

Singular genuine rigidity

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Definitions

For an *isometric immersion*

$$f: M^n \rightarrow \mathbb{R}^{n+p}$$

of an n -dimensional connected Riemannian manifold M^n with codimension p into flat Euclidean space, we denote its *second fundamental form* by

$$\alpha^f: TM \times TM \rightarrow T_f^\perp M, \quad \alpha^f(X, Y) := (\tilde{\nabla}_X Y)_{T_f^\perp M},$$

and its *normal connection* by

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Any Riemannian manifold M^n can be isometrically embedded in \mathbb{R}^{n+p} , for sufficiently large codimension p .

First key problem: (Only known for a few manifolds)

For a given Riemannian manifold M^n , what is (locally or globally) the lowest possible codimension $p = p(M^n)$ in Nash Theorem?

Uniqueness: rigidity

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(In particular, rigidity implies that $p = p(M^n)$).

If otherwise, we say that f is *(isometrically) deformable*.

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Describe (in some sense) all deformations of

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General deformation problem: (even more difficult...)

Describe all isometric immersions of M^n into \mathbb{R}^{n+q} for a given codimension q ... for 'small' codimension q ($q < n$?).

Local rigidity results: Hypersurfaces ($p = 1$)

The *type number* of $f: M^n \rightarrow \mathbb{R}^{n+1}$ at $x \in M^n$ is the integer

$$\tau(x) = \text{rank } \alpha^f(x).$$

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Any immersed hypersurface with $\tau \geq 3$ is rigid.

FACT. By the Gauss equation,

$$\tau(x) \leq 1 \iff K_M(x) = 0.$$

Using the Gauss Parametrization (Sbrana), it is easy to see that the set of (local) flat hypersurfaces of \mathbb{R}^{n+1} can be naturally identified with $C^\infty(\mathbb{R}, \mathbb{R}^{n+1})$: the smooth curves in \mathbb{R}^{n+1} .

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They are divided in 4 types of $(n - 2)$ -ruled hypersurfaces:

- **surface-like** (products): deform as surfaces in \mathbb{R}^3 or \mathbb{S}^3 ;
- **(n-1)-ruled**: moduli space of deformations = $C^\infty(\mathbb{R}, \mathbb{R})$;
- **continuous type**: moduli space of deformations = \mathbb{R} ;
- **discrete type**: exactly one more deformation: moduli \mathbb{Z}_2 .

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“Generically”, any submanifold with $p \leq n$ and $n \geq 6$ is rigid.

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These are algebraic results, with little geometry...

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This kind of “dishonest” deformations must be discarded since we are reducing the codimension of the problem, and for that purpose we introduce the following general definition:

Isometric extensions

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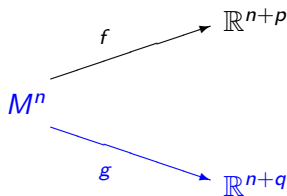
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$$M^n \xrightarrow{f} \mathbb{R}^{n+p}$$

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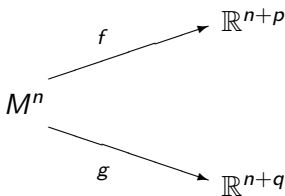


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We say that a pair of isometric immersions $f: M^n \rightarrow \mathbb{R}^{n+p}$ and $g: M^n \rightarrow \mathbb{R}^{n+q}$ *extends isometrically* when there are:

- a Riemannian manifold N^{n+r} with $r \geq 1$,
- an isometric embedding $j: M^n \hookrightarrow N^{n+r}$,
- and isometric immersions $F: N^{n+r} \rightarrow \mathbb{R}^{n+p}$

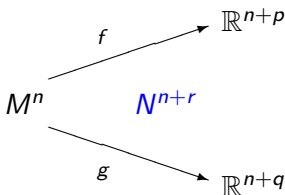


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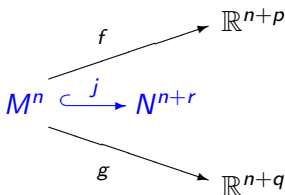


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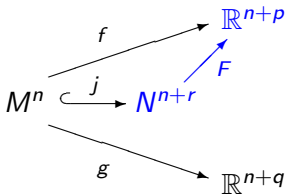


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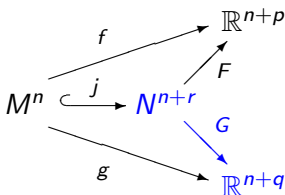


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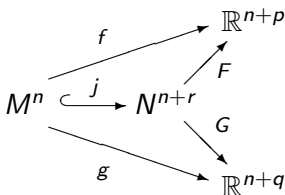
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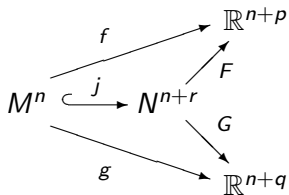
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such that $f = F \circ j$ and $g = G \circ j$.

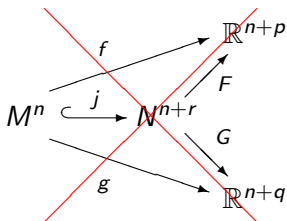


commutes!

Genuine deformations

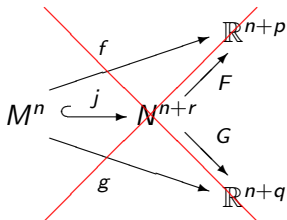


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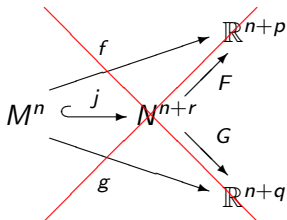


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Definition (Genuine deformation)

Given a submanifold $f: M^n \rightarrow \mathbb{R}^{n+p}$ we say that $g: M^n \rightarrow \mathbb{R}^{n+q}$ is a *genuine deformation* of f if there is no open subset $U \subset M^n$ along which the restrictions $f|_U$ and $g|_U$ extend isometrically.

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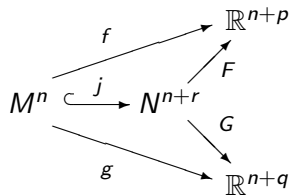
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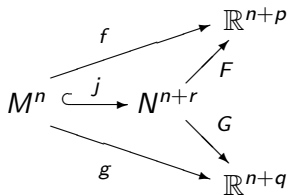
Remark

This is a symmetric concept: for pairs and sets.

Genuine rigidity

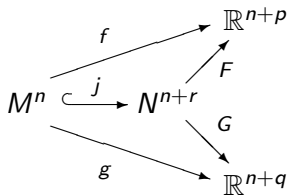


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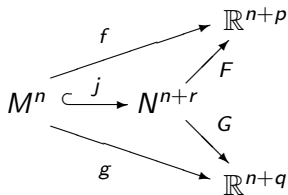
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Definition (Genuine rigidity)

Given $f: M^n \rightarrow \mathbb{R}^{n+p}$ and a positive integer q , we say that f is *genuinely rigid* in \mathbb{R}^{n+q} if for any $g: M^n \rightarrow \mathbb{R}^{n+q}$ there is an open dense subset $U \subset M^n$ such that $f|_U$ and $g|_U$ extend isometrically.

That is, g is nowhere a genuine deformation of f .

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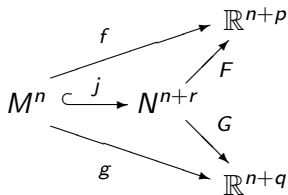
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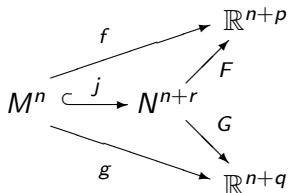
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Remark

- If $r = p = q$, we recover the concept of *isometric rigidity*.
- If $r = p < q$, we recover the concept of *compositions*.

The main result

Assume that the submanifolds $f: M^n \rightarrow \mathbb{R}^{n+p}$ and $g: M^n \rightarrow \mathbb{R}^{n+q}$ are *mutually D^d -ruled*, that is, D^d is a d -dimensional integrable distribution of M^n whose leaves are mapped by f and g onto open subsets of affine subspaces.

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Set

$$L_D = \text{span}\{\alpha^f(D, TM)\} \quad \text{and} \quad L'_D = \text{span}\{\alpha^g(D, TM)\}.$$

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Set

$$L_D = \text{span}\{\alpha^f(D, TM)\} \quad \text{and} \quad L'_D = \text{span}\{\alpha^g(D, TM)\}.$$

Then, by the Gauss equation, the map $\mathcal{T}_D: L_D \rightarrow L'_D$ given by

$$\mathcal{T}_D(\alpha^f(Y, X)) = \alpha^g(Y, X), \quad Y \in D^d, X \in TM,$$

is a well defined **bundle isometry**.

The main result: local in nature

Theorem (Dajczer & F—, 2004)

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Let $g: M^n \rightarrow \mathbb{R}^{n+q}$ be a genuine deformation of $f: M^n \rightarrow \mathbb{R}^{n+p}$, with $p + q < n$ and $\min\{p, q\} \leq 5$.

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Let $g: M^n \rightarrow \mathbb{R}^{n+q}$ be a genuine deformation of $f: M^n \rightarrow \mathbb{R}^{n+p}$, with $p + q < n$ and $\min\{p, q\} \leq 5$. Then, locally f and g are mutually D^d -ruled, with $d \geq n - p - q + 3 \dim L_D$.

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Remark

The above result generalizes and unifies several rigidity results and all the ones we know about compositions.

\exists genuine deformation \Rightarrow mutually ruled

Thus, for a pair of genuine deformations we have:

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Thus, for a pair of genuine deformations we have:

$$T_f^\perp M = L_D \oplus L_D^\perp$$

$$L_D := \text{span} \{ \alpha^f(D, TM) \}$$

$$T_g^\perp M = L'_D \oplus L'^\perp_D$$

$$L'_D := \text{span} \{ \alpha^g(D, TM) \}$$

∃ genuine deformation \Rightarrow mutually ruled

Thus, for a pair of genuine deformations we have:

$$T_f^\perp M = L_D \oplus L_D^\perp$$

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$$\begin{array}{c} \mathcal{T}_D \\ (\nabla^\perp)_{L_D} = (\nabla^\perp)_{L'_D} \\ (\alpha^f)_{L_D} = (\alpha^g)_{L'_D} \end{array}$$



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Moreover, the estimate $d \geq n - p - q + 3 \dim L_D$ is sharp.

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But: it is very hard to obtain global applications from this our result!

Genuine conformal rigidity

Analogously, given a manifold M^n with a *conformal* structure, we may want to understand its *conformal* immersions $f: M^n \rightarrow \mathbb{R}^{n+p}$.

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In 2010, together with Ruy Tojeiro (UFSCar), we extended several of these conformal results by means of a genuine *conformal* rigidity framework, giving them a unified character. Actually, we obtained precisely the conformal version of the genuine rigidity theorem above, with its corresponding corollaries.

Global rigidity

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Theorem (Sacksteder, 1962)

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Theorem (translating Sacksteder...)

Any compact Euclidean hypersurface is genuinely rigid.

Global rigidity

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Any pair of isometric immersions of a compact Riemannian manifold M^n into \mathbb{R}^{n+2} , $n \geq 5$, must extend isometrically along each connected component of an open dense subset of M^n

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Why “**maybe singularly**”?? The introduction of singularities in this result was shown to be a [necessary condition](#) in a recent local classification we made with Guilherme Freitas (2017).

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Singularities are not only **necessary**, but also **natural** for the problem, deeply **simplify the theory**, and are **easy to deal with** for ruled submanifolds, as we do when we classify flat and ruled surfaces in \mathbb{R}^3 .

Introducing singularities

Our purpose in this joint work with Felipe Guimarães (IMPA) was then to introduce singularities in the genuine rigidity theory, mainly with the double purpose of obtaining **new global** rigidity results and to **unify** the known ones.

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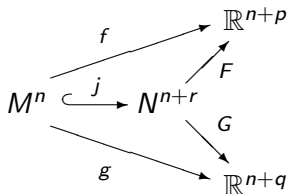
Yet, we got some nice surprises for the local theory...

Definitions

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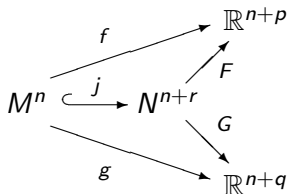
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We say that f and g *singularly extend isometrically* if the diagram commutes, where j is an isometric embedding as before, and F and G are isometric immersions, except maybe on M^n .

Definitions

Analogously, we say that g is a *strong genuine deformation* of f if there is no open subset where they singularly extend isometrically.

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With these definitions we show:

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Let $f: M^n \rightarrow \mathbb{R}^{n+p}$ and $\hat{f}: M^n \rightarrow \mathbb{R}^{n+q}$ be isometric immersions of a compact Riemannian manifold with $p+q < n$ and $\min\{p, q\} \leq 5$.

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The proof actually works untouched for compact (or complete and bounded) submanifolds in hyperbolic space, and with a little bit of extra work we obtained a similar result for **complete** submanifolds in the sphere.

Global consequences

For $p + q \leq 4$ this unifies Sacksteder and Dajczer-Gromoll theorems above, and gives a new global result for $\{p, q\} = \{1, 3\}$:

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Corollary

Any compact (or complete and bounded) submanifold M^n of \mathbb{R}^{n+p} is singularly genuinely rigid in \mathbb{R}^{n+q} for $q \leq 4 - p$.

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Corollary

Let M^n be a compact manifold whose k -th Pontrjagin class satisfies that $[p_k] \neq 0$ for some $k > \frac{3}{4}(p+q)$. Then, any analytic immersion $f : M^n \rightarrow \mathbb{R}^{n+p}$ (with the induced metric) is singularly genuinely rigid in \mathbb{R}^{n+q} in the C^∞ -category.

Local analysis: the form ϕ_T

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$$\tau : L^\ell \subset T_f^\perp M \rightarrow \hat{L}^\ell \subset T_{\hat{f}}^\perp M$$

be a vector bundle isometry that preserves the second fundamental forms and the normal connections restricted to the rank ℓ vector normal subbundles L and \hat{L} . Extend τ as Id on TM : $\bar{\tau} = Id \oplus \tau$.

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Let $\phi_\tau : TM \times (TM \oplus L) \rightarrow L^\perp \times \hat{L}^\perp$ be the bilinear form

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where $\tilde{\nabla}$ stands for the connection in Euclidean space and $L^\perp \times \hat{L}^\perp$ is endowed with the semi-Riemannian metric $\langle, \rangle = \langle, \rangle|_{L^\perp} - \langle, \rangle|_{\hat{L}^\perp}$.

The form ϕ_T and the main local result

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Then, $D \subset TM$ and locally f and \hat{f} are *mutually $\mathcal{O}(D)$ -ruled*.

The key advantage here over other local rigidity results: it deals with [easily to construct null sets](#) instead of nullity distributions.

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Recall that $Y \in TM$ is a (left) *regular element* of ϕ_τ if

$$\text{rank}(\phi_\tau^Y) = i(\phi_\tau) := \max\{\text{rank}(\phi_\tau^X) : X \in TM\},$$

where $\phi_\tau^X = \phi_\tau(X, \cdot)$. Denote by $RE(\phi_\tau) \subset TM$ the open dense subset of regular elements of ϕ_τ , and $K_Y = \ker(\phi_\tau^Y)$ for $Y \in RE(\phi_\tau)$.

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Corollary

If \hat{f} is a strongly genuine deformation of f , then locally they are *mutually $\mathcal{O}(D_Y^d)$ -ruled* $\forall Y \in RE(\phi_\tau)$, where $D_Y^d = \ker(\phi_\tau^Y) \subset TM$ and $d = n + \ell - i(\phi_\tau) \geq n - p - q + 3\ell$.

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Of course, we also obtain local rigidity under any circumstance that excludes ruled submanifolds. For example:

Corollary

Any immersed submanifold M^n of \mathbb{R}^{n+p} with $Ric > 0$ is singularly genuinely rigid in \mathbb{R}^{n+q} for any $q < n - p$.

All the computations

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Proof:

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Proof:..... Just kidding!! :oP ■

Thanks!!!!

ℒ.

There's nothing here.

Really!

Really really!

Really really reaaaally!

:op

