diagram of spectra of the following sort

(0.4)
$$\begin{array}{ccc} A \vee B & \xrightarrow{h_1} & C \\ \downarrow i & & \downarrow j \\ A \cup_f B \wedge I^+ & \xrightarrow{h_2} & D \end{array}$$

where f: $B \to A$ is a map of spectra and i is the natural inclusion on A and includes B as $B \times \{1\} \subset B \wedge I^+ \subset A \cup_f B \wedge I^+$.

Now let g: $\Sigma B \to D$ be any map of spectra, and let $h_2(g)$: $A \cup_f B \wedge I^+ \to D$ be the composite

$$h_2(g): A \cup_f B \wedge I^+ \xrightarrow[p]{} \left(A \cup_f B \wedge I^+\right) \vee \Sigma B \xrightarrow[h_2 \vee g]{} D$$

where the first component of p is the identity, and the second component of p is the collapse map

$$p: A \cup_f B \wedge I^+ \to A \cup_f B \wedge I^+/A \vee B = \Sigma B.$$

We then have a strictly commutative diagram

(0.5)
$$\begin{array}{ccc} A \vee B & \xrightarrow{h_1} & C \\ \downarrow i & & \downarrow j \\ A \cup_f B \wedge I^+ & \xrightarrow{h_2(g)} & D. \end{array}$$

PROPOSITION 0.6. *Let h_c: $C(i) \to C(j)$ and $h_c(g)$: $C(i) \to C(j)$ be the maps of homotopy cofibers of spectra induced by the commutativity of diagrams* (0.4) *and* (0.5) *respectively. Then $h_c(g)$ is homotopic to the sum of h_c and the composite $C(i) \underset{r}{\to} \Sigma B \underset{g}{\to} D \to D \cup_j \hat{C} = C(j)$, where the map r is homotopy equivalence given by the natural collapse map*

$$C(i) = \left(A \cup_f B \wedge I^+\right) \cup_i \widehat{A \vee B} \to A \cup_f B \wedge I^+/A \vee B = \Sigma B.$$

An important type of spectrum that we will be dealing with is a Thom spectrum $T(f)$ associated to a map f: $X \to BO$. (This spectrum will sometimes be denoted $T(X)$.) If X comes equipped with a filtration $X_0 \hookrightarrow \cdots \hookrightarrow X_{k-1} \hookrightarrow X_k \hookrightarrow \cdots X$ and factorizations of the restrictions of f to X_k as maps f_k: $X_k \to BO(j_k)$ for some nondecreasing sequence $\{j_k\colon k > 0\}$, then one can define a Thom (pre)spectrum as Stong does in [30], and then obtain a strict Ω-spectrum by the above procedure. Given any X of the homotopy type of a CW complex, one can take a skeletal filtration of X, but the resulting Thom spectrum will depend on the choice of CW decomposition and (homotopy) factorization of f (albeit the homotopy type of the Thom spectrum is independent of such choices). In [29] L. G. Lewis described a canonical way of defining $T(f)$ given simply a map f: $X \to BO$. As far as we are concerned, the upshot of this study is that given a strictly commutative diagram

(0.7)
$$\begin{tikzcd} A \arrow[r, "h_1"] \arrow[d, "i"'] & B \arrow[d, "j"] & \\ Y \arrow[r, "h_2"'] & X \arrow[r, "f"'] & BO \end{tikzcd}$$

we get an induced *strictly* commutative diagram of Thom spectra

(0.8)
$$\begin{tikzcd} TA \arrow[r, "Th_1"] \arrow[d, "Ti"'] & TB \arrow[d, "Tj"] \\ TY \arrow[r, "Th_2"'] & TX. \end{tikzcd}$$

Definition (0.9). We say that diagram (0.7) has *trivial Thomification* if the induced map of mapping cones of spectra (coming from 0.8)

$$TY \cup_{Ti} \widehat{TA} \to TX \cup_{Tj} \widehat{TB}$$

is null homotopic.

The following is standard obstruction theory.

PROPOSITION 0.10. *If diagram* (0.8) *has trivial Thomification then there exists a map of spectra* $g\colon TY \to TB$ *that homotopy lifts* Th_2 *and homotopy extends* Th_1 (i.e., $Tg \circ Ti \cong Th_1$ *and* $Tj \circ Tg \simeq Th_2$).

In the work to follow we will often study the k-dimensional homotopy type of a space or spectrum. To do this coherently we now establish some terminology.

Definition 0.11. 1. A *k-dimensional map* $f\colon X \to Y$ (between spaces or spectra) is a diagram of the sort

$$\begin{tikzcd} & X' \arrow[dl, "h"'] \arrow[dr, "f'"] & \\ X & & Y \end{tikzcd}$$

where h is a k-connected map. Also f will be referred to as a map "through dimension k".

2. Two k dimensional maps $f_1\colon X \to Y$ and $f_2\colon X \to Y$ are *homotopic* if

$$f_1 = \begin{tikzcd} & X_1' \arrow[dl, "h_1"'] \arrow[dr, "f_1'"] & \\ X & & Y \end{tikzcd}$$

and

$$f_2 = \begin{tikzcd} & X_2' \arrow[dl, "h_2"'] \arrow[dr, "f_2'"] & \\ X & & Y \end{tikzcd}$$

and there exist a space X'' and two k-connected maps $g_1\colon X'' \to X_1'$ and $g_2\colon X'' \to X_2'$ so that $h_1 \circ g_1 \simeq h_2 \circ g_2$ and $f_1' \circ g_1 \simeq f_2' \circ g_2$.

Remark. Observe that by using homotopy pullbacks we can compose two k-dimensional maps (and get a k-dimensional map) in a canonical way.

We will also have several occasions to study the $2k$-dimensional homotopy type of a k-connected space X. Recall that the Freudenthal suspension theorem yields that the natural inclusion $X \hookrightarrow \Omega^\infty\Sigma^\infty X$ (which is the zero$^{\text{th}}$ space of the suspension spectrum $\Sigma^\infty X$) is a $2k$-dimensional homotopy equivalence. So studying the $2k$-homotopy type of X is equivalent to studying the $2k$-homotopy type of $\Sigma^\infty X$ and so we will often not distinguish between these two problems in our notation.

A special, and important case of this situation is the study of the k-dimensional homotopy type of Eilenberg-MacLane spaces and spectra. We end this section by collecting some rather standard results about this situation.

Let K_n be a space that has the same k-dimensional homotopy type as an Eilenberg-MacLane space $K(\mathbf{Z}_2, n)$, $k > n$. The following is an easy exercise.

PROPOSITION 0.12. *Let $[X, K_n]_r$ denote the homotopy classes of r dimensional maps $X \to K_n$. Then for $n < r < k$ there is a bijective correspondence $[X, K_n]_r \leftrightarrow H^n(X; \mathbf{Z}_2)$.*

Note. The analogous statement holds for spectra as well.

Now suppose that K is a spectrum of the same k-dimensional homotopy type as the Eilenberg-MacLane spectrum $K(\mathbf{Z}_2)$.

Let X be a q-connected space (or spectrum) and let $X \wedge K$ denote the smash product spectrum.

PROPOSITION 0.13. a. *$X \wedge K$ has the same $(q + k)$-dimensional homotopy type as the wedge of Eilenberg-MacLane spectra $\bigvee_w \Sigma^{|w|}K(\mathbf{Z}_2)$, where w runs over a $\mathbf{Z}_2$-vector space basis for $\overline{H}_*(X; \mathbf{Z}_2)$ and $|w|$ denotes the dimension of w.*

b. *Suppose Y is also q-connected and let f: $X \wedge K \to Y \wedge K$ be any m-dimensional map of spectra $(m > k + q)$ so that in cohomology, f^*: $H^r(Y \wedge K) \to H^r(X \wedge K)$ $(r < m)$ maps A-module generators to A-module generators, where A is the* mod 2 *Steenrod algebra. Then the homotopy cofiber of f has the same $q + k - 1$ dimensional homotopy type as a wedge of Eilenberg-MacLane spectra. (See the remark below.)*

c. *Let f: $X \wedge K \to Y \wedge K$ be as in part* b *and suppose further that f_*: $H_r(X \wedge K) \to H_r(Y \wedge K)$ is a monomorphism for each $r < m$. Then there is a $(k + q - 1)$-dimensional "retraction map"* g: *$Y \wedge K \to X \wedge K$ so that the composite $g \circ f$: $X \wedge K \to X \wedge K$ is a $(k + q - 1)$-dimensional homotopy equivalence.*

Remark. By the homotopy cofiber of an m-dimensional map f: $A \to B$ given by the diagram

$$\begin{tikzcd} & A' \arrow[dl, "h"'] \arrow[dr, "f'"] & \\ A & & B \end{tikzcd}$$

we mean the mapping cone $B \cup_{f'} A'$. By abuse of notation we denote this by

$C(f)$. Notice we then have a "homotopy cofibration sequence through dimension m", $A \xrightarrow[f]{} B \to C(f)$. This definition is relevant to part b above.

This concludes our collection of elementary facts, notation, and terminology. We now proceed with the proof of the immersion conjecture.

1. A reduction to a lemma

As mentioned in the introduction our main objects of study are the Brown-Peterson universal normal bundle spaces, BO/I_n. We begin by recalling their theorem [8].

THEOREM 1.1. *For each $n \geq 0$ there exists a* CW *complex BO/I_n of cohomological dimension n together with a map ρ: $BO/I_n \to BO$ satisfying the following properties*:

a. *$H^*(BO/I_n) = H^*(BO)/I_n$ where I_n is the ideal of relations among the Stiefel-Whitney classes of normal bundles of n-manifolds, as described in the introduction.*

b. *ρ^*: $H^*(BO) \to H^*(BO/I_n) = H^*(BO)/I_n$ is the natural projection.*

c. *If M^n is any n-manifold and ν_M: $M \to BO$ classifies its stable normal bundle, then there exists a map $\tilde{\nu}_M$: $M \to BO/I_n$ making the following diagram homotopy commute*:

$$\begin{tikzcd} & BO/I_n \arrow[d, "\rho"] \\ M \arrow[ur, "\tilde{\nu}_M"] \arrow[r, "\nu_M"'] & BO. \end{tikzcd}$$

In [8] Brown and Peterson conjectured the following result, which as observed in the introduction implies the truth of the immersion conjecture.

THEOREM A. *There is a homotopy lifting ρ_n: $BO/I_n \to BO(n - \alpha(n))$ of ρ: $BO/I_n \to BO$.*

The object of this section is to state a crucial lemma and show how Theorem A follows from it. In order to state this lemma we first adopt some notational conventions.

Fix n and let $k = n - \alpha(n)$. Assume that $BO(k)$ and BO have cellular decompositions so that the inclusion i: $BO(k) \to BO$ is a cofibration. Finally, if X is a space, let X^I be the space of paths α: $I \to X$.

Define the pull-back space P_n by the rule

$$P_n = \{(x, y, \alpha) \in BO/I_n \times BO(k) \times BO^I \colon \alpha(0) = i(y) \text{ and } \alpha(1) = \rho(x)\}.$$

Define maps $P_n \to BO/I_n$ and $P_n \to BO(k)$ by projecting onto the first and second coordinates, respectively. Observe that

$$\begin{tikzcd} P_n \arrow[r] \arrow[d] & BO(k) \arrow[d, "i"] \\ BO/I_n \arrow[r, "\rho"'] & BO \end{tikzcd}$$

is a homotopy pull-back diagram. In particular there is a canonical homotopy E: $P \times I \to BO$ making this diagram commute. E is given by the formula

$$E((x, y, \alpha), t) = \alpha(t). \tag{1.2}$$

The following is our main lemma.

LEMMA B. *There exists a space X_n together with a map h_n: $X_n \to P$ satisfying the following properties*:

1. *If f_n and g_n are the compositions f_n: $X_n \xrightarrow[h_n]{} P \to BO(k)$ and g_n: $X_n \xrightarrow[h_n]{} P \to BO/I_n$, then there is a splitting map of 2-local Thom spectra σ_n: $MO/I_n \to TX_n$. That is, $1 \cong Tg_n \circ \sigma_n$: $MO/I_n \to TX_n \to MO/I_n$.*

2. *The following diagram of 2-local Thom spectra homotopy commutes*:

$$\begin{tikzcd} TX_n \arrow[r, "Th_n"] \arrow[d, "Tg_n"'] & TP_n \\ MO/I_n \arrow[r, "\sigma_n"'] & TX_n. \arrow[u, "Th_n"'] \end{tikzcd}$$

Remark. Observe that the composition $MO/I_n \xrightarrow[\sigma_n]{} TX_n \xrightarrow[Tf_n]{} MO(k)$ is a lifting of $T\rho$: $MO/I_n \to MO$. The fact that such a lifting exists was proved by Brown and Peterson in [8].

Our goal in this section is to prove that Lemma B implies that this lifting "de-Thom-ifies". More specifically, we will show that Lemma B implies the following strengthening of Theorem A.

THEOREM A′. *There is a lifting ρ_n: $BO/I_n \to BO(k)$ of ρ: $BO/I_n \to BO$ that makes the following diagram of 2-local Thom spectra homotopy commute*:

$$\begin{tikzcd} TX_n \arrow[rr, "Tf_n"] \arrow[dr, "Tg_n"'] & & MO(k) \\ & MO/I_n. \arrow[ur, "T\rho_n"'] & \end{tikzcd}$$

For the rest of this section we will assume the validity of Lemma B. Theorem A′ will then be proved in four steps. We begin in Section 1.a by recalling some results of Brown and Peterson [8] and of Snaith [24] that we will use extensively not only in our proof that Lemma B ⇒ Theorem A, but throughout the entire paper. In Section 1.b we make some preliminary observations and describe some inductive assumptions that will be used to prove Theorem A′. The inductive step is completed in Section 1.c for the case k is odd modulo a proof of a lemma which is done in 1.d. This case is technically easier than when k is even since when k is even $\pi_1 BO$ acts nontrivially on the homotopy fiber of the map $i\colon BO(k) \to BO$. We describe how to handle this technical problem in Section 1.e.

1.a. *De-Thom-ification obstructions and the stable splitting of BO.* As mentioned above, in order to prove Theorem A′ we will show that the composition $MO/I_n \xrightarrow[\sigma_n]{} TX_n \xrightarrow[Tf_n]{} MO(k)$ of the maps in Lemma B "de-Thom-ifies". In order to do this we will need results concerning how the cohomology of a Thom spectrum is changed when a cohomology class in the base space is killed. This problem has been studied by Mahowald [17], Browder [2], and by Brown and Peterson [8]. We now recall Brown and Peterson's results.

Suppose $f\colon B \to BO$ is a map which induces an isomorphism in homotopy groups through dimension k. Let V be a graded Z_2-vector space with $V_q = 0$ for $q \le k$, and let $K(V)$ be the corresponding Eilenberg-MacLane spectrum of type $K(Z_2)$ with the property that $\pi_*(K(V)) \cong V$. Represent $K(V)$ as an Ω-spectrum made up of spaces $\{K(V)_q\}$. Let $\gamma\colon B \to K(V)_1$ represent a sum of cohomology classes and let B' be the homotopy fibre of α. Thus we have a two-stage Postnikov system.

$$\begin{tikzcd} B' \arrow[r, "i"] & B \arrow[r] \arrow[d, "\gamma"] & BO \\ & K(V)_1. & \end{tikzcd}$$

Let T and T' denote the Thom spectra of the stable bundles classified by f and $f \circ i$ respectively. The cohomology $H^*(T/T')$ can, in a range of dimensions, be described as follows.

Let $A(BO)$ be the semi-tensor product of the Steenrod algebra $\mathscr{A}$ with $H^*(BO)$. That is

$$A(BO) = A \otimes H^*(BO)$$

with the algebra structure defined by

$$(a \otimes u)(b \otimes v) = \Sigma ab_i' \otimes (\chi(b_i'')u)v$$

where Δ is the Cartan coproduct $\Delta(b) = \Sigma b_i' \otimes b_i''$ in A. As in [8] we denote $a \otimes u$ by $a \circ u$.

Consider the homomorphism

$$\psi\colon (A(BO) \otimes V)^q \to H^{q+1}(T/T')$$

given by $\psi(a \circ u \otimes v) = a(u \cup \phi(\gamma^*(v_1))$ where $v_1 \in H^*(K(V)_1)$ corresponds to $v \in V$ and ϕ is the relative Thom isomorphism. In [8] Brown and Peterson proved the following.

THEOREM 1.3. *The map* $\psi\colon (A(BO) \otimes V)^q \to H^{q+1}(T/T')$ *is an isomorphism for* $q \leq 2k$.

The isomorphism $\psi\colon A(BO) \otimes V \to H^*(T/T')$ can be realized in a more geometric manner as follows. (See § 6 of [8].)

Consider the map

$$1 \times \gamma\colon B \to B \times K(V)_1.$$

This induces a map of pairs

$$c\colon (B, B') \to (B \times K(V)_1, B)$$

which Brown and Peterson showed induces cohomology isomorphism through dimension $2k + 1$.

Now an easy calculation (done in [8]) shows that in these dimensions the map $f\colon B \to BO$ induces a cohomology isomorphism:

$$c^*\colon H^*(BO \times K(V)_1, BO) \xrightarrow{\cong} H^*(B \times K(V)_1, B) \xrightarrow{\cong} H^*(B, B').$$

The induced map of Thom spectra

$$T(c)\colon T/T' \to T \wedge K(V)_1$$

therefore induces an isomorphism

$$\begin{aligned} T(c)^*\colon (\mathscr{A}(BO) \times V)^q \to H^{q+1}(MO \wedge K(V)_1) &\cong H^{q+1}(T \wedge K(V)_1) \\ &\xrightarrow{\cong} H^{q+1}(T/T') \end{aligned}$$

for $q \leq 2k$. It is also easy to verify that $T(c)^* = \psi$ as defined above.

We also remark that Brown and Peterson's verification of this does not depend on $K(V)_1$ being Eilenberg-MacLane. Indeed

$$T(c)^*\colon H^{q+1}(T \wedge K(V)_1) \to H^{q+1}(T/T')$$

would be an isomorphism for $q \leq 2k$ if $K(V)_1$ were replaced by any k-connected space Z, and B' were the homotopy fiber of a map $\gamma\colon B \to Z$. We leave it to the reader to verify this.

Notice that if ζ is a stable vector bundle over a space X classified by a map $g\colon X \to BO$ then $H^*(T\zeta)$ can be thought of as an $A(BO)$ module via the action

$$(a \circ \beta)(\phi(\alpha)) = a\phi(\alpha \cup g^*(\beta))$$

where $a \in \mathscr{A}$, $\beta \in H^*BO$, $\alpha \in H^*(X)$, and $\phi\colon H^*(X) \to H^*(T\zeta)$ is the Thom-isomorphism. Notice furthermore that this action is natural with respect to bundle maps, and also that in the situation described above the composition homomorphism

$$(A(BO) \otimes V)^q \to H^{q+1}(T/T') \to H^{q+1}(T)$$

is $A(BO)$-linear.

We observe that Brown and Peterson's results can be generalized slightly. More specifically let QBO be the infinite loop space $QBO = \varinjlim_N \Omega^N\Sigma^N BO$. Let $\xi\colon QBO \to BO$ be the map induced by the infinite loop space structure of BO, and let $TQBO$ be the associated Thom spectrum. Now suppose we have a 2-stage Postnikov tower over QBO:

$$\begin{tikzcd} Y' \arrow[r] & Y \arrow[r, "g"] \arrow[d, "\alpha"] & QBO \\ & K(V) & \end{tikzcd}$$

where g is an $[n/2]$-equivalence and $K(V)$ is $[n/2]$-connected. Then Brown and Peterson's techniques go through verbatim to yield n-dimensional homotopy equivalences

$$c(\alpha)\colon Y/Y' \to K(V) \wedge QBO^+ \quad \text{and} \tag{1.4}$$
$$Tc(\alpha)\colon TY/TY' \to K(V) \wedge TQBO.$$

Notice we can define the semi-tensor product algebra

$$A(QBO) = A \otimes H^*(QBO)$$

analogously to how $A(BO)$ was defined. Observe that $H^*(TY)$ is an $A(QBO)$-module. We leave the details of this generalization to the reader.

We will need to know more about the homotopy type of QBO. In particular we will need the following theorem of V. Snaith [24].

THEOREM 1.5. *For every integer q there is a homotopy equivalence of suspension spectra*

$$\Sigma^\infty BO \simeq \Sigma^\infty BO(2q) \vee \Sigma^\infty BO/BO(2q)$$

and therefore, there is an equivalence

$$QBO \simeq QBO(2q) \times Q(BO/BO(2q)).$$

If we are only concerned about stable homotopy equivalences through a fixed range of dimensions we can even say more.

THEOREM 1.6. *For every integer m there is a $2m$-dimensional homotopy equivalence*

$$QBO \simeq QBO(m) \times BO/BO(m).$$

This theorem will easily follow from our next observation.

LEMMA 1.7. *For every integer m the quotient space $BO/BO(m)$ has the $2m+1$-dimensional homotopy type of a product of Eilenberg-MacLane spaces of type $K(\mathbf{Z}_2, q)$ with $q > m$.*

Proof. Since $BO/BO(m)$ is m-connected Lemma 1.7 is a statement about its stable range and we may, without loss of generality, deal with suspension spectra.

Consider the cofibration sequence

$$BO(m+1)/BO(m) \to BO/BO(m) \to BO/BO(m+1).$$

Through dimension $2m+1$, $BO(m+1)/BO(m)$ is homotopy equivalent to $\Sigma^{m+1}MO$, which is a wedge of $K(\mathbf{Z}_2)$'s. Since the above cofibration sequence induces a short exact sequence in cohomology, we therefore get a splitting through dimension $2m+1$:

$$\Sigma^\infty BO/BO(m) \simeq \Sigma^{m+1}MO \vee \Sigma^\infty BO/BO(m+1).$$

By continuing in this manner (by next splitting $\Sigma^\infty BO/BO(m+1)$) we prove the lemma.

Proof of 1.6. When m is even, 1.6 follows from 1.5 since $BO/BO(m)$, being m-connected, is $2m$-homotopy equivalent to $Q(BO/BO(m))$.

If m is odd, say $m = 2q+1$, then through dimension $2m+1$ the following cofibration sequences:

$$BO(2q) \to BO(2q+1) \to BO(2q+1)/BO(2q)$$

and

$$BO(2q+2)/BO(2q+1) \to BO/BO(2q+1) \to BO/BO(2q+2)$$

split, since in this range $BO(2q+1)/BO(2q)$ and $BO(2q+2)/BO(2q+1)$ are Eilenberg-MacLane. Now 1.6 follows easily from 1.5.

Observe that we can now prove the following stable version of Theorem A.

COROLLARY 1.8. *There exists a stable map* r_m: $\Sigma^\infty BO/I_n \to \Sigma^\infty BO(k)$ *that homotopy lifts* $\Sigma^\infty\rho$: $\Sigma^\infty BO/I_n \to \Sigma^\infty BO$.

Proof. For $n \leq 3$ we leave this as an exercise for the reader. For $n \geq 4$, n is less than $2k$ $(= 2(n - \alpha(n))$; so 1.1 and 1.7 imply that the obstructions to finding r_n are purely cohomological. But all such cohomological obstructions vanish by Theorem 1.1, part a.

1.b. *Preliminary observations.* We are now ready to begin our proof that Lemma B implies Theorem A. In this section we make some preliminary observations and set up an inductive argument. The inductive step will be completed in the next section.

The central idea is to play off the Thom spectrum level information given in the hypotheses of Lemma B against the Snaith splitting (1.6).

More specifically, consider the homotopy

$$E \circ h_n\colon X_n \times I \to P \times I \to BO$$

where E is the homotopy of (1.2). Note that $E \circ h_n$ is a homotopy from $i \circ f_n$: $X_n \to BO(k) \to BO$ to $\rho \circ g_n$: $X_n \to BO/I_n \to BO$. This defines a homotopy

$$H\colon X_n \times I \xrightarrow[E \circ h_n]{} BO \hookrightarrow QBO \xrightarrow{s} QBO(k)$$

where s: $QBO \to QBO(k)$ is the Snaith splitting, (1.6). Notice that since $QBO(k) \underset{Qi}{\hookrightarrow} QBO$ is a cofibration we can choose the splitting s so that $s \circ Qi$: $QBO(k) \to QBO(k)$ is the identity. Observe that the restriction of H to $X_n \times 0$ is Qf_n: $X_n \underset{f_n}{\to} BO(k) \hookrightarrow QBO(k)$.

This homotopy defines a map of the mapping cylinder, which, by the abuse of notation we can also call H:

$$H\colon BO/I_n \cup_{g_n} X_n \times I \to QBO(k)$$

where H restricted to BO/I_n is the composition $Q\rho_n$: $BO/I_n \underset{\rho}{\to} BO \hookrightarrow QBO \underset{s}{\to} QBO(k)$.

We shall inductively construct liftings up a Postnikov tower, using the Snaith splitting to show that Lemma B implies the following.

THEOREM A″. *There exists a map* ρ_n: $BO/I_n \to BO(k)$ *together with a homotopy* G: $BO/I_n \times I \to QBO(k)$ *between* $Q\rho_n$: $BO/I_n \underset{\rho}{\to} BO \subset QBO \underset{s}{\to} QBO(k)$ *and* $BO/I_n \underset{\rho_n}{\to} BO(k) \subset QBO(k)$ *that satisfy the follow-*

ing property. The map of pairs

$$G \cup H\colon \left(I \times BO/I_n \cup_{g_n} X_n \times I,\ BO/I_n \amalg X_n\right) \to (QBO(k), BO(k))$$

induces a null homotopic map of quotients of Thom spectra

$$\Sigma TX \simeq T\left(I \times BO/I_n \cup_g X \times I\right)/MO/I_n \vee TX_n \to TQBO(k)/MO(k).$$

Here BO/I_n is included in $I \times BO/I_n$ as $1 \times BO/I_n$ and X_n is included in $X_n \times I$ as $X_n \times 1$.

Remark. Notice that Theorem A″ clearly implies Theorem A′ and thus Theorem A. Our first step in proving this is the following.

LEMMA 1.9. *The map of pairs*

$$H\colon \left(BO/I_n \cup_g X \times I, X_n\right) \to (QBO(k), BO(k))$$

has null homotopic Thom-ification. That is, the induced map of quotients of Thom spectra

$$T\left(BO/I_n \cup_g X \times I\right)/TX_n \to TQBO(k)/MO(k)$$

is null homotopic.

Proof. The commutative diagram

$$\begin{tikzcd} X_n \times 1 \arrow[r, "f_n"] \arrow[d] & BO(k) \arrow[d] \\ BO/I_n \cup X \times I \arrow[r, "H"'] & QBO(k) \end{tikzcd}$$

strictly factors as

$$\begin{tikzcd} X_n \times 1 \arrow[r, "h_n"'] \arrow[d, hook] & P \times 1 \arrow[r] \arrow[d, hook] & BO(k) \arrow[d] \\ BO/I_n \cup_g X_n \times I \arrow[r, "1 \cup h_n \times 1"'] & BO/I_n \cup P \times I \arrow[r, "\overline{E}"'] & QBO(k) \end{tikzcd}$$

where $\overline{E}$ is induced by the homotopy

$$P \times I \underset{E}{\to} BO \to QBO(k).$$

Thus, it is sufficient to prove that the map of pairs

$$\left(BO/I_n \cup_g X \times I, X\right) \to \left(BO/I_n \cup P \times I, P\right)$$

induces the trivial map of quotients of Thom spectra

$$\Phi\colon T(BO/I_n \cup X \times I)/TX \to T(BO/I_n \cup P \times I)/TP.$$

To do this observe that the hypotheses of Lemma B yield a homotopy commutative diagram

$$\begin{tikzcd}
T(BO/I_n \cup X \times I)/TX \arrow[rr, "\Phi"] \arrow[d] & & T(BO/I_n \cup P_n \times I)/TP_n \arrow[d] \\
\Sigma TX_n \arrow[rr, "\Sigma Th_n"'] \arrow[dr, "\Sigma Tg_n"'] & & \Sigma TP_n \\
& \Sigma MO/I_n. \arrow[ur, "Th_n \circ \sigma_n"'] &
\end{tikzcd}$$

But since the left hand vertical map forms a cofibration sequence, the composition

$$T(BO/I_n \cup X \times I/TX) \underset{\Phi}{\to} T(BO/I_n \cup P \times I)/TP \to \Sigma TP$$

is null homotopic. Lemma 1.9 will therefore be a corollary of the following observation.

LEMMA 1.10. *There is a homotopy splitting*

$$\Sigma TP_n \simeq T(BO/I_n \cup_g X \times I)/TP_n \vee \Sigma MO/I_n.$$

Proof. The splitting is given by the composition $\Sigma MO/I_n \underset{\sigma_n}{\to} \Sigma TX_n \underset{Th_n}{\to} \Sigma TP_n$.

To complete the proof that Lemma B implies Theorem A we will use a Moore-Postnikov tower for the map $BO(k) \to BO$. We will work first with the case k odd because in this case the fibration sequence

$$V_k \to BO(k) \to BO$$

is well-known to be simple. That is, $\pi_1 BO = \mathbf{Z}_2$ acts trivially on $\pi_* V_k$. (Here $V_k = \varinjlim_r V_{r,r-k}$, where $V_{m,j}$ is the Stiefel manifold of j-frames in $\mathbf{R}^m$.) We will describe the necessary modifications for our argument for the case k even in Section 1.d.

Now for technical reasons (that will become clear later) we will begin our Postnikov tower by killing all the cohomology in $BO/BO(k)$ immediately, rather than beginning by just killing the Euler class. More specifically: Let $B_0 = BO$ and let B_1 be the homotopy fiber of the map

$$k_1\colon BO \xrightarrow[\Pi w_{k+r}]{} \prod_{r \geq 1} K(\mathbf{Z}_2, k+r) = K_1,$$

where w_j is the j^{th} Stiefel-Whitney class. We then have a map $BO(k) \to B_1$ which still induces a nilpotent homotopy fibration (i.e., $\pi_1 B_1$ acts nilpotently on $\pi_*(B_1, BO(k))$. We can therefore construct a Postnikov tower for the homotopy fibration $BO(k) \to B_1$. So we have a tower of homotopy fibration sequences

$$\begin{array}{ccccccccccc} BO(k) & \to & \cdots & \to & B_i & \to B_{i-1} & \to & \cdots & \to & B_1 & \to B_0 \\ & & & & \downarrow{\scriptstyle k_i} & & & & & \downarrow{\scriptstyle k_1} & \\ & & & & K_i & & & & & K_1 & \end{array}$$

where each K_i is a product of Eilenberg-MacLane spaces. Now for $n > 3$, n is less than or equal to $2k = 2(n - \alpha(n))$, and it is well known that $\pi_q V_k$ and therefore $\pi_q(B_1, BO(k))$ are finite abelian 2-groups for $q \leq n$. (See [16] for example.) By modifying this tower if necessary, we may then assume that through dimension n each K_i is a product of Eilenberg-MacLane spaces of the form $K(\mathbf{Z}_2, q)$. (See [14], [17] for a discussion of modified Moore-Postnikov towers.)

We will have to keep careful track of various homotopies involved in liftings up this tower so that we will need more explicit descriptions of the spaces B_i in this tower. We inductively assume that through the i^{th} stage our tower

$$\begin{array}{ccccc} B_i & \to B_{i-1} & \to & \cdots & \to B_0 \\ \downarrow{\scriptstyle k_i} & & & & \\ K_i & & & & \end{array}$$

has the additional properties that not only are $B_j \to B_{j-1} \xrightarrow[k_j]{} K_j$ *homotopy* fibration sequences, but the maps e_j: $BO(k) \to B_j$ are assumed to be *cofibrations*, and that the k-invariants k_j: $B_{j-1} \to K_j$ strictly factor as a composition

$$B_j \to B_j/BO(k) \xrightarrow[\bar{k}_j]{} K_{j+1}.$$

We now show how B_{i+1} is constructed so that these additional properties are satisfied. Since, by assumption e_i: $BO(k) \to B_i$ is a cofibration, we can define our k-variant k_{i+1}: $B_i \to K_{i+1}$ to factor as $B_i \to B_i/BO(k) \xrightarrow[\bar{k}_i]{} K_{i+1}$. Let $\overline{B}_{i+1}$ be the homotopy fiber of k_i. That is,

$$\overline{B}_{i+1} = \{(x, \alpha) \in B_i \times K_{i+1}^I \quad \text{such that } \alpha(0) = k_i(x) \text{ and } \alpha(1) = *\}$$

where $* \in K_{i+1}$ is a distinguished basepoint. We have an obvious map $\bar{e}_{i+1}$: $BO(k) \to \overline{B}_{i+1}$ given by $\bar{e}_{i+1}(z) = (e_i(z), \varepsilon_*)$, where ε_* is the constant path at $* \in K_{i+1}$. Finally, define

$$B_{i+1} = \overline{B}_{i+1} \cup_{\bar{e}_{i+1}} BO(k) \times I$$

and define e_i: $BC(k) \to B_{i+1}$ to be the inclusion as $BO(k) \times 1$. Notice that we can define $B_{i+1} \to B_i$ by $B_{i+1} \underset{r}{\to} \overline{B}_{i+1} \to B_i$ where r is the obvious retraction. Under this map e_{i+1} strictly lifts e_i. This completes the inductive step and allows us to obtain a tower of homotopy fibrations

$$BO(k) \to \cdots \to B_1 \to B_{i-1} \to \cdots \to B_0 = BO$$

with the properties described above.

Before we begin our proof of Theorem A″ we make some observations about this tower.

First, notice that the composition $QB_i \to QBO \underset{s}{\to} QBO(k)$ is a stable splitting of the map e_i: $BO(k) \to B_i$. Thus, through dimension n, $QB_i \simeq QBO(k) \times B_i/BO(k)$. Furthermore, the map j: $B_i \to QBO(k) \times B_i/BO(k)$ given by the product of the composition $B_i \subset QB_i \to QBO \underset{s}{\to} QBO(k)$ and the projection $B_i \to B_i/BO(k)$ is homotopic to the stabilization $B_i \subset QB_i$.

Also by our construction, for $i > 0$ the composition $B_i \to BO \to BO/BO(k)$ induces the zero homomorphism in cohomology and is therefore null homotopic through dimension n. Thus $QB_i \to QBO$ factors homotopically through $QBO(k)$, and so if TQB_i is the Thom spectrum of the bundle classified by $QB_i \to QBO \underset{\xi}{\to} BO$ then $TQB_i \simeq TQBO(k) \wedge B_i/BO(k)^+$.

Let j': $B_i' \to QBO(k) \times B_i/BO(k)$ be j: $B_i \to QBO(k) \times B_i/BO(k)$ made into a fibration. That is, $B_i' = \{(x, \alpha) \in B_i \times (QBO(k) \times B_i/BO(k))^I$ such that $\alpha(0) = j(x)\}$ and j': $B_i' \to QBO(k) \times B_i/BO(k)$ is given by $j'(x, \alpha) = \alpha(1)$.

Finally let $F_i \to QBO(k)$ be the restriction of the fibration j': $B_i' \to QBO(k) \times B_i/BO(k)$ over $QBO(k)$. Clearly the composition

$$BO(k) \overset{e_i}{\hookrightarrow} B_i \hookrightarrow B_i'$$

has image in F_i. Call the induced map u_i: $BO(k) \to F_i$.

We now have the machinery to prove Theorem A″.

We begin the proof (and end this section) by stating our inductive assumptions.

Inductive assumptions 1.11. There exists a homotopy lifting ρ_i: $BO/I_n \to B_i'$ of ρ with the following properties.

(1) The diagram

$$\begin{tikzcd}
BO/I_n \amalg X_n \arrow[rr, "\rho_i \amalg f_{n,i}"] \arrow[d, hook] & & B_i' \arrow[d] \\
BO/I_n \cup_g X_n \times I \arrow[r, "H"'] & QBO(k) \arrow[r, hook] & QBO(k) \times B_i/BO(k)
\end{tikzcd}$$

strictly commutes, where $f_{n,i}\colon X_n \to B_i'$ is the composition $X_n \xrightarrow[f_n]{} BO(k) \xrightarrow[e_i]{} B_i \hookrightarrow B_i'$.

(2) The map of pairs

$$(BO/I_n \cup X \times I, BO/I_n \amalg X_n) \to (QBO(k) \times B_i/BO(k), B_i')$$

has trivial Thom-ification.

Remarks. (1) We leave the proof of the fact that such a map $\rho_0\colon BO/I_n \to B_0' \simeq BO$ exists as an exercise for the reader.

(2) It is clear that once the inductive step is complete we will be able to conclude the truth of Theorem A′.

1.c. *Lemma* B $\Rightarrow$ *Theorem* A. In this section we complete the proof of Theorem A″ in the case k is odd by completing the inductive step. So throughout this section we will operate under induction assumptions (1.11).

First observe that since $F_i \to QBO(k)$ is the restriction of the fibration $j_i'\colon B_i' \to QBO(k) \times B_i/BO(k)$, by property (1) of the inductive assumptions, ρ_i has its image in F_i.

LEMMA 1.12. *The map of pairs* H: $(BO/I_n \cup X_n \times I,\ BO/I_n \amalg X_n) \to (QBO(k), F)$ *has trivial Thom-ification.*

Proof. For notational ease, we write $TH\colon \Sigma TX_n \to TQBO(k) \cup \widehat{TF}$ for the induced map of quotients

$$T([BO/I_n \cup X_n \times I] \cup (BO/I_n \amalg X_n) \times I)/MO/I_n \vee TX_n$$
$$\to T(QBO(k) \cup F \times I)/TF.$$

To study this map, first observe that F_i is the fiber of the fibration

$$B_i' \to QBO(k) \times B_i/BO(k) \to B_i/BO(k)$$

where the second map is the projection. Hence by the results of Section 1.a we have a homotopy cofibration sequence of Thom spectra, through dimension n,

$$TF_i \to TB_i' \to B_i/BO(k) \wedge MO.$$

Similarly we have a cofibration sequence

$$TQBO(k) \subset T(QBO(k) \times B_i/BO(k)) = TQBO(k) \wedge B_i/BO(k)^+$$
$$\to TQBO(k) \wedge B_i/BO(k).$$

Moreover, since $B_i/BO(k)$ is $[n/2]$-connected, through dimension n, $B_i/BO(k) \wedge TQBO(k) \simeq B_i/BO(k) \wedge TQBO$.

Thus, if by the notation QB_i we now mean the space $QBO(k) \times B_i/BO(k)$, we have a commutative diagram of n-dimensional homotopy cofibration sequences.

$$\begin{array}{ccccc} TF_i & \to & TB_i' & \longrightarrow & B_i/BO(k) \wedge MO \\ \downarrow & & \downarrow & & \downarrow \\ TQBO(k) & \to & TQB_i & \longrightarrow & B_i/BO(k) \wedge TQBO \\ \downarrow & & \downarrow & & \downarrow \\ TQBO(k) \cup \widehat{TF} & \to & TQB_i \cup \widehat{TB_i} & \to & B_i/BO(k) \wedge TQBO/MO. \end{array}$$

Now by our inductive hypotheses we know that the composition $\Sigma TX_n \underset{TH}{\to} TQBO(k) \cup \widehat{TF} \to TQB_i \cup \widehat{TB}_i$ is null. Thus it is enough to show that through dimension n, $TQBO(k) \cup \widehat{TF}$ splits off of $TQB_i \cup \widehat{TB}_i$. Such a splitting is guaranteed by a splitting map $B_i/BO(k) \wedge TQBO/MO \to TQB_i \cup \widehat{TB}_i$, which is supplied by the composition

$$\begin{aligned} B_i/BO(k) \wedge TQBO/MO &\to B_i/BO(k) \wedge TQBO \\ &\simeq_n B_i/BO(k) \wedge TQBO(k) \subset B_i/BO(k)^+ \wedge TQBO(k) \\ &= TQB_i \to TQB_i \cup \widehat{TB}_i. \end{aligned}$$

The first map is given by the splitting of Eilenberg-MacLane spectra $TQBO \simeq MO \vee TQBO/MO$, and by $\simeq_n$ we mean n-dimensional homotopy equivalence.

Our goal is to lift the map $\rho_i\colon BO/I_n \to F_i$ to B_{i+1} so as to satisfy the inductive hypotheses. Toward this end let $\overline{R}_{i+1}$ be the (strict) pullback of the fibration $B_i' \to QBO(k) \times B_i/BO(k)$ over $QBO(k) \times B_{i+1}/BO(k)$ so we have a strictly commutative diagram of pullbacks.

$$\begin{array}{ccccc} F_i & \to & \overline{R}_{i+1} & \longrightarrow & B_i' \\ \downarrow & & \downarrow & & \downarrow \\ QBO(k) & \to & QBO(k) \times B_{i+1}/BO(k) & \to & QBO(k) \times B_i/BO(k) \\ & & \downarrow \simeq & & \\ & & QB_{i+1}. & & \end{array}$$

We therefore have a strictly commutative diagram:

$$\begin{array}{ccccc} BO/I_n \amalg X_n & \longrightarrow & F_i & \longrightarrow & \overline{R}_{i+1} \\ \downarrow & & \downarrow & & \downarrow \\ BO/I_n \cup X_n \times I & \to & QBO(k) & \to & QB_{i+1}, \end{array}$$

and by the above lemma, the induced map of pairs

$$(BO/I_n \cup X_n \times I,\ BO/I_n \amalg X_n) \to (QB_{i+1}, \overline{R}_{i+1})$$

has trivial Thomification. (Again here, by the symbol QB_{i+1}, I mean $QBO(k) \times B_{i+1}/BO(k)$).

In order to refine this map of pairs to complete the inductive step, we study the space $\overline{R}_{i+1}$ and its relation to B_{i+1} more carefully. We recall how these spaces were defined.

$$\overline{B}_{i+1} = \{(x, \alpha) \in B_i \times K_{i+1}^I \text{ such that } \alpha(0) = k_i(x) \text{ and } \alpha(1) = *\},$$

$\bar{e}_{i+1}$: $BO(k) \to \overline{B}_{i+1}$ is given by $\bar{e}_{i+1}(z) = (e_i(z), \varepsilon_*)$,

$$B_{i+1} = \overline{B}_{i+1} \cup_{\bar{e}_{i+1}} BO(k) \times I$$

and e_{i+1}: $BO(k) \to B_{i+1}$ is the inclusion of $BO(k)$ as $BO(k) \times 1$.

Now similarly

$$B_i' = \{(x, \alpha) \in B_i \times (QBO(k) \times B_i/BO(k))^I \text{ such that } \alpha(0) = j(x)\}$$

where j: $B_i \to QBO(k) \times B_i/BO(k)$ and so therefore

$$F_i = \{(x, \alpha) \in B_i' \text{ such that } \alpha(1) \in QBO(k)\}, \quad \text{and}$$

$$\overline{R}_{i+1} = \{(x, \alpha, y) \in B_i \times (QBO(k) \times B_i/BO(k))^I \times (B_{i+1}/BO(k) \times QBO(k))$$

such that $\alpha(0) = j(x)$ and $\alpha_2(1) = p_{i+1}(y)\}$, where α_2 is the projection of α onto $B_i/BO(k)$ and p_{i+1} is induced by the map $B_{i+1}/BO(k) \to B_i/BO(k)$.

Now the composition $B_{i+1}/BO(k) \xrightarrow[p_{i+1}]{} B_i/BO(k) \xrightarrow[k_{i+1}']{} K_{i+1}$ is null homotopic; so let

$$N: \widehat{B_{i+1}/BO(k)} \to K_{i+1}$$

be a based null homotopy, where $\hat{Y}$ denotes the cone on Y. For each triple $(x, \alpha, y) \in \overline{R}_{i+1}$ let $\bar{\alpha}$: $I \to K_{i+1}$ be the path given by the composition $I \xrightarrow[\alpha]{} QBO(k) \times B_i/BO(k) \to B_i/BO(k) \xrightarrow[k_i']{} K_{i+1}$. Also, by taking the adjoint of N, n: $B_{i+1}/BO(k) \to K_{i+1}^I$ we then have n_y: $I \to K_{i+1}$, a path in K_{i+1} from $\bar{\alpha}(1) = k_i' \circ p_{i+1}(y)$ to the base point $* \in K_{i+1}$. Finally let α': $I \to K_{i+1}$ be the path sum of $\bar{\alpha}$ and n_y. The correspondence

$$(x, \alpha, y) \to (x, \alpha')$$

defines a map

$$r_{i+1}: \overline{R}_{i+1} \to \overline{B}_{i+1}$$

so that the diagram

$$\begin{tikzcd} BO(k) \arrow[r, "u_i"] \arrow[dr, "\bar{e}_{i+1}"'] & F_i \arrow[r] & \bar{R}_{i+1} \arrow[dl, "r_{i+1}"] \\ & \bar{B}_{i+1} & \end{tikzcd}$$

strictly commutes. (Recall both u_i and $\bar{e}_{i+1}$ are defined by formulas of the form $z \to (l_i(z), \varepsilon)$ where ε is a constant path.) Let $\bar{v}_i$: $BO(k) \to \bar{R}_{i+1}$ be the composition $BO(k) \xrightarrow{u_i} F_i \to \bar{R}_{i+1}$. If we then let $R_{i+1} = \bar{R}_{i+1} \cup_{\bar{v}_i} BO(k) \times I$ and let v_i: $BO(k) = BO(k) \times 1 \subset R_{i+1}$ be the inclusion, we have a strictly commutative diagram:

Diagram 1.12.

$$\begin{tikzcd} R_{i+1} \arrow[r] \arrow[d] & B_{i+1} \arrow[d] \\ QBO(k) \times R_{i+1}/BO(k) \arrow[r] & QBO(k) \times B_{i+1}/BO(k). \end{tikzcd}$$

Our next goal is to change the map $R_{i+1} \to QBO(k) \times R_{i+1}/BO(k)$ so that it homotopy lifts the fibration map $\bar{R}_{i+1} \to QB_{i+1} = QBO(k) \times B_{i+1}/BO(k)$. Then we will show that the composition $BO/I_n \to F_i \to \bar{R}_{i+1}$ can be adjusted if necessary to obtain a map of pairs $(BO/I_n \cup X_n \times I,\ X_n \amalg BO/I_n) \to (QBO(k) \times R_{i+1}/BO(k), R_{i+1})$ that has trivial Thomification. To do this, we first need to study the relationship between R_{i+1} and B_{i+1} more carefully.

Since the diagram

$$\begin{tikzcd} R_{i+1} \arrow[r] \arrow[d] & B_i \arrow[d] \\ QB_{i+1} \arrow[r] & QB_i \end{tikzcd}$$

is a homotopy pullback square, the homotopy fiber of $R_{i+1} \to B_i$ is equivalent to the homotopy fiber of $QB_{i+1} \to QB_i$, which is $\Omega K_{i+1} \wedge BO^+$ through dimension n. Since the fiber of $B_{i+1} \to B_i$ is ΩK_{i+1}, a quick diagram chase will yield that the fiber of

$$R_{i+1} \to B_{i+1}$$

is $\Omega K_{i+1} \wedge BO$. Similarly one can check that the stable fiber of $\Sigma^\infty R_{i+1} \to$

$\Sigma^\infty B_{i+1}$ is $(\Omega K_i \wedge BO) \wedge BO^+$ and that the induced map from the fiber to the stable fiber $\Omega K_{i+1} \wedge BO \to \Omega K_{i+1} \wedge BO \wedge BO^+$ induces a surjection in cohomology so that there is a splitting $\Omega K_{i+1} \wedge BO \wedge BO^+ \underset{s}{\to} \Omega K_{i+1} \wedge BO$. Also observe that the fibration $R_{i+1} \to B_{i+1}$ has a section since the composition $B_{i+1} \to B_i \to K_{i+1} \wedge BO^+$ is null homotopic.

Thus $\Sigma^\infty R_{i+1} \simeq \Sigma^\infty B_{i+1} \vee \Omega K_{i+1} \wedge BO \wedge BO^+$ and the composition

$$R_{i+1} \to \Omega K_{i+1} \wedge BO \wedge BO^+ \underset{s}{\to} \Omega K_{i+1} \wedge BO$$

yields a homotopy equivalence $R_{i+1} \simeq B_{i+1} \times \Omega K_{i+1} \wedge BO$. The following is immediate.

LEMMA 1.13. a. $R_{i+1} \simeq B_{i+1} \times \Omega K_{i+1} \wedge BO$, *where the projection of this homotopy equivalence onto* B_{i+1} *is the map* $R_{i+1} \to B_{i+1}$ *of diagram* 1.12.

b. $QR_{i+1} \simeq QB_{i+1} \times (\Omega K_{i+1} \wedge BO) \wedge BO^+$
$\simeq QBO(k) \times B_{i+1}/BO(k) \times (\Omega K_{i+1} \wedge BO) \wedge BO^+$, *through dimension* n.

c. $R_{i+1}/BO(k) \simeq_n B_{i+1}/BO(k) \vee (\Omega K_{i+1} \wedge BO) \wedge BO^+$.

d. *The fibration* $\overline{R}_{i+1} \to QB_{i+1} = QBO(k) \times B_{i+1}/BO(k)$ *described above is homotopic, via the homotopy equivalence in* (a) *to the Pontrjagin product of the inclusion* $B_{i+1} \to QB_{i+1}$ *and the map* $\Omega K_{i+1} \wedge BO \hookrightarrow \Omega K_{i+1} \wedge BO^+ \hookrightarrow QB_{i+1}$, *where the second map in this composition is the inclusion of the fiber of* $QB_{i+1} \to QB_i$.

From now on let the symbol QR_{i+1} denote the space

$$QBO(k) \times B_{i+1}/BO(k) \times (\Omega K_{i+1} \wedge BO) \wedge BO^+.$$

We can then define a new map $\bar{q}$: $R_{i+1} \to QR_{i+1}$ as the product of the above fibration map $\overline{R}_{i+1} \to QB_{i+1}$ and the composition $R_{i+1} \to R_{i+1}/B_{i+1} \xrightarrow{\simeq} (\Omega K_{i+1} \wedge BO) \wedge BO^+$. Now let q: $R'_{i+1} \to QR_{i+1}$ denote this map turned into a fibration in the canonical way. Similarly let the symbol QB_{i+1} denote the space $QBO(k) \times B_{i+1}/BO(k)$ and let $B'_{i+1} \to QB_{i+1}$ be the stabilization map turned into a fibration. By 1.13 and the definition of $\bar{q}$ one can easily construct a map $QR_{i+1} \to QB_{i+1}$ so that we have a strictly commutative diagram of fibrations

$$\begin{tikzcd} R'_{i+1} \arrow[r] \arrow[d, "q"'] & B'_{i+1} \arrow[d] \\ QR_{i+1} \arrow[r] & QB_{i+1}. \end{tikzcd}$$

Hence, the inductive step will be complete once we prove

THEOREM 1.14. *There exists a map ρ'_{i+1}: $BO/I_n \to R'_{i+1}$ in the homotopy class of $\bar{\rho}_{i+1}$: $BO/I_n \underset{\rho_i}{\to} F_i \to \overline{R}_{i+1} \subset R'_{i+1}$ with the following properties*:

a. *The diagram*

$$\begin{tikzcd} BO/I_n \amalg X_n \arrow[r, "\rho'_{i+1} \amalg f'_{i+1}"] \arrow[d] & R'_{i+1} \arrow[d, "q"] \\ BO/I_n \cup X \times I \underset{H}{\to} QBO(k) \subset & QR_{i+1} \end{tikzcd}$$

strictly commutes, where f'_{i+1}: $X_n \to R'_{i+1}$ is the composition $X_n \underset{f}{\to} BO(k) \underset{v_{i+1}}{\to} R_{i+1} \subset R'_{i+1}$.

b. *The map of pairs $(BO/I_n \cup X_n \times I,\ BO/I_n \amalg X_n) \to (QR_{i+1}, R'_{i+1})$ has trivial Thomification.*

Proof. Since

$$R'_{i+1} = \Big\{(z,\alpha) \in R_{i+1} \times \big(QBO(k) \times B_{i+1}/BO(k)$$

$$\times (\Omega K_{i+1} \wedge BO) \wedge BO^+\big)^I \text{ such that } \alpha(0) = \bar{q}(z)\Big\}$$

where $\bar{q}$: $R_{i+1} \to QR_{i+1}$ is the map described earlier, and since $R_{i+1} = \overline{R}_{i+1} \cup_{\bar{v}_i} BO(k) \times I$, $\overline{R}_{i+1}$ sits naturally in R'_{i+1} as a subspace. With the definitions of $\overline{R}_{i+1}$ and B_{i+1} given above, it is an easy exercise to construct an explicit retraction r: $R'_{i+1} \to \overline{R}_{i+1}$ that maps $BO(k) = BO(k) \times 1 \subset R_{i+1} \subset R'_{i+1}$ to $\overline{R}_{i+1}$ via $\bar{v}_i$, and so that the following diagram strictly commutes:

$$\begin{tikzcd} R'_{i+1} \arrow[r, "r"] \arrow[d, "q"'] & \overline{R}_{i+1} \arrow[d] \\ QR_{i+1} \arrow[r, "\text{project}"'] & QB_{i+1}. \end{tikzcd}$$

Our first main step in proving Theorem 1.14 is showing the following:

LEMMA 1.15. *The map of pairs H: $(BO/I_n \cup_{g_n} X_n \times I,\ BO/I_n \amalg X_n) \to (QB_{i+1}, \overline{R}_{i+1})$ lifts to a map of pairs*

$$H_N\colon \big(BO/I_n \cup_{g_n} X_n \times I,\ BO/I_n \amalg X_n\big) \to \big(QR_{i+1}, R'_{i+1}\big).$$

Proof. Observe that the diagram

$$\begin{tikzcd} X_n \arrow[r, "f_n"'] \arrow[d, hook] \arrow[rrrr, bend left, "f'_{i+1}"] & BO(k) \arrow[r, hook] & \overline{R}_{i+1} \cup_{v_i} BO(k) \times I = R_{i+1} \arrow[r, phantom, "\subset"] & & R'_{i+1} \arrow[d, "q"] \\ BO/I_n \cup_{g_n} X \times I \arrow[r] & QBO(k) \arrow[rrr, hook] & & & QR_{i+1} \end{tikzcd}$$

strictly commutes. However, the diagram

$$\text{(*)} \quad \begin{tikzcd} BO/I_n \arrow[r, "\bar{\rho}_{i+1}"'] \arrow[d, hook] & \overline{R}_{i+1} \arrow[r, hook] \arrow[d] & R'_{i+1} \arrow[d, "q"] \\ BO/I \cup X \times I \arrow[r] & QBO(k) \arrow[r] & QR_{i+1} \end{tikzcd}$$

need *not* strictly commute, although the outside rectangle of the following diagram does commute:

$$\begin{tikzcd} BO/I_n \arrow[r] \arrow[d, hook] & R'_{i+1} \arrow[r, "r"] \arrow[d, "q"] & \overline{R}_{i+1} \arrow[d] \\ BO/I \cup_g X \times I \arrow[r] & QR_{i+1} \arrow[r] & QB_{i+1}. \end{tikzcd}$$

However, diagram (*) does homotopy commute. To see this it is enough by Lemma 1.13, part d to show that the composition

$$\begin{aligned} BO/I_n \to R'_{i+1} \to QR_{i+1} \simeq QB_{i+1} \times (\Omega K_{i+1} \wedge BO) \wedge BO^+ \\ \underset{\text{project}}{\to} \Omega K_{i+1} \wedge BO \wedge BO^+ \end{aligned}$$

is null homotopic. Now this is a cohomology question, and so by the Thom isomorphism we need only check that

$$MO/I_n \to TR'_{i+1} \to TQR_{i+1} \simeq TQB_{i+1} \wedge [\Omega K_{i+1} \wedge BO \wedge BO^+]^+$$

actually factors through TQB_{i+1}. This, however, follows because since the map of pairs $(BO/I_n \cup_g X \times I,\ BO/I_n \amalg X) \to (QB_{i+1}, \overline{R}_{i+1})$ has trivial Thom-ification, the map $TX_n \to T\overline{R}_{i+1} \simeq TR'_{i+1}$ is homotopic to the map $TX \underset{Tg}{\to} MO/I_n \to TR'_{i+1}$, and thus $MO/I_n \to TR_{i+1}$ is homotopic to $MO/I_n \underset{\sigma_n}{\to} TX_n \to TR_{i+1}$ which factors through TQB_{i+1}.

So let $N\colon BO/I_n \times I \to \Omega K_{i+1} \wedge BO \wedge BO^+$ be a null homotopy of the composition

$$BO/I_n \to \bar{R}_{i+1} \hookrightarrow R'_{i+1} \to QR_{i+1} = QB_{i+1} \times \Omega K_{i+1} \wedge BO \wedge BO^+ \to \Omega K_{i+1} \wedge BO \wedge BO^+.$$

We can then define a new map

$$H_N\colon BO/I_n \cup_g X \times I \to QR_{i+1} = QB_{i+1} \times \Omega K_{i+1} \wedge BO \wedge BO^+$$

to be the product of $H\colon BO/I_n \cup_g X \times I \to QBO(k) \subset QB_{i+1}$ and of

$$BO/I_n \cup_g X \times I \xrightarrow[1\cup(g\times 1)]{} BO/I_n \times I \xrightarrow[N]{} \Omega K_{i+1} \wedge BO \wedge BO^+.$$

We then have a strictly commutative diagram

$$\begin{tikzcd} BO/I\amalg X \arrow[r, "\rho_{i+1}\amalg f'_{i+1}"'] \arrow[d, hook] & R'_{i+1} \arrow[d, "q"] \\ BO/I \cup X \times I \arrow[r, "H_N"'] & QR_{i+1} \end{tikzcd}$$

which, when composed with the diagram

$$\begin{tikzcd} R'_{i+1} \arrow[r] \arrow[d, "q"'] & \bar{R}_{i+1} \arrow[d] \\ QR_{i+1} \arrow[r, "\text{project}"'] & QB_{i+1} \end{tikzcd}$$

is the map of pairs $H\colon (BO/I_n \cup X \times I,\ BO/I_n\amalg X) \to (QB_{i+1}, \bar{R}_{i+1})$ with which we began. This proves Lemma 1.15.

We now proceed with the proof of Theorem 1.14.

By Lemma 1.15 the composite of maps of quotients of Thom spectra

$$\Sigma TX_n = T(BO/I_n \cup X \times I)/MO/I_n \vee TX \xrightarrow[TH_N]{} T\big(QR_{i+1} \cup_q R'_{i+1} \times I\big)/TR'_{i+1} \to T\big(QB_{i+1} \cup \bar{R}_{i+1} \times I\big)/T\bar{R}_{i+1}$$

is homotopic to the quotient map induced by the map of pairs

$$H\colon \big(BO/I_n \cup X \times I,\ BO/I_n\amalg X_n\big) \to \big(QB_{i+1}, \bar{R}_{i+1}\big)$$

and hence is null homotopic. Thus the above map of quotients which, for brevity we write as

$$TH_N\colon \Sigma TX_n \to TQR_{i+1}/TR'_{i+1}$$

lifts to a map

$$\Sigma TX \to TQB_{i+1} \wedge (\Omega K_{i+1} \wedge BO \wedge BO^{+}) \simeq {}_{n}TQBO \wedge (\Omega K_{i+1} \wedge BO \wedge BO^{+})$$

which is the stable fiber of $TQR_{i+1}/TR'_{i+1} \to TQB_{i+1}/T\overline{R}_{i+1} \to TQB_{i+1}/T\overline{R}_{i+1}$.

Notice that $H^{*}(TQBO \wedge (\Omega K_{i+1} \wedge BO \wedge BO^{+}))$ is a free module over $A(QBO) = A \otimes H^{*}QBO$ (which was defined in Section 1.a).

The following is the last technical result needed to prove Theorem 1.14.

LEMMA 1.16. *We can choose the lifting*

$$\Sigma TX \to TQBO \wedge (\Omega K_{i+1} \wedge BO \wedge BO^{+})$$

to be of the form

$$\Sigma TX \xrightarrow[Tg]{} \Sigma MO/I_n \xrightarrow[\alpha]{} TQBO \wedge (\Omega K_{i+1} \wedge BO \wedge BO^{+})$$

where in cohomology, α^{*} is $A(QBO)$-linear.

The proof of this lemma is somewhat involved and we therefore postpone it until Section 1.d. So assuming its validity for now we proceed to prove Theorem 1.14.

Define a map

$$\begin{aligned} T\overline{H}\colon T(BO/I_n \cup X_n \times I) \to TQR_{i+1} &= TQB_{i+1} \wedge (\Omega K_{i+1} \wedge BO \wedge BO^{+})^{+} \\ &= TQB_{i+1} \vee (TQB_{i+1} \wedge \Omega K_{i+1} \wedge BO \wedge BO^{+}) \end{aligned}$$

to be the wedge of the map $TH_N\colon T(BO/I_n \cup X_n \times I) \to TQB_{i+1}$ and the map

$$\begin{aligned} T(BO/I_n \cup X_n \times I) &\to T(BO/I_n \cup X_n \times I)/MO/I \vee TX = \Sigma TX_n \\ &\xrightarrow[Tg_n]{} \Sigma MO/I_n \xrightarrow[\alpha]{} TQB_{i+1} \wedge \Omega K_{i+1} \wedge BO \wedge BO^{+}. \end{aligned}$$

Then, by the lemma, the map of pairs

$$(T(BO/I_n \cup X \times I),\ MO/I_n \vee TX) \xrightarrow{(T\overline{H},\, T\rho_{i+1} \vee Tf'_{i+1})} (TQR_{i+1}, TR'_{i+1})$$

is null homotopic. But since α is $A(QBO)$-linear in cohomology, we can "de-Thomify" $T\overline{H}$ in the following manner:

$$\alpha\colon \Sigma MO/I_n \to TQBO \wedge \Omega K_{i+1} \wedge BO \wedge BO^{+}$$

is the unique (up to homotopy) $A(QBO)$-linear map induced by the composition

$$\beta\colon \Sigma MO/I_n \xrightarrow[\alpha]{} TQBO \wedge (\Omega K_{i+1} \wedge BO \wedge BO^{+}) \xrightarrow[\text{project}]{} \Omega K_{i+1} \wedge BO \wedge BO^{+}.$$

Let β also represent the Thom isomorphic image

$$\beta\colon \Sigma BO/I_n \to \Omega K_{i+1} \wedge (BO \wedge BO^{+}).$$

Define

$$\overline{H}\colon BO/I_n \cup X \times I \to QR_{i+1} = QB_{i+1} \times \Omega K_{i+1} \wedge BO \wedge BO^{+}$$

to be the product of H: $BO/I_n \cup_g X \times I \to QBO(k) \subset QB_{i+1}$ and the sum of the homotopies

$$BO/I_n \cup_g X \times I \xrightarrow[1\cup(g\times 1)]{} BO/I_n \times I \underset{N}{\to} \Omega K_{i+1} \wedge BO \wedge BO^+ \quad \text{and}$$

$$BO/I_n \cup_g X \times I \longrightarrow BO/I_n \times I \to \Sigma BO/I_n \underset{\beta}{\to} \Omega K_{i+1} \wedge BO \wedge BO^+.$$

The sum of these homotopies factors through a homotopy we call

$$w\colon BO/I_n \times I \to \Omega K_{i+1} \wedge BO \wedge BO^+.$$

Notice we then have a strictly commutative diagram

$$\begin{tikzcd} BO/I_n \amalg X \arrow[r, "\rho_{i+1}\amalg f'_{i+1}"'] \arrow[d] & R_{i+1} \arrow[d] \\ BO/I_n \cup X \times I \arrow[r, "\overline{H}"'] & QR_{i+1}. \end{tikzcd}$$

We claim that the Thomification of the induced map of pairs is, up to homotopy, the sum of the map

$$TH_N\colon \Sigma TX \to TQR_{i+1}/TR'_{i+1}$$

and the composite

$$\bar{\alpha}\colon \Sigma TX \underset{Tg}{\to} \Sigma MO/I_n \underset{\alpha}{\to} TQBO \wedge (\Omega K_{i+1} \wedge BO \wedge BO^+) \to TQR_{i+1}/TR'_i$$

which, by Lemma 1.16 is null homotopic.

We now verify this claim. To ease notation let L denote

$$\Omega K_{i+1} \wedge BO \wedge BO^+.$$

Recall that

$$\overline{H}\colon BO/I_n \cup_g X_n \times I \to QR_{i+1} = QB_{i+1} \times L$$

is given by the product of H: $BO/I_n \cup_g X_n \times I \to QBO(k) \subset QB_{i+1}$ and the composition w': $BO/I_n \cup_g X_n \times I \xrightarrow[1\cup g_n\times 1]{} BO/I_n \times I \underset{w}{\to} L$. Thus on the Thom spectrum level,

$$T\overline{H}\colon T(BO/I_n \cup_{g_n} X_n \times I) \to TQR_{i+1} = TQB_{i+1} \wedge L^+$$

is given by the composition

(1.17)

$$T\overline{H}\colon T(BO/I_n \cup_{g_n} X_n \times I) \xrightarrow[\Delta]{} T(BO/I_n \cup X_n \times I) \wedge (BO/I_n \cup X_n \times I)^+$$
$$\xrightarrow[TH\wedge w']{} TQB_{i+1} \wedge L^+.$$

Now the homotopy w: $BO/I_n \times I \to L$, being the sum of the null homotopy N: $\widehat{BO}/I_n \to L$ and the homotopy $BO/I_n \times I \to \widehat{BO}/I_n \to \Sigma BO/I_n \underset{\beta}{\to} L$, strictly factors as the composition

$$w\colon BO/I_n \times I \to \widehat{BO}/I_n \underset{p_1}{\to} \widehat{BO}/I_n \vee \Sigma BO/I_n \xrightarrow[N \vee \beta]{} L$$

where p_1: $\widehat{BO}/I_n \to \widehat{BO}/I_n \vee \Sigma BO/I_n$ is obtained by identifying $BO/I_n \times \{\frac{1}{2}\} \subset \widehat{BO}/I_n$ to the basepoint.

Now the map

$$\begin{aligned} 1 \wedge p_1\colon & (T(BO/I_n \cup X_n \times I)) \wedge (\widehat{BO}/I_n)^+ \\ & \to T(BO/I_n \cup X_n \times I) \wedge (\widehat{BO}/I_n \vee \Sigma BO/I_n)^+ \\ & = T(BO/I_n \cup X_n \times I) \wedge \widehat{BO}/I_n^+ \vee T(BO/I_n \cup X_n \times I) \wedge \Sigma BO/I_n \end{aligned}$$

is clearly homotopic, by a homotopy that fixes $T(BO/I_n \cup X_n \times I) \wedge BO/I_n \hookrightarrow T(BO/I_n \cup X_n \times I) \wedge \widehat{BO}/I_n$, to the map

$$\begin{aligned} \bar{p}\colon & T(BO/I_n \cup X_n \times I) \wedge \widehat{BO}/I_n^+ \\ & \to T(BO/I_n \cup X_n \times I) \wedge BO/I_n^+ \vee T(BO/I_n \cup X_n \times I) \wedge \Sigma BO/I_n \end{aligned}$$

which, on the first component is the identity, and on the second component is the collapse map obtained by identifying $T(BO/I_n \cup X_n \times I) \wedge BO/I_n \hookrightarrow T(BO/I_n \cup X_n \times I) \wedge \widehat{BO}/I_n$ to a point.

Thus by the decomposition of $T\overline{H}$ given above (1.17) we have that $T\overline{H}$ is homotopic, via a homotopy that fixes $MO/I_n \vee TX_n \subset T(BO/I_n \cup_{g_n} X_n \times I)$ to the composition which for reasons which will become apparent below, we call, $TH_N(\alpha')$:

$$\begin{aligned} TH_N(\alpha')\colon & T(BO/I_n \cup_{g_n} X_n \times I) \\ & \xrightarrow[\Delta]{} T(BO/I_n \cup X_n \times I) \wedge (BO/I_n \cup X_n \times I)^+ \\ & \xrightarrow[1 \wedge (1 \cup g_n \times 1)]{} T(BO/I_n \cup X_n \times I) \wedge \widehat{BO}/I_n^+ \\ & \xrightarrow[\bar{p}]{} T(BO/I_n \cup X_n \times I) \wedge \overline{BO/I_n}^+ \\ & \qquad \vee T(BO/I_n \cup X_n \times I) \wedge \Sigma BO/I_n \\ & \xrightarrow[(TH \wedge N) \vee (TH \wedge \beta)]{} TQB_{i+1} \wedge L^+. \end{aligned}$$

Now

$$\begin{aligned}(TH \wedge N) \circ \Delta\colon T(BO/I_n \cup_{g_n} X_n \times I) &\xrightarrow{\Delta} T(BO/I_n \cup X_n \times I) \\ &\quad \wedge (BO/I_n \cup X_n \times I^+)^+ \\ &\xrightarrow{TH \wedge N} TQB_{i+1} \wedge L^+\end{aligned}$$

is, by definition, $TH_N\colon T(BO/I_n \cup X_n \times I) \to TQB_{i+1} \wedge L^+$. Thus the above composite $TH_N(\beta)$ is *equal* to the composite map

$$\begin{aligned}TH_N(\alpha')\colon T\big(BO/I_n \cup_{g_n} X_n \times I\big) &\xrightarrow[p]{} T\big(BO/I_n \cup_{g_n} X_n \times I\big) \vee \Sigma TX_n \\ &\xrightarrow[TH_N \vee \alpha']{} TQB_{i+1} \wedge L^+\end{aligned}$$

where p is the identity on the first component, and the map that collapses $MO/I_n \vee TX_n \subset T(BO/I_n \cup X_n \times I)$ to a point on the second component, and where $\alpha'\colon \Sigma TX_n \to TQB_{i+1} \wedge L$ is the composite

$$\begin{aligned}\alpha'\colon \Sigma TX_n &= T\big(BO/I_n \cup X_n \times I\big)\big/MO/I_n \vee TX_n \\ &\xrightarrow[\Delta]{} T(BO/I_n \cup X_n \times I) \wedge \Sigma X_n^+ \\ &\xrightarrow[1 \wedge g_n]{} T(BO/I_n \cup X_n \times I) \wedge \Sigma BO/I_n \xrightarrow[TH \wedge \beta]{} TQB_{i+1} \wedge L.\end{aligned}$$

See Proposition 0.6 for the justification of the notation $TH_N(\alpha')$. Now since the map $\alpha\colon \Sigma MO/I_n \to TQB_{i+1} \wedge L^+$ is the unique (up to homotopy) $A(QBO)$-linear map that extends $\beta\colon \Sigma MO/I_n \to L^+$, and since α' above is $A(QBO)$-linear, then α' is homotopic to

$$\bar{\alpha} = \alpha \circ Tg_n\colon \Sigma TX_n \to \Sigma MO/I_n \to TQB_{i+1} \wedge L.$$

Now let $TH_N(\bar{\alpha})\colon T(BO/I_n \cup X_n \times I) \to TQB_{i+1} \wedge L^+$ be the map defined as $TH_n(\alpha')$ was, except with $\bar{\alpha}$ replacing α'. The upshot of the above analysis is that the maps of stable cofibers induced by the strict commutativity of the following two diagrams are homotopic.

$$\begin{array}{ccc} MO/I_n \wedge TX_n & \xrightarrow[T\rho_{i+1} \vee Tf'_{i+1}]{} & TR_{i+1} \\ \downarrow & & \downarrow \\ T(BO/I_n \cup_{g_n} X_n \times I) & \xrightarrow[TH]{} & TQR_{i+1} \end{array}$$

$$\begin{array}{ccc} MO/I_n \vee TX_n & \xrightarrow[T\rho_{i+1} \vee Tf'_{i+1}]{} & TR_{i+1} \\ \downarrow & & \downarrow \\ T(BO/I_n \cup X_n \times I) & \xrightarrow[TH_N(\bar{\alpha})]{} & TQR_{i+1} \end{array}$$

The claim now follows from Proposition 0.6.

To prove the theorem, however (and thereby complete the inductive step), we wish to modify ρ_{i+1}: $BO/I_n \to R'_{i+1}$ and replace $\overline{H}$ by H and still obtain the same (homotopic) map of pairs of Thom spectra. This is easily done as follows:

By the homotopy lifting property for the fibration $R'_{i+1} \to QR_{i+1}$, we can find a homotopy

$$\tilde{w}\colon BO/I_n \times I \to R'_{i+1}$$

with the properties 1. $\tilde{w}|_{BO/I_n \times 0} = \rho_{i+1}$ and 2. $\tilde{w}$ lifts the homotopy

$$BO/I_n \times I \to QR_{i+1} = QB_{i+1} \times \Omega K_{i+1} \wedge BO \wedge BO^+$$

given by the product of the maps

$$BO/I_n \times I \xrightarrow[\text{project}]{} BO/I_n \xrightarrow[\rho_{i+1}]{} R'_{i+1} \to QB_{i+1} \quad \text{and}$$

$$w\colon BO/I_n \times I \to \Omega K_{i+1} \wedge BO^+ \wedge BO.$$

Finally, let $\rho'_{i+1} = \tilde{w}|_{BO/I_n \times 1}$. Then the following diagram strictly commutes:

$$\begin{tikzcd}
BO/I_n \amalg X \arrow[rr, "\rho'_{i+1} \amalg f'_{i+1}"] \arrow[d, hook] & & R'_{i+1} \arrow[d] \\
BO/I_n \cup_g X_n \times I \arrow[r, "H"'] & QBO(k) \arrow[r, hook] & QR_{i+1},
\end{tikzcd}$$

and induces a map of pairs which is homotopic to the one induced by the diagram

$$\begin{tikzcd}
BO/I_n \amalg X_n \arrow[r, "\rho_{i+1} \amalg f'_{i+1}"'] \arrow[d, hook] & R'_{i+1} \arrow[d] \\
BO/I_n \cup X_n \times I \arrow[r, "\overline{H}"'] & QR_{i+1}
\end{tikzcd}$$

and thus by the above claim induces a null homotopic map of quotients of Thom spectra. This, modulo 1.16, completes the proof of Theorem 1.14 and hence the inductive step in our proof of Theorem A″ when k is odd.

1.d. *Proof of Lemma 1.16*. The proof of 1.16 is rather technical; so we begin by describing the idea of the argument. As above, let L denote $(\Omega K_{i+1} \wedge BO) \wedge BO^+$. In order to construct a map $\alpha\colon \Sigma MO/I_n \to TQBO \wedge L$ satisfying 1.16 we first recall how liftings $\phi\colon \Sigma TX_n \to TQBO \wedge L$ of the map

$TH_N\colon \Sigma TX_n \to TQR_{i+1}/TR'_{i+1}$ arise. Such liftings come from null homotopies of the composition $TH\colon \Sigma TX_n \overset{TH_N}{\to} TQR_{i+1}/TR'_{i+1} \to TQB_{i+1}/TR_{i+1}$. Now when TH is viewed as the Thomification of the map of pairs described earlier, such a null homotopy also determines a lifting $G\colon T(BO/I_n \cup X_n \times I) \to TR'_{i+1}$ that extends $T\rho_{i+1} \vee Tf'_{i+1}\colon MO/I_n \vee TX_n \to TR'_{i+1}$, so that the composition

$$T(BO/I_n \cup X_n \times I) \underset{G}{\to} TR'_{i+1} \to TQR_{i+1} \xrightarrow[\text{project}]{} TQB_{i+1}$$

is homotopic rel$(MO/I_n \vee TX_n)$ to TH. Thus, roughly speaking, such liftings ϕ of TH_N and G of TH carry the same information. That is, both reflect a null homotopy of the map of pairs determined by TH.

Now by 1.13, there are n-dimensional equivalences $TR_{i+1} \simeq TB_i \vee L$ and $TQR_{i+1} \simeq TQB_{i+1} \vee TQBO \wedge L$. Using the fact that both H^*L and $H^*(TQBO \wedge L)$ are free modules over $A(BO)$ and $A(QBO)$ respectively, we will be able to choose a lifting G of TH that induces an $A(BO)$-module homomorphism. We will then see that this translates to our being able to choose a lifting $\phi\colon \Sigma TX_n \to TQBO \wedge L$ of TH_N which is $A(QBO)$-linear in cohomology. The fact that ϕ can be chosen to factor through an $A(BO)$-linear map $\alpha\colon \Sigma MO/I_n \to TQBO \wedge L$ will follow easily from Lemma 1.9 which says that the map of pairs which we also called $TH\colon MO/I_n \cup \widehat{TX}_n \to TQBO(k)/MO(k)$ is null.

In order to make these ideas more precise, we begin by describing precisely how one obtains a lifting $\phi\colon \Sigma TX_n \to TQBO \wedge L$ of TH_N from a map $G\colon T(BO/I_n \cup X_n \times I) \to TR'_{i+1}$ that extends $T\rho_{i+1} \vee Tf'_{i+1}$ so that the composite

$$T(BO/I_n \cup X_n \times I) \overset{G}{\to} TR'_{i+1} \xrightarrow{Tq} TQR_{i+1} \to TQB_{i+1}$$

is homotopic, rel$(MO/I_n \vee TX_n)$ to TH.

Let $\mathscr{G}\colon T(BO/I_n \cup_g X_n \times I) \to TQR_{i+1}$ be the composite of the first two maps in this composition. A lifting $\Sigma TX_n \to TQBO \wedge L \simeq TQR_{i+1}/TQB_{i+1}$ is obtained by measuring the difference between $\mathscr{G}$ and TH_N. More precisely, observe that the projections of $\mathscr{G}$ onto TQR_{i+1}/TQB_{i+1} strictly factors through a map

$$\mathscr{G}_1\colon MO/I_n \cup \widehat{TX}_n \to TQR_{i+1}/TQB_{i+1}$$

since $\mathscr{G}$ restricted to TX_n has image in $MO(k) \subset TQB_{i+1}$. Similarly, as constructed above, the projection of $TH_N\colon T(BO/I_n \cup X \times I) \to TQR_{i+1}$ to TQR_{i+1}/TQB_{i+1} factors through the null homotopy

$$\eta\colon \widehat{MO}/I_n \underset{T\Delta}{\to} MO \wedge \widehat{BO}/I_n \xrightarrow[i \wedge N]{} TQBO \wedge L$$

where $T\Delta$ is the Thom-ification of the diagonal map Δ: $BO/I_n \to BO \times BO/I_n$, and i is the usual inclusion.

Combining $\mathscr{G}_1$ and η we get a map

$$\bar{\alpha}\colon \Sigma TX_n = \underbrace{MO}/I_n \cup \widehat{TX}_n \xrightarrow[\eta \cup \mathscr{G}_1]{} TQR_{i+1}/TQB_{i+1} \simeq TQBO \wedge L.$$

Standard techniques verify that $\bar{\alpha}$ lifts TH_N: $\Sigma TX_n \to TQR_{i+1}/TR'_{i+1}$.

We now show that our original lifting map G: $T(BO/I_n \cup X \times I) \to TR'_{i+1}$ can be chosen so that the resulting map $\bar{\alpha}$ is $A(QBO)$-linear in cohomology. To do this we will find a lifting G as above and a map J: $BO/I_n \cup \hat{X}_n \to L$ so that the induced map $\mathscr{G}_1$: $MO/I_n \cup \widehat{TX}_n \to TQBO \wedge L$ is given, up to relative homotopy, by

$$MO/I_n \cup {}_g\widehat{TX} \xrightarrow[T\Delta]{} MO \wedge \left(BO/I_n \cup {}_g\hat{X}_n\right) \xrightarrow[i \wedge J]{} TQBO \wedge L.$$

Once this is accomplished, by the definition of η above we will have that $\bar{\alpha} = \eta \cup \mathscr{G}_1$ factors as

$$\bar{\alpha}\colon \Sigma TX = \underbrace{MO}/I_n \cup \widehat{TX}_n \xrightarrow[T\Delta]{} MO \wedge \left(\underbrace{BO}/I_n \cup \hat{X}_n\right) \xrightarrow[i \wedge (N \cup J)]{} TQBO \wedge L,$$

and hence is $AQBO$-linear in cohomology.

To construct such a lifting G, we first let $\bar{G}$: $T(BO/I_n \cup X_n \times I) \to TF_i \to T\bar{R}_{i+1} \to TR'_{i+1}$ be any map induced from a null homotopy of the Thom-ification of the map of pairs resulting from the commutative diagram

$$\begin{array}{ccc} BO/I_n \amalg X_n & \longrightarrow & F_i \\ \big\downarrow & & \big\downarrow \\ BO/I_n \cup X_n \times I & \xrightarrow[H]{} & QBO(k). \end{array} \qquad \text{(See Prop. 0.10.)}$$

Notice that when projected to

$$\begin{aligned} TQR_{i+1} &= TQB_{i+1} \vee TQB_{i+1} \wedge L \\ &= TQBO(k) \wedge B_{i+1}/BO(k)^+ \vee TQB_{i+1} \wedge L, \end{aligned}$$

the image of $\bar{G}$ lies in $TQBO(k) \vee TQB_{i+1} \wedge L$. Its projection onto $TQBO(k)$ is homotopic, rel$(MO/I_n \vee TX_n)$ to TH and its projection on $TQBO \wedge L = TQR_{i+1}/TQB_{i+1}$ factors as a composition

$$\begin{aligned} MO/I_n \cup \widehat{TX} \xrightarrow[\gamma]{} TR_{i+1}/TB_{i+1} &= TB_{i+1} \wedge \Omega K_{i+1} \wedge BO \\ &\xrightarrow[T\Delta]{} TB_{i+1} \wedge B^+_{i+1} \wedge \Omega K_{i+1} \wedge BO \\ &\to TQB_{i+1} \wedge \Omega K_{i+1} \wedge BO \wedge BO^+, \end{aligned}$$

where the last map is induced by the classification map $B_{i+1} \to BO$. Now using the splitting $\Sigma TX \simeq \Sigma MO/I_n \vee MO/I_n \cup \widehat{TX}$ we can extend γ to a map

$$\gamma': \Sigma TX \to TR_{i+1}/TB_{i+1}$$

in such a way as to be homotopically trivial on $\Sigma MO/I_n$.

Now define γ_1: $BO/I_n \cup \hat{X} \to \Omega K_{i+1} \wedge BO$ to be the composition

$$\gamma_1: BO/I_n \cup \hat{X} \subset MO \wedge (BO/I_n \cup \hat{X}) \underset{\Phi}{\to} MO \wedge T(BO/I_n \cup \hat{X})$$
$$\xrightarrow[1\wedge\gamma]{} MO \wedge TB_{i+1} \wedge \Omega K_{i+1} \wedge BO \underset{\pi}{\to} \Omega K_{i+1} \wedge BO,$$

where Φ is a homotopy equivalence induced by the Thom isomorphism in cohomology, and where π is the obvious projection map.

Finally consider the map

$$\bar{\gamma}: MO/I_n \cup \widehat{TX} \to TR_{i+1}/TB_{i+1} = TB_{i+1} \wedge \Omega K_{i+1} \wedge BO \simeq_n MO \wedge \Omega K_{i+1} \wedge BO$$

defined to be the composition

$$\bar{\gamma}: MO/I_n \cup \widehat{TX}_n \underset{T\Delta}{\to} MO \wedge (BO/I_n \cup \hat{X}_n)^+ \underset{1\wedge\gamma_1}{\to} MO \wedge \Omega K_{i+1} \wedge BO.$$

Also, similar to what was done above, we may extend $\bar{\gamma}$ to a map $\bar{\gamma}'$: $\Sigma TX_n \to MO \wedge \Omega K_{i+1} \wedge BO$ that is homotopically trivially on $\Sigma MO/I_n$.

Now observe that by Lemma 1.13 there is a splitting

$$\lambda: MO \wedge \Omega K_{i+1} \wedge BO \to TR_{i+1}.$$

Using this splitting we can define a map

$$G: T(BO/I_n \cup X \times I) \to TR'_{i+1}$$

to be the sum of $\overline{G}$ and the composition

$$T(BO/I_n \cup X_n \times I) \to T(BO/I_n \cup X_n \times I)/MO/I_n \cup TX_n$$
$$= \Sigma TX_n \xrightarrow{\bar{\gamma}' - \gamma'} MO \wedge \Omega K_{i+1} \wedge BO \underset{\lambda}{\to} TR'_{i+1}.$$

Notice that G extends $T\rho_{i+1} \vee Tf'_{i+1}$: $MO/I_n \vee TX_n \to TR'_{i+1}$. Furthermore, observe that the map of pairs induced by G,

$$(T(BO/I_n \cup X \times I), TX_n) \to (TR'_{i+1}, TB_{i+1})$$

is homotopic to $\bar{\gamma}$, and hence by the definition of $\bar{\gamma}$ the induced map of pairs

$$\mathscr{G}_1: (TBO/I_n \cup X_n \times I, TX_n) \to (TQR_{i+1}, TQB_{i+1}) \simeq TQBO \wedge L$$

factors up to relative homotopy as the composition

$$\mathscr{G}_1: MO/I_n \cup \widehat{TX}_n \underset{T\Delta}{\to} MO/I_n \wedge (BO/I_n \cup \hat{X}_n)^+ \xrightarrow[i\wedge\Delta(\gamma_1)]{} TQBO \wedge L$$

where $\Delta(\gamma_1)$ is the composition

$$BO/I_n \cup \hat{X}_n \underset{\Delta}{\rightarrow} BO^+ \wedge BO/I_n \cup \hat{X}_n \xrightarrow[1\wedge\gamma_1]{} BO^+ \wedge \Omega K_{i+1} \wedge BO = L.$$

Thus, by the remarks above, we need only show that the projection of the composition

$$\mathscr{G}\colon T(BO/I_n \cup X \times I) \underset{G}{\rightarrow} TR'_{i+1} \underset{q}{\rightarrow} TQR_{i+1}$$

to TQB_{i+1} is homotopic, rel($TX \vee MO/I_n$), to TH: $T(BO/I_n \cup X_n \times I) \rightarrow TQBO(k) \subset TQB_{i+1}$.

To do this it is sufficient to show that $\mathscr{G}$ is homotopic rel($TX_n \vee MO/I_n$) to the composition $T(BO/I_n \cup X_n \times I) \underset{\bar{G}}{\rightarrow} TR'_{i+1} \underset{q}{\rightarrow} TQR_{i+1} \xrightarrow[\text{project}]{} TQB_{i+1}$, since by assumption this is relatively homotopic to TH. This in turn will follow once we observe that G and $\bar{G}$ are homotopic maps from $T(BO/I_n \cup X_n \times I)$ to TR'_{i+1}, since the projection of any homotopy between them to TQB_{i+1} is an appropriate relative homotopy. To see that G and $\bar{G}$ are homotopic recall that their difference homotopy factors as the composition

$$T(BO/I_n \cup X_n \times I) \longrightarrow T(BO/I_n \cup X_n \times I)/MO/I_n \vee TX_n \underset{=}{\rightarrow} \Sigma TX_n$$

$$\xrightarrow[\gamma' - \bar{\gamma}']{} MO \wedge \Omega K_{i+1} \wedge BO \underset{\lambda}{\rightarrow} TR'_{i+1}.$$

Observe that the first map in this composition is clearly null and hence so is the composition.

Thus $\mathscr{G}$ is homotopic rel($MO/I_n \cup TX$) to TH and as seen before thus allows us to conclude that there is a lifting

$$\tilde{\alpha}\colon \Sigma TX \rightarrow TQR_{i+1}/TQB_{i+1} = TQB_{i+1} \wedge L$$

that is $A(QBO)$-linear in cohomology. To finish the proof of the lemma, however, we need such a lifting that factors through an $A(BO)$-linear map $\Sigma MO/I_n \rightarrow TQBO \wedge L$.

To do this recall that we have a strictly commutative diagram

$$\begin{array}{ccccc}
X_n & \xrightarrow{f_n} & BO(k) & \hookrightarrow & R'_{i+1} \\
\downarrow & & \downarrow & & \downarrow \\
BO/I_n \cup_g X \times I & \rightarrow & QBO(k) & \hookrightarrow & QR_{i+1}
\end{array}$$

and therefore by Lemma 1.9 the induced map of quotients of Thom spectra is null homotopic. Hence the restriction of $\tilde{\alpha}$ to $MO/I_n \cup \widehat{TX}$ when projected to TQR_{i+1}/TR'_{i+1} is null and thus lifts to TR'_{i+1} and in particular to

$TR'_{i+1}/TB_{i+1} = MO \wedge \Omega K_i \wedge BO \hookrightarrow TQBO \wedge \Omega K_{i+1} \wedge BO$. But the inclusion $TQBO \wedge \Omega K_{i+1} \wedge BO \hookrightarrow TQBO \wedge (\Omega K_{i+1} \wedge BO) \wedge BO^+$ is induced by the diagonal on BO and is a split, $A(QBO)$-free wedge summand. The existence of an appropriate map α: $\Sigma MO/I_n \to TQB_{i+1} \wedge L$ that is $AQBO$-linear and that satisfies Lemma 1.16 is now immediate.

This now completes the proof of Theorem 1.14 and therefore Theorem A″ when k is odd.

1.e. *The nonorientable case.* In this section we discuss the modifications necessary in the above proof of Theorem A″ in the case when $k = n - \alpha(n)$ is even. In this case, as was pointed out earlier, the homotopy fibration sequence $V_k \to BO(k) \to BO$ is nonorientable. That is, $\pi_1 BO = \mathbf{Z}_2$ acts nontrivially on $\pi_* V_k$. In particular, $\pi_k V_k = \mathbf{Z}$ has the unique nontrivial $\mathbf{Z}_2$ action on it. Thus the first nonzero k-invariant, the Euler class, is a twisted cohomology class $\chi \in H^{k+1}(BO; \{\mathbf{Z}\})$. A standard construction yields that χ can be classified by a map of homotopy fibrations

$$\begin{tikzcd} BO(k) \arrow[r] \arrow[d] & RP^\infty \arrow[d] \\ BO \arrow[r] & S^\infty \times_{\mathbf{Z}_2} K(\mathbf{Z}, k+1). \end{tikzcd}$$

(See for example Nussbaum [22].) In this diagram $K(\mathbf{Z}, k+1)$ is assumed to have a cellular $\mathbf{Z}_2$ action, and the map $RP^\infty \to S^\infty \times_{\mathbf{Z}_2} K(\mathbf{Z}, k+1)$ is given by thinking of RP^∞ as $S^\infty \times_{\mathbf{Z}_2} *$, and then mapping $*$ to a fixed point of $K(\mathbf{Z}, k+1)$.

Now it is folklore that lifting problems from BO to $BO(k)$ are 2-primary problems through dimension $2k - 1$. This has been made precise by Brown in [27]. We shall therefore assume that all fibrations have been fiberwise localized at 2.

Thus to begin our inductive argument for the proof of Theorem A″ we let B_1 be the homotopy pull-back for the diagram

$$\begin{tikzcd} B_1 \arrow[rr] \arrow[d, "p"'] & & RP^\infty \arrow[d] \\ BO \arrow[r, "k_0"'] & S^\infty \times_{\mathbf{Z}_2} K(\mathbf{Z}_{(2)}, k+1) \times & \prod\limits_{q > k+1} K(\mathbf{Z}_2, q) \end{tikzcd}$$

where $\mathbf{Z}_{(2)}$ denotes the 2-local integers, and where k_0 is the product of the localized twisted Euler class and the Stiefel-Whitney classes:

$$w_q\colon BO \to BO/BO(k) \to K(\mathbf{Z}_2, q) \quad \text{for } q > k+1.$$

As in the last section we may assume that we have a lifting e_1: $BO(k) \to B_1$ that is a cofibration. Moreover we have an obvious product map j: $B_1 \to QBO(k) \times B_1/BO(k)$ that is equivalent through dimension n to the stabilization map $B_1 \subset QB_1$. Let

$$j'\colon B_1' \to QBO(k) \times B_1/BO(k)$$

be j, made into a fibration in the canonical way. To begin our inductive argument we prove the following (compare with inductive assumptions 1.11):

LEMMA 1.17. *There exists a homotopy lifting* ρ_1: $BO/I_n \to B_1'$ *of* ρ *with the following properties*:

(1) *The diagram*

$$\begin{tikzcd}
BO/I_n \amalg X_n \arrow[rr, "\rho_1 \amalg f_{n,1}"] \arrow[d, hook] & & B_1' \arrow[d] \\
BO/I_n \cup_g X_n \times I \arrow[r, "H"'] & QBO(k) \subset & QBO(k) \times B_1/BO(k)
\end{tikzcd}$$

strictly commutes, where $f_{n,1}$ *is the composition* $X_n \xrightarrow[f_n]{} BO(k) \xrightarrow[e]{} B_1'$.

(2) *The induced map of pairs*

$$(BO/I_n \cup X \times I,\ BO/I_n \amalg X_n) \to \big(QBO(k) \times B_1/BO(k), B_1'\big)$$

has trivial Thomification.

Proof. Recall the map $\bar{E}$: $BO/I_n \cup P_n \times I \to QBO(k)$ used in the proof of Lemma 1.9. This yielded a strictly commutative diagram

(1.18)
$$\begin{tikzcd}
X_n \arrow[r, "h_n"] \arrow[d] & P_n \arrow[r] \arrow[d, hook] & BO(k) \arrow[d, hook] \\
BO/I_n \cup_g X_n \times I \arrow[r] & BO/I_n \cup P \times I \arrow[r, "\bar{E}"] & QBO(k).
\end{tikzcd}$$

Define C to be the canonical homotopy pull-back for the square

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & B_1 \arrow[d] \\
BO/I_n \cup P \times I \arrow[r, "E"'] & BO
\end{tikzcd}$$

where E is as in Section 1.a. As a point-set,

$$C = \{(x, y, \alpha) \in (BO/I_n \cup P \times I) \times B_1 \times BO^I\colon \alpha(0) = E(x),\ \alpha(1) = p(y)\}.$$

Define the map $r\colon P_n \to C$ by $r(z) = (\iota(z), \bar{e}(z), \varepsilon)$ where ι is the inclusion $P_n = P_n \times 1 \subset BO/I_n \cup P_n \times I$, $\bar{e}$ is the composition $P_n \to BO(k) \xrightarrow{e} B_1$, and ε is the constant path at $E \circ \iota(z)$. One can easily verify that $r\colon P \to C$ is well-defined. Now let

$$j\colon C' \to BO/I_n \cup P \times I$$

be $C \to BO/I_n \cup P \times I$ made into a fibration in the canonical way. Now r still denotes the map $r\colon P \to C \subset C'$ and r still lifts the inclusion $\iota\colon P \subset BO/I_n \cup P \times I$ and the map $\bar{e}\colon P \to B_1$.

Thus by diagram (1.1), $r \circ h_n\colon X_n \to C$ lifts the composition $X_n \subset BO/I_n \cup_g X_n \times I \xrightarrow[h_n]{} BO/I_n \cup P_n \times I$ and the map $f_{n,1}\colon X_n \to B_1$. So to prove the lemma it is sufficient to construct a lifting $q\colon BO/I_n \to C'$ of the inclusion $BO/I_n \subset BO/I_n \cup P \times I$ so that the diagram

$$\begin{tikzcd}
BO/I_n \amalg X_n \arrow[r, "q \amalg r \circ h_n"] \arrow[d] & C' \arrow[d] \\
BO/I_n \cup X_n \times I \arrow[r, "h_n"'] & BO/I_n \cup P \times I
\end{tikzcd}$$

has trivial Thomification.

To do this, first recall that Brown proved in [27] that the twisted Euler class $\chi \in H^{k+1}(BO/I_n; \{\mathbf{Z}\})$ is zero. Now since the Stiefel-Whitney classes, $w_q \in H^*(BO/I_n; \mathbf{Z}_2)$ for $q > k + 1$ are also zero, the fibration $p\colon C' \to BO/I_n \cup P \times I$ is trivial. That is, there exists a fiber homotopy equivalence

$$\begin{tikzcd}
C' \arrow[r, "\phi"'] \arrow[dr] & (BO/I_n \cup P_n \times I) \times K(\mathbf{Z}, k) \times \prod_{q>k} K(\mathbf{Z}_2, q) \arrow[d] \\
& BO/I_n \cup P_n \times I.
\end{tikzcd}$$

For ease of notation write $B \times K = (BO/I_n \cup P \times I) \times K(\mathbf{Z}, k) \times \prod K(\mathbf{Z}_2, q)$. By use of this homotopy equivalence we can construct the appropriate $q\colon BO/I_n \to B \times K$ that lifts the inclusion $u\colon BO/I_n \subset B$.

Such liftings are of the form $q = j \times \alpha\colon BO/I_n \to B \times K$.

We now show that we can choose the appropriate map α.

We first study the Thom-ification of the map of pairs

$$(BO/I_n \cup X_n \times I,\ X_n \amalg BO/I_n) \xrightarrow[(h_n,\ \phi \circ r \circ h_n \amalg u)]{} (B, B \times K).$$

This can be thought of as a map

$$\beta\colon \Sigma TX_n \to \Sigma TB \wedge K \simeq \Sigma MO \wedge K.$$

(The last homotopy equivalence is through dimension $n + 1$.)

Now β can be described as follows. The map $\phi \circ r \circ h_n\colon X_n \to B \times K$ is a product of the map

$$\bar{g}\colon X_n \subset BO/I_n \cup X_n \times I \to BO/I_n \cup P \times I = B$$

and a map $\bar{\beta}\colon X_n \to K$. Also, β is the suspension of the composition

$$\beta\colon TX_n \xrightarrow[T(\bar{g} \wedge \bar{\beta})]{} TB \wedge K^+ \to TB \wedge K \simeq MO \wedge K.$$

By the proof of Lemma 1.9, the composition $MO/I_n \cup \widehat{TX}_n \to \Sigma TX_n \xrightarrow[\beta]{} \Sigma MO \wedge K$ is trivial, so that β factors as

$$\begin{tikzcd} TX_n \arrow[rr, "\beta"] \arrow[dr, "Tg"'] & & MO \wedge K \\ & MO/I_n. \arrow[ur, "\bar{\alpha}"'] & \end{tikzcd}$$

Now $H^k(MO \wedge K; \mathbf{Z}_2) = \mathbf{Z}_2$ is generated by a class, say x. The Thom isomorphic image of $\beta^*(x)$ in $H^k(X_n; \mathbf{Z}_2)$ is the mod 2 reduction of $\bar{\beta} \in H^k(X_n; \mathbf{Z})$. Since $g_n^*\colon H^*(BO/I_n; \mathbf{Z}_2) \to H^*(X_n; \mathbf{Z}_2)$ is a monomorphism, the Bockstein Sq^1 applied to the Thom isomorphic image α' of $\bar{\alpha}^*(x)$ in $H^*(BO/I_n; \mathbf{Z}_2)$ is zero. Thus α' is the mod 2 reduction of an integral class $\alpha \in H^k(BO/I_n; \mathbf{Z})$. We may therefore define

$$q = j \times \alpha\colon BO/I_n \to B \times K.$$

It is now clear by its construction that the Thomification of the map of pairs

$$(BO/I_n \cup X \times I,\ X_n \amalg BO/I_n) \xrightarrow[(h_n,\ \phi r h_n \amalg q)]{} (B, B \times K)$$

is trivial, which completes the proof of Lemma 1.17.

To complete the proof of Theorem A″ in this case, observe that the 2-localized homotopy fiber of the map

$$BO(k) \to B_1$$

has finite 2-torsion homotopy groups through dimension n. It is in particular a nilpotent homotopy fibration. We may then construct a modified Postnikov tower of principal fibrations

$$BO(k) \to \cdots \to B_i \to B_{i-1} \to \cdots \to B_1$$

with fibers that are products of $K(\mathbf{Z}_2, q)$'s. The inductive argument in Sections 1.b, 1.c, and 1.d then goes through verbatim to complete the proof of Theorem A″. We leave the verification to the reader.

2. The homotopy of BO/I_n and a proof of Lemma B when $n = 2^i - 1$

Our proof of Lemma B will be by induction on n. In order to complete the inductive step we will need first to study the tower of fibrations converging to BO/I_n constructed by Brown and Peterson [8]. We do this in Section 2.a. In Section 2.b we state our inductive assumptions for the proof of Lemma B and complete the inductive step in the case when n is of the form $2^i - 1$.

2.a. *A study of* BO/I_n. To begin our study of the Brown-Peterson construction of BO/I_n, we first study their Thom spectra MO/I_n. It turns out that MO/I_n is a wedge of Brown-Gitler spectra $B(k)$; so we begin by recalling some properties of these spectra [3].

The spectra $B(k)$ were originally defined as the inverse limit of a tower of fibrations of spectra

$$\text{(2.1)}\qquad \begin{array}{ccccccccc} & & \Sigma^{-1}H_{k,q} & & & & \Sigma^{-1}H_{k,1} & & \\ & & \downarrow{\scriptstyle i_q} & & & & \downarrow{\scriptstyle i_1} & & \\ \cdots & \to & E_{k,q} & \xrightarrow[p_q]{} & E_{k,q-1} & \cdots \xrightarrow[p_2]{} & E_{k,1} & \xrightarrow[p_1]{} & H_{k,0} = K(\mathbf{Z}_2) \\ & & & & \downarrow{\scriptstyle s_q} & & & & \downarrow{\scriptstyle s_1} \\ & & & & H_{k,q} & & & & H_{k,1} \end{array}$$

where $H_{k,q}$ is the cofibre of $p_q\colon E_{k,q} \to E_{k,q-1}$ and is a wedge of suspensions of $K(\mathbf{Z}_2)$.

This tower was shown to satisfy the following properties:

(2.2) a. $H^*(B(k) = \varinjlim_q H^*(E_{k,q}) = A/A\{\chi(Sq^i)\colon i > k\}$ as A-modules.

b. If X is any CW complex, then the homotopy exact sequence

$$\pi_r(H_{r,q} \wedge X) \xrightarrow{i_{q*}} \pi_r(E_{k,q} \wedge X) \xrightarrow{p_{q*}} \pi_r(E_{k,q-1} \wedge X)$$

is

1. split, short exact if $r < 2k$,
2. short exact (but not necessarily split) if $r = 2k$.

3. p_{q*} is surjective if $r = 2k + 1$.

Now define the spectrum

$$MO/I_n = \bigvee_{\omega} \Sigma^{|\omega|} B(\overline{n - |\omega|})$$

where if m is an integer, $\overline{m}$ denotes the greatest integer $\leq m/2$, and where the wedge is taken over all monomials $b_\omega \in \pi_q(MO)$ such that $q \leq n$. (Recall that $\pi_* MO = Z_2[b_k\colon\ k \neq 2^i - 1]$.) This notation will be justified later. One can therefore use (2.1) to construct a Postnikov tower for MO/I_n (see Prop. 2.4 of [8]).

(2.3)
$$\begin{tikzcd}
 & \Sigma^{-1}K_q \arrow[d, "i_q"] & & & \Sigma^{-1}K_1 \arrow[d, "i_1"] & \\
MO/I_n \cdots \arrow[r] & Y_q \arrow[r, "p_q"'] & Y_{q-1} \cdots \arrow[r, "p_2"'] \arrow[d, "s_q"] & \cdots & Y_1 \arrow[r, "p_1"'] & Y_0 = MO \arrow[d, "s_1"] \\
 & & K_q & & & K_1
\end{tikzcd}$$

where $Y_q = \bigvee_{\omega} \Sigma^{|\omega|} E_{\overline{n-|\omega|},\, q}$, $K_q = \bigvee_{\omega} \Sigma^{|\omega|} H_{\overline{n-|\omega|},\, q}$, and the maps i_q, p_q, and s_q here denote (by abuse of notation) the wedge of the corresponding maps in the Postnikov towers (2.1) for the appropriate Brown-Gitler spectra.

Corollary 2.4. *In the Postnikov tower* 2.3 *for* MO/I_n *the homomorphism*

$$p_{q*}\colon \pi_r(Y_q \wedge X) \to \pi_r(Y_{q-1} \wedge X)$$

is surjective for all $r \leq n$, *where* X *is any* CW *complex.*

Proof. This follows from (2.2.b).

In [8], Brown and Peterson essentially realized Postnikov tower 2.3 on the base space level. That is, they proved the following:

Theorem 2.5. *There is a tower of fibrations of spaces*

$$\begin{tikzcd}
 & \Omega K_l' \arrow[d, "j_l"] & & & \Omega K_1' \arrow[d, "j_1"] & \\
\cdots \arrow[r] & B_l \arrow[r, "\pi_l"'] & B_{l-1} \arrow[r] \arrow[d, "\kappa_l"] & \cdots \arrow[r] & B_1 \arrow[r, "\pi_1"'] & B_0 = BO \arrow[d, "\kappa_l"] \\
 & & K_l' & & & K_1'
\end{tikzcd}$$

where the spaces K_l' are products of Eilenberg-MacLane spaces of type $K(\mathbf{Z}_2, q)$, having the following properties.

1. *If T_l is the Thom spectrum of the stable vector bundle over B_l classified by the composition $\pi_1 \circ \cdots \circ \pi_{l-1} \circ \pi_l$: $B_l \to BO$, then there exist maps f_l: $T_l \to Y_l$ which induce isomorphisms in $\mathbf{Z}_2$-homology through dimension n.*

2. *The diagrams*

$$\begin{array}{ccc} T_l & \xrightarrow{T\pi_l} & T_{l-1} \\ {\scriptstyle f_l}\downarrow & & \downarrow{\scriptstyle f_{l-1}} \\ Y_l & \xrightarrow[p_l]{} & Y_{l-1}. \end{array}$$

commute up to homotopy.

3. *If $BO/I_n = \varprojlim_l B_l$, and ρ: $BO/I_n \to BO$ is induced by the π_i's, then $H^*(BO/I_n) \simeq H^*(BO)/I_n$, where I_n is the ideal described in the introduction. Furthermore the maps f_l induce a* mod 2 *equivalence of spectra*

$$f: T(BO/I_n) \to MO/I_n.$$

(Because of this equivalence we shall no longer distinguish between the 2-localization of the Thom spectrum $T(BO/I_n)$ and MO/I_n.)

4. *The map s_q: $Y_q \to K_q$, which can be viewed as a sum of cohomology classes of $H^*(Y_q)$, has the Thom isomorphic image of the class κ_q: $B_q \to K_q'$ as a summand.*

The following result is a very important corollary of the properties of this Postnikov tower for BO/I_n.

Definition 2.6. Let X be any finite CW complex and h: $X \to BO$ any map. The pair (X, h) is said to be *quasi-normal of dimension n* if there exists an n-dimensional manifold M^n and a map g: $M^n \to X$ satisfying

1. The composition $h \circ g$: $M^n \to BO$ classifies the stable normal bundle of M^n, and
2. g^*: $H^*(X) \to H^*(M)$ is injective.

Corollary 2.7. *If (X, h) is quasi-normal of dimension n, then the map h: $X \to BO$ lifts to BO/I_n. In fact if h_{l-1}: $X \to B_{l-1}$ is any lifting of h to the $(l-1)^{\text{st}}$ stage of Postnikov tower* 2.5, *then there exists a lifting h_l: $X \to B_l$ such that $\pi_l \circ h_l \simeq h_{l-1}$.*

Proof. Inductively, assume there is a lifting $h_{l-1}\colon X \to B_{l-1}$ of h. The obstruction to finding a lift h_l of h_{l-1} is the composition

$$r\colon X \xrightarrow[h_{l-1}]{} B_{l-1} \xrightarrow[\kappa_l]{} K_l'$$

which, since K_l' is a product of $K(Z_2, q)$'s, lies in $H^*(X)$. Since $g^*\colon H^*(X) \to H^*(M)$ is one-to-one it is enough to show that $g^*(r) = 0$.

Now $g^*(r)$ is the obstruction to the existence of a lifting $g_l\colon M \to B_l$ of the map $g_{l-1} = h_{l-1} \circ g\colon M \to B_{l-1}$, which is in turn a lifting of the stable normal bundle map $\nu_M\colon h \circ g\colon M \to X \to BO$. The map g_{l-1} therefore induces a map of Thom spectra

$$T(\nu_M) \xrightarrow{Tg_{l-1}} T_{l-1}$$

which by S-duality yields an element

$$[Tg_{l-1}] \in \pi_n(T_{l-1} \wedge M) \simeq \pi_n(Y_{l-1} \wedge M).$$

By Corollary 2.4, $[Tg_{l-1}]$ is in the image of $p_{l*}\colon \pi_n(Y_l \wedge M) \to \pi_n(Y_{l-1} \wedge M)$ and thus by exactness the class $s_{l*}[Tg_{l-1}] \in \pi_n(K_l \wedge M)$ is zero.

By S-duality this implies that the composition

$$T(\nu_M) \xrightarrow[Tg_{l-1}]{} T_{l-1} \xrightarrow[f_{l-1}]{} Y_{l-1} \xrightarrow[s_l]{} K_l$$

is null homotopic. By the Thom isomorphism theorem and property 4 of the Postnikov tower in Theorem 2.5, this implies that the map

$$M \xrightarrow[g_{l-1}]{} B_{l-1} \xrightarrow[\kappa_l]{} K_l'$$

is null homotopic. But this composition represents $g^*(r)$, which therefore must be zero. Again, since g^* is injective we can therefore conclude that the obstruction r is zero.

Remarks. 1. Notice that Corollary 2.7 in particular implies that every stable normal bundle map $\nu_M\colon M^n \to BO$ lifts to BO/I_n, as remarked earlier.

2. Other examples of quasi-normal bundles of dimension n are the pairs $(BO/I_n, \rho)$ and $(BO/I_r \times BO/I_{n-r}, \rho \times \rho)$. This implies there exist pairings

$$BO/I_r \times BO/I_{n-r} \to BO/I_n$$

living over the Whitney sum pairing of BO.

3. A less trivial example is the following. Consider the configuration space

$$F(R^2, k) = \left\{(x_1, \dots x_k) \in (R^2)^k \quad \text{such that } x_i \neq x_j \text{ if } i \neq j\right\}$$

Both $F(R^2, k)$ and $R^k = (R^1)^k$ are acted upon by the symmetric group Σ_k by permuting coordinates. Let γ_k be the vector bundle

$$F(R^2, k) \otimes_{\Sigma_k} R^k \to F(R^2, k)/\Sigma_k.$$

In [7] Brown and Peterson proved that these bundles have the following properties:

a. The mod 2 cohomological dimension of $F(R^2, k)/\Sigma_k$ is $k - \alpha(k)$.

b. Localized at the prime 2 the Thom spectrum $T\gamma_k$ is homotopy equivalent to the Brown-Gitler spectrum $B(\bar{k})$.

c. In [7, proof of Theorem B] Brown and Peterson constructed a manifold Q_k and a bundle map $g\colon \nu_{Q_k} \to \gamma_k$ so that

$$g^*\colon H^*\big(F(R^2, k)/\Sigma_k\big) \to H^*(Q_k)$$

is injective. In our terminology this implies that the pair $(F(R^2, k)/\Sigma_k, \gamma_k)$ is quasi-normal of dimension k.

From now on we ease the notation by letting $F_k = F(R^2, k)/\Sigma_k$. Observe that properties a and c of the bundle γ_k imply the following:

Corollary 2.8. *The classifying map $\gamma_k\colon F_k \to BO$ lifts to maps $\gamma_k\colon F_k \to BO(k - \alpha(k))$ and $\tilde{\gamma}_k\colon F_k \to BO/I_k$.*

The bundles γ_k over F_k will be used to construct the spaces X_n of Lemma A.

2.b. *The inductive assumptions and the proof of Lemma* B *in dimensions of the form* $2^i - 1$. We are now ready to prove Lemma B. Let $m \geq 4$ be a fixed integer. We make the following inductive assumptions. (We leave the verification of these properties for the case $m \leq 3$ as an exercise for the reader, after first observing that BO/I_1 is 2-locally contractible and $BO/I_2 \simeq BO/I_3 \simeq {}_2RP^2$.)

(2.9) For every integer $k < m$ there exist a space

$$X_k = \coprod_{\omega} M_\omega \times F_{k-|\omega|}$$

where the disjoint union is taken over all monomials $b_\omega \in \pi_* MO$ of dimension $\leq k$ and where M_ω is a manifold representing b_ω, together with a map

$$h_k\colon X_k \to P_k$$

satisfying the following properties.

1. The map $f_k\colon X_k \xrightarrow{h_k} P_k \to BO(k - \alpha(k))$ is induced, up to homotopy, by compositions of the form

$$M_\omega \times F_{k-|\omega|} \xrightarrow{\nu_\omega \times \gamma_{k-|\omega|}} BO(|\omega| - \alpha|\omega|) \times BO(k - |\omega| - \alpha(k - |\omega|))$$
$$\xrightarrow[m]{} BO(k - (\alpha|\omega| + \alpha(k - |\omega|))) \to BO(k - \alpha(k)).$$

Here m is the Whitney product pairing and $\nu_\omega\colon M_\omega \to BO(|\omega| - \alpha|\omega|)$ classifies an $(|\omega| - \alpha|\omega|)$-dimensional normal bundle of M_ω, with the property that if b_ω is decomposable, say $b_\omega = b_{\omega_1} b_{\omega_2}$, then $M_\omega = M_{\omega_1} \times M_{\omega_2}$ and $\nu_\omega = \nu_{\omega_1} \times \nu_{\omega_2}$.

2. Let g_k: $X_k \longrightarrow BO/I_k$ be the composition $X_k \xrightarrow[h_k]{} P_k \longrightarrow BO/I_k$, and let $\tilde{\sigma}_k$: $\bigvee_\omega \Sigma^{|\omega|} B(\overline{k - |\omega|}) \longrightarrow TX_k$ be the map of 2-local spectra defined to be the wedge of the maps

$$\Sigma^{|\omega|} B(\overline{k - |\omega|}) \simeq S^{|\omega|} \wedge B(\overline{k - |\omega|}) \xrightarrow{\tau \wedge 1} T\nu_\omega \wedge B(\overline{k - |\omega|}) \simeq T\nu_\omega \wedge T(\gamma_{|\omega|})$$

where τ: $S^{|\omega|} \to T\nu_\omega$ is given by the Thom-Pontrjagin construction. We then have

a. $Tg_k \circ \tilde{\sigma}_k$: $\bigvee_\omega \Sigma^{|\omega|} B(\overline{k - |\omega|}) \to TX_k \to MO/I_k$ is a mod 2 homotopy equivalence, and

b. If β_k is a 2-local homotopy inverse to $Tg_k \circ \tilde{\sigma}_k$, and if we define

$$\sigma_k = \tilde{\sigma}_k \circ \beta_k\colon MO/I_k \to TX_k,$$

then $Th_k \simeq Th_k \circ \sigma_k \circ Tg_k$: $TX_k \to TP_k$.

Remark. We are in particular inductively assuming the truth of Lemma B for $k < m$. Indeed, by the results of Section 1, inductive hypotheses 1–3 above imply the existence of a map ρ_k: $BO/I_k \to BO(k - \alpha(k))$ making the following diagram commute:

$$\begin{tikzcd}
 & & MO(k - \alpha(k)) \arrow[d] \\
TX_k \arrow[r, "Tg_k"'] \arrow[urr, "Tf_k"] & MO/I_k \arrow[r, "T\rho"'] \arrow[ur, "T\rho_k"] & MO
\end{tikzcd}$$

We begin completing the inductive step with the following observations. Consider the map of 2-local spectra

$$\bar{\zeta}\colon \bigvee_{|\omega| \leq m} S^{|\omega|} \wedge B(\overline{m - |\omega|}) \to MO(m - \alpha(m))$$

defined as follows. If $|\omega| < m$, let the restriction of $\bar{\zeta}$ to $S^{|\omega|} \wedge B(\overline{m - |\omega|})$ be given by the composition

$$S^{|\omega|} \wedge B(\overline{m - |\omega|}) \xrightarrow{\tau \wedge 1} T\nu_\omega \wedge B(\overline{m - |\omega|}) \simeq T(\nu_\omega) \wedge T(F_{m-|\omega|})$$

$$\xrightarrow[T\nu_\omega \wedge T\gamma_{m-|\omega|}]{} MO(|\omega| - \alpha|\omega|)$$

$$\wedge MO(m - |\omega| - \alpha(m - |\omega|)) \xrightarrow[Tm]{} MO(m - \alpha(m)).$$

If $|\omega| = m$ and b_ω is decomposable, say $b_\omega = b_{\omega_1} b_{\omega_2}$, define the restriction of $\bar{\zeta}$ to $S^{|\omega|}$ to be the composition

$$S^{|\omega|} = S^{|\omega_1|} \wedge S^{|\omega_2|} \xrightarrow[\tau \wedge \tau]{} TM_{\omega_1} \wedge TM_{\omega_2} \xrightarrow[T\nu_{\omega_1} \wedge T\nu_{\omega_2}]{}$$

$$MO(|\omega_1| - \alpha|\omega_1|) \wedge MO(|\omega_2| - \alpha|\omega_2|) \xrightarrow[T\mathrm{m}]{} MO(m - \alpha(m)).$$

Finally, if $|\omega| = m$ and b_ω is indecomposable, i.e. $b_\omega = b_m$, let the restriction of $\bar{\zeta}$ to S^m represent a class in $\pi_m^S(MO(m - \alpha(m))$ that projects to $b_m \in \pi_m(MO)$. Such a class exists by R. Brown's theorem [9, §1.3].

Now if h: $T(BO/I_m) = MO/I_m \to \bigvee_{|\omega| \leq m} S^{|\omega|} \wedge B(\overline{m - |\omega|})$ is any 2-local homotopy equivalence we can then form a map of Thom spectra

$$\zeta_n\colon MO/I_m \underset{h}{\overset{\simeq}{\to}} \bigvee_{|\omega| \leq m} S^{|\omega|} \wedge B(\overline{m - |\omega|}) \underset{\zeta}{\to} MO(m - \alpha(m)).$$

We observe that one can construct 2-local equivalences

$$h\colon MO/I_m \to \bigvee S^{|\omega|} \wedge B(\overline{m - |\omega|})$$

in the following manner.

First notice that for any pair of integers $r, s > 0$ such that $r + s = m$, the pair $(BO/I_r \times BO/I_s, \rho \times \rho)$ is quasi-normal of dimension m. We can therefore find a lifting

$$\mu_{r,s}\colon BO/I_r \times BO/I_s \to BO/I_m$$

of

$$\rho \times \rho\colon BO/I_r \times BO/I_s \to BO \times BO \underset{\mathrm{m}}{\to} BO.$$

Let $\mu = \{\mu_{r,s}\colon r + s = m\}$ be a collection of such pairings. This collection μ induces a mod 2 equivalence h_μ: $MO/I_m \to \bigvee S^{|\omega|} \wedge B(m - |\omega|)$ as follows.

$$\text{Let } \tilde{X}_m = \coprod_{\substack{\omega \leq m \\ \omega \neq (m)}} M_\omega \times F_{m-|\omega|}$$

where the union is taken over all monomials $b_\omega \in \pi_q MO$ such that $q \leq m$ except the ring generator (if m is not of the form $2^i - 1$).

Define $\tilde{g}_\mu$: $\tilde{X}_m \to BO/I_m$ to be the disjoint union of the maps

$$M_\omega \times F_{m-|\omega|} \xrightarrow[\tilde{\nu}_\omega \times \tilde{\gamma}_{m-|\omega|}]{} BO/I_{|\omega|} \times BO/I_{m-|\omega|} \underset{\mu}{\to} BO/I_m$$

if $|\omega| < m$, where $\tilde{\nu}_\omega$ and $\tilde{\gamma}_{m-|\omega|}$ are liftings of the stable normal bundle map ν_ω and the bundle map $\gamma_{m-|\omega|}$ respectively, defined by the restrictions of $g_{|\omega|}$ and $g_{m-|\omega|}$ to M_ω and $F_{m-|\omega|}$.

If $|\omega| = m$ and b_ω is decomposable, say $b_\omega = b_{i_1} \cdot b_{i_2} \cdots b_{i_r}$ where $i_1 \le i_2 \le \cdots \le i_r$, define the restriction of $\tilde{g}_\mu$ to M_ω to be the following composition.

$$M_\omega = M_{i_1} \times M_{\omega'} \xrightarrow[\tilde{\nu}_{i_1} \times \tilde{\nu}_{\omega'}]{} BO/I_{i_1} \times BO/I_{|\omega'|} \xrightarrow[\mu_{i_1, |\omega'|}]{} BO/I_m$$

where

$$b_{\omega'} = b_{i_2} \cdots b_{i_r}.$$

Let

$$\tilde{h}_\mu \colon \bigvee_{\substack{|\omega| \le m \\ \omega \ne (m)}} S^{|\omega|} \wedge B\big(\overline{m - |\omega|}\big) \to MO/I_m$$

be the composition

$$\bigvee_{\substack{|\omega| \le m \\ \omega \ne (m)}} S^{|\omega|} \wedge B\big(\overline{m - |\omega|}\big) \xrightarrow[\vee \tau \wedge 1]{} \vee T\nu_\omega \wedge B(m - |\omega|) = T\tilde{X}_m \xrightarrow[T\tilde{g}_\mu]{} MO/I_m.$$

Observe that Theorem 2.5 implies that 2-locally, the cofibre of $\tilde{h}_\mu$ is a sphere S^m if m is not of the form $2^i - 1$, and that $\tilde{h}_\mu$ is a mod 2 equivalence if $m = 2^i - 1$. Furthermore, if $m \ne 2^i - 1$ the projection of MO/I_m onto the cofibre of $\tilde{h}_\mu$ has a retraction $r_m \colon S^m \to MO/I_m$. Define

$$h_\mu \colon \bigvee_{|\omega| \le m} S^{|\omega|} \wedge B\big(\overline{m - |\omega|}\big) \to MO/I_m$$

to be equal to $\tilde{h}_\mu$ if $m = 2^i - 1$, and equal to the wedge of $\tilde{h}_\mu$ and r_m if $m \ne 2^i - 1$. Clearly h_μ is a mod 2 equivalence.

Now recall also that the disjoint union

$$\mathscr{F} = \coprod_{j \ge 0} F(R^2, j)/\Sigma_j = \coprod_{j \ge 0} F_j$$

is a $\mathscr{C}_2$-space in the sense of May [21] and in particular a homotopy commutative, associative H-space (where $F(R^2, 0)/\Sigma_0$ is by definition, a disjoint basepoint). The multiplication

$$p \colon \mathscr{F} \times \mathscr{F} \to \mathscr{F}$$

induces pairings $p_{r,s} \colon F_r \times F_s \to F_{r+s}$ and therefore pairings $\pi_{r,s} \colon X_r \times X_s \to \tilde{X}_{r+s}$ defined as the disjoint union of maps

$$(M_{\omega_1} \times F_{r - |\omega_1|}) \times (M_{\omega_2} \times F_{s - |\omega_2|}) \to (M_{\omega_1} \times M_{\omega_2}) \times F_{r - |\omega_1|} \times F_{s - |\omega_2|}$$
$$\xrightarrow[1 \times p]{} M_{\omega_1 + \omega_2} \times F_{r + s - |\omega_1 + \omega_2|}.$$

LEMMA 2.10. *There exists a collection of pairings*

$$\mu = \{\mu_{r,m-r}: BO/I_r \times BO/I_{m-r} \to BO/I_m,\ r \geq 1\}$$

that lift the maps $\rho \times \rho$: $BO/I_r \times BO/I_{m-r} \to BO$, *and which makes the following diagram homotopy commute*:

$$\begin{tikzcd} X_r \times X_{m-r} \arrow[r, "\pi_{r,m-r}"] \arrow[d, "g_r \times g_{m-r}"] & \tilde{X}_m \arrow[d, "\tilde{g}_\mu"] \\ BO/I_r \times BO/I_{m-r} \arrow[r, "\mu"'] & BO/I_m. \end{tikzcd}$$

The proof of Lemma 2.10 is somewhat involved and technical and is postponed until Section 4.

We may now complete the inductive step in the case m is of the form $m = 2^i - 1$. Notice that in this case there are no ring generators of $\pi_* MO$ in dimensions m. We may therefore let

$$X_m = \tilde{X}_m.$$

In order to define h_m: $X_m \to P_m$ observe that since each P_k is the pull-back of i: $BO(k - \alpha(k)) \to BO$ and ρ: $BO/I_k \to BO$, then the pairings

$$\mu_{r,m-r}: BO/I_r \times BO/I_{m-r} \to BO/I_m$$

induce pairings

$$\nu_{r,m-r}: P_r \times P_{m-r} \to P_m$$

that homotopy lift $\mu_{r,m-r}$ and the Whitney sum pairing $BO(r - \alpha(r)) \times BO(m - r - \alpha(m - r)) \to BO(m - \alpha(m))$. So define

$$h_m: X_m \to P_m$$

by defining its restriction to $M_\omega \times F_{m-|\omega|}$ to be

$$M_\omega \times F_{m-|\omega|} \xrightarrow[h_{|\omega|} \times h_{m-|\omega|}]{} P_{|\omega|} \times P_{m-|\omega|} \xrightarrow[\nu_{|\omega|, m-|\omega|}]{} P_m$$

if $|\omega| < m$, and if $|\omega| = m$ and $M_\omega = M_{i_1} \times M_{\omega'}$ as above, then define the restriction of h_m to M_ω to be the composition

$$M_\omega = M_{i_1} \times M_{\omega'} \xrightarrow[h_{i_1} \times h_{m-i_1}]{} P_{i_1} \times P_{m-i_1} \xrightarrow[\nu_{i_1, m-i_1}]{} P_m.$$

We now check that the pair (X_m, h_m) satisfies inductive hypotheses (2.9). The fact that the map f_j: $X_m \xrightarrow[h_m]{} P_m \to BO(m - \alpha(m))$ satisfies property (1) of (2.9) is clear because as noted above the pairings $\nu_{r,m-r}$ lift the Whitney sum pairings on the $BO(k)$'s. The fact that property 2.a is also satisfied is similarly clear and we leave its verification to the reader.

To verify property 2.b first observe that since the pairings $\nu_{r,m-r}$ lift the pairings $\mu_{m,m-r}$ on the BO/I_k's, the map $g_m\colon X_m \underset{h_m}{\to} P_m \to BO/I_m$ is homotopic to $\tilde{g}_\mu\colon \tilde{X}_m = X_m \to BO/I_m$. Thus by the commutativity of the diagram in Lemma 2.10 and the definition of the splitting $\sigma_m\colon MO/I_m \to TX_m$, we have the following homotopy commutative diagrams:

(*Remark*. The bottom square homotopy commutes since by the pull-back property, the two maps $T(h_m \circ \pi_{r,m-r})$ and $T(\nu_{r,m-r} \circ (h_r \times h_{m-r}))$ differ by a map from $TX_r \wedge TX_{m-r}$ to $\Sigma^{-1}MO$ that induces an $A(BO)$-homomorphism. But since $H^*(\Sigma^{-1}MO)$ is a cyclic $A(BO)$-module generated by a class $u \in H^{-1}(\Sigma^{-1}MO)$, there are no such maps.)

$$\begin{array}{ccc} MO/I_r \wedge MO/I_{m-r} & \xrightarrow{T\mu_{r,m-r}} & MO/I_m \\ \Big\downarrow{\scriptstyle \sigma_r \wedge \sigma_{m-r}} & & \Big\downarrow{\scriptstyle \sigma_m} \\ TX_r \wedge TX_{m-r} & \xrightarrow{T\pi} & TX_m \\ \Big\downarrow{\scriptstyle Th_r \wedge Th_{m-r}} & & \Big\downarrow{\scriptstyle Th_m} \\ TP_r \wedge TP_{m-r} & \xrightarrow[T\nu_{r,m-r}]{} & TP_m. \end{array}$$

To verify property 2.b, first restrict Th_m to $T(M_\omega \times F_{m-|\omega|}) = T(M_\omega) \wedge B(m - |\omega|)$ where $|\omega| = r < m$. By definition the restriction of Th_m to this spectrum is $T\nu_{r,m-r} \circ Th_r \wedge Th_{m-r}$, which by inductive assumption is $T\nu_{r,m-r} \circ Th_r \wedge Th_{m-r} \circ \sigma_r \wedge \sigma_{m-r} \circ Tg_r \wedge Tg_{m-r}$. But this is homotopic, by the commutativity of the above diagram to $Th_m \circ \sigma_m \circ T\mu_{r,m-r} \circ Tg_r \wedge Tg_{m-r}$ which, by the definition of g_m is homotopic to $Th_m \circ \sigma_m \circ Tg_m$. Thus when restricted to $T(M_\omega \times F_{m-|\omega|})$ for $|\omega| < m$, Th_m and $Th_m \circ \sigma_m \circ Tg_m$ agree. The fact that they agree when restricted to TM_ω when $|\omega| = m$ is proved the same way since M_ω is decomposable. Details are left to the reader.

Hence (X_m, h_m) satisfies the inductive hypotheses and therefore modulo 2.10, the proof of Lemma B in the case $m = 2^i - 1$ is now complete.

Part II

3. The proof of Lemma B in dimensions not equal to $2^i - 1$

In this section we complete the proof of Lemma B by completing the inductive step in the case $m \neq 2^i - 1$. So in this section we will continue to operate under inductive assumptions (2.9).

The added complication when $m \neq 2^i - 1$ is that there is a ring generator in dimension m in $\pi_*(MO) = Z_2[b_r\colon r \neq 2^i - 1]$. Our goal is to identify this problem as an explicit obstruction that we can calculate and show to be zero. On the Thom spectrum level this is easy because by the splitting of MO/I_m described earlier, a sphere representing an indecomposable in $\pi_*(MO)$ splits off of MO/I_m. In Section 3.a we will show that we can correspondingly remove a disk from BO/I_m and obtain a complex $\widetilde{BO}/I_m$ whose Thom spectrum does not involve any indecomposable cobordism classes in dimension m. Then using techniques analogous to those used in the case $m = 2^i - 1$ (when there are no m-dimensional indecomposables in $\pi_*(MO)$) we show that the restriction of ρ to $\widetilde{BO}/I_m$ lifts to a map $\tilde{\rho}_m\colon \widetilde{BO}/I_m \to BO(m - \alpha(m))$. Also in Section 3.a we show that to complete the inductive step it is sufficient to prove that $\tilde{\rho}_m$ can be extended over this "indecomposable" cell. We then observe that for trivial reasons, this extension can be done after suspending, and also on the Thom spectrum level. In Section 3.b we prove, modulo a lemma, that the required unstable extension of $\tilde{\rho}_m$ exists. In Section 3.c we prove the lemma.

3.a *The space* $\widetilde{BO}/I_m$. Consider the spectrum

$$\widetilde{MO}/I_m = \bigvee_{\substack{|\omega| \leq m \\ \omega \neq (m)}} \Sigma^{|\omega|} \overline{B(m - |\omega|)}$$

so that

$$MO/I_m \simeq \widetilde{MO}/I_m \vee S^m.$$

We shall show first that there is a subcomplex

$$\widetilde{BO}/I_m \subset BO/I_m$$

whose Thom spectrum is 2-locally homotopy equivalent to $\widetilde{MO}/I_m$. Then, using techniques similar to the ones used in the case when $m = 2^i - 1$, we will show that the restriction of ρ to $\widetilde{BO}/I_m$ lifts to $BO(m - \alpha(m))$.

We construct the space $\widetilde{BO}/I_m$ as follows. By standard cell-attaching techniques we may take the $(m-1)$-skeleton of BO/I_m and attach just enough of the m-cells to form a subcomplex

$$BO/I_m^{(m-1)} \subset BO/I_m$$

with the following properties:

1. $H_q(BO/I_m^{(m-1)}; Z_2) \xrightarrow{\cong} H_q(BO/I_m; Z_2)$ for $q \leq m - 1$.
2. $H_q(BO/I_m^{(m-1)}; Z_2) = 0$ for $q \geq m$.

Suppose $b_\omega \in \pi_m MO$ is a decomposable element. Say $b_\omega = b_{i_1} \cdot b_{\omega'}$ as before. Define the manifold M_ω to be

$$M_\omega = M_{i_1} \times M_{\omega'}.$$

Let $\tilde{M}_\omega$ be the space obtained by removing an m-disk from M_ω. So $M_\omega = \tilde{M}_\omega \cup D^m$. $\tilde{M}_\omega$ is homotopy equivalent to an $m-1$ dimensional CW complex M_ω^{m-1}. Thus there exists a map

$$\alpha_\omega \colon S^{m-1} \to M_\omega^{m-1}$$

so that M_ω is homotopy equivalent to the mapping cone

$$M_\omega \simeq M_\omega^{m-1} \bigcup_{\alpha_\omega} D_\omega^m.$$

Consider the map $g_\omega \colon M_\omega \to BO/I_m$ given by the composition

$$g_\omega \colon M_\omega = M_{i_1} \times M'_\omega \xrightarrow[g_{i_1} \times g_{\omega'}]{} BO/I_{|i_1|} \times BO/I_{|\omega'|} \xrightarrow[\mu]{} BO/I_m.$$

Now let $\tilde{g}_\omega \colon M_\omega^{m-1} \to BO/I_m$ be the restriction of g_ω. Clearly $\tilde{g}_\omega$ factors through $BO/I_m^{(m-1)}$. We may then define a map

$$\beta_\omega \colon S^{m-1} \xrightarrow[\alpha_\omega]{} M_\omega^{m-1} \xrightarrow[\tilde{g}_\omega]{} BO/I_m^{(m-1)}$$

and can thus define a complex

$$\widetilde{BO}/I_m = BO/I_m^{(m-1)} \cup \left(\bigcup_{\omega \neq (m)} D_\omega^m \right)$$

where the union is taken over all m-dimensional classes $b_\omega \neq b_m$ in $\pi_m MO$, and the disk D_ω^m is attached to $BO/I_m^{(m-1)}$ via the map β_ω.

Define a map

$$i_m \colon \widetilde{BO}/I_m \to BO/I_m$$

to be equal to the inclusion when restricted to $BO/I_m^{(m-1)}$, and when restricted to the disk D_ω^m to be the composition

$$D_\omega^m \hookrightarrow M_\omega^{m-1} \bigcup_{\alpha_\omega} D_\omega^m \simeq M_\omega \xrightarrow[g_\omega]{} BO/I_m.$$

Clearly i_m is well-defined, and may, without loss of generality be assumed to be an inclusion of cell complexes.

Now an easy calculation shows that the composition of Thom spectrum maps

$$T(\widetilde{BO}/I_m) \xrightarrow{T(i_m)} MO/I_m \simeq \bigvee_\omega \Sigma^{|\omega|} B(\overline{m - |\omega|})$$
$$\longrightarrow \bigvee_{\omega \neq (m)} \Sigma^{|\omega|} B(\overline{m - |\omega|}) \to \widetilde{MO}/I_m$$

induces an isomorphism of mod 2-cohomology groups, and is therefore a 2-local equivalence.

Observe that we may write $BO/I_m \simeq \widetilde{BO}/I_m \cup D^m$ in the following manner.

Let N_m be any m-manifold that represents an indecomposable element in $\pi_m(MO)$.
Let

$$\nu_m\colon N_m \to BO/I_m$$

be a lifting of the stable normal bundle map of N_m. Now as before we may give N_m a CW decomposition so that

$$N_m = N_m^{m-1} \bigcup_{\alpha_m} D^m.$$

Since $\widetilde{BO}/I_m$ contains the $(m-1)$-skeleton of BO/I_m, the restriction of ν_m to N_m^{m-1} factors through a map

$$\tilde{\nu}_m\colon N_m^{m-1} \to \widetilde{BO}/I_m.$$

Let

$$\beta = \tilde{\nu}_m \circ \alpha_m\colon S^{m-1} \to N_m^{m-1} \to \widetilde{BO}/I_m.$$

A direct cohomological observation yields that

$$i_m \cup \nu_m|_{D^m}\colon \widetilde{BO}/I_m \bigcup_{\beta} D^m \to BO/I_m$$

is a mod 2 homological equivalence.

Now let $\tilde{\rho}\colon \widetilde{BO}/I_m \to BO$ be the restriction of ρ. We now proceed to show that $\tilde{\rho}$ lifts to a map

$$\tilde{\rho}_m\colon \widetilde{BO}/I_m \to BO(m-\alpha(m)).$$

To do this, we first study the space

$$\tilde{X}_m = \coprod_{\omega \neq (m)} M_\omega \times F_{m-|\omega|}.$$

In the case considered in the last section, when $m = 2^i - 1$, $\tilde{X}_m = X_m$ and we constructed a map $h_m\colon X_m \to P_m$. This same construction yields a map in this case,

$$\tilde{h}_m\colon \tilde{X}_m \to P_m.$$

Now let $\tilde{P}_m$ be the homotopy pull-back for the square

$$\begin{tikzcd} \tilde{P}_m \arrow[r] \arrow[d] & BO(m-\alpha(m)) \arrow[d] \\ \widetilde{BO}/I_m \arrow[r, "\rho"'] & BO. \end{tikzcd}$$

In this setting, a simple obstruction theoretic argument yields that the pairings of Lemma 2.10, $\mu_{r,m-r}\colon BO/I_r \times BO/I_{m-r} \to BO/I_m$ in fact factor through pairings

$$\tilde{\mu}_{r,m-r}\colon BO/I_r \times BO/I_{m-1} \to \widetilde{BO}/I_m.$$

This implies that the pairings $\tilde{\nu}_{r,m-r}\colon P_r \times P_{m-r} \to P_m$ factor through pairings

$$\tilde{\nu}_{r,m-r}\colon P_r \times P_{m-r} \to \tilde{P}_m.$$

Now by construction, $\tilde{h}_m\colon \tilde{X}_m \to P_m$, when restricted to any single connected component, factors through such a pairing, and hence $\tilde{h}_m$ factors through a map, which by abuse we also call $\tilde{h}_m$,

$$\tilde{h}_m\colon \tilde{X}_m \to \tilde{P}_m.$$

Lemma 3.1. *The map $\tilde{h}_m\colon \tilde{X}_m \to \tilde{P}_m$ satisfies the following properties*:

1. *if $\tilde{g}_m\colon \tilde{X}_m \to \widetilde{BO}/I_m$ is the composition $\tilde{X}_m \xrightarrow{\tilde{h}_m} \tilde{P}_m \to \widetilde{BO}/I_m$, then $T\tilde{g}_m$ has a splitting $\tilde{\delta}_m\colon \widetilde{MO}/I_m \to T\tilde{X}_m$.*
2. *The following diagram of 2-local Thom spectra homotopy commutes*:

$$\begin{array}{ccc} T\tilde{X}_m & \xrightarrow{T\tilde{h}_m} & T\tilde{P}_m \\ {\scriptstyle T\tilde{g}_m}\downarrow & & \uparrow{\scriptstyle T\tilde{g}_m} \\ \widetilde{MO}/I_m & \xrightarrow[\tilde{\delta}_m]{} & T\tilde{X}_m. \end{array}$$

Proof. Notice that this lemma is the precise analogue of Lemma B with $\tilde{X}_m$ replacing X_m, $\tilde{P}_m$ replacing P_m, etc. The proof of Lemma B in the case $m = 2^i - 1$, so that $X_m = \tilde{X}_m$, given in Section 2, goes through verbatim to prove this lemma.

Now the proof that Lemma B implies Theorem A (and in fact Theorem A′) given in Section 1 also goes through verbatim with $\tilde{X}_m$ replacing X_m, etc. to imply the following.

Lemma 3.2. *There exists a map $\tilde{\rho}_m\colon \widetilde{BO}/I_m \to BO(m - \alpha(m))$ that lifts $\tilde{\rho}$ and that makes the following diagram of Thom spectra homotopy commute*:

$$\begin{array}{ccc} T\tilde{X}_m & \xrightarrow{T\tilde{f}_m} & MO(m - \alpha(m)) \\ & {\scriptstyle T\tilde{g}_m}\searrow \quad \nearrow{\scriptstyle T\tilde{\rho}_m} & \\ & \widetilde{MO}/I_m & \end{array}$$

Here $\tilde{f}_m\colon \tilde{X}_m \to BO(m - \alpha(m))$ is the composition $\tilde{f}_m\colon \tilde{X}_m \xrightarrow{\tilde{h}_m} \tilde{P}_m \to BO(m - \alpha(m))$.

Remark. Since $\tilde{\delta}_m\colon T(\widetilde{BO}/I_m) \to TX_m$ splits $T\tilde{g}_m\colon TX_m \to T(\widetilde{BO}/I_m)$, the induced 2-local Thom spectrum map

$$T\tilde{\rho}_m\colon T\left(\widetilde{BO}/I_m\right) \to MO(m-\alpha(m))$$

can be described explicitly as follows:

$$T\tilde{\rho}_m \cong T\tilde{f}_m \circ \tilde{\delta}_m\colon T\left(\widetilde{BO}/I_m\right) \to T\tilde{X}_m \to MO(m-\alpha(m)).$$

Our goal in the remainder of Section 3 is to prove the following.

THEOREM 3.3. *There exists a map $\rho_m\colon BO/I_m \to BO(m-\alpha(m))$ making the following diagram homotopy commute*:

$$\begin{tikzcd}
\widetilde{BO}/I_m \arrow[r, "\tilde{\rho}_m"] \arrow[d, hook] & BO(m-\alpha(m)) \arrow[d] \\
BO/I_m \arrow[ur, "\rho_m"] \arrow[r, "\rho"'] & BO.
\end{tikzcd}$$

Before we begin the proof of this theorem we first show how it will allow us to complete the inductive step in the proof of Lemma B. So assume the truth of Theorem 3.3 for now.

Let M_m be any manifold that represents an indecomposable class in $\pi_m(MO)$, and let $\tilde{\nu}_m\colon M_m \to BO/I_m$ be any lifting of the stable normal bundle map. Also, let $\nu_m = \rho_m \circ \tilde{\nu}_m\colon M_m \to BO/I_m \to BO(m-\alpha(m))$. Furthermore we define

$$X_m = \tilde{X}_m \amalg M_m$$

and the map $h_m\colon X_m \to P_m$ to be the disjoint union of $\tilde{h}_m$ and the composition $M_m \xrightarrow{\tilde{\nu}_m} BO/I_m \xrightarrow{s_m} P_m$, where s_m is a section of $P_m \to BO/I_m$ that exists by the pull-back property. The fact that the pair (X_m, h_m) satisfies our inductive hypotheses 2.9 is clear, and its verification is left to the reader.

We are therefore reduced to showing that the map $\tilde{\rho}_m$ extends over the final cell D^m making up BO/I_m in such a way that the diagram in the statement of Theorem 3.3 homotopy commutes. We end Section 3.a by proving that this can be done on the Thom spectrum level and on the base space level after suspending. That is, we will prove the following.

PROPOSITION 3.4. a. *There exists a map $t_m\colon MO/I_m \to MO(m-\alpha(m))$ making the following diagram*

$$\begin{tikzcd}
\widetilde{MO}/I_m \arrow[r, "T\tilde{\rho}_m"] \arrow[d, phantom, "\cap"] & MO(m-\alpha(m) \arrow[d] \\
MO/I_m \arrow[ur, "t_m"] \arrow[r, "T\rho"'] & MO
\end{tikzcd}$$

homotopy commute.

b. *There exists a stable map* r_m: $\Sigma^\infty BO/I_m \to \Sigma^\infty BO(m-\alpha(m))$ *making the diagram*

$$\begin{tikzcd} \Sigma^\infty \widetilde{BO}/I_m \arrow[r, "\Sigma^\infty \rho_m"] \arrow[d, phantom, "\cap"] & \Sigma^\infty BO(m-\alpha(m)) \arrow[d] \\ \Sigma^\infty BO/I_m \arrow[ur, "r_m"] \arrow[r, "\Sigma^\infty \rho"'] & \Sigma^\infty BO \end{tikzcd}$$

homotopy commute. Here Σ^∞ *is the functor that assigns to a space its associated suspension spectrum.*

Proof. a. Since $MO/I_m \simeq MO/I_m \vee S^m$, part a follows from R. Brown's result [9] which states that the homomorphism $\pi_m^s(MO(m-\alpha(m)) \to \pi_m(MO)$ is surjective.

b. By Lemma 1.7 the obstruction to finding r_m is cohomological. But by the definition of the ideal $I_m \subset H^*BO$ and the fact that the homomorphism $H^*(BO/I_m, \widetilde{BO}/I_m) \to H^*(BO/I_m)$ is injective, all such cohomological obstructions vanish.

3.b. *Extending the map* $\tilde{\rho}_m$. Recall that in order to complete the proof of Lemma B we must prove Theorem 3.3. That is, we must show that the map $\tilde{\rho}_m$: $\widetilde{BO}/I_m \to BO(m-\alpha(m))$ extends to all of BO/I_m in a particular manner. Proposition 3.4 says that such an extension exists stably (after suspending) and on the level of Thom spectra. Our scheme is to use the Moore-Postnikov tower for the map $\tilde{\rho}_m$ and the results of the previous section to prove that this stable extension of ρ_m desuspends to yield an unstable map ρ_m whose Thom-ification is the extension of $T\tilde{\rho}_m$ given in 3.4. This will be done by an inductive procedure which we will set up in this section. We will also reduce the inductive step to proving a certain lemma which we leave for section 3.c.

We begin by observing that the map $\tilde{\rho}_m$: $\widetilde{BO}/I_m \to BO(m-\alpha(m))$ defines a map

$$\tilde{q}\colon \widetilde{BO}/I_m \to P_m$$

so that the composition $\widetilde{BO}/I_m \underset{\tilde{q}}{\to} P \to BO/I_m$ is the inclusion. Notice that in this setting, the existence of a map ρ_m: $BO/I_m \to BO(m-\alpha(m))$ satisfying Theorem 3.3 would be implied by the existence of an extension of $\tilde{q}$ to a map q: $BO/I_m \to P$ with the property that the composition $BO/I_m \to P \to BO/I_m$ is homotopic to the identity.

We observe that when localized at an odd prime p or at the rationals there is no problem in finding such an extension. We are therefore dealing with a 2-local problem, so from here on we assume that all spaces and spectra are localized at 2. (We leave the verification of these observations as an exercise for the reader. These localization questions have been studied in detail by Brown [27].)

Define $\tilde{Q}$ to be the homotopy fiber of the map $P \to S^m$ defined by the composition

$$P \to BO/I_m \to BO/I_m/\widetilde{BO}/I_m \simeq S^m.$$

Now the homotopy fiber of the inclusion $\tilde{Q} \to P$ is ΩS^m. We may therefore define the class $\alpha \in \pi_{m-1}(\tilde{Q})$ to be the composition

$$\alpha \colon S^{m-1} \subset \Omega S^m \to \tilde{Q}.$$

Finally, define the space Q to be the mapping cone

$$Q = \tilde{Q} \bigcup_{\alpha} D^m.$$

We observe that there is a natural map $Q \to P$ extending the inclusion $\tilde{Q} \to P$. Furthermore since the composition $\widetilde{BO}/I_m \xrightarrow[\tilde{q}]{} P \to S^m$ is null homotopic, $\tilde{q}$ lifts to a map which, by abuse of notation we also call $\tilde{q}\colon \widetilde{BO}/I_m \to \tilde{Q}$. Thus Theorem 3.3 (and therefore Lemma B) will follow from:

THEOREM 3.5. *There exists a map* $q\colon BO/I_m \to Q$ *making the following diagram homotopy commute*:

$$\begin{tikzcd}
\widetilde{BO}/I_m \arrow[d, "\tilde{q}"'] & \subset & BO/I_m \arrow[d, "q"] \arrow[drr, "="] & & \\
\tilde{Q} & \subset & Q \arrow[r] & P \arrow[r] & BO/I_m.
\end{tikzcd}$$

Proof. Our proof will be by an inductive process using the Moore-Postnikov tower for the map $\tilde{q}\colon \widetilde{BO}/I_m \to \tilde{Q}$:

$$(3.6)\qquad \begin{tikzcd}
\widetilde{BO}/I_m \arrow[r] & \cdots \arrow[r] & \tilde{Q}_j \arrow[r] & \tilde{Q}_{j-1} \arrow[r] & \cdots \arrow[r] & \tilde{Q}_1 \arrow[r] & \tilde{Q}_0 = \tilde{Q} \\
 & & L_j \arrow[u] & L_{j-1} \arrow[u] & & L_1. \arrow[u] &
\end{tikzcd}$$

Now let $\mathscr{F}$ be the homotopy fiber of the map $\tilde{q}\colon \widetilde{BO}/I_m \to \tilde{Q}$. An easy calculation shows that after localizing at the rationals we have an isomorphism

$$\pi_{r-1}(\mathscr{F})_0 \cong \pi_r\left(V_{m-\alpha(m)}\right)_0$$

for $r \leq m-1$. (Here $V_{m-\alpha(m)}$ is the fiber of the map $BO(m-\alpha(m)) \to BO$.)

Now for $m-\alpha(m)$ odd these groups are zero so that in this range $\pi_{r-1}\mathscr{F}$ is a finite 2-group. (Recall all spaces are localized at 2.) When $m-\alpha(m)$ is even we have that $\pi_{m-\alpha(m)-1}(\mathscr{F}) \cong \mathbf{Z}_{(2)}$, the 2-local integers, and for $m-\alpha(m) \leq r-1 \leq m-1$, $\pi_{r-1}(\mathscr{F})$ is a finite 2-group.

Assume for now that $m - \alpha(m)$ is odd. As we did in Section 1 we may modify tower (3.6) if necessary so that there exists an integer N such that for $i \le N$ each map $\tilde{Q}_i \to \tilde{Q}_{i-1}$ is a principal fibration with fibre L_i a product of Eilenberg-MacLane spaces of type $K(\mathbf{Z}_2, n)$ with $n \le m - 2$, and that for $i > N$, L_i is $m - 2$ connected, and so the pairs (Q_{i-1}, Q_i) are $m - 1$ connected. By abuse of notation we shall assume that tower (3.6) represents the first N stages of the modified Postnikov tower for the fibration $\tilde{q}$: $\widetilde{BO/I_m} \to \tilde{Q}$.

Our goal is to use this tower to produce a map $\bar{\alpha}$: $S^{m-1} \to \widetilde{BO}/I_m$ that lifts α: $S^{m-1} \to \tilde{Q}$. Once this was done we would obtain a map

$$\widetilde{BO}/I_m \bigcup_{\bar{\alpha}} D^m \to \tilde{Q} \bigcup_{\alpha} D^m = Q$$

extending $\tilde{q}$. Now since the composition $BO/I_m \cup_{\bar{\alpha}} D^m \to Q \to BO/I_m$ would be a homotopy equivalence, this would allow us to construct a map q: $BO/I_m \to Q$ satisfying the requirements of Theorem 3.5. Since we are thus only interested in lifting an $(m - 1)$-dimensional homotopy class we are justified in only considering the first N stages of the modified Postnikov tower for the map $\tilde{q}$.

To prove the existence of the map $\bar{\alpha}$: $S^{m-1} \to \widetilde{BO}/I_m$ we make the following inductive assumptions.

(3.7) 1. There is a lifting α_{j-1}: $S^{m-1} \to \tilde{Q}_{j-1}$ of $\alpha_0 = \alpha$: $S^{m-1} \to \tilde{Q}_0 = \tilde{Q}$.

2. If $Q_{j-1} = \tilde{Q}_{j-1} \cup_{\alpha_{j-1}} D^m$, then there exists a *stable* map q_{j-1}: $\Sigma^\infty BO/I_m \to \Sigma^\infty Q_{j-1}$ that satisfies the following properties.

a. The following diagram homotopy commutes:

$$\begin{tikzcd}
\widetilde{BO}/I_m \arrow[r, hook] \arrow[d, "\tilde{q}_{j-1}"] & BO/I_m \arrow[d, "q_{j-1}"] \arrow[drr, "="] & & \\
\tilde{Q}_{j-1} \arrow[r, hook] & Q_{j-1} \arrow[r] & Q_0 \arrow[r] & BO/I_m
\end{tikzcd}$$

where $\tilde{q}_j$: $\widetilde{BO}/I_m \to \tilde{Q}_{j-1}$ is given by tower (3.6).

b. Let $d_m \in H_m BO/I_m$ be the Thom isomorphic image of the class represented by the sphere in the splitting

$$MO/I_m \simeq \widetilde{MO}/I_m \vee S^m.$$

Then if Φ: $H_* Q_{j-1} \to H_* TQ_{j-1}$ is the Thom isomorphism,

$$\Phi\big(q_{j-1*}(d_m)\big) \in H_m TQ_{j-1}$$

is spherical.

Remark. These inductive assumptions are taken to be vacuous if $j = 0$.

Before we complete the inductive step, we make some observations about what the inductive hypotheses immediately imply.

Observation 3.8. There exists a map of spectra τ_{j-1}: $MO/I_m \to TQ_{j-1}$ so that the composition $MO/I_m \xrightarrow[\tau_{j-1}]{} TQ_{j-1} \to TQ \to MO/I_m$ is the identity, and so that the following diagram commutes:

$$\begin{array}{ccc} H_*MO/I_m & \xrightarrow{\tau_{j-1*}} & H_*TQ_{j-1} \\ \cong \uparrow \Phi & & \cong \uparrow \Phi \\ H_*BO/I_m & \xrightarrow[q_{j-1*}]{} & H_*Q_{j-1}. \end{array}$$

Proof. The class $\Phi q_{j-1*}(d_m) \in H_m TQ_{j-1}$ is assumed to be spherical; so let

$$\sigma_{j-1}\colon S^m \to TQ_{j-1}$$

be a map that represents this class. Notice that by inductive assumption 2.a we may assume that σ_{j-1} lifts the class represented by the sphere in the splitting $MO/I_m \simeq \widetilde{MO}/I_m \vee S^m$. Define τ_{j-1} to be the composition

$$\tau_{j-1}\colon MO/I_m \simeq \widetilde{MO}/I_m \vee S^m \xrightarrow[T\tilde{q}_{j-1} \vee \sigma_{j-1}]{} TQ_{j-1}.$$

Clearly τ_{j-1} satisfies the required properties.

Now given any stable vector bundle ζ over a base space B, $H^*(B)$ can be given the structure of a *right* A-module by the following rule. If $x \subset H^*(B)$, define

$$x\,Sq^n = \Phi^{-1}(\chi(Sq^n)(\Phi(x)))$$

where again Φ: $H^*B \to H^*(T(\zeta))$ is the Thom isomorphism.

Observation 3.9. The homomorphism q_{j-1}^*: $H^*Q_{j-1} \to H^*BO/I_m$ is a homomorphism of both left and *right* A-modules.

Proof. This follows immediately from 3.8.

Observation 3.10. The homomorphism q_{j-1}^*: $H^*Q_{j-1} \to H^*(BO/I_m)$ is a homomorphism of $H^*(BO)$-modules.

Proof. This follows from observation 3.9 and the fact that if x is any cohomology class in the base space of a vector bundle and w_j is the j^{th} Stiefel-Whitney class of this bundle, then an easy exercise yields

$$x \cup w_j = \sum_{k \le j} (\chi(Sq^k)(x))\chi(Sq^{j-k}).$$

Observation 3.11. *The homomorphism* τ_{j-1}^*: $HTQ_{j-1} \to HMO/I_m$ *is* $A(BO)$*-linear.*

Proof. This follows from 3.8 and 3.10 and the definition of the $A(BO)$-module structure of the cohomology of Thom spectra.

We now proceed to complete the inductive step in our proof of Theorem 3.5.

Now since the fibrations $\tilde{Q}_j \to \tilde{Q}_{j-1}$ of tower (3.6) are assumed to be principal, there exist k invariant maps $\tilde{\gamma}_j$: $\tilde{Q}_{j-1} \to L_{j,1}$ with fiber $\tilde{Q}_j$. Here $L_{j,1}$ is a product of Eilenberg-MacLane spaces with the property that $L_j = \Omega L_{j,1}$. Notice furthermore that these k-invariants factor through $\tilde{Q}_{j-1}/\widetilde{BO}/I_m$ so that there is a map $\bar{\gamma}_j$: $\tilde{Q}_{j-1}/\widetilde{BO}/I_m \to L_{j,1}$ so that $\tilde{\gamma}_j$ is the composition

$$\tilde{\gamma}_j\colon \tilde{Q}_{j-1} \to \tilde{Q}_{j-1}/\widetilde{BO}/I_m \xrightarrow[\bar{\gamma}_j]{} L_{j-1}.$$

Now notice that we have a stable equivalence of cofibers

$$\Sigma^\infty \tilde{Q}_{j-1}/\widetilde{BO}/I_m \simeq \Sigma^\infty Q_{j-1}/BO/I_m \tag{3.12}$$

and since these spectra are $m/2$-connected, the projection map $\Sigma^\infty Q_{j-1} \to \Sigma^\infty Q_{j-1}/BO/I_m$ desuspends to give an extension $Q_{j-1} \to \tilde{Q}_{j-1}/\widetilde{BO}/I_m$ of the projection from $\tilde{Q}_{j-1}$. We may therefore define an extension

$$\gamma_j\colon Q_{j-1} \to L_{j,1}$$

to be the composition

$$\gamma_j\colon Q_{j-1} \to \tilde{Q}_{j-1}/\widetilde{BO}/I_m \xrightarrow[\bar{\gamma}_j]{} L_{j,1}.$$

For $j \geq 1$ we define Q_j to be the homotopy fiber of γ_j. (Recall Q_0 was previously defined to be Q.)

Notice that the map of pairs $(Q_j, \tilde{Q}_j) \to (Q_{j-1}, \tilde{Q}_{j-1})$ is by construction a homotopy equivalence. Thus

$$\pi_m\big(Q_j, \tilde{Q}_j\big) \cong \pi_m\big(Q_{j-1}, \tilde{Q}_{j-1}\big) \cong \mathbf{Z}$$

generated by a class $\alpha_{j-1} \in \pi_{m-1}(\tilde{Q}_{j-1})$. Thus α_{j-1} lifts to a map α_j: $S^{m-1} \to \tilde{Q}_j$ and it is immediately verified that

$$Q_j \simeq \tilde{Q}_j \bigcup_{\alpha_j} D^m.$$

(Again, for $j = 0$, α_0 is taken to be α: $S^{m-1} \to \tilde{Q}$.)

To complete the inductive step we must construct a stable map q_j: $BO/I_m \to Q_j$ that satisfies certain properties. *In what follows we will* *assume that* $j > 0$. At the end of Section 3.c we will describe the modification of the argument necessary for $j = 0$.

The following is the first step in constructing the map q_j.

PROPOSITION 3.13. a. *There exists a stable lifting* q_j': $\Sigma^\infty BO/I_m \to \Sigma^\infty Q_j$ *of* q_{j-1} *that extends* $\tilde{q}_j$: $\widetilde{BO/I_m} \to \tilde{Q}_j \subset Q_j$.

b. *There exists a lifting* τ_j: $MO/I_m \to TQ_j$ *of* τ_{j-1} *that extends* $T\tilde{q}_j$: $\widetilde{MO/I_m} \to \widetilde{TQ_j} \subset TQ_j$.

Remark. After this proposition is proved we will then have to show that q_j' can be adjusted to make a map q_j that satisfies the inductive hypothesis (3.7 : 2.b). That is, $\Phi(q_{j*}(d_m)) \in H_m TQ_j$ is spherical.

Proof of 3.13. Recall that the k invariant γ_j: $Q_{j-1} \to L_{j,1}$ factors through $Q_{j-1}/BO/I_m$ stably, and so the composite $BO/I_m \xrightarrow[q_{j-1}]{} Q_{j-1} \to L_{j,1}$ is null. Now let $c(\gamma_j)$: $Q_{j-1} \to L_{j,1} \wedge BO^+$ be the composition

$$c(\gamma_j)\colon Q_{j-1} \xrightarrow[\gamma_j \times f_{j-1}]{} L_{j,1} \times BO \longrightarrow L_{j,1} \times BO/L_{j,1} \times *$$

where f_{j-1} is the classifying map. By the results of Section 1.a, through dimension m, $c(\gamma_j)$ is the projection onto the cofiber $Q_{j-1} \to Q_{j-1}/Q_j$.

Now since q_{j-1}^* is a homomorphism of $H^*(BO)$-modules (3.10), the composition

$$BO/I_m \xrightarrow[q_{j-1}]{} Q_{j-1} \xrightarrow[c(\gamma_j)]{} L_{j,1} \wedge BO^+$$

is zero in cohomology, and therefore is null homotopic. Thus q_{j-1} lifts to a map q_j': $BO/I_m \to Q_j$. Moreover, since the inclusion $\widetilde{BO/I_m} \subset BO/I_m$ induces a surjection in cohomology, q_j' may be chosen so that it extends $\tilde{q}_j$: $\widetilde{BO/I_m} \to \tilde{Q}_j \subset Q_j$.

The existence of a map τ_j: $MO/I_m \to TQ_j$ satisfying 3.13 is proved the same way, using the fact that τ_{j-1}^*: $H^*TQ_{j-1} \to H^*MO/I_m$ is $A(BO)$-linear (3.11). Details are left to the reader. This completes the proof of Prop. 3.13.

We now proceed to show that the stable map q_j': $BO/I_m \to Q_j$ can be modified to a map q_j so as to satisfy the inductive hypotheses (i.e. such that $\Phi q_{j*}(d_m)$ is spherical). In order to do this, we shall introduce an intermediate space F and study stable maps from BO/I_m to F. This is done for technical reasons that will become apparent later.

Consider the following diagram of fibration sequences:

$$\begin{tikzcd} F' \arrow[r] \arrow[d] & Q_{j-1} \arrow[r, "p'"] \arrow[d, "="] & Q/B \arrow[d, "i"] \\ F \arrow[r] & Q_{j-1} \arrow[r, "p"'] & Q/B \wedge K. \end{tikzcd}$$

Here Q/B is the quotient $\tilde{Q}_{j-1}/\widetilde{BO/I_m}$, which as observed before (3.12) is stably homotopy equivalent to the stable quotient $Q_{j-1}/BO/I_m$. The map p' is an extension of the projection $\tilde{Q}_{j-1} \to Q/B$ that stably is the projection map $\Sigma^\infty Q_{j-1} \to \Sigma^\infty(Q_{j-1}/BO/I_m)$. As defined above, the k-invariant γ_j: $Q_{j-1} \to L_{j,1}$ factors through p'. The space $Q/B \wedge K$ is defined to be a product of Eilenberg-MacLane spaces that corresponds to the spectrum $Q/B \wedge K(\mathbf{Z}_2)$. More precisely

$$Q/B \wedge K = \varinjlim_{q} \Omega^q\big(Q/B \wedge K(\mathbf{Z}_2, q)\big).$$

The map i: $Q/B \to Q/B \wedge K$ is the usual inclusion i: $Q/B = Q/B \wedge S^0 \to Q/B \wedge K$. The map p: $Q_{j-1} \to Q/B \wedge K$ is defined to be $i \circ p'$. Finally, the spaces F' and F are defined to be the homotopy fibers of p' and p respectively.

Now observe that the k-invariant γ_j: $Q_{j-1} \to L_{j,1}$ not only factors through p', but also factors (up to homotopy) through p: $Q_{j-1} \to Q/B \wedge K$ since $L_{j,1}$ is Eilenberg-MacLane, and since i: $Q/B \to Q/B \wedge K$ induces a surjection in cohomology. Thus we have a homotopy lifting

$$l\colon F \to Q_j$$

of the inclusion $F \to Q_{j-1}$. Notice, moreover, that we may assume without loss of generality that the map $\tilde{q}_j$: $\widetilde{BO/I_m} \to \tilde{Q}_j \to Q_j$ in fact factors through F'. That is, there is a map $\tilde{g}$: $\widetilde{BO/I_m} \to F'$ so that $\tilde{q}_j$ is the composition

$$\tilde{q}_j\colon \widetilde{BO/I_m} \to F \xrightarrow[l]{} Q_j.$$

Thus in order to complete the inductive step it suffices to prove the following.

Proposition 3.14. *There is a stable map* g: $\Sigma^\infty BO/I_m \to \Sigma^\infty F$ *that extends* $\tilde{g}$ *and lifts* q_{j-1}: $\Sigma^\infty BO/I_m \to \Sigma^\infty Q_{j-1}$, *and so that the Thom isomorphic image,* $\Phi(g_*(d_m)) \in H_m TF$ *is spherical.*

Proof. We first observe that the stable cofiber of the map $F \to Q_{j-1}$ is, through dimension m, equivalent to $Q/B \wedge K \wedge BO^+$, by the results of Section 1.a; and that the projection map $Q_{j-1} \to Q_{j-1}/F \simeq Q/B \wedge K \wedge BO^+$ is given by

$$c(p)\colon Q_{j-1} \xrightarrow[p \times f_{j-1}]{} Q/B \wedge K \times BO \xrightarrow[\text{project}]{} Q/B \wedge K \wedge BO^+$$

where f_{j-1} is the classifying map. Thus to obtain a stable lifting

$$g'\colon \Sigma^\infty BO/I_m \to \Sigma^\infty F$$

of q_{j-1}, we need to show that the stable composition

$$BO/I_m \to Q_{j-1} \to Q/B \wedge K \wedge BO^+$$

is null homotopic. To do this, recall that by 3.10 the homomorphism q_{j-1}^* is $H^*(BO)$-linear. Since q_{j-1}^* is surjective, this says that the inclusion of the kernel, which is given by $p'^*\colon H^*(Q/B) \to H^*(Q_{j-1})$, is $H^*(BO)$-linear. By the definition of the $H^*(BO)$-structure and of the map $p\colon Q_{j-1} \to Q/B \wedge K$, this says that the following diagram homotopy commutes:

$$\begin{tikzcd} Q_{j-1} \arrow[r, "p"] \arrow[dr, "c(p)"'] & Q/B \wedge K \arrow[d, "\Delta"] \\ & Q/B \wedge BO^+ \wedge K \end{tikzcd}$$

where Δ is defined as follows. Consider the map of pairs

$$1 \times f_{j-1}\colon \left(\tilde{Q}_{j-1}, \widetilde{BO}/I_m\right) \to \left(\tilde{Q}_{j-1} \times BO, \widetilde{BO}/I_m \times BO\right).$$

This induces a stable map of quotients $Q/B \to Q/B \wedge BO^+$. Δ is the smash of this map with the identity of K.

Thus, since $p \circ q_{j-1}\colon BO/I_m \to Q_{j-1} \to Q/B \wedge K$ is null homotopic, so is the composition $c(p) \circ q_{j-1}$. Therefore there is a stable lifting $g'\colon BO/I_m \to F$ of q_{j-1}. Moreover, since the inclusion $\widetilde{BO}/I_m \to BO/I_m$ induces a surjection in cohomology, g' may be chosen so that it extends $\tilde{g}\colon \widetilde{BO}/I_m \to F$.

We now proceed to study possible choices of such liftings $g'\colon BO/I_m \to F$.

For ease of notation let $F/\tilde{B}$ denote the quotient $F/\widetilde{BO}/I_m$, $F/B = \tilde{F}/\widetilde{BO}/I_m$, and let $T(F/\tilde{B})$ and $T(F/B)$ denote the quotients of the corresponding Thom spectra. Here $\tilde{F}$ is the pullback of $F \to Q_{j-1}$ over $\tilde{Q}_{j-1} \hookrightarrow Q_{j-1}$. Observe that the existence of a stable extension g' of $\tilde{g}$ allows us to construct a map of quotients

$$\varepsilon(g')\colon S^m = BO/I_m/\widetilde{BO}/I_m \to F/\tilde{B}.$$

Observe that the projection onto the cofiber of $\varepsilon(g')$ is a retraction $r(g')\colon F/\tilde{B} \to F/B$ of the inclusion $i\colon F/B = \tilde{F}/\widetilde{BO}/I_m \to F/\tilde{B}$.

Now suppose for the moment that we can find a stable lifting g' as above so that $\epsilon(g')\colon S^m \to F/\tilde{B}$ has the property that $\Phi\varepsilon(g')_*[S^m] \in H_mT(F/\tilde{B})$ is spherical. Under these conditions we then let $g = g'\colon \Sigma^\infty BO/I_m \to \Sigma^\infty F$.

We now show that g satisfies 3.14, that is, that $\Phi(g_*(d_m)) \in H_mTF$ is spherical.

First observe that the splitting $MO/I_m \simeq \vee \Sigma^{|\omega|}B(\overline{m - |\omega|})$ implies that $H_m(MO/I_m)$ is all spherical, since for $|\omega| < m$ the homological dimension of $\Sigma^{|\omega|}B(\overline{m - |\omega|})$ is $m - \alpha(\overline{m - |\omega|}) < m$. Thus the image of $\tilde{g}_*$: $H_m(\widetilde{BO}/I_m) \to H_m F$ consists entirely of classes with spherical Thom isomorphic image in $H_m TF$. Now consider the following commutative diagram:

$$\begin{array}{ccc} BO/I_m & \xrightarrow{\quad g \quad} & F \\ \downarrow & & \downarrow \\ S^m = BO/I_m/\widetilde{BO}/I_m & \xrightarrow[\varepsilon(g)]{} & F/\tilde{B}. \end{array}$$

Since $\Phi\varepsilon(g)_*[S^m] \in H_m TF/\tilde{B}$ is spherical we have that the entire image of the composition $H_m BO/I_m \to H_m F \to H_m F/\tilde{B}$ has spherical Thom isomorphic image. Now since $T\tilde{g}_*$: $\pi_* \widetilde{MO}/I_m \to \pi_* TF$ is a monomorphism, and thus $\pi_* TF \to \pi_* T(F/\tilde{B})$ is an epimorphism, there is a class $\sigma \in \pi_m TF$ whose Hurewicz image $[\sigma] \in H_m TF$ has the property that $[\sigma] - \Phi(g_*(d_m)) \in H_m TF$ projects to zero in $H_m T(F/\tilde{B})$. Thus $[\sigma] - \Phi(g_*(d_m))$ is in the image of $\tilde{g}_*$: $H_m \widetilde{MO}/I_m \to H_m TF$, and hence is spherical. Thus $\Phi g_*(d_m)$ is spherical as well. Hence g satisfies 3.14.

Thus we have reduced 3.14 to proving the following:

LEMMA 3.15. *There exists a class $\varepsilon' \in \pi_m^s F/\tilde{B}$ satisfying the following properties*:

a. *The wedge map $i \vee \varepsilon'$: $F/B \vee S^m \to F/\tilde{B}$ is a homotopy equivalence, where i is the inclusion of $F/B = \tilde{F}/\widetilde{BO}/I_m$ into $F/\tilde{B}$.*

b. *Let $r(\varepsilon')$: $F/\tilde{B} \to F/B$ be the retraction defined by this homotopy equivalence, and let*

$$g\colon \Sigma^\infty BO/I_m \to \Sigma^\infty F$$

be the inclusion of the stable fiber of the composition $F \to F/\tilde{B} \xrightarrow[r(\varepsilon')]{} F/B$. Then g lifts q_{j-1}: $BO/I_m \to Q_{j-1}$.

c. *$\Phi\varepsilon'_*[S^m] \in H_m T(F/\tilde{B})$ is spherical.*

In order to prove this lemma we will describe a necessary and sufficient condition or a stable (relative) homotopy class to have spherical Thom isomorphic image.

So let X be any space, $A \hookrightarrow X$ a subspace (possibly A is empty), and let f: $X \to BO$ be a map that classifies stable bundles over X and A whose Thom spectra we denote by TX and TA. We let $T(X/A)$ be the quotient TX/TA.

Let Δ: $T(X/A) \to X/A \wedge MO$ be the Thom-ification of the map of pairs

$$(X, A) \xrightarrow[1\times f]{} (X \times BO, A \times BO)$$

(If $A = \varnothing$, Δ: $TX \to X^+ \wedge MO$).

Let $\overline{M} = MO/S^0$ be the cofiber of the unit u: $S^0 \to MO$ and define

$$\overline{\Delta}: T(X/A) \to X/A \wedge \overline{M}$$

to be the composition $\overline{\Delta}$: $T(X/A) \xrightarrow{\Delta} X/A \wedge MO \xrightarrow{\text{project}} X/A \wedge \overline{M}$. (Again if $A = \varnothing$, $\overline{\Delta}$: $TX \to X^+ \wedge \overline{M}$.) Finally, define

$$U(X/A) \to T(X/A)$$

to be the inclusion of the stable fiber of $\overline{\Delta}$: $T(X/A) \to X/A \wedge \overline{M}O$. Notice that $U(X/A)$ is the stable pull-back for the square

$$\begin{array}{ccc} U(X/A) & \longrightarrow & T(X/A) \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ X/A & \xrightarrow[i]{} & X/A \wedge MO \end{array} \qquad \text{or, if } A = \varnothing, \qquad \begin{array}{ccc} U(X) & \longrightarrow & T(X) \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ X^+ & \xrightarrow[i]{} & X^+ \wedge MO \end{array}$$

where $i = 1 \wedge u$: $X/A = X/A \wedge S^0 \to X/A \wedge MO$.

LEMMA 3.16. *A class $\alpha \in \pi_m^s(X/A)$ has spherical Thom isomorphic image in $H_m T(X/A)$ if and only if it lifts to an element $\alpha' \in \pi_m^s U(X/A)$.*

Proof. Suppose $\alpha \in \pi_m^s(X/A)$ lifts to $\alpha' \in \pi_m^s U(X/A)$. Let $\beta \in \pi_m T(X/A)$ be the image of α' under the map in the above diagram. By the commutativity of the diagram $\Delta_*[\beta] = [\alpha] \otimes u \in H_m(X/A) \otimes H_0 MO \subset H_m(X/A \wedge MO)$. By the definition of the Thom isomorphism (cup product with the Thom class in cohomology) it is then clear that α and β have Thom isomorphic Hurewicz images in $H_m(X/A)$ and $H_m T(X/A)$ respectively.

Conversely, suppose that $\alpha \in \pi_m^s(X/A)$ has spherical Thom isomorphic image in $H_m T(X/A)$. That is, suppose there is a class $\beta \in \pi_m T(X/A))$ so that $[\beta] = \Phi_*[\alpha] \in H_m T(X/A)$. Thus in cohomology, the homomorphisms α^*: $H^* X/A \to H^*(S^m)$ and β^*: $H^* T(X/A) \to H^*(S^m)$ are Thom isomorphic images of each other.

Now as observed in the calculations made to prove observations 3.9–3.11, this means that α^* is a homomorphism of both left and right A-modules, and therefore α^* is a homomorphism of $H^*(BO)$-modules, and β^* is a homomorphism of $A(BO)$-modules. Now since the $A(BO)$-structure of $H^*(S^m)$ is trivial this implies that if $w \in H^r BO$ with $r \geq 1$, and if $x \in H^q X/A$ is any class, then under composition $H^*(X/A \wedge MO) \xrightarrow[\Delta]{} H^* T(X/A) \xrightarrow[\beta^*]{} H^* S^m$, $x \otimes (w \cup u)$ maps to zero. Dually this clearly says that $[\beta] \in H_m T(X/A)$ maps under Δ_* to $i_*[\alpha] = \alpha \otimes u \in H_m(X/A \wedge MO)$. Furthermore, since $X/A \wedge MO$ is Eilenberg-MacLane, this says that $\Delta_* \beta = i_* \alpha \in \pi_m(X/A \wedge MO)$. By the pull-back property of the above diagram this says that α and β lift to a class $\alpha' \in \pi_m^s U(X/A)$. This completes the proof of Lemma 3.16.

Observe that by this lemma, condition c in Lemma 3.15 can be replaced by the following.

Condition 3.15.c′. $\varepsilon' \in \pi_m^s(F/\tilde{B})$ lifts to $\pi_m^s U(F/\tilde{B})$.
This is the form in which we will prove Lemma 3.15.

The next major step in our proof of 3.15 is to reduce it to the existence of a class $\gamma \in \pi_m U(Q/\tilde{B})$ satisfying certain properties. (This will be Lemma 3.17). In order to do this we make use of the spectrum C defined to be the *stable* fiber of the map p: $Q_{j-1} \to Q/B \wedge K$, and let $\tilde{C}$ be the stable fiber of the restriction of p to $\tilde{Q}_{j-1}$. By the strict commutativity of the diagram

$$\begin{tikzcd} Q_{j-1} \arrow[r] \arrow[d, "="'] & Q/B \arrow[d, "i"] \\ Q_{j-1} \arrow[r] & Q/B \wedge K \end{tikzcd}$$

we have an induced stable map of stable fibers

$$j\colon BO/I_m \to C.$$

Similarly, by the strict commutativity of the diagram

$$\begin{tikzcd} \tilde{Q}_{j-1} \arrow[r, "p"] \arrow[d, "="'] & Q/B \wedge K \arrow[d, "\Delta"] \\ \tilde{Q}_{j-1} \arrow[r, "c(p)"'] & Q/B \wedge BO^+ \wedge K \end{tikzcd}$$

we have an induced map of stable fibers

$$u\colon \tilde{C} \to \tilde{F},$$

and hence a map of cofibers $\tilde{C}/\tilde{B} \to \tilde{F}/\tilde{B} \to F/\tilde{B}$.

Notice that in the above diagram if we replaced $\tilde{Q}_{j-1}$ by Q_{j-1} then the diagram would only homotopy commute, and a choice of homotopies would yield a choice of extensions of u: $\tilde{C} \to \tilde{F}$ to a map $C \to F$. We will have to choose this extension carefully.

Notice that the composition $BO/I_m \xrightarrow{j} C \to Q_{j-1} \to BO/I_m$ is homotopic to the identity, and therefore induces a splitting $C \simeq BO/I_m \vee C/B$, where $C/B = \tilde{C}/\widetilde{BO}/I_m \simeq C/BO/I_m \simeq \Sigma^{-1}Q/B \wedge K/S^0$. The map $C/B \to F/B$ is given by the composition

$$\begin{aligned} C/B &\simeq \Sigma^{-1}[Q/B \wedge K/Q/B \wedge S^0] \\ &\xrightarrow{\Delta} \Sigma^{-1}[Q/B \wedge BO^+ \wedge K/Q/B \wedge S^0] \simeq F/B. \end{aligned}$$

Similarly, let TC be the stable fiber of the map Tp: $TQ \to T(Q/B) \wedge K$, and let $T\tilde{C}$ be the fiber of the restriction of Tp to $T\tilde{Q}$. Then a similar analysis shows that the induced map $Tj\colon MO/I_m \to TC$ yields a splitting $TC \simeq MO/I_m \vee T(C/B)$, where $T(C/B) = TC/MO/I_m \simeq \Sigma^{-1}T(Q/B) \wedge K/S^0$. Notice further that we have a Thom isomorphism $\Phi\colon H^*C \to H^*TC$ induced by the Thom isomorphism $\Phi\colon H^*BO/I_m \to H^*MO/I_m$ and $\Phi\colon H^*Q/B \to H^*T(Q/B)$. Hence the maps $j\colon BO/I_m \to C$ and $Tj\colon MO/I_m \to TC$ preserve the Thom isomorphism.

Also observe that by the strict commutativity of the square

$$\begin{tikzcd}
T\tilde{Q}_{j-1} \arrow[r, "Tp"] \arrow[d, "="'] & TQ/B \wedge K \arrow[d, "\Delta \wedge 1"] \\
T\tilde{Q}_{j-1} \arrow[r, "Tc(p)"'] & Q/B \wedge MO \wedge K
\end{tikzcd}$$

there is a canonical map, $T\tilde{C} \to T\tilde{F}$, and hence of cofibers $T(\tilde{C}/\tilde{B}) \to T(F/B)$ which is given by the composition

$$\begin{aligned}
T(\tilde{C}/\tilde{B}) &\cong \Sigma^{-1}\left[T(Q/B) \wedge K/T(Q/B) \wedge S^0\right] \\
&\xrightarrow{\Delta} \Sigma^{-1}\left[Q/B \wedge MO \wedge K/T(Q/B) \wedge S^0\right] \simeq T(F/B).
\end{aligned}$$

Notice that by the naturality of these constructions, the induced maps $\tilde{C} \to \tilde{F}$ and $T\tilde{C} \to T\tilde{F}$ preserve the Thom isomorphism.

We can summarize these preservations of Thom isomorphism statements as follows. The Thom isomorphism $\Phi\colon H^*C \to H^*TC$ induces an $A(BO)$-module structure on H^*TC, and therefore a map $\Delta\colon TC \to C^+ \wedge MO$. Then $U(C)$, may be defined as the stable pull-back for the square

$$\begin{tikzcd}
U(C) \arrow[r] \arrow[d] & T(C) \arrow[d, "\Delta"] \\
C^+ \arrow[r, "i"'] & C^+ \wedge MO,
\end{tikzcd}$$

and $U(\tilde{C})$ is defined similarly. The above compatibility of the Thom isomorphisms coming from the maps $j\colon BO/I_m \to C$, $Tj\colon MO/I_m \to TC$, and from $\tilde{C} \to \tilde{F}$ and $T\tilde{C} \to T\tilde{F}$ implies that we get induced maps of the appropriate pull-back squares, and in particular maps $U(BO/I_m) \to U(C)$ and $U(\tilde{C}) \to U(\tilde{F})$ that make the following diagram homotopy commute:

$$\begin{tikzcd}
 & & U(\widetilde{BO/I_m}) \arrow[dll, "U(\bar{g})"'] \arrow[r] \arrow[d] & U(BO/I_m) \arrow[d] \\
U(F) \supset & U(\tilde{F}) & U(\tilde{C}) \arrow[l] \arrow[r] & U(C).
\end{tikzcd}$$

We therefore have an induced map of quotients $U(F/\tilde{B}) \to U(F/\tilde{C})$.

Now consider the map $\varepsilon\colon S^m = B/\tilde{B} \to Q/\tilde{B}$ and its projection to $Q/\tilde{C}$, $\varepsilon_C\colon S^m \to Q/\tilde{B} \to Q/\tilde{C}$. These maps are induced by the strict commutativity of the diagram

$$\begin{tikzcd} \widetilde{BO}/I_m \arrow[r, hook] \arrow[d] & BO/I_m \arrow[d, "j"] \arrow[dr, "q_{j-1}"] & \\ \tilde{C} \arrow[r] & C \arrow[r, "q_c"'] & Q_{j-1} \end{tikzcd}$$

where q_c is the inclusion of the stable fiber of p: $Q_{j-1} \to Q/B \wedge K$.

Our goal is to show that ε_c: $S^m \to Q/\tilde{C}$ lifts to a map $\bar{\gamma}$: $S^m \to U(F/\tilde{B})$. Once this is done we would let ε': $S^m \to F/\tilde{B}$ be the composition $S^m \xrightarrow{\bar{\gamma}} U(F/\tilde{B}) \to F/\tilde{B}$. Clearly ε' satisfies conditions a and c′ of Lemma 3.15. We now show that ε' satisfies condition 3.15.b.

Let $r(\varepsilon')$: $F/\tilde{B} \to F/B$ and $r_c(\varepsilon')$: $F/\tilde{C} \to F/C$ be the projections onto the cofiber of ε' and ε'_c respectively. By assumption ε'_c lifts ε_c so the following diagram homotopy commutes:

$$\begin{tikzcd} F \arrow[r] \arrow[d, "="] & F/\tilde{B} \arrow[r, "r(\varepsilon')"'] \arrow[d] & F/B \arrow[d] \\ F \arrow[r] \arrow[d] & F/\tilde{C} \arrow[r, "r_c(\varepsilon')"'] \arrow[d] & F/C \arrow[d] \\ Q_{j-1} \arrow[r] & Q/\tilde{C} \arrow[r, "r_c"'] & Q/C \end{tikzcd}$$

where r_c is the projection onto the cofiber of ε_c. Thus if we let g: $BO/I_m \to F$ and g_c: $C \to F$ be the inclusions of the stable fibers of the above maps $F \to F/B$ and $F \to F/C$ respectively, then the following diagram homotopy commutes:

$$\begin{tikzcd} BO/I_m \arrow[r, "j"'] \arrow[d, "g"'] & C \arrow[r, phantom, "="] \arrow[d, "g_c"] & C \arrow[d, "q_c"] \\ F \arrow[r, "="'] & F \arrow[r] & Q_{j-1}. \end{tikzcd}$$

But since $q_c \circ j \simeq q_{j-1}$: $BO/I_m \to Q_{j-1}$, condition 3.15.b is therefore satisfied.

Hence we are reduced to showing that the map ε_c: $S^m \xrightarrow[\varepsilon]{} Q/\tilde{B} \to Q/\tilde{C}$ lifts to $U(F/\tilde{B})$.

To construct such a lifting, we study the map $U(F/\tilde{B}) \to U(Q/\tilde{B})$ in more detail. Notice that the cofiber of this map is $U(Q/F) \simeq U(Q/B \wedge BO^+ \wedge K) \simeq Q/B \wedge U(BO) \wedge K$. The projection onto the cofiber, δ: $U(Q/B) \to Q/B \wedge U(BO) \wedge K$ is the map of stable fibers induced by the strict commutativity of the diagram

$$\begin{array}{ccccccc}
TQ/\tilde{B} & \xrightarrow{\Delta} & Q/\tilde{B} \wedge MO & \xhookrightarrow{i} & Q/\tilde{B} \wedge MO \wedge K & \xrightarrow{r \wedge 1} & Q/B \wedge MO \wedge K \\
\downarrow{\scriptstyle \bar{\Delta}} & & & & \downarrow{\scriptstyle 1 \wedge \bar{\Delta} \wedge 1} & & \downarrow{\scriptstyle 1 \wedge \bar{\Delta} \wedge 1} \\
Q/\tilde{B} \wedge \overline{M} & \xrightarrow[\Delta \wedge 1]{} & Q/\tilde{B} \wedge BO^+ \wedge \overline{M} & \xhookrightarrow[i]{} & Q/\tilde{B} \wedge BO^+ \wedge \overline{M} \wedge K & \xrightarrow[r \wedge 1]{} & Q/B \wedge BO^+ \wedge \overline{M} \wedge K,
\end{array}$$

where r denotes the projection onto the cofiber of ε: $S^m \to Q/\tilde{B}$. Thus δ factors as the composition

$$\delta\colon U(Q/\tilde{B}) \xrightarrow[\tilde{\delta}]{} Q/\tilde{B} \wedge U(BO) \wedge K \xrightarrow[r \wedge 1]{} Q/B \wedge U(BO) \wedge K.$$

Now an easy calculation shows that $H_0(U(BO)) \simeq \mathbf{Z}/2 \simeq \pi_0(U(BO) \wedge K)$. Let v: $S^0 \to U(BO) \wedge K$ be the generator, and consider the class

$$\varepsilon \wedge v\colon S^m = S^m \wedge S^0 \to Q/\tilde{B} \wedge U(BO) \wedge K.$$

The following lemma is the tool needed to construct the required lifting of $S^m \xrightarrow{\varepsilon} Q/\tilde{B} \to Q/\tilde{C}$ to $U(F/\tilde{C})$. We postpone its proof until the next section.

LEMMA 3.17. *$\varepsilon \wedge v$: $S^m \to Q/\tilde{B} \wedge U(BO) \wedge K$ lifts to a class*

$$\gamma \in \pi_m U(Q/\tilde{B}).$$

Assuming 3.17, we now show that $S^m \xrightarrow{\varepsilon} Q/\tilde{B} \to Q/\tilde{C}$ lifts to $U(F/\tilde{B})$, and therefore to $U(F/\tilde{C})$.

Let $\gamma \in \pi_m U(Q/\tilde{B})$ be as in 3.17. Since $(r \wedge 1)_*(\varepsilon \wedge v) = 0$ in $\pi_m(Q/B \wedge K \wedge U(BO))$, then by the factorization of δ given above, $\delta_*(\gamma) = 0$. Thus γ lifts to a class $\bar{\gamma} \in \pi_m U(F/\tilde{B})$. We now show that $\bar{\gamma}$ is a lifting of $S^m \xrightarrow[\varepsilon]{} Q/\tilde{B} \to Q/\tilde{C}$.

To see this, observe that since the compositions $U(Q/\tilde{B}) \to T(Q/\tilde{B}) \xrightarrow[i\Delta]{} Q/\tilde{B} \wedge MO \wedge K$ and

$$U(Q/\tilde{B}) \xrightarrow[\tilde{\delta}]{} Q/\tilde{B} \wedge K \wedge UBO \to Q/\tilde{B} \wedge MO \wedge K$$

are homotopic, we have that the composite $S^m \xrightarrow[\gamma]{} U(Q/\tilde{B}) \to T(Q/\tilde{B}) \xrightarrow[i\Delta]{} Q/\tilde{B} \wedge MO \wedge K$ is homotopic to $\varepsilon \wedge u$: $S^m \wedge S^0 \to Q/\tilde{B} \wedge MO \wedge K$, where $u \in \pi_0(MO \wedge K) = \mathbf{Z}/2$ is the generator. But this clearly implies that the composition $S^m \xrightarrow[\gamma]{} U(Q/\tilde{B}) \to TQ/\tilde{B}$ represents the Thom isomorphic image, $\Phi[\varepsilon] \in H_m TQ/\tilde{B}$. Hence γ, and therefore $\bar{\gamma}$, lifts a class $\bar{\varepsilon} \in \pi_m Q/\tilde{B}$ that has the same Hurewicz image in $H_m Q/\tilde{B}$ as does $\varepsilon \in \pi_m Q/\tilde{B}$.

Now $Q/\tilde{C} \simeq Q/C \vee S^m \simeq (Q/B \wedge K) \vee S^m$. Thus the composition $S^m \xrightarrow[\varepsilon - \bar{\varepsilon}]{} Q/\tilde{B} \to Q/\tilde{C}$ is null homotopic. Hence the composition $S^m \xrightarrow[\bar{\gamma}]{} U(F/\tilde{B}) \to$

$U(F/\tilde{C})$ lifts the composite $S^m \xrightarrow[\varepsilon]{} Q/\tilde{B} \to Q/\tilde{C}$, which as observed above, completes the proof of Lemma 3.15.

3.c *Proof of Lemma* 3.17 *and the case* $j = 0$. We begin by proving Lemma 3.17. This is the final step in the proof of 3.15, which was the last loose end in the completion of the inductive step in the proof of Theorem 3.5 in the case $j > 0$. We will end this section by proving that the inductive hypotheses hold in the case $j = 0$.

Proof of 3.17. Recall that $\tilde{\delta}$: $U(Q/\tilde{B}) \to Q/\tilde{B} \wedge U(BO) \wedge K$ is the map of fibers induced by the strict commutativity of the diagram

$$\begin{array}{ccc} TQ/\tilde{B} & \xrightarrow{\Delta} & Q/\tilde{B} \wedge MO \wedge K \\ \Big\downarrow{\scriptstyle \bar{\Delta}} & & \Big\downarrow{\scriptstyle 1 \times \bar{\Delta} \wedge 1} \\ Q/\tilde{B} \wedge \overline{M} & \xrightarrow[i(\Delta \wedge 1)]{} & Q/\tilde{B} \wedge BO^+ \wedge \overline{M} \wedge K \end{array} \tag{3.18}$$

Consider another map, $\tilde{\partial}$: $U(Q/\tilde{B}) \to Q/\tilde{B} \wedge U(BO) \wedge K$ defined to be the map of fibers induced by the strict commutativity of the diagram

$$\begin{array}{ccc} TQ/\tilde{B} & \xrightarrow{i\Delta} & Q/\tilde{B} \wedge MO \wedge K \\ \Big\downarrow{\scriptstyle \bar{\Delta}} & & \Big\downarrow{\scriptstyle 1 \wedge \bar{\Delta} \wedge 1} \\ Q/\tilde{B} \wedge \overline{M} & \xrightarrow[i(1 \wedge \Delta)]{} & Q/\tilde{B} \wedge BO^+ \wedge \overline{M} \wedge K \end{array} \tag{3.19}$$

Since the two maps $\Delta_l = i(\Delta \wedge 1)$ and $\Delta_r = i(1 \wedge \Delta)$ from $Q/\tilde{B} \wedge \overline{M} \wedge K$ to $Q/\tilde{B} \wedge BO^+ \wedge \overline{M} \wedge K$ are *not* the same, the two induced maps of *fibers*, $\tilde{\delta}$ and $\tilde{\partial}$, are *not* homotopic. Our goal then is to find a subspace $Z \hookrightarrow UQ/\tilde{B}$ with the following properties:

($3.19\frac{1}{2}$) a. The restrictions of $\tilde{\delta}$ and $\tilde{\partial}$ to Z are homotopic maps $Z \to Q/\tilde{B} \wedge U(BO) \wedge K$, and

b. There is an element $T\varepsilon \in \pi_m TQ/\tilde{B}$ whose Hurewicz image is the Thom isomorphic image, $\Phi_*[\varepsilon] \in H_m TQ/\tilde{B}$, and such that $T\varepsilon$ lifts to an element in $\pi_m Z$. (Note: By 3.16 we know that $T\varepsilon$ lifts to $\pi_m U(Q/\tilde{B})$.)

Before we construct a space Z with these properties, we will show how it will allow us to produce the required lifting of $\varepsilon \wedge v$: $S^m \wedge S^0 \to Q/\tilde{B} \wedge U(BO) \wedge K$ to $\pi_m U(Q/\tilde{B})$.

By property b, $T\varepsilon \in \pi_m T(Q/\tilde{B})$ lifts to $\pi_m Z$. Choose any such lifting and let γ: $S^m \to U(Q/\tilde{B})$ be its image in $\pi_m U(Q/\tilde{B})$. We claim that $\tilde{\delta}_*(\gamma) = \varepsilon \wedge v$.

To prove this, by property a, it is sufficient to show that $\tilde{\partial}_*(\gamma) = \varepsilon \wedge v$. To see this, notice that diagram (3.19) strictly factors into the diagram

(3.20)

$$\begin{tikzcd}
T(Q/\tilde{B}) \arrow[r, "i\Delta"] \arrow[d, "\bar{\Delta}"'] & Q/\tilde{B} \wedge MO \wedge K \arrow[r, "="] \arrow[d, "\text{project}"] & Q/\tilde{B} \wedge MO \wedge K \arrow[d, "1 \wedge \bar{\Delta} \wedge 1"] \\
Q/\tilde{B} \wedge \bar{M} \arrow[r, "i"'] & Q/\tilde{B} \wedge \bar{M} \wedge K \arrow[r, "\Delta_r"', "(=1 \wedge \Delta \wedge 1)"' below] & Q/\tilde{B} \wedge BO^+ \wedge \bar{M} \wedge K.
\end{tikzcd}$$

Thus the induced map of fibers, $\tilde{\partial}$, factors as a composition

$$(3.20\tfrac{1}{2}) \qquad \tilde{\partial}\colon U(Q/\tilde{B}) \xrightarrow[\xi]{} Q/\tilde{B} \wedge S^0 \wedge K \xrightarrow[1 \wedge v \wedge 1]{} Q/\tilde{B} \wedge U(BO) \wedge K.$$

The map ξ in this composition is clearly given up to homotopy as the composite

$$\xi\colon U(Q/\tilde{B}) \xrightarrow[j]{} T(Q/\tilde{B}) \xrightarrow[\Delta]{} Q/\tilde{B} \wedge MO \xrightarrow[1 \wedge u]{} Q/\tilde{B} \wedge K$$

where u represents the Thom class, and j is the inclusion of the stable fiber of $\bar{\Delta}$. We now compute:

$$\begin{aligned}
\tilde{\partial}_*(\gamma) &= (1 \wedge v \wedge 1)_*(1 \wedge u)_* \Delta_* j_*(\gamma) \\
&= (1 \wedge v \wedge 1)_*(1 \wedge u)_* \Delta_*(T\varepsilon), \quad \text{by the definition of } \gamma, \\
&= (1 \wedge v \wedge 1)_*(\varepsilon \wedge 1), \quad \text{since } [T\varepsilon] = \Phi_*[\varepsilon], \\
&= \varepsilon \wedge v.
\end{aligned}$$

We are therefore reduced to constructing a subspace $Z \subset U(Q/\tilde{B})$ satisfying properties a and b above.

Let the space Z be the stable pull-back for the diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & Q/\tilde{B} \wedge S^0 \wedge K \arrow[d, "1 \wedge v \wedge 1"] \\
U(Q/\tilde{B}) \arrow[r, "\tilde{\delta}"'] & Q/\tilde{B} \wedge UBO \wedge K.
\end{tikzcd}$$

By the factorization of $\tilde{\partial}$ given above, the restrictions of $\tilde{\delta}$ and $\tilde{\partial}$ to Z are clearly homotopic. We now proceed to verify that there is an appropriate element $T\varepsilon \in \pi_m TQ/\tilde{B}$ that lifts to $\pi_m Z$.

To do this we first study the map

$$\tilde{\delta}_1\colon U(Q/\tilde{B}) \wedge K \xrightarrow[\tilde{\delta} \wedge 1]{} (Q/\tilde{B} \wedge UBO \wedge K) \wedge K \xrightarrow[1 \wedge \mu]{} Q/\tilde{B} \wedge UBO \wedge K,$$

where $\mu\colon K \wedge K \to K$ is the cup product pairing. Next, $\tilde{\partial}_1\colon U(Q/\tilde{B}) \wedge K \to Q/\tilde{B} \wedge UBO \wedge K$ is defined similarly. Notice that $\tilde{\delta}$ and $\tilde{\partial}$ factor as $\tilde{\delta}_1 \circ i$ and $\tilde{\partial}_1 \circ i\colon U(Q/\tilde{B}) \to U(Q/\tilde{B}) \wedge K \to Q/\tilde{B} \wedge UBO \wedge K$, respectively.

We first show how to produce liftings in $\pi_m U(Q/\tilde{B}) \wedge K$ in the kernel of the difference homomorphism $(\tilde{\delta}_1 - \tilde{\partial}_1)_*$. To do this consider the following strictly commutative diagram:

$$\begin{tikzcd} TQ/\tilde{B} \wedge K \arrow[r, "\Delta \wedge 1"] \arrow[d, "\Delta \wedge 1"'] & Q/\tilde{B} \wedge MO \wedge K \arrow[d, "\Delta_r \wedge 1"] \\ Q/\tilde{B} \wedge MO \wedge K \arrow[r, "\Delta_l \wedge 1"'] & Q/\tilde{B} \wedge BO^+ \wedge MO \wedge K. \end{tikzcd}$$

By standard considerations, $\Delta_*: H_*TQ/\tilde{B} \to H_*Q/\tilde{B} \wedge MO$ is a monomorphism, and its image is precisely the kernel of the difference homomorphism

$$(\Delta_r - \Delta_l)_*: H_*(Q/\tilde{B} \wedge MO) \to H_*(Q/\tilde{B} \wedge BO^+ \wedge MO).$$

Similarly, define the vector space $V \subset H_*(Q/\tilde{B} \wedge \overline{M})$ to be the kernel of the difference homomorphism

$$(\Delta_r - \Delta_l)_*: H_*(Q/\tilde{B} \wedge \overline{M}) \to H_*(Q/\tilde{B} \wedge BO^+ \wedge \overline{M}).$$

Then define $K(V)$ to be the wedge of Eilenberg-MacLane spectra so that $\pi_* K(V) = V$. Notice that there is a canonically defined map j: $K(V) \to Q/\tilde{B} \wedge \overline{M} \wedge K$ so that the compositions $(\Delta_l \wedge 1) \circ j$ and $(\Delta_r \wedge 1) \circ j$: $K(V) \to Q/\tilde{B} \wedge \overline{M} \wedge K \to Q/\tilde{B} \wedge BO^+ \wedge \overline{M} \wedge K$ are homotopic.

Now observe that since the vector spaces $H_*TQ/\tilde{B}$ and V are both kernels of the appropriate difference homomorphisms, we get an induced map

$$d: TQ/\tilde{B} \wedge K \to K(V)$$

so that $\overline{\Delta} \wedge 1$: $TQ/\tilde{B} \wedge K \to Q/\tilde{B} \wedge \overline{M} \wedge K$ factors as $TQ/\tilde{B} \wedge K \xrightarrow[d]{} K(V) \xrightarrow[j]{} Q/\tilde{B} \wedge \overline{M} \wedge K$.

We define W to be the stable homotopy fiber of d: $TQ/\tilde{B} \wedge K \to K(V)$. This factorization of $i\overline{\Delta}$ allows us to construct a map of stable fibers $W \to U(Q/\tilde{B}) \wedge K$. (Notice there are many choices of such maps.)

LEMMA 3.21. *There is a map of stable fibers ϕ: $W \to U(Q/\tilde{B}) \wedge K$ so that through dimension m, $\tilde{\partial}_1 \circ \phi$ and $\tilde{\delta}_1 \circ \phi$ are homotopic.*

Proof. Observe that d: $TQ/\tilde{B} \wedge K \to K(V)$ is a map between Eilenberg-MacLane spectra that in cohomology preserves Steenrod algebra generators. Thus W is Eilenberg-MacLane. If D: $\Sigma^{-1}K(V) \to W$ is the connecting map in the cofibration sequence $W \to TQ/\tilde{B} \wedge K \xrightarrow{d} K(V)$ we then have a splitting $W \simeq \mathrm{Im}(D) \vee \mathrm{Ker}(d)$, where $\mathrm{Im}(D)$ is an Eilenberg-MacLane spectrum with $\pi_*\mathrm{Im}(D) = \mathrm{Image}(D_*$: $\pi_*\Sigma^{-1}K(V) \to \pi_*W)$. Similarly $\mathrm{Ker}(d)$ is Eilenberg-MacLane with $\pi_*\mathrm{Ker}(d) = \mathrm{kernel}(d_*$: $\pi_*TQ/\tilde{B} \wedge K \to \pi_*K(V))$. We will choose our map ϕ: $W \to U(Q/\tilde{B}) \wedge K$ to be a map of stable fibers that preserves Steenrod algebra generators in cohomology (and hence is determined by its induced homomorphism in homotopy groups). Notice that such maps ϕ agree when restricted to $\mathrm{Im}(D)$ since they all must induce a homotopy commutative diagram

$$\begin{tikzcd}
 & \Sigma^{-1}K(V) \arrow[r, "j"] \arrow[dl] \arrow[d, "D"] & \Sigma^{-1}Q/\tilde{B} \wedge \overline{M} \wedge K \arrow[r, "\Delta_r - \Delta_l"] \arrow[d] & \Sigma^{-1}Q/\tilde{B} \wedge BO^+ \wedge \overline{M} \wedge K \arrow[d] \\
\mathrm{Im}(D) \arrow[r] & W \arrow[r, "\phi"] & U(Q/\tilde{B}) \wedge K \arrow[r, "\tilde{\delta}_1 - \tilde{\partial}_1"] & \Sigma^{-1}Q/\tilde{B} \wedge U(BO) \wedge K,
\end{tikzcd}$$

where the vertical maps are the connecting maps from the cofibration sequences defining the stable fibers W, $U(Q/\tilde{B}) \wedge K$ and $Q/\tilde{B} \wedge U(BO) \wedge K$, respectively. Notice also that by the definition of $K(V)$, the horizontal composition $(\Delta_r - \Delta_l) \circ j$ in the top row is null homotopic. Thus for any such map of stable fibers ϕ, the restrictions of $\tilde{\delta}_1\phi$ and $\tilde{\partial}_1\phi$ to $\mathrm{Im}(D)$ are homotopic. Thus, by the splitting $W \simeq \mathrm{Im}(D) \vee \mathrm{Ker}(d)$ we are reduced to proving the following:

Claim. There is a homotopy lifting σ: $\mathrm{Ker}(d) \to U(Q/\tilde{B}) \wedge K$ of the inclusion i: $\mathrm{Ker}(d) \to T(Q/\tilde{B}) \wedge K$ so that through dimension m,

$$(\tilde{\delta}_1 \circ \sigma)_* = (\tilde{\partial}_1 \circ \sigma)_*: \pi_*\mathrm{Ker}(d) \to \pi_*(Q/\tilde{B} \wedge UBO \wedge K).$$

Proof. We will study the difference homomorphism

$$(\tilde{\delta}_1 - \tilde{\partial}_1)_*: \pi_*U(Q/\tilde{B}) \wedge K \to \pi_*Q/\tilde{B} \wedge U(BO) \wedge K$$

in terms of "algebraic mapping cones." More specifically, for any space X let $C_*(X)$ be the $\mathbf{Z}/2$-singular chain complex of X, and let f: $X \to Y$ be any map. Recall that the algebraic mapping cone, $C_*(f)$ of the induced chain map f_*: $C_*(X) \to C_*(Y)$ is the chain complex with n-chains $C_n(f) = C_n(Y) \oplus C_{n-1}(X)$, with boundary map ∂_f: $C_n(f) \to C_{n-1}(f)$ given by the formula $\partial_f(a, b) = (\partial_Y(a) + f_*(b), \partial_X(b))$, where ∂_X and ∂_Y are the boundary maps in $C_*(X)$ and $C_*(Y)$ respectively. Recall that the mapping cone $M(f)$ of the map f: $X \to Y$ has homology $H_*M(f) = H_*(C_*(f), \partial_f)$. Recall furthermore that this construction is natural in the sense that if we have a strictly commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g_1"'] & Y \arrow[d, "g_2"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

then the induced map of mapping cones, $m(g)$: $M(f) \to M(f')$ can be calculated in homology via the chain map

(3.22)

$$g_{2_*} \oplus g_{1_*}: C_*(f) = C_*(Y) \oplus C_{*-1}(X) \to C_*(Y') \oplus C_{*-1}(X') = C_*(f').$$

We now apply this construction to study the maps $\tilde{\delta}_1$ and

$$\tilde{\partial}_1\colon U(Q/\tilde{B}) \wedge K \to Q/\tilde{B} \wedge U(BO) \wedge K.$$

So consider the strictly commutative diagrams

$$\begin{array}{ccc} TQ/\tilde{B} & \xrightarrow{\bar{\Delta}} & Q/\tilde{B} \wedge \bar{T}Q \\ \Big\downarrow \Delta & & \Big\downarrow \Delta_l = \Delta \wedge 1 \\ Q/\tilde{B} \wedge TQ & \xrightarrow{1\wedge\bar{\Delta}} & Q/\tilde{B} \wedge Q^+ \wedge \bar{T}Q \end{array} \qquad \begin{array}{ccc} TQ/\tilde{B} & \xrightarrow{\bar{\Delta}} & Q/\tilde{B} \wedge \bar{T}Q \\ \Big\downarrow \Delta & & \Big\downarrow \Delta_r = 1 \wedge \bar{\Delta} \\ Q/\tilde{B} \wedge TQ & \xrightarrow{1\wedge\bar{\Delta}} & Q/B \wedge Q^+ \wedge \bar{T}Q, \end{array}$$

where $Q = Q_{j-1}$ and $\bar{T}Q$ is the cofiber of the Thom map $S^0 \to TQ$. Since the classifying map $Q \to BO$ is $m/2$-connected, as is $Q/\tilde{B}$, we have natural m-dimensional equivalences of mapping cones $M(\bar{\Delta}) \simeq \Sigma U(Q/\tilde{B})$ and $M(1 \wedge \bar{\Delta}) \simeq \Sigma Q/\tilde{B} \wedge U(BO)$. Moreover, with respect to these equivalences, the induced maps of mapping cones $m(\Delta_l) \wedge 1$ and

$$m(\Delta_r) \wedge 1\colon M(\bar{\Delta}) \wedge K \to M(1 \wedge \bar{\Delta}) \wedge K$$

are $\tilde{\delta}_1$ and $\tilde{\partial}_1$ respectively. Thus to compute $(\tilde{\delta}_1 - \tilde{\partial}_1)_*$ in homotopy groups we are reduced to computing $m(\Delta_l)_* - m(\Delta_r)_*$ in homology, which we will do using algebraic mapping cones. However, for reasons that will become apparent, we will use the following slight modification of the singular complex for our calculation.

Replace $\bar{\Delta}\colon TQ \to Q^+ \wedge \bar{T}Q$ and $1 \wedge \bar{\Delta}\colon Q^+ \wedge TQ \to Q^+ \wedge Q^+ \wedge \bar{T}Q$ by the inclusions into their mapping cylinders, and replace Δ_l and $\Delta_r\colon Q^+ \wedge \bar{T}Q \to Q^+ \wedge Q^+ \wedge \bar{T}Q$ by the induced canonical map of mapping cylinders. Make the same replacements for the restrictions $\bar{\Delta}\colon \widetilde{MO}/I_m \to \widetilde{BO}/I_m^+ \wedge TQ$, $1 \wedge \bar{\Delta}\colon \widetilde{BO}/I_m^+ \wedge TQ \to \widetilde{BO}/I_m^+ \wedge Q^+ \wedge \bar{T}Q$, and Δ_l and $\Delta_r\colon \widetilde{BO}/I_m^+ \wedge \bar{T}Q \to B\tilde{Q}/I_m \wedge Q^+ \wedge \bar{T}Q$. By abuse of notation we will not change the names of these maps. Indeed these changes are made only so that we may assume that the induced map of singular chains $\bar{\Delta}_*\colon C_*(TQ) \to C_*(Q^+ \wedge \bar{T}Q)$ and its restriction $\bar{\Delta}_*\colon C_*(\widetilde{MO}/I_m) \to C_*(BO/I_m^+ \wedge \bar{T}Q)$ are monomorphisms.

The following observation will allow us to complete the proof of the claim.

Observation 3.23. Let $Z_*(X) \hookrightarrow C_*(X)$ denote the cycles, and $p\colon Z_*(X) \to H_*(X)$ the projection map. Then there are sections $t_1\colon H_*(TQ/\tilde{B}) \to Z_*(TQ/\tilde{B})$ and $t_2\colon H_*(Q/\tilde{B} \wedge \bar{T}Q) \to Z_*(Q/\tilde{B} \wedge \bar{T}Q)$ of these projections that make the following diagram commute:

$$\begin{array}{ccc} Z_*(TQ/\tilde{B}) & \xrightarrow[\bar{\Delta}_*]{} & Z_*(Q/\tilde{B} \wedge \bar{T}Q) \\ \Big\uparrow t_1 & & \Big\uparrow t_2 \\ H_*(TQ/\tilde{B}) & \xrightarrow[\bar{\Delta}_*]{} & H_*(Q/\tilde{B} \wedge \bar{T}Q). \end{array}$$

Proof. Let $\overline{\Delta}'_*$: $Z'_*(TQ) \to Z_*(Q^+ \wedge \overline{T}Q)$ be the inclusion of those cycles that are homologous to cycles in the image of $\overline{\Delta}_*$: $Z_*(TQ) \to Z_*(Q^+ \wedge \overline{T}Q)$. Since $\overline{\Delta}_*$: $H_*(TQ) \to H_*(Q^+ \wedge \overline{T}Q)$ is injective in positive dimensions, the projection p: $Z'_*(TQ) \to H_*(Q^+ \wedge \overline{T}Q)$ restricts to a projection p': $Z'_*(TQ) \to H_*(TQ)$. By letting $Z'_*(TQ/\tilde{B})$ be the quotient of the inclusion $Z_*(\widetilde{MO}/I_m) \to Z_*(TQ) \hookrightarrow Z'_*(TQ)$, we have an induced projection of quotients

$$p'\colon Z'_*(TQ/\tilde{B}) \to H_*\left(TQ/H_*\widetilde{MO}/I_m\right) = H_*(TQ/\tilde{B}).$$

We will first show that p' has a section s', and p: $Z_*(Q/\tilde{B} \wedge \overline{T}Q) \to H_*(Q/\tilde{B} \wedge \overline{T}Q)$ a section s, that are compatible under the inclusion $\overline{\Delta}'_*$: $Z'_*(TQ/\tilde{B}) \to Z_*(Q/\tilde{B} \wedge \overline{T}Q)$.

Let s_1: $H_*(\widetilde{MO}/I_m) \to Z_*(\widetilde{MO}/I_m)$ be any section of the projection p: $Z_*(\widetilde{MO}/I_m) \to H_*(\widetilde{MO}/I_m)$. Recall that p is a surjection of $\mathbf{Z}/2$-vector spaces so that such a section exists. Since $\overline{\Delta}_*$: $H_*(\widetilde{MO}/I_m) \to H_*(\widetilde{BO}/I_m^+ \wedge \overline{T}Q)$ is injective in positive dimensions, s_1 can be extended to a section

$$s_2\colon H_*\left(\widetilde{BO}/I_m^+ \wedge \overline{T}Q\right) \to Z_*\left(\widetilde{BO}/I_m^+ \wedge \overline{T}Q\right).$$

Similarly, since $\tilde{q}_*$: $H_*(\widetilde{BO}/I_m^+ \wedge \overline{T}Q) \to H_*(Q^+ \wedge \overline{T}Q)$ is injective, s_2 can be extended to a section s_3: $H_*(Q^+ \wedge \overline{T}Q) \to Z_*(Q^+ \wedge \overline{T}Q)$.

Now observe that the image of the composition $H_*(TQ) \xrightarrow{\overline{\Delta}_*} H_*(Q^+ \wedge \overline{T}Q) \xrightarrow{s_3} Z_*(Q^+ \wedge \overline{T}Q)$ lies in the image of $\overline{\Delta}'_*$: $Z'_*(TQ) \to Z_*(Q^+ \wedge \overline{T}Q)$. Hence this composition factors uniquely through a map s_4: $H_*(TQ) \to Z'_*(TQ)$. Note that s_4 is a section of p': $Z'_*(TQ) \to H_*(TQ)$ in positive dimensions, since s_3 is a section, and since $\overline{\Delta}_*$: $H_*TQ \to H_*(Q^+ \wedge \overline{T}Q)$ is injective in positive dimensions. Notice furthermore that since the compositions $\tilde{q}_* \circ \overline{\Delta}_*$: $H_*\widetilde{MO}/I_m \to H_*(\widetilde{BO}/I_m^+ \wedge \overline{T}Q) \to H_*(Q^+ \wedge \overline{T}Q)$ and $\overline{\Delta}_* \circ T\tilde{q}_*$: $H_*\widetilde{MO}/I_m \to H_*(TQ) \to H_*(Q^+ \wedge \overline{T}Q)$ are equal, then by the properties of the sections s_i described above, we have that the compositions

$$s_4 \circ T\tilde{q}_*\colon H_*\widetilde{MO}/I_m \to H_*TQ \to Z'_*(TQ)$$

and $\tilde{q}_* \circ s_1$: $H_*\widetilde{MO}/I_m \to Z_*(\widetilde{MO}/I_m) \to Z'_*(TQ)$ are equal. Hence we can pass to quotients and get sections

$$\begin{aligned} s'\colon H_*(TQ/\tilde{B}) = H_*(TQ)/H_*\left(\widetilde{MO}/I_m\right) &\to Z'_*(TQ)/Z_*\left(\widetilde{MO}/I_m\right) \\ &= Z'_*(TQ/\tilde{B}) \end{aligned}$$

and

$$\begin{aligned} s\colon H_*(Q/\tilde{B} \wedge \overline{T}Q) &= H_*(Q^+ \wedge \overline{T}Q)/H_*\left(\widetilde{BO}/I_m^+ \wedge \overline{T}Q\right) \\ &\to Z_*(Q^+ \wedge \overline{T}Q)/Z_*\left(\widetilde{BO}/I_m^+ \wedge \overline{T}Q\right) \to Z_*(Q/\tilde{B} \wedge \overline{T}Q) \end{aligned}$$

that make the following diagram commute:

$$\begin{array}{ccc} Z'_*(TQ/\tilde{B}) & \xrightarrow[\bar{\Delta}'_*]{} & Z_*(Q/\tilde{B} \wedge \bar{T}Q) \\ \uparrow s' & & \uparrow s \\ H_*(TQ/\tilde{B}) & \xrightarrow[\bar{\Delta}_*]{} & H_*(Q/\tilde{B} \wedge \bar{T}Q). \end{array}$$

We now adjust s' so that it has image in the singular cycles $Z_*(TQ/\tilde{B}) \hookrightarrow Z'_*(TQ/\tilde{B})$. To do this consider the following diagram of short exact sequences:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B_*(TQ/\tilde{B}) & \longrightarrow & B'_*(TQ/\tilde{B}) & \xrightarrow{\pi} & B'_*/B & \longrightarrow & 0 \\ & & \downarrow j & & \downarrow j' & & \downarrow & & \\ 0 & \to & Z_*(TQ/\tilde{B}) & \longrightarrow & Z'_*(TQ/\tilde{B}) & \xrightarrow{\pi} & Z'_*/Z & \longrightarrow & 0 \\ & & \downarrow p & & \downarrow p' & & \downarrow & & \\ 0 & \to & H_*(TQ/\tilde{B}) & \xrightarrow{=} & H_*(TQ/\tilde{B}) & \longrightarrow & 0 & \xrightarrow{=} & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where $B_*(TQ/\tilde{B})$ and $B'_*(TQ/\tilde{B})$ denote the kernels of p and p'. Notice that $B'_*/B = Z'_*/Z_*$. Notice also that π: $B'_*(TQ/\tilde{B}) \to B'_*/B$ is a surjection of $\mathbf{Z}/2$-vector spaces and so it has a section e_1: $B'_*/B \to B'_*(TQ/\tilde{B})$. Let e: $Z'_*/Z_* \to Z'_*(TQ/\tilde{B})$ be the induced section of π: $Z'_*(TQ/\tilde{B}) \to Z'_*/Z_*$; namely,

$$e\colon Z'_*/Z_* = B'_*/B_* \xrightarrow{e_1} B'_*(TQ/B) \xrightarrow{j'} Z'_*(TQ/\tilde{B}).$$

Now let s': $H_*(TQ/\tilde{B}) \to Z'_*(TQ/\tilde{B})$ be as above. Define t_1 to be the difference $t_1 = s' - e \circ \pi \circ s'$: $H_*(TQ/\tilde{B}) \to Z_*(TQ/\tilde{B})$. Notice that $\pi \circ t_1 = \pi s' - \pi e \pi s' = \pi s' - \pi s' = 0$, so that t_1 takes values in $Z_*(TQ/\tilde{B})$. Moreover t_1 is a section of p because $p \circ t_1 = p' \circ t_1 = p's' - p'e\pi s' = 1 - p'j'e_1\pi s = 1$, since $p'j' = 0$.

Now since $\bar{\Delta}'_*$: $Z'_*(TQ/\tilde{B}) \to Z_*(Q/\tilde{B} \wedge \bar{T}Q)$ is monic, there is an extension $\bar{\pi}$: $Z_*(Q/\tilde{B} \wedge \bar{T}Q) \to Z'_*/Z_*$ of π: $Z'_*(TQ/\tilde{B}) \to Z'_*/Z_*$. Let f: $Z'_*/Z \to Z_*(Q/\tilde{B} \wedge \bar{T}Q)$ be the composition $f = \bar{\Delta}'_* \circ e$: $Z'_*/Z \to Z'_*(TQ/\tilde{B}) \to Z_*(Q/\tilde{B} \wedge \bar{T}Q)$. Finally, define t_2 to be the difference $t_2 = s - f \circ \bar{\pi} \circ s$: $H_*(Q/\tilde{B} \wedge \bar{T}Q) \to Z_*(Q/\tilde{B} \wedge \bar{T}Q)$, where s is as above. Reasoning like that above proves that t_2 is a section. The fact that t_1 and t_2 satisfy

3.23 now follows from the compatibility of the sections s' and s, and the definitions of e and f.

We now use 3.23 to complete the proof of the claim. Consider the homomorphism

$$\sigma_*: \pi_{*-1}(\operatorname{Ker}(d)) \to C_*(\bar{\Delta}) = C_*(Q/\tilde{B} \wedge \bar{T}Q) \oplus C_{*-1}(TQ/\tilde{B}),$$

defined by $\sigma_*(x) = (0, t_{1*}i_*(x))$, where t_1 is as in 3.23 and i_* is induced by the inclusion i: $\operatorname{Ker}(d) \to TQ/\tilde{B} \wedge K$. Now since the image of i_*: $\pi_*\operatorname{Ker}(d) \to \pi_*(TQ/\tilde{B} \wedge K) = H_*(TQ/\tilde{B})$ lies in the kernel of $\bar{\Delta}_*$: $H_*(TQ/\tilde{B}) \to H_*(Q/\tilde{B} \wedge \bar{M}) = Y_*(Q/\tilde{B} \wedge \bar{T}Q)$, through dimension m, then by 3.23 and the definition of the boundary map $\partial_{\bar{\Delta}}$ of the algebraic mapping cone $C_*(\bar{\Delta})$, we have that $\partial_{\bar{\Delta}} \circ \sigma_* = 0$. Hence σ_* induces a map σ_*: $\pi_{*-1}\operatorname{Ker}(d) \to H_*(M(\bar{\Delta}))$ which lifts the inclusion i_*: $\pi_{*-1}\operatorname{Ker}(d) \to H_*(TQ/\tilde{B})$. Furthermore, on the chain level we can compute $m(\Delta_l)_* \circ \sigma_*$ by formula 3.22; namely,

$$m(\Delta_l)_* \circ \sigma_*(x) = m(\Delta_l)_*(0, t_{1*}i_*(x)) = (0, \Delta_* t_{1*} i_*(x)).$$

Similarly, $m(\Delta_r)_* \circ \sigma_*(x)$ also equals $(0, \Delta_* t_{1*} i_*(x))$. Thus

$$(m(\Delta_l)_* - m(\Delta_r)_*) \circ \sigma_* = 0.$$

By the above remarks, these properties translate to the existence of a homomorphism σ_*: $\pi_*\operatorname{Ker}(d) \to \pi_*(U(Q/\tilde{B}) \wedge K)$ which lifts i_*: $\pi_*\operatorname{Ker}(d) \to \pi_*(TQ/\tilde{B} \wedge K)$ and which, through dimension m, has the property that $(\tilde{\delta}_{1*} - \tilde{\partial}_{1*}) \circ \sigma_* = 0$. Now by realizing this homomorphism of homotopy groups by a map of Eilenberg-MacLane spectra, σ: $\operatorname{Ker}(d) \to U(Q/\tilde{B}) \wedge K$, the claim (and hence the lemma) is proved.

We now gather some implications of Lemma 3.21 so that we may use them to complete the proof Lemma 3.17.

Consider the map j: $U(Q/\tilde{B}) \wedge K \to (Q/\tilde{B} \wedge K) \vee (TQ/\tilde{B} \wedge K)$ given by the definition of $U(Q/\tilde{B})$ as the stable pull-back of Δ: $TQ/\tilde{B} \to Q/\tilde{B} \wedge MO$ and i: $Q/\tilde{B} \to Q/\tilde{B} \wedge MO$. Let $\operatorname{Im}(j) \subset \pi_*(Q/\tilde{B} \wedge K) \oplus \pi_*(TQ/\tilde{B} \wedge K)$ be the image of the induced map in homotopy groups. (Note: These maps i and j are not related to the maps of the same name used in the proof of 3.21.)

Corollary 3.24. *The composition* $\ker(\tilde{\delta}_1 - \tilde{\partial}_1)_* \subset \pi_* UQ/\tilde{B} \wedge K \to \operatorname{Im}(j)$ *is surjective.*

Proof. Notice that by the proof of 3.16, the map j_*: $\pi_* U(Q/\tilde{B}) \wedge K = H_* U(Q/\tilde{B}) \to H_*(Q/\tilde{B}) \oplus H_*(TQ/\tilde{B}) = \pi_* Q/\tilde{B} \wedge K \oplus \pi_* TQ/\tilde{B} \wedge K$ is given by the composition j_*: $H_* U(Q/\tilde{B}) \underset{j_2}{\to} H_* TQ/\tilde{B} \underset{\Phi \oplus 1}{\to} H_* Q/\tilde{B} \oplus H_* TQ/\tilde{B}$, where j_2 is the 2^{nd} component of j_* and Φ is the Thom isomorphism. Thus it is sufficient to show that the composition $\ker(\tilde{\delta}_1 - \tilde{\partial}_1)_* \subset \pi_* UQ/\tilde{B} \wedge K \to \operatorname{Im}(j_2)$ is surjective.

Now by the exact sequence for the cofibration $UQ/\tilde{B} \wedge K \xrightarrow{j_2} TQ/\tilde{B} \wedge K \xrightarrow{\bar{\Delta}} Q/\tilde{B} \wedge \overline{M} \wedge K$ we see that $\operatorname{Im}(j_2) = \ker(\bar{\Delta}_*)$ which is equal to $\ker d_*$. (This is true because by its definition, $\bar{\Delta}$ factors through d: $TQ/\tilde{B} \wedge K \to K(V)$, and the inclusion $\pi_* K(V) \to \pi_* Q/\tilde{B} \wedge \overline{M} \wedge K$ is monic.) 3.24 now follows immediately from 3.21.

Our goal is to obtain a result analogous to 3.24 with $\pi_* U(Q/\tilde{B})$ replacing $\pi_*(UQ/\tilde{B} \wedge K)$. (See 3.27 below.) To do this we introduce an intermediate space $U'(Q/\tilde{B})$ defined to be the pull-back for the diagram

$$\begin{tikzcd}
U'(Q/\tilde{B}) \arrow[r] \arrow[d] & T(Q/\tilde{B}) \arrow[d, "i\Delta"] \\
Q/\tilde{B} \arrow[r, "i"'] & Q/\tilde{B} \wedge MO \wedge K.
\end{tikzcd}$$

Notice that we have the following obvious maps of pull-back squares

(3.25)

$$\begin{tikzcd}
U(Q/\tilde{B}) \arrow[r] \arrow[d] & TQ/\tilde{B} \arrow[d, "\Delta"] & {} \arrow[r, "i_1"'] & {} & U'Q/\tilde{B} \arrow[r] \arrow[d] & TQ/\tilde{B} \arrow[d, "i\Delta"] & {} \arrow[r, "i_2"'] & {} & UQ/\tilde{B} \wedge K \arrow[r] \arrow[d] & TQ/\tilde{B} \wedge K \arrow[d, "\Delta \wedge 1"] \\
Q/\tilde{B} \arrow[r, "i"'] & Q/\tilde{B} \wedge MO & & & Q/\tilde{B} \arrow[r, "i"'] & Q/\tilde{B} \wedge MO \wedge K & & & Q/\tilde{B} \wedge K \arrow[r] & Q/\tilde{B} \wedge MO \wedge K
\end{tikzcd}$$

where i_1 is induced by the inclusion i: $Q/\tilde{B} \wedge MO \hookrightarrow Q/\tilde{B} \wedge MO \wedge K$ and i_2 is induced by the inclusions i: $TQ/\tilde{B} \hookrightarrow TQ/\tilde{B} \wedge K$ and i: $Q/\tilde{B} \to Q/\tilde{B} \wedge K$. Notice that $i_2 \circ i_1 = i$: $UQ/\tilde{B} \to U(Q/\tilde{B}) \wedge K$.

Let j': $U'(Q/\tilde{B}) \to Q/\tilde{B} \vee TQ/\tilde{B}$ be the map given by the definition of $U'(Q/\tilde{B})$ as a pull-back spectrum as above. Consider the image of the composite $i \circ j' = j \circ i_2$: $\pi_* U'Q/\tilde{B} \to \pi_*(Q/\tilde{B} \wedge K) \oplus \pi_*(TQ/\tilde{B} \wedge K)$.

Corollary 3.26. *The composition* $\ker[(\tilde{\delta}_1 - \tilde{\partial}_1) \circ i_2]_* \hookrightarrow \pi_* U'(Q/\tilde{B}) \to \operatorname{Im}(j \circ i_2) = \operatorname{Im}(i \circ j')$ *is surjective.*

Proof. This follows from 3.24 and a diagram chase using the long exact sequences in homotopy groups induced by the pull-back squares in (3.25).

Now let $\bar{j}$: $U(Q/\tilde{B}) \to Q/\tilde{B} \vee TQ/\tilde{B}$ be the map given by the definition of $U(Q/\tilde{B})$ as a pull-back. Consider the image of the composite $(i \circ \bar{j})_* = (j \circ i)_*$: $\pi_* UQ/\tilde{B} \to \pi_* Q/\tilde{B} \wedge K \oplus \pi_* TQ/\tilde{B} \wedge K$. Our goal is to prove the following:

Proposition 3.27. *The composition* $\ker(\tilde{\delta} - \tilde{\partial})_* \subset \pi_* UQ/\tilde{B} \to \operatorname{Im}(i \circ \bar{j})_* = \operatorname{Im}(j \circ i)_*$ *is surjective.*

Remark. Once we prove 3.27 our proof of Lemma 3.17 will be complete. To see this, recall that by our inductive assumptions, the class $\varepsilon \in \pi_m Q/\tilde{B}$ has the property that $\Phi_*[\varepsilon] \in H_*TQ/\tilde{B}$ is spherical. Hence by 3.16, ε lifts to $\pi_m U(Q/\tilde{B})$. Moreover, by the proof of 3.16 $[\varepsilon] + \Phi_*[\varepsilon] \in H_*Q/\tilde{B} \oplus H_*T(Q/\tilde{B}) = \pi_*(Q/\tilde{B} \wedge K) \oplus \pi_*(T(Q/\tilde{B}) \wedge K)$ lies in the image of

$$(i \circ \bar{j})_*: \pi_*UQ/\tilde{B} \to \pi_*Q/\tilde{B} \oplus \pi_*TQ/\tilde{B}$$
$$\to \pi_*((Q/\tilde{B}) \wedge K) \oplus \pi_*(TQ/\tilde{B} \wedge K).$$

By 3.27 we can therefore find an element $\gamma \in \ker(\tilde{\delta}_* - \tilde{\partial}_*) \subset \pi_*U(Q/\tilde{B})$ in the preimage of $[\varepsilon] + \Phi_*[\varepsilon]$. Since $\tilde{\delta}_*(\gamma) = \tilde{\partial}_*(\gamma)$, then by the factorization of $\tilde{\partial}$ given in $3.20\frac{1}{2}$, $\tilde{\delta}_*(\gamma) = (1 \wedge v \wedge 1)_*(\xi_*(\gamma)) \in \pi_*(Q/\tilde{B} \wedge U(BO) \wedge K)$. By the pull-back property of the space Z, this means that $\gamma \subset \pi_m U(Q/\tilde{B})$ lifts to $\pi_m Z$. If $T\varepsilon \in \pi_m TQ/\tilde{B}$ is the projection of this class in $\pi_*T(Q/\tilde{B})$, then the above analysis shows that the space Z and the class $T\varepsilon \in \pi_m TQ/\tilde{B}$ satisfy properties $3.19\frac{1}{2}$. This, as observed above, will complete the proof of 3.17.

Proof of 3.27. This proposition will follow from 3.26, the fact that MO is Eilenberg-MacLane, and several diagram chases. We now describe this argument, leaving some of the details to the reader.

Let $C \subset \pi_*(\Sigma^{-1}Q/\tilde{B} \wedge MO \wedge K)$ be the kernel of the homomorphism

$$\pi_*(\Sigma^{-1}Q/\tilde{B} \wedge MO \wedge K) \xrightarrow{\Delta_l - \Delta_r} \pi_*(\Sigma^{-1}Q/\tilde{B} \wedge BO^+ \wedge MO \wedge K)$$
$$\xrightarrow{D_*} \pi_*(Q/\tilde{B} \wedge U(BO) \wedge K),$$

where D_* is the connecting map in the long exact sequence coming from the pull-back square defining $Q/\tilde{B} \wedge U(BO)$ (and thus $Q/\tilde{B} \wedge U(BO) \wedge K$). Notice that since $Q/\tilde{B} \wedge U(BO) \wedge K$ is Eilenberg-MacLane, the Hurewicz map h: $\pi_*(Q/\tilde{B} \wedge U(BO) \wedge K) \to H_*(Q/\tilde{B} \wedge U/BO) \wedge K)$ is monic, and thus C is also the kernel of

$$h \circ D_* \circ (\Delta_l - \Delta_r): \pi_*(\Sigma^{-1}Q/\tilde{B} \wedge MO \wedge K)$$
$$\to H_*(Q/\tilde{B} \wedge U(BO) \wedge K).$$

Now let $\bar{K}$ be the cofiber of the generator $S^0 \to K$ of $\pi_0 K = \mathbf{Z}_2$, and let $\bar{C} \hookrightarrow \pi_*(\Sigma^{-1}Q/\tilde{B} \wedge MO \wedge \bar{K})$ be the kernel of the composition

$$h \circ D_* \circ (\Delta_l - \Delta_r): \pi_*(\Sigma^{-1}Q/\tilde{B} \wedge MO \wedge \bar{K}) \to \pi_*(Q/\tilde{B} \wedge U(BO) \wedge \bar{K})$$
$$\to H_*(Q/\tilde{B} \wedge U(BO) \wedge \bar{K}).$$

An easy exercise using the fact that MO is Eilenberg-MacLane shows that the natural projection p: $C \to \bar{C}$ (induced by the projection $K \to \bar{K}$) is surjective.

Now notice that we have a cofibration sequence

$$\Sigma^{-1}Q/\tilde{B} \wedge MO \xrightarrow{i} \Sigma^{-1}Q/\tilde{B} \wedge MO \wedge K \to \Sigma^{-1}Q/\tilde{B} \wedge MO \wedge \bar{K},$$

and thus by diagram 3.25, we have an induced cofibration sequence $U(Q/\tilde{B}) \xrightarrow{i_1} U'(Q/\tilde{B}) \to \Sigma^{-1}Q/\tilde{B} \wedge MO \wedge \overline{K}$. If we let $\operatorname{Im}(C) \subset \pi_* U'Q/\tilde{B}$ be the image of C under the connecting map $\pi_*\Sigma^{-1}Q/\tilde{B} \wedge MO \wedge K \to \pi_* U'(Q/\tilde{B})$ we then have the following commutative diagram:

$$\begin{tikzcd} & C \arrow[d] \arrow[drr, "p"] & & \\ & \operatorname{Im}(C) \arrow[rr, "p_1"] \arrow[d] & & \overline{C} \arrow[d, "="] \\ \ker[(\tilde{\delta}_1 - \tilde{\partial}_1) \circ i_2] \arrow[rrr, "p_2"'] & & & \overline{C} \arrow[r, hook] & \pi_*\Sigma^{-1}Q/\tilde{B} \wedge MO \wedge \overline{K}, \end{tikzcd}$$

where p_1 and p_2 are both induced by the projection onto the cofiber $U'(Q/\tilde{B}) \to \Sigma^{-1}Q/\tilde{B} \wedge MO \wedge \overline{K}$ of i_1: $UQ/\tilde{B} \to U'(Q/\tilde{B})$. Notice that since p is surjective, so are p_1 and p_2. Notice furthermore that by 3.26, $\ker[(\tilde{\delta}_1 - \tilde{\partial}_1) \circ i_2] \to \operatorname{Im}(j \circ i_2)$ is surjective, and that by a diagram chase, $\operatorname{Im}(C)$ lies in the kernel of this map. Because of these facts, the following is immediate.

Claim. The composition $\ker(p_2) \hookrightarrow \ker[(\tilde{\delta}_1 - \tilde{\partial}_1) \circ i_2] \to \operatorname{Im}(j \circ i_2)$ is surjective.

To complete the proof of 3.27 notice that if $x \in \ker(p_2) \hookrightarrow \ker[(\tilde{\delta}_1 - \tilde{\partial}_1) \circ i_2] \hookrightarrow \pi_* U'Q/\tilde{B}$, then its projection to $\pi_*\Sigma^{-1}Q/\tilde{B} \wedge MO \wedge \overline{K}$ is given by $p_2(x) = 0$. By the exact sequence for the cofibration sequence $U(Q/\tilde{B}) \xrightarrow{i_1} U'(Q/\tilde{B}) \to \Sigma^{-1}Q/\tilde{B} \wedge MO \wedge \overline{K}$, there is an element $\bar{x} \in \pi_* U(Q/\tilde{B})$ with $i_{1*}(\bar{x}) = x \in \pi_* U'(Q/\tilde{B})$. Now since

$$(\tilde{\delta} - \tilde{\partial}) = (\tilde{\delta}_1 - \tilde{\partial}_1) \circ i = (\tilde{\delta}_1 - \tilde{\partial}_1) \circ i_2 \circ i_1,$$

we have that $(\tilde{\delta} - \tilde{\partial})_*(\bar{x}) = (\tilde{\delta}_1 - \tilde{\partial}_1) \circ i_2(x) = 0$. Thus we have observed that every element in $\ker(p_2) \subset \pi_* U'(Q/\tilde{B})$ is in the image of $\ker(\tilde{\delta} - \tilde{\partial})_* \subset \pi_* U(Q/\tilde{B})$ under i_1. Now 3.27 follows from the claim. As proved above, this completes the proof of Lemma 3.17.

We have now finished the proof of Lemma 3.15, and thereby, Proposition 3.14, which in turn completes the inductive step in the proof of Theorem 3.5 in the case $j > 0$ and $m - \alpha(m)$ odd. We now study the case $j = 0$.

In this case our goal is to construct a stable map q_0: $BO/I_m \to Q_0 = Q$ that satisfies the following properties:

1. q_0 extends $\tilde{q}_0$: $\widetilde{BO/I_m} \to \tilde{Q}$.
2. The composition $BO/I_m \xrightarrow[q_0]{} Q \to P_m \to BO/I_m$ is homotopic to the identity, and
3. $\Phi q_{0_*}(d_m) \in H_m TQ$ is spherical.

Now recall that Q was constructed out of the pull-back space P as $Q = \tilde{Q} \cup_\alpha D^m$, where $\tilde{Q}$ was defined to be the homotopy fiber of the composition

$P \to BO/I_m \to BO/I_m/\widetilde{BO}/I_m = S^m$, and $\alpha\colon S^{m-1} \to \tilde{Q}$ was defined to be the composition $\alpha\colon S^{m-1} \hookrightarrow \Omega S^m \to \tilde{Q}$. By this construction and the fact that BO/I_m is m-dimensional, an easy obstruction theoretic argument shows that it is sufficient to construct a stable map $q\colon BO/I_m \to P$ satisfying the above conditions with P replacing Q, since the pair (P, Q) is m-connected.

To do this, as argued above, it is sufficient to find a class $\varepsilon \in \pi_m^s(U(P/\tilde{B}))$ lifting the sphere in the splitting $U(B/\tilde{B}) \simeq S^m \vee \Sigma^{m-1}\overline{M}$. Here $P/\tilde{B}$ and $B/\tilde{B}$ denote the cofibers of $\tilde{q}\colon \widetilde{BO}/I_m \to P$ and $\widetilde{BO}/\mathrm{I}_\mathrm{m} \hookrightarrow \mathrm{BO}/\mathrm{I}_\mathrm{m}$ *respectively.*

To construct ε, recall from 3.4 that there is a stable lifting $r_m\colon BO/I_m \to BO(m - \alpha(m))$ of $\rho\colon BO/I_m \to BO$ that extends $\tilde{\rho}_m\colon \widetilde{BO}/I_m \to BO(m - \alpha(m))$. Also there is a map of Thom spectra $t_m\colon MO/I_m \to MO(m - \alpha(m))$ lifting $T\rho$ and extending $T\rho_m$. Now since $H_*BO(m - \alpha(m)) \to H_*BO$ is injective, the maps r_m and t_m must preserve the Thom isomorphism. Thus by 3.16 there is a class $\varepsilon' \in \pi_m U(BO(m - \alpha(m))/\tilde{B})$ whose projection to $\pi_m U(BO/\tilde{B})$ agrees with the projection of the sphere in the splitting $U(B/\tilde{B}) \simeq S^m \vee \Sigma^{m-1}\overline{M}$ to $U(BO/\tilde{B})$. Thus we will be done once we prove the following:

LEMMA 3.28. *The space $U(P/\tilde{B})$ has the m-dimensional stable homotopy type of the stable pull-back for the diagram*

$$\begin{array}{ccc} U(P/\tilde{B}) & \longrightarrow & U(B/\tilde{B}) \\ \downarrow & & \downarrow \\ U(BO(m-\alpha(m))/\tilde{B}) & \longrightarrow & U(BO/\tilde{B}). \end{array}$$

Proof. The cofiber of the left hand vertical arrow is $U(BO(m - \alpha(m))/P)$, and of the right hand arrow is $U(BO/B)$. We need to check that these cofibers have the same m-dimensional homotopy type.

To see this, recall that since the maps $\rho\colon BO/I_m \to BO$ and $BO(m - \alpha(m)) \to BO$ are both $m/2$-connected, then the map $P \to BO(m - \alpha(m))$ is also $m/2$-connected. Now the results of Section 1.a imply that the induced maps of cofibers $BO(m - \alpha(m))/P \to BO/BO/I_m$ and $MO(m - \alpha(m))/TP \to MO/MO/I_m$ are therefore m-connected. Since these maps preserve the Thom isomorphism in cohomology, this implies that the induced map $U(BO/(m - \alpha(m))/P) \to U(BO/BO/I_m)$ is an m-equivalence. This proves 3.21.

This then completes the inductive step in the proof of Theorem 3.5 in the case $j = 0$ and $m - \alpha(m)$ is odd, and hence the proof of 3.5 is complete when $m - \alpha(m)$ is odd.

In the case when $m - \alpha(m)$ is even the added complication is that the fiber of the map $\tilde{q}\colon \widetilde{BO}/I_m \to \tilde{Q}$ has one infinite homotopy group, namely $\pi_{m-\alpha(m)-1}(\mathscr{F}) \simeq \mathbf{Z}_{(2)}$, the 2-local integers. (See the comments concerning $\pi_*(\mathscr{F})$

after the statement of 3.5.) This implies that we cannot modify Postnikov tower (3.6) so that the fibers are products of $K(Z_2, n)$'s in this dimension. Observe, however, that Theorem 3.5 only involves lifting an $(m-1)$-dimensional homotopy class, and for $m - \alpha(m)$ even, $m - 1 > m - \alpha(m)$. Thus this homotopy group represents no obstruction to carrying out the program described above for the case $m - \alpha(m)$ odd. We leave details to the interested reader.

This completes our proof of 3.5, and therefore modulo the proof of Lemma 2.10 which we provide in the next section, the proof of Lemma B (and hence the immersion conjecture) is now complete.

4. Proof of Lemma 2.10

We begin by restating (and renumbering) Lemma 2.10.

THEOREM 4.1. *There exists a collection of pairings*

$$\mu = \{\mu_{r,m-r}\colon BO/I_r \times BO/I_{m-r} \to BO/I_m,\ r \geq 1\}$$

that lift the maps $\rho \times \rho\colon BO/I_r \times BO/I_{m-r} \to BO$, *and which make the following diagrams homotopy commute*:

$$\begin{tikzcd} X_r \times X_{m-r} \arrow[r, "\pi_{r,m-r}"] \arrow[d, "g_r \times g_{m-r}"'] & \tilde{X}_m \arrow[d, "g_\mu"] \\ BO/I_r \times BO/I_{m-r} \arrow[r, "\mu"'] & BO/I_m. \end{tikzcd}$$

Proof. This theorem is proved by induction on m. So assume its validity for integers $< m$. Our first step in proving 4.1 for the integer m is to study in detail the maps g_r, $r < m$, the pairings $\pi_{r,m-r}$, and their Thom-ifications.

Our first observation is that the homotopy associativity and commutativity of the collection of pairings $p = \{p_{r,s}\colon F_r \times F_s \to F_{r+s}\}$ imply certain associativity and commutativity properties of the collection $\pi = \{\pi_{r,m-r}\colon X_r \times X_{m-r} \to \tilde{X}_m\}$. We describe these properties as follows:

Define

$$DX_m = \coprod_{r=1}^{m-1} X_r \times X_{m-r}, \quad \text{and}$$

$$CX_m = \coprod_{\substack{i+j+k=m \\ i,j,k>0}} X_i \times X_j \times X_k \amalg DX_m.$$

We can define two maps, $L_X, R_X\colon CX_m \to DX_m$ in the following manner. When restricted to $X_i \times X_j \times X_k$, L_X is defined to be $\pi_{i,j} \times 1\colon (X_i \times X_j) \times X_k \to X_{i+j} \times X_k \hookrightarrow DX_m$. Also, R_X is defined to be

$$1 \times \pi_{j,k}\colon X_i \times (X_j \times X_k) \to X_i \times X_{j+k} \hookrightarrow DX_m.$$

When restricted to $DX_m \hookrightarrow CX_m$, L_X is defined to be the identity map. When restricted to components of $X_r \times X_s \hookrightarrow DX_m$ of the form $(M_\omega \times F_{r-|\omega|}) \times (M_\nu \times F_{s-|\nu|})$ with either $0 < |\omega| < r$ or $|\nu| < s$, R_X is defined to be the "switch" map

$$\begin{aligned}(M_\omega \times F_{r-|\omega|}) \times (M_\nu \times F_{s-|\nu|}) &\to (M_\omega \times F_{s-|\nu|}) \times (M_\nu \times F_{r-|\omega|}) \\ &\hookrightarrow X_{s+|\omega|-|\nu|} \times X_{r+|\nu|-|\omega|} \hookrightarrow DX_m.\end{aligned}$$

On the components $F_r \times M_\nu$ with $|\nu| = s$, R_X is the switch map, $F_r \times M_\nu \to M_\nu \times F_r \hookrightarrow X_s \times X_r \hookrightarrow DX_m$. On all other components of $X_r \times X_s \hookrightarrow DX_m$, R_X is defined to be the identity.

The associativity and commutativity properties of π referred to above can be stated by saying that the two compositions

$$\pi \circ L_x \quad \text{and} \quad \pi \circ R_X\colon CX_m \to DX_m \to \tilde{X}_m$$

are homotopic.

Let $E(X)$ be the homotopy co-equalizer (i.e., mapping torus) of L_X and R_X. Then

$$E(X) = DX_m \cup CX_m \times I/\sim$$

where if $(z, t) \in CX_m \times I$ then we make the identification $(z, 0) \sim L_X(z)$, and $(z, 1) \sim R_X(z)$. Let $r\colon DX_m \to E(\tilde{X})$ be the inclusion. Then there is a canonical homotopy between $r \circ L_X$ and $r \circ R_X$. Notice that since $\pi \circ L_X \simeq \pi \circ R_X$, we therefore have a map $h\colon E(X) \to \tilde{X}_m$ that extends $\pi\colon DX_m \to \tilde{X}_m$. Moreover, the map h has a section $j_r\colon \tilde{X}_m \to E(X)$ defined as follows:

Consider the inclusion $j\colon \tilde{X}_m \hookrightarrow DX_m$ defined on the components of the form $M_\omega \times F_{m-|\omega|}$, $|\omega| < m$, to be the inclusion

$$M_\omega \times F_{m-|\omega|} \hookrightarrow X_{|\omega|} \times X_{m-|\omega|} \hookrightarrow DX_m.$$

On the components of the form M_ω, $|\omega| = m$, j is defined as follows. Recall that $M_\omega \subset \tilde{X}_m$ with $|\omega| = m$ implies that M_ω is decomposable. Say $M_\omega = M_{i_1} \times \cdots \times M_{i_k}$ with $i_1 \leq \cdots \leq i_k$ and each M_{i_q} indecomposable. Write $M_\omega = M_{i_1} \times M_{\omega'}$. Then the restriction of j to M is the inclusion

$$M_\omega = M_{i_1} \times M_{\omega'} \hookrightarrow X_{i_1} \times X_{|\omega'|} \hookrightarrow DX_m.$$

Finally $j_r\colon \tilde{X}_m \to E(X)$ is defined to be the composition $\tilde{X}_m \underset{j}{\to} DX_m \underset{r}{\to} E(X)$. Clearly $h \circ j_r$ is homotopic to the identity.

LEMMA 4.2. *The following diagram homotopy commutes*:

$$\begin{tikzcd} DX_m \arrow[r, "\pi"] \arrow[dr, "r"'] & \tilde{X}_m \arrow[d, "j_r"] \\ & E(X). \end{tikzcd}$$

Proof. We first show that $j_r \circ \pi$ and r are homotopic when restricted to a component of the form $(M_\omega \times F_{k-|\omega|}) \times (M_\nu \times F_{m-k-|\nu|}) \subset X_k \times X_{m-k} \subset DX_m$.

Now since $r\colon DX_m \to E(X)$ equalizes L_X and R_X the restriction of r to this space is homotopic to the restriction of r to this space thought of as a component of $X_{|\omega|} \times X_{m-|\omega|} \subset DX_m$ via the inclusion

$$M_\omega \times F_{k-|\omega|} \times M_\nu \times F_{m-k-|\nu|} \hookrightarrow X_{|\omega|} \times X_{k-|\omega|} \times X_{|\nu|} \times X_{m-k-|\nu|}$$
$$\underset{1\times\pi}{\to} X_{|\omega|} \times X_{m-|\omega|}.$$

By the homotopy commutativity and associativity of π, this is homotopic to the composition

$$M_\omega \times F_{k-|\omega|} \times M_\nu \times F_{m-k-|\nu|} \underset{=}{\to} M_\omega \times M_\nu \times F_{k-|\omega|} \times F_{m-k-|\eta|}$$
$$\hookrightarrow X_{|\omega|} \times X_{|\nu|} \times X_{k-|\omega|} \times X_{m-k-|\nu|}$$
$$\underset{1\times\pi}{\to} X_{|\omega|} \times X_{|\nu|} \times X_{m-|\omega|-|\nu|} \underset{1\times\pi}{\to} X_{|\omega|} \times X_{m-k-|\omega|} \underset{r}{\to} E(X).$$

But again, since r equalizes L_X and R_X this is homotopic to the composition

$$M_\omega \times F_{k-|\omega|} \times M_\nu \times F_{m-k-|\omega|} = M_\omega \times M_\nu \times F_{k-|\omega|} \times F_{m-k-|\nu|}$$
$$\hookrightarrow X_{|\omega|} \times X_{|\nu|} \times X_{k-|\omega|} \times X_{m-k-|\nu|}$$
$$\underset{\pi\times\pi}{\to} X_{|\omega+\nu|} \times X_{m-|\nu|-|\omega|} \underset{r}{\to} E(X).$$

Now by a check of definitions, this last composition is the restriction of $j_r \circ \pi$ to this space. The verification that r and $j_r \circ \pi$ are homotopic maps when restricted to the other components of DX_m is done in the same way and is left to the reader.

We will now study the homotopy type of the Thom spectra of some of these spaces.

Let $D_m = \coprod_{r=1}^m BO/I_r \times BO/I_{m-r}$. The map $g_r \times g_{m-r}\colon X_r \times X_{m-r} \to BO/I_r \times BO/I_{m-r}$ defines a map $g\colon DX_m \to D_m$.

Let R and L be maps $CX_m \to D_m$ defined to be the compositions $g \circ R_X$ and $g \circ L_X$ respectively. Finally define $E(B)$ to be the mapping torus of R and L. We then have a canonical map of mapping tori $e(g)\colon E(X) \to E(B)$ that extends $g\colon DX_m \to D_m$.

Now let $\mu_0\colon D_m \to BO$ be defined via the product maps $\rho \times \rho\colon BO/I_r \times BO/I_{m-r} \to BO$. It is clear that the two compositions $\mu_0 \circ R$ and $\mu_0 \circ L$ from $C(X_m)$ to BO are homotopic and hence μ_0 extends to a map $e(\mu_0)\colon E(B) \to BO$. Let $TE(B)$ and $TE(X)$ be the associated Thom spectra.

Finally, let $D(\sigma)\colon TD_m \to TDX_m$ be the splitting of $Tg\colon TDX_m \to TD_m$ defined by the splittings $\sigma_r \wedge \sigma_{m-r}\colon MO/I_r \wedge MO/I_{m-r} \to TX_r \wedge TX_{m-r} \hookrightarrow TDX_m$.

LEMMA 4.3. *There is a map of spectra* $E(\sigma)$: $TE(B) \to TE(X)$ *making the following diagram homotopy commute*:

$$\begin{array}{ccc} TD_m & \longrightarrow & TE(B) \\ \downarrow{\scriptstyle D(\sigma)} & & \downarrow{\scriptstyle E(\sigma)} \\ TDX_m & \longrightarrow & TE(X). \end{array}$$

Proof. Consider the self map of Thom spectra $C(\sigma)$: $TCX_m \to TCX_m$ defined via the compositions

$$TX_i \wedge TX_j \wedge TX_k \xrightarrow[Tg \wedge Tg \wedge Tg]{} MO/I_i \wedge MO/I_j \wedge MO/I_k$$
$$\xrightarrow[\sigma \wedge \sigma \wedge \sigma]{} TX_i \wedge TX_j \wedge TX_k \hookrightarrow TCX_m.$$

By our inductive assumption about the maps Tg and σ, one can, by a simple check of the definitions, verify that the following diagram homotopy commutes:

(4.4)
$$\begin{array}{ccc} TCX_m & \overset{TL}{\underset{TR}{\rightrightarrows}} & TD_m \\ \downarrow{\scriptstyle C(\sigma)} & & \downarrow{\scriptstyle D(\sigma)} \\ TCX_m & \overset{TL_X}{\underset{TR_x}{\rightrightarrows}} & TDX_m. \end{array}$$

Now since homotopy co-equalizers (mapping tori) are preserved under passing to Thom spectra, the homotopy commutativity of this diagram allows us to find a map $E(\sigma)$: $TE(B) \to TE(X)$ of mapping tori that extends $D(\sigma)$.

We now continue our proof of Theorem 4.1. To do this consider the tower of fibrations converging to BO/I_m given in Theorem 2.5:

$$\begin{array}{ccccccccc} BO/I_m \to & \cdots & \to B_i \to & B_{i-1} & \to \cdots \to & B_0 & = BO \\ & & & \downarrow{\scriptstyle \kappa_i} & & \downarrow{\scriptstyle \kappa_1} & \\ & & & K_i & & K_1 & \end{array}$$

(The k-invariant spaces K_i are called K_i' in 2.5.)

Let μ_0: $D_m \to B_0 = BO$ be the map defined above. As observed earlier $\mu_0 \circ R$ and $\mu_0 \circ L$ are homotopic maps from CX_m to B_0.

Inductively assume there is a map μ_{i-1}: $D_m \to B_{i-1}$ that lifts μ_0 and so that $\mu_{i-1} \circ R$ and $\mu_{i-1} \circ L$ are homotopic.

Let $g(\mu_{i-1})$: $\tilde{X}_m \to B_{i-1}$ be the composition

$$g(\mu_{i-1}) \colon \tilde{X}_m \underset{j}{\hookrightarrow} DX_m \underset{g}{\to} D_m \underset{\mu_{i-1}}{\to} B_{i-1}.$$

Since $\mu_{i-1} \circ R$ is homotopic to $\mu_{i-1} \circ L$, then by Lemma 4.2 the following diagram homotopy commutes:

$$\begin{tikzcd} DX_m \arrow[r, "\pi"] \arrow[d, "g"] & \tilde{X}_m \arrow[d, "g(\mu_{i-1})"] \\ D_m \arrow[r, "\mu_{i-1}"'] & B_{i-1}. \end{tikzcd}$$

Thus to prove Theorem 4.1 it suffices to complete the inductive step. We therefore now proceed to construct an appropriate lifting $\mu_i\colon D_m \to B_i$.

For Y one of the spaces D_m, DX_m, CX_m, or $\tilde{X}_m$, let $PY \to Y$ be the pull-back of the fibration $B_i \to B_{i-1}$ over the map $Y \to B_{i-1}$. Observe that all of the spaces Y in question are quasi-normal of dimension m and thus the fibrations $PY \to Y$ are trivial fibrations with fiber ΩK_i. Our goal is to carefully choose trivializations so as to obtain sections $Y \to PY$ so that the composition $\mu_i\colon D_n \to PD_n \to B_i$ satisfies $\mu_i \circ R \cong \mu_i \circ L$.

Now since the maps $\pi \circ L_X$ and $\pi \circ R_X$; $CX_m \to DX_m \to X_m$ are homotopic, we have maps of fibrations

$$PR_X \text{ and } PL_X\colon PCX_m \to PDX_m, \quad \text{and}$$

$$P\pi\colon PDX_m \to P\tilde{X}_m$$

that cover R_X, L_X, and π respectively, and so that the compositions $P\pi \circ PR_X$ and $P\pi \circ PL_X$ are homotopic.

Define $PE(X)$ to be the mapping torus of PR_X and $PL_X\colon PCX_m \to PDX_m$, so that there is a map $Ph\colon PE(X) \to P\tilde{X}_m$ that extends $P\pi\colon PDX_m \to P\tilde{X}_m$.

By the pull-back properties of these spaces we can find maps of these fibrations so that the following diagram homotopy commutes:

$$(4.5)\qquad \begin{tikzcd} PCX_m \arrow[r, shift left, "PR_X"] \arrow[r, shift right, "PL_X"'] \arrow[d, "="'] & PDX_m \arrow[r] \arrow[d, "Pg"] \arrow[rr, bend left, "P\pi"] & PE(X) \arrow[r] \arrow[d, "Pe(g)"] & P\tilde{X}_m \arrow[r, "Pg(\mu_{i-1})"'] & B_i \\ PCX_m \arrow[r, shift left, "PR"] \arrow[r, shift right, "PL"'] & PD_m \arrow[r] \arrow[urrr, bend right, "P(\mu_{i-1})."'] & PE(B) & & \end{tikzcd}$$

Here PR and PL are defined to be $Pg \circ PR_X$ and $Pg \circ PL_X$ respectively, and $PE(B)$ is defined to be the mapping torus of PR and PL.

COROLLARY 4.6. *Let Y be any space and f_1: $PD_m \to Y$ and f_2: $P\tilde{X}_m \to Y$ any maps such that the following diagram homotopy commutes*:

$$\begin{array}{ccc} PDX_m & \xrightarrow{P\pi} & P\tilde{X}_m \\ \downarrow{\scriptstyle Pg} & & \downarrow{\scriptstyle f_2} \\ PD_m & \xrightarrow[f_1]{} & Y. \end{array}$$

Then $f_1 \circ PR$ and $f_1 \circ PL$: $PCX_m \to PD_m \to Y$ are homotopic maps.

Proof. This follows by the commutativity of diagram (4.5) and the mapping torus properties of $PE(X)$.

We will use this, and the following last bit of technical machinery to prove Theorem 4.1.

LEMMA 4.7. *The pull back fibrations $PD_m \to D_m$ and $P\tilde{X}_m \to \tilde{X}_m$ have trivializations α: $P\tilde{X}_m \to \Omega K_i$ and β: $PD_m \to \Omega K_i$ so that the following diagram homotopy commutes*:

$$\begin{array}{ccc} PDX_m & \xrightarrow{P\pi} & P\tilde{X}_m \\ \downarrow{\scriptstyle Pg} & & \downarrow{\scriptstyle \alpha} \\ PD_m & \xrightarrow[\beta]{} & \Omega K_i. \end{array}$$

Proof. As mentioned above, the pull-back fibration $P\tilde{X}_m \to \tilde{X}_m$ is trivial with fiber ΩK_i. We make a careful choice of trivialization α: $P\tilde{X}_m \to \Omega K_i$ as follows. Let $S_1 \hookrightarrow H_*(PDX_m)$ be the kernel of Pg_*: $H_*(PDX_m) \to H_*(PD_m)$. Let $S_2 \hookrightarrow H_*(P\tilde{X}_m)$ be the image of S_1 under $P\pi_*$: $H_*PDX_m \to H_*P\tilde{X}_m$. Thus S_1 and S_2 are sub-A^*-comodules of H_*PDX_m and $H_*P\tilde{X}_m$ respectively.

We let α: $P\tilde{X}_m \to \Omega K_i$ be any trivialization so that in homology, α_* is trivial on $S_2 \subset H_*P\tilde{X}_m$. To see that we can find such a trivialization, it is sufficient to prove that the image of the inclusion $H_*(\Omega K_i) \to H_*P\tilde{X}_m$ intersects S_2 trivially (i.e., the intersection is zero). Even stronger, by the definition of S_2, it is sufficient to show that the image of the inclusion of the homology of the fiber $H_*(\Omega K_i) \to H_*(PDX_m)$ intersects S_1 trivially. But by the definition of S_1 this will be true if and only if the pull-back fibration $PD_m \to D_m$ is trivial. This, however, was observed above. Thus we can find such a trivialization α: $P\tilde{X}_m \to \Omega K_i$.

Notice that since the composition $S_1 \hookrightarrow H_*PDX_m \xrightarrow{P\pi_*} H_*P\tilde{X}_m \xrightarrow{\alpha_*} H_*(\Omega K_i)$ is zero, the fact that the sequence $0 \to S_1 \to H_*PDX_m \xrightarrow{Pg_*} H_*PD_m \to 0$ is an exact sequence of A^*-comodules, implies that there is a map $\beta\colon PD_m \to \Omega K_i$ making the diagram in the statement of 4.7 homotopy commute.

We now complete the proof of Theorem 4.1.

Let $\alpha\colon P\tilde{X}_m \to \Omega K_i$ and $\beta\colon PD_m \to \Omega K_i$ be trivializations as in 4.7. By Corollary 4.6 the compositions $\beta \circ PR\colon PCX_m \to PD_m \to \Omega K_i$ and

$$\beta \circ PL\colon PCX_m \to PD_m \to \Omega K_i$$

are homotopic trivializations. Therefore with respect to these trivializations, the maps PR and PL are given (up to homotopy) as

$$CX_m \times \Omega K_i \underset{L\times 1}{\overset{R\times 1}{\rightrightarrows}} D_m \times \Omega K_i.$$

These trivializations therefore induce sections $s\colon D_m \to PD_m$ and $\bar{s}\colon CX_m \to PCX_m$ so that the following diagram homotopy commutes:

$$\begin{tikzcd} PCX_m \arrow[r, shift left, "PL"] \arrow[r, shift right, "PR"'] & PD_m \\ CX_m \arrow[u, "\bar{s}"] \arrow[r, shift left, "L"] \arrow[r, shift right, "R"'] & D_m \arrow[u, "s"'] \end{tikzcd}$$

Define $\mu_i\colon D_m \to B_i$ to be the composition

$$\mu_i\colon D_m \underset{s}{\to} PD_m \underset{P(\mu_{i-1})}{\to} B_i.$$

We need to show that the composites $\mu_i \circ R$ and $\mu_i \circ L$ are homotopic.

By the commutativity of this diagram $\mu_i \circ R = P(\mu_{i-1}) \circ s \circ R \simeq P(\mu_{i-1}) \circ PR \circ \bar{s}$. Similarly $\mu_i \circ L \simeq P(\mu_{i-1}) \circ PL \circ \bar{s}$. Now by the commutativity of diagram (4.5), $P(\mu_{i-1}) \circ PR$ and $P(\mu_{i-1}) \circ PL$ factor as $Pg(\mu_{i-1}) \circ P\pi \circ PR_X$ and $Pg(\mu_{i-1}) \circ P\pi \circ PL_X$ respectively. But since $P\pi \circ R_X$ and $P\pi \circ L_X$ are homotopic so, therefore, are $\mu_i \circ R$ and $\mu_i \circ L$.

This completes the inductive step in the proof of Theorem 4.1.

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