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A local and global splitting result for real Kähler Euclidean submanifolds

By

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Abstract. We show that if a real Kähler Euclidean submanifold is as far as possible of being minimal, then it should split locally as a product of hypersurfaces almost everywhere, possibly in lower codimension. In addition, if the manifold is complete, simply connected and has constant nullity, it should split globally as a product of surfaces in $\mathbb{R}^3$ and an Euclidean factor. Several applications are also given.

The series of papers [8], [9], [11] and [12] was devoted to the study of the relative nullity distribution Δ of an isometric immersion $f\colon M^m \to \mathbb{R}^{m+p}$, that is, $\Delta = \operatorname{Ker}\alpha$ is the nullity space of the second fundamental form α of f, when the m-dimensional (connected) Riemannian manifold M^m is assumed to have nonpositive sectional curvature. Since, along any open subset where $\nu = \dim \Delta$ is constant, the relative nullity is a smooth integrable distribution with totally geodesic leaves in both M^m and $\mathbb{R}^{m+p}$, the positiveness of ν imposes strong restrictions on the submanifold structure.

In [10] we gave analogous results to the main ones in [8] and [11] when f is a *real Kähler Euclidean submanifold*, that is, $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ is a (real) isometric immersion of a Kähler manifold, but with the reversed sign on weaker curvatures, namely, when M^{2n} has either nonnegative Ricci curvature or nonnegative holomorphic sectional curvature. It was shown that, with either curvature assumption, $\nu \geqq 2n - 2p$ and, if equality holds everywhere, then f should split locally as a product of p real Kähler hypersurfaces almost everywhere, the splitting being global when M^{2n} is complete. The purpose of this note is to give a general theorem for real Kähler Euclidean submanifolds that, in particular, allows us to further extend these results.

Recall that such a real Kähler Euclidean submanifold f is said to be *pluriharmonic* when every holomorphic curve in M^{2n} is mapped by f onto a minimal surface in $\mathbb{R}^{2n+p}$. It is

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easy to see that f is pluriharmonic if and only if the J-invariant subspace $\Delta_J(x)$ of T_xM given by

$$\Delta_J(x) = \{X \in T_xM : \alpha(X, JY) = \alpha(JX, Y), \quad \forall\, Y \in T_xM\}$$

satisfies that $\Delta_J(x) = T_xM$ for all $x \in M^{2n}$. Due to this property, we define the *index of pluriharmonic nullity of f at x* by

$$\nu_J(x) = \dim_{\mathbb{C}} \Delta_J(x).$$

It is an interesting fact that f is pluriharmonic, i.e., $\nu_J \equiv n$, if and only if it is minimal; cf. [7] and Remark 8 in [10]. On the other side, our aim here is to show that, when f is as far as possible of being pluriharmonic, it should split locally as a product of $n - \mu/2$ hypersurfaces almost everywhere, possibly in lower codimension. Here, μ stands for *index of nullity* of M^{2n}, that is, the (even) dimension of the nullity space of its curvature tensor R: $\mu = \dim\{X \in TM : R(X, \cdot\,) = 0\}$. More precisely, we have:

Theorem 1. *Let f: $M^{2n} \to \mathbb{R}^{2n+p}$ be any real Kähler Euclidean submanifold. Then, $\nu_J \geqq n - p$. Assume further that equality holds everywhere and that M^{2n} is nowhere flat. Then, along each connected component W_λ of an open dense subset $W \subset M^{2n}$, μ is constant and there are $r = n - \mu/2 \geqq 1$ nowhere flat real Kähler Euclidean hypersurfaces f_i: $M_i^{2n_i} \to \mathbb{R}^{2n_i+1}$, $1 \leqq i \leqq r$, such that W_λ is the Riemannian product*

$$W_\lambda = M_1^{2n_1} \times \cdots \times M_r^{2n_r}.$$

In addition, there is an open subset $V \subset \mathbb{R}^{2n+r}$ with $(f_1 \times \cdots \times f_r)(W_\lambda) \subset V$, and an isometric immersion h: $V \subset \mathbb{R}^{2n+r} \to \mathbb{R}^{2n+p}$ such that $f|_{W_\lambda} = h \circ (f_1 \times \cdots \times f_r)$.

In the above situation, we say that $\hat{f} = f_1 \times \cdots \times f_r : W_\lambda \to \mathbb{R}^{2n+r}$ *splits as a product of r hypersurfaces*. It is easy to see that any nowhere flat real Kähler Euclidean hypersurface $M^{2m} \subset \mathbb{R}^{2m+1}$ must have $\nu \equiv \mu \equiv 2m - 2$. Hence, they are locally classified by means of the Gauss parametrization in terms of pseudoholomorphic surfaces in the sphere; cf. [1], [3], [4], [11] and [12]. It turns out that the only complete real Kähler Euclidean hypersurfaces are the products of a complete orientable surface in $\mathbb{R}^3$ and an Euclidean factor ([13]).

As an easy consequence of Theorem 1, we can now extend the main results in [10]:

Corollary 2. *Let f: $M^{2n} \to \mathbb{R}^{2n+p}$ be an isometric immersion of a Kähler manifold with either nonnegative Ricci curvature or nonnegative holomorphic sectional curvature. Then, the index of relative nullity ν of f satisfies that $\nu \geqq 2n - 2p$. Moreover:*

i) *If $\nu \equiv 2n - 2p$, then there is an open dense subset $W \subset M^{2n}$ such that $f|_W$ splits locally as a product of p nowhere flat real Kähler Euclidean hypersurfaces with nonnegative Ricci curvature;*

ii) *If $\nu \equiv 2n - 2p + 1$, then there is an open dense subset $W \subset M^{2n}$ such that $f|_W = h \circ \hat{f}$, where $\hat{f}$: $M^{2n} \to \mathbb{R}^{2n+p-1}$ splits locally as a product of $p - 1$ nowhere flat real Kähler Euclidean hypersurfaces with nonnegative Ricci curvature, and h is an isometric immersion h: $V \subset \mathbb{R}^{2n+p-1} \to \mathbb{R}^{2n+p}$ with $\hat{f}(W) \subset V$.*

Part ii) is precisely the positive answer to the question posed in [10], that is, the analogous result to the main one in [12] holds for real Kälher Euclidean submanifolds but with the *reversed* sign on weaker curvatures. We point out that, since the flat immersion h in part ii) has relative nullity of codimension one, we conclude that in either part the submanifold can be locally described using the Gauss parametrization for each factor, thus given a complete local representation.

As a simple application, we obtain the following for submanifolds of the cylinder over a sphere $\mathbb{S}^m_c$ of constant sectional curvature c, that strongly generalizes Corollary 5 in [8] for $c > 0$ since no assumptions on curvatures are needed:

Corollary 3. *Let $M^{2n} \subset \mathbb{S}^m_c \times \mathbb{R}^k$ be a real Kähler submanifold, with $6n \geqq 2m+3k+2$. Then, $k = 2s$ is even, $m = 3(n-s)-1$ and M^{2n} is an open subset of $\mathbb{S}^2_{c_1} \times \cdots \times \mathbb{S}^2_{c_{n-s}} \times \mathbb{C}^s$, where $1/c = 1/c_1 + \cdots + 1/c_{n-s}$. In particular, the only Kälher submanifolds $M^{2n} \subset \mathbb{S}^{3n-1}_c$ are the (open subsets of) products of umbilical 2-spheres, $\mathbb{S}^2_{c_1} \times \cdots \times \mathbb{S}^2_{c_n} \subset \mathbb{S}^{3n-1}_c \subset \mathbb{R}^{3n}$.*

The presence of a positive definite shape operator clearly implies that $\nu_J = 0$. So:

Corollary 4. *Let $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ be an analytic isometric immersion of a compact Kähler manifold with $p \leqq n$. Then, $p = n$ and f splits globally as a product of n compact surfaces in $\mathbb{R}^3$.*

Since the shape operator in the direction of the mean curvature vector of any Euclidean submanifold with positive Ricci curvature is positive definite, we also obtain:

Corollary 5. *Let $N \subset \mathbb{R}^{2n+p}$ be an immersed Euclidean submanifold with positive Ricci curvature, with $p \leqq n$. Then, N has no immersed Kähler submanifold $M^{2n} \subset N$, unless $p = n$ and $M^{2n} \subset N \subset \mathbb{R}^{3n}$ splits as a product of n surfaces in $\mathbb{R}^3$.*

The open dense subset W in Theorem 1 is necessary due to the fact that the nullity μ of the manifold does not have to be constant, and because of the "jumping" of the relative nullity between the hypersurface factors, phenomena that indeed occur. When these are controlled, we get global consequences. To state them, we need a slightly weaker notion of compositions than the one considered in Theorem 1 (see [5]):

Definition. Given an isometric immersion $g\colon M^m \to \mathbb{R}^{m+r}$ we say that an isometric immersion $f\colon M^m \to \mathbb{R}^{m+p}$, $p \geqq r$, is a *composition* of g when there is an isometric embedding $g'\colon M^m \hookrightarrow N_0^{m+r}$ into a flat manifold N_0^{m+r}, a local isometry $j\colon N_0^{m+r} \to \mathbb{R}^{m+r}$ satisfying $g = j \circ g'$ and an isometric immersion $h\colon N_0^{m+r} \to \mathbb{R}^{m+p}$ such that $f = h \circ g'$.

Observe that, along any open subset $U \subset M^m$ where g is an embedding, there exists an isometric immersion $h\colon V \subset \mathbb{R}^{m+r} \to \mathbb{R}^{m+p}$ of a tubular neighborhood V of $g(U)$ such that $f = h \circ g$. In particular, this holds globally if g itself is an embedding.

Theorem 6. *Let* $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ *be a complete simply connected nonflat real Kähler Euclidean submanifold. Assume that* $\nu_J \equiv n - p$ *and that* μ *is constant. Then, there are* $r = n - \mu/2 \geqq 1$ *complete orientable nowhere flat surfaces* $f_i : M_i^2 \to \mathbb{R}^3$ *such that*

$$M^{2n} = M_1^2 \times \cdots \times M_r^2 \times \mathbb{C}^{n-r},$$

and f *is a composition of the product immersion* $f_1 \times \cdots \times f_r \times I$*, where* $I\colon \mathbb{C}^{n-r} \to \mathbb{R}^{2n-2r}$ *is the identity map.*

In particular, if M^{2n} is compact, then each $M_i^2 \subset \mathbb{R}^3$ is a strictly convex surface, hence embedded. So in this case we conclude that $r = n = p$ and $f = f_1 \times \cdots \times f_n$ splits globally as a product of n surfaces in $\mathbb{R}^3$. We can also extend the global results in [10]:

Corollary 7. *Let* $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ *be an isometric immersion of a complete simply connected Kähler manifold with either nonnegative Ricci curvature or nonnegative holomorphic sectional curvature. Then* $\nu \geqq 2n - 2p$*. Moreover:*

i) *It holds that* $\nu \equiv 2n - 2p$ *if and only if we have the global splittings*

$$M^{2n} = M_1^2 \times \ldots \times M_p^2 \times \mathbb{C}^{n-p} \quad \text{and} \quad f = f_1 \times \cdots \times f_p \times I,$$

where $f_i : M_i^2 \to \mathbb{R}^3$, $1 \leqq i \leqq p$*, is a complete oriented surface of positive curvature;*

ii) *It holds that* $\nu \equiv 2n - 2p + 1$ *if and only if we have the global splittings*

$$M^{2n} = M_1^2 \times \ldots \times M_{p-1}^2 \times \mathbb{C}^{n-p+1} \quad \text{and} \quad f = h \circ (f_1 \times \cdots \times f_{p-1} \times I),$$

where $f_i : M_i^2 \to \mathbb{R}^3$, $1 \leqq i \leqq p-1$*, is a complete oriented surface of positive curvature, and* h *is an isometric immersion* $h\colon V \subset \mathbb{R}^{2n+p-1} \to \mathbb{R}^{2n+p}$.

Further applications of our main results here will be given in [14].

The proofs. Let us first fix some notations. From now on, $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ will be an isometric immersion of the Kähler manifold M^{2n} into the Euclidean space. Fix a point $x \in M^{2n}$. The second fundamental form $\alpha = \alpha(x)$ of f at x is the symmetric bilinear map

$$\alpha : T_xM \times T_xM \to T_x^{\perp}M = N,$$

where T_xM is the real tangent space of M^{2n}, and $(T_x^{\perp}M, \langle\, ,\rangle)$ the normal space of f at x. Extend α bilinearly over $\mathbb{C}$, and still denote it by $\alpha : (T_xM)\otimes\mathbb{C} \times (T_xM)\otimes\mathbb{C} \to N\otimes\mathbb{C}$. Let V be the space of type (1, 0) tangent vectors at x, that is, V is the complex subspace of $(T_xM)\otimes\mathbb{C}$ given by $V = \{v - iJv : v \in T_xM\}$. Thus, $(T_xM)\otimes\mathbb{C} \cong V \oplus \overline{V}$. Write

$$H = \alpha|_{V\times\overline{V}} \quad \text{and} \quad S = \alpha|_{V\times V}$$

for the (1, 1) and (2, 0) parts of α, respectively. Then $S : V \times V \to N\otimes\mathbb{C}$ is a symmetric complex bilinear map, while $H : V \times \overline{V} \to N\otimes\mathbb{C}$ is Hermitian, that is,

$$H(Y, \overline{X}) = \overline{H(X, \overline{Y})}\,, \quad \forall\, X, Y \in V.$$

For the sake of simplicity, we write S_{XY} for $S(X,Y)$ and $H_{X\overline{Y}}$ for $H(X,\overline{Y})$. Let us also extend the inner product $\langle , \rangle$ on N bilinearly over $\mathbb{C}$ to $N \otimes \mathbb{C}$, and still denote it by $\langle , \rangle$. We have the following key result from [10]:

Lemma 8. *Let $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ be a real Kähler Euclidean submanifold, $p \geqq 1$. Take $x \in M^{2n}$, and let V, N, H, S be as above. Then, $\nu_J(x) \geqq n-p$. Moreover, if $\nu_J(x) = n-p$, there exists a basis $\{e_1,\ldots,e_n\}$ of V such that $H_{e_i\overline{e_j}} = S_{e_ie_j} = 0$, if either $i \neq j$ or $i = j > p$. Moreover, for $1 \leqq i \leqq p$, $H_{e_i\overline{e_i}} \neq 0$, $S_{e_ie_i}$ is collinear to $w_i = H_{e_i\overline{e_i}}/|H_{e_i\overline{e_i}}|$, and $\{w_1,\ldots,w_p\}$ is an orthonormal basis of the normal space N at x.*

Assume from now on that $\nu_J(x) = n-p$, for all $x \in M^{2n}$. Our purpose first is to realize M^{2n} as an Euclidean submanifold in its optimal codimension by means of the Fundamental Theorem for submanifolds.

By Lemma 8, $\nu \geqq 2n-2p$ and there exists a tangent diagonalizing frame $\{e_1,\ldots,e_p\}$ of type $(1,0)$ vectors in $\Delta^{\perp}_{1,0} = V \cap (\Delta^{\perp} \otimes \mathbb{C})$ at each x. Note that this frame is unique up to permutations and scalings. In other words, the set $\{[e_1],\ldots,[e_p]\}$ is unique in the symmetric power $S^n(\mathbb{P}(\Delta^{\perp}_{1,0}))$ of the projectified holomorphic $\Delta^{\perp}_{1,0}$ bundle. Hence, since H is smooth, in a sufficiently small neighborhood U of x, we can take a smooth frame $\{e_1,\ldots,e_n\}$ such that it has the diagonalization property of Lemma 8 at each point in U. In particular, we also obtain from Lemma 8 the smooth orthonormal normal frame $\{w_1,\ldots,w_p\}$.

For each $1 \leqq i \leqq p$, consider the *shape tensor* A_{w_i} on M^{2n} defined by $\langle A_{w_i}X, Y\rangle = \langle \alpha_f(X,Y), w_i\rangle$, and set

$$V_i = \operatorname{Im} A_{w_i}. \tag{1}$$

By Lemma 8, $1 \leqq \dim V_i \leqq 2$ and

$$V_1 \oplus \cdots \oplus V_p = \Delta^{\perp}. \tag{2}$$

Let Γ be the nullity space of the curvature tensor R of M^{2n}, i.e., $\Gamma = \{X \in TM : R(X,\cdot) = 0\}$ and $\mu = \dim\Gamma$ the index of nullity of M^{2n}. We have that $2n-2r = \mu \geqq \nu \geqq 2n-2p$, where $2r = \operatorname{rank} R$. Changing the order of the base $\{e_1,\ldots,e_p\}$ if necessary, we thus can assume that

$$\dim V_j = 2, \quad \forall 1 \leqq j \leqq r, \quad \text{and} \quad \dim V_s = 1, \quad \forall\, r+1 \leqq s \leqq p. \tag{3}$$

Hence, $\nu = 2n-p-r$ and

$$V_1 \oplus \cdots \oplus V_r = \Gamma^{\perp}. \tag{4}$$

If μ is constant, which is always the case on each connected component of an open dense subset of M^{2n}, the decompositions (2) and (4) are smooth. Let $L \subset T^{\perp}M$ be the normal subbundle of rank r spanned by $\{w_1,\ldots,w_r\}$ and $\pi\colon T^{\perp}M \to L$ its orthogonal projection. Set

$$\hat{\nabla}^{\perp} = \pi \circ \nabla^{\perp}|_{TM\times L} : TM \times L \to L \quad \text{and} \quad \hat{\alpha} = \pi \circ \alpha\colon TM \times TM \to L.$$

Proposition 9. *If M^{2n} is simply connected and μ is constant, there is an isometric immersion $\hat{f}\colon M^{2n} \to \mathbb{R}^{2n+r}$ whose second fundamental form is $\hat{\alpha} = \tau \circ \alpha$, for some* *parallel bundle isometry* $\tau: L \to T^{\perp}_{\hat f}M$. *In particular, the nullities of* $\hat f$ *satisfy that* $\hat\mu \equiv \hat\nu \equiv 2\hat\nu_J \equiv 2n - 2r$.

Proof. The last assertion follows from the construction of $\hat\alpha$ and (4). For the first one, we apply the Fundamental Theorem of submanifolds to the pair $(\hat\alpha, \hat\nabla^{\perp})$. We have to show that it satisfies Gauss, Codazzi and Ricci equations for flat ambient space. It is clear that $\hat\alpha$ satisfies the Gauss equation since we took from α precisely the normal directions of rank one that does not contribute to the curvature.

Fix the following index conventions:

$$1 \leqq j, k \leqq r, \quad r+1 \leqq s \leqq p, \quad 1 \leqq i \leqq p.$$

Recall that the Codazzi equation for the shape operator A_{w_i} of f is

$$\nabla_X(A_{w_i}Y) - A_{w_i}\nabla_X Y - A_{\nabla^{\perp}_X w_i}Y = \nabla_Y(A_{w_i}X) - A_{w_i}\nabla_Y X - A_{\nabla^{\perp}_Y w_i}X. \tag{5}$$

Taking in (5) $X, Y \in V_i^{\perp} = \mathrm{Ker}\, A_{w_i}$ we easily obtain using (2) that

$$A_{w_j}(\psi_{ij}(X)Y - \psi_{ij}(Y)X) = 0, \ \ \forall\, X, Y \in V_i^{\perp},$$

where ψ_{ij} is defined by $\psi_{ij}(X) = \langle \nabla^{\perp}_X w_i, w_j\rangle$. Suppose that $\psi_{ij}|_{V_i^{\perp}} \neq 0$. Then, the above implies that $V_i^{\perp} \cap \mathrm{Ker}\,\psi_{ij} \subset \mathrm{Ker}\, A_{w_j} = V_j^{\perp}$. This is a contradiction to (2) since $\dim V_j = 2$. Therefore,

$$V_i^{\perp} = \mathrm{Ker}\, A_{w_i} \subset \mathrm{Ker}\,\psi_{ij}.$$

Since $\psi_{ij} = -\psi_{ji}$ if $i = k \leqq r$, we have that $TM = (V_j \cap V_k)^{\perp} = V_j^{\perp} + V_k^{\perp} \subset \mathrm{Ker}\,\psi_{jk}$. Hence,

$$\psi_{jk} = 0, \tag{6}$$

i.e., w_j is parallel in L, and, by dimension reasons,

$$\psi_{js} = 0, \quad \text{or} \quad \mathrm{Ker}\,\psi_{js} = \mathrm{Ker}\, A_{w_s}. \tag{7}$$

In particular, $A_{w_i}(\psi_{ij}(X)Y - \psi_{ij}(Y)X) = 0$, or, equivalently, $A_{\nabla^{\perp}_X w_j}Y = A_{\nabla^{\perp}_Y w_j}X$, for all $X, Y \in TM$. From (6) and (5) for $i = j$ we conclude that $(\hat\alpha, \hat\nabla^{\perp})$ verifies Codazzi equation.

The Ricci equation for f says that $[A_{w_j}, A_{w_k}] = \sum_s \psi_{js} \wedge \psi_{ks} = 0$, where the last equality follows from (7). Therefore, by (6) we conclude that $(\hat\alpha, \hat\nabla^{\perp})$ satisfies Ricci equation. □

Proof of Theorem 1. The estimate on ν_J follows from the first part of Lemma 8. Assume that $\nu_J \equiv n-p$ everywhere. Consider any simply connected open subset $U \subset M^{2n}$ where μ is constant, and let $\hat f$ be the isometric immersion given by Proposition 9 when applied to $f|_U$. Proposition 10 part ii) in [10] says that $\hat f$ splits locally as a product of r

nowhere flat real Kähler Euclidean hypersurfaces almost everywhere in U. Therefore, all we have to show is that $f|_U$ is a composition of $\hat{f}$.

With the notations of Lemma 8 and Proposition 9, let

$$T = \operatorname{span}\{(\tilde{\nabla}_X \xi)_{TM\oplus L}\} : X \in TM,\ \xi \in L^{\perp}\},$$

where $\tilde{\nabla}$ stands for the usual connection in $\mathbb{R}^{2n+p}$ and $(v)_{TM\oplus L}$ denotes the orthogonal projection of $v \in \mathbb{R}^{2n+p}$ onto $TM \oplus L$. By Proposition 8 in [5], all we have to show is that $T \cap L = 0$, since it is clear from Lemma 8, (2) and (4) that α decomposes regularly.

Indeed, if $\eta \in T$, there are $a_l \in \mathbb{R}$ such that $\eta = \sum_l a_l(\tilde{\nabla}_{X_l}\xi_l)_{TM\oplus L}$. Writing $\xi_l = \sum_{s=r+1}^{p} b_{sl} w_s$, we have that

$$\eta = \left(\sum_{s=r+1}^{p} \tilde{\nabla}_{Y_s} w_s \right)_{TM\oplus L}$$

where $Y_s = \sum_l a_l b_{ls} X_l$. If, in addition, $\eta \in L$, from (2) we get that $Y_s \in \operatorname{Ker} A_{w_s}$. But then we conclude from (7) that $\eta = 0$.

P r o o f o f C o r o l l a r y 2. It is easy to check that $\nu \geqq 2\nu_J$ under either curvature assumption (see the proof of Theorem 1 in [10]). Hence, $\nu_J = n - p$ everywhere, and $r = p$ or $r = p - 1$ in *i*) or *ii*), respectively. The result then follows from Theorem 1.

P r o o f o f C o r o l l a r y 3. Considering $\mathbb{S}^m_c \times \mathbb{R}^k \subset \mathbb{R}^{m+k+1}$, that has a positive semi-definite second fundamental form A, we have that $\nu_J \geqq n - (m + k + 1 - 2n) \geqq k/2$. If $\nu_J > k/2$, then there is $X \in \Delta_J$ such that $\langle AX, X\rangle > 0$. But then $\langle AJX, JX\rangle < 0$, a contradiction. So, $2\nu_J = k = 2s$, $m = 3(n - s) - 1$, and Δ_J is always parallel to the $\mathbb{R}^k$ factor. Thus, $M^{2n} \subset N^{2n-2s} \times \mathbb{C}^s$, for some $N^{2(n-s)} \subset \mathbb{S}^{3(n-s)-1}$ that is locally a product of $n - s$ surfaces in $\mathbb{R}^3$. This can only occurs if each surface is itself (an open subset of) a two dimensional sphere.

P r o o f o f T h e o r e m 6. We have that $\nu = 2n - p - r$ is also constant. Hence, by Proposition 9 we have the isometric immersion $\hat{f} : M^{2n} \to \mathbb{R}^{2n+r}$ with $\hat{\nu}_J = n - r$ and $\hat{\nu} = 2n - 2r$ everywhere. We claim that the relative nullity $\hat{\Delta}$ of $\hat{f}$ is, in fact, parallel in $\mathbb{R}^{2n+r}$.

Apply Theorem 1 to $\hat{f}$, and consider a connected component W_λ where $\hat{f}$ splits as a product of nowhere flat hypersurfaces. Take one of these hypersurfaces, say, $\hat{f}_1$, and assume it is not locally a product of a surface and an Euclidean factor. So, there is a vector T in the relative nullity $\hat{\Delta}_1 \subset \hat{\Delta}$ of $\hat{f}_1$ such that $C_T \neq 0$, where $C_T : \hat{\Delta}_1^{\perp} \to \hat{\Delta}_1^{\perp}$ is the splitting tensor of $\hat{\Delta}_1$, i.e., $C_T X = -(\nabla_X T)_{\hat{\Delta}_1^{\perp}}$. Observe that $\hat{\Delta}_1^{\perp}$ has dimension two and is J-invariant, since $\hat{\Delta}_1$ coincides with the nullity of the curvature tensor of $\hat{f}_1$. By the product structure of $\hat{f}$ in W_λ, the splitting tensor $\hat{C}_T : \hat{\Delta}^{\perp} \to \hat{\Delta}^{\perp}$ of $\hat{f}$ in the direction of T is zero except on $\hat{\Delta}_1^{\perp}$, where it equals C_T. On the other hand, since $\hat{\nu}$ is constant and M^{2n} is complete, the leaves of $\hat{\Delta}$ are complete. Since $\hat{C}_{JT} = J\hat{C}_T$, we easily check that there are $a, b \in R$ such that $\hat{C}_{aT+bJT} = (aI + bJ)\hat{C}_T$ has a nonzero real eigenvalue. But this is in contradiction with Lemma 1.8 in [6]. We conclude that each hypersurface factor in W_λ is locally a product of a surface and an Euclidean factor. Therefore, $\hat{C}_T = 0$ for any $T \in \hat{\Delta}$, that is, $M^{2n} = N^{2r} \times \mathbb{C}^{n-r}$ and $\hat{f} = \hat{f}' \times I$ split for some complete real Kähler submanifold $\hat{f}' : N^{2r} \to \mathbb{R}^{3r}$ and the claim is proved.

We end the proof applying Proposition 10 part iii) in [10] to $\hat{f}'$, since we then get that $\hat{f}'$ splits globally as a product of nowhere flat orientable surfaces in $\mathbb{R}^3$.

Proof of Corollary 7. Again, we have everywhere that $\nu_J = n - p$, and $r = p$ or $r = p - 1$ in i) or ii), respectively. Theorem 6 then gives the result, since $\hat{f}$ is an embedding because the surfaces are strictly convex.

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