

# Singular genuine rigidity

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# Definitions

For an *isometric immersion*

$$f: M^n \rightarrow \mathbb{R}^{n+p}$$

of an  $n$ -dimensional connected Riemannian manifold  $M^n$  with codimension  $p$  into flat Euclidean space, we denote its *second fundamental form* by

$$\alpha^f: TM \times TM \rightarrow T_f^\perp M, \quad \alpha^f(X, Y) := (\tilde{\nabla}_X Y)_{T_f^\perp M},$$

and its *normal connection* by

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## Theorem (Nash, 1956)

*Any Riemannian manifold  $M^n$  can be isometrically embedded in  $\mathbb{R}^{n+p}$ , for sufficiently large codimension  $p$ .*

**First key problem:** (Only known for a few manifolds)

For a given Riemannian manifold  $M^n$ , what is (locally or globally) the lowest possible codimension  $p = p(M^n)$  in Nash Theorem?

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(In particular, rigidity implies that  $p = p(M^n)$ ).

If otherwise, we say that  $f$  is *(isometrically) deformable*.

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Describe (in some sense) all deformations of

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General deformation problem: (even more difficult...)

Describe all isometric immersions of  $M^n$  into  $\mathbb{R}^{n+q}$  for a given codimension  $q$ ... for 'small' codimension  $q$  ( $q < n$  ?).

# Local rigidity results: Hypersurfaces ( $p = 1$ )

The *type number* of  $f: M^n \rightarrow \mathbb{R}^{n+1}$  at  $x \in M^n$  is the integer

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**Theorem (Beez, 1876 & Killing, 1885)**

*Any immersed hypersurface with  $\tau \geq 3$  is rigid.*

FACT. By the Gauss equation,

$$\tau(x) \leq 1 \iff K_M(x) = 0.$$

Using the Gauss Parametrization (Sbrana), it is easy to see that the set of (local) flat hypersurfaces of  $\mathbb{R}^{n+1}$  can be naturally identified with  $C^\infty(\mathbb{R}, \mathbb{R}^{n+1})$ : the smooth curves in  $\mathbb{R}^{n+1}$ .

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They are divided in 4 types of  $(n - 2)$ -ruled hypersurfaces:

- **surface-like** (products): deform as surfaces in  $\mathbb{R}^3$  or  $\mathbb{S}^3$ ;
- **(n-1)-ruled**: moduli space of deformations =  $C^\infty(\mathbb{R}, \mathbb{R})$ ;
- **continuous type**: moduli space of deformations =  $\mathbb{R}$ ;
- **discrete type**: exactly one more deformation: moduli  $\mathbb{Z}_2$ .

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*“Generically”, any submanifold with  $p \leq n$  and  $n \geq 6$  is rigid.*

These are algebraic results, with little geometry...

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Let  $N^{n+1} \subset \mathbb{R}^{n+2}$  be a Sbrana-Cartan (i.e., deformable and classified) hypersurface and

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This kind of “dishonest” deformations must be discarded since we are reducing the codimension of the problem, and for that purpose we introduce the following general definition:

# Isometric extensions

## Definition (Isometric extensions)

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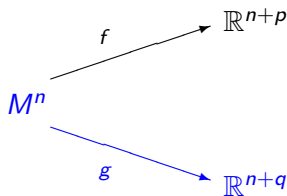
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$$M^n \xrightarrow{f} \mathbb{R}^{n+p}$$

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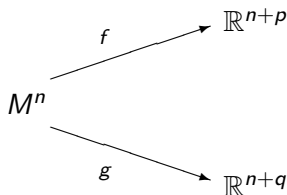


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- a Riemannian manifold  $N^{n+r}$  with  $r \geq 1$ ,
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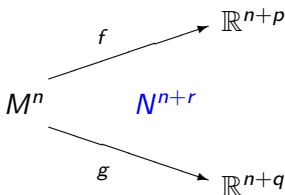


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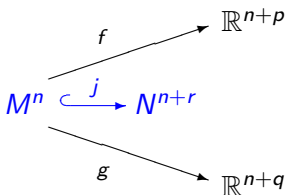


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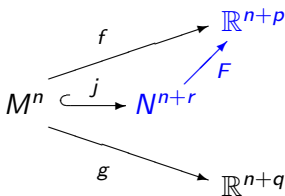


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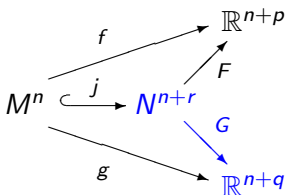


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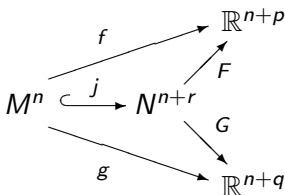
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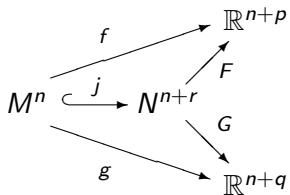
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such that  $f = F \circ j$  and  $g = G \circ j$ .

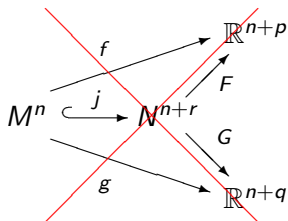


commutes!

# Genuine deformations

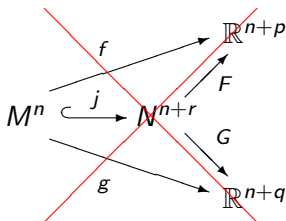


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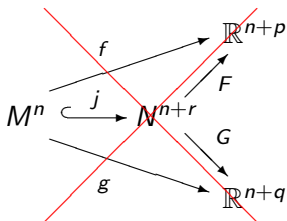
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## Definition (Genuine deformation)

Given a submanifold  $f: M^n \rightarrow \mathbb{R}^{n+p}$  we say that  $g: M^n \rightarrow \mathbb{R}^{n+q}$  is a *genuine deformation* of  $f$  there is no open subset  $U \subset M^n$  along which the restrictions  $f|_U$  and  $g|_U$  extend isometrically.



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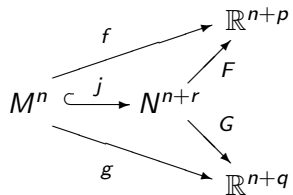
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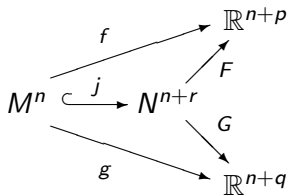
## Remark

This is a symmetric concept: for pairs and sets.

# Genuine rigidity

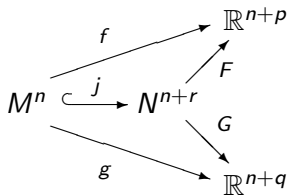


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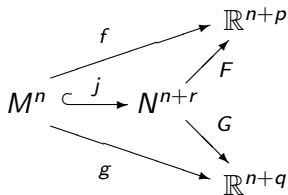
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## Definition (Genuine rigidity)

Given  $f: M^n \rightarrow \mathbb{R}^{n+p}$  and a positive integer  $q$ , we say that  $f$  is *genuinely rigid* in  $\mathbb{R}^{n+q}$  if for any  $g: M^n \rightarrow \mathbb{R}^{n+q}$  there is an open dense subset  $U \subset M^n$  such that  $f|_U$  and  $g|_U$  extend isometrically.

That is,  $g$  is nowhere a genuine deformation of  $f$ .

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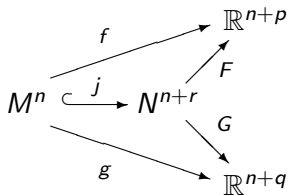
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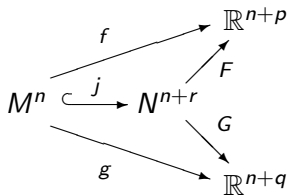
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## Remark

- If  $r = p = q$ , we recover the concept of *isometric rigidity*.
- If  $r = p < q$ , we recover the concept of *compositions*.

# The main result

Assume that the submanifolds  $f: M^n \rightarrow \mathbb{R}^{n+p}$  and  $g: M^n \rightarrow \mathbb{R}^{n+q}$  are *mutually  $D^d$ -ruled*, that is,  $D^d$  is a  $d$ -dimensional integrable distribution of  $M^n$  whose leaves are mapped by  $f$  and  $g$  onto open subsets of affine subspaces.



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$$L_D = \text{span}\{\alpha^f(D, TM)\} \quad \text{and} \quad L'_D = \text{span}\{\alpha^g(D, TM)\}.$$

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Set

$$L_D = \text{span}\{\alpha^f(D, TM)\} \quad \text{and} \quad L'_D = \text{span}\{\alpha^g(D, TM)\}.$$

Then, by the Gauss equation, the map  $\mathcal{T}_D: L_D \rightarrow L'_D$  given by

$$\mathcal{T}_D(\alpha^f(Y, X)) = \alpha^g(Y, X), \quad Y \in D^d, X \in TM,$$

is a well defined **bundle isometry**.

# The main result: local in nature

Theorem (Dajczer & F—, 2004)

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Let  $g: M^n \rightarrow \mathbb{R}^{n+q}$  be a genuine deformation of  $f: M^n \rightarrow \mathbb{R}^{n+p}$ , with  $p + q < n$  and  $\min\{p, q\} \leq 5$ .

# The main result: local in nature

Theorem (Dajczer & F—, 2004)

Let  $g: M^n \rightarrow \mathbb{R}^{n+q}$  be a genuine deformation of  $f: M^n \rightarrow \mathbb{R}^{n+p}$ , with  $p + q < n$  and  $\min\{p, q\} \leq 5$ . Then, locally  $f$  and  $g$  are mutually  $D^d$ -ruled, with  $d \geq n - p - q + 3 \dim L_D$ .

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## Remark

The above result generalizes and unifies several rigidity results and all the ones we know about compositions.

# $\exists$ genuine deformation $\Rightarrow$ mutually ruled

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Moreover, the estimate  $d \geq n - p - q + 3 \dim L_D$  is sharp.

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**But:** it is very hard to obtain global applications from this our result!

# Genuine conformal rigidity

Analogously, given a manifold  $M^n$  with a *conformal* structure, we may want to understand its *conformal* immersions  $f: M^n \rightarrow \mathbb{R}^{n+p}$ .

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In 2010, together with Ruy Tojeiro (UFSCar), we extended several of these conformal results by means of a genuine *conformal* rigidity framework, giving them a unified character. Actually, we obtained precisely the conformal version of the genuine rigidity theorem above, with its corresponding corollaries.

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Theorem (translating Sacksteder...)

*Any compact Euclidean hypersurface is genuinely rigid.*

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Why “**maybe singularly**”?? The introduction of singularities in this result was shown to be a [necessary condition](#) in a recent local classification we made with Guilherme Freitas (2017).



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*We should not try to avoid singularities  
when working with ruled extensions!*

Singularities are not only **necessary**, but also **natural** for the problem, deeply **simplify the theory**, and are **easy to deal with** for ruled submanifolds, as we do when we classify flat and ruled surfaces in  $\mathbb{R}^3$ .

# Introducing singularities

Our purpose in this joint work with Felipe Guimarães (IMPA) was then to introduce singularities in the genuine rigidity theory, mainly with the double purpose of obtaining **new global** rigidity results and to **unify** the known ones.

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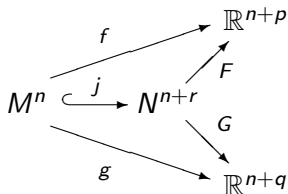
Yet, we got some nice surprises for the local theory...

# Definitions

All the definitions for introducing singularities are the natural ones, adapting the regular case:

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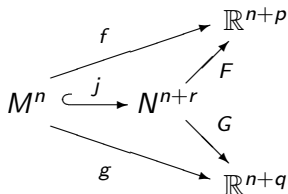
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We say that  $f$  and  $g$  *singularly extend isometrically* if the diagram commutes, where  $j$  is an isometric embedding as before, and  $F$  and  $G$  are isometric immersions, except maybe on  $M^n$ .



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With these definitions we show:

# Global statements

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The proof actually works untouched for compact (or complete and bounded) submanifolds in hyperbolic space, and with a little bit of extra work we obtained a similar result for **complete** submanifolds in the sphere.



# Global consequences

For  $p + q \leq 4$  this unifies Sacksteder and Dajczer-Gromoll theorems above, and states that the only way to isometrically immerse a compact Euclidean hypersurface in codimension 3 is as a composition:

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## Corollary

*Any compact (or complete and bounded) submanifold  $M^n$  of  $\mathbb{R}^{n+p}$  is singularly genuinely rigid in  $\mathbb{R}^{n+q}$  for  $q \leq 4 - p$ .*

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## Corollary

*Let  $M^n$  be a compact manifold whose  $k$ -th Pontrjagin class satisfies that  $[p_k] \neq 0$  for some  $k > \frac{3}{4}(p + q - 3)$ . Then, any analytic immersion  $f : M^n \rightarrow \mathbb{R}^{n+p}$  (with the induced metric) is singularly genuinely rigid in  $\mathbb{R}^{n+q}$  in the  $C^\infty$ -category.*

# Local analysis: the form $\phi_T$

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Let

$$\tau : L^\ell \subset T_f^\perp M \rightarrow \hat{L}^\ell \subset T_{\hat{f}}^\perp M$$

be a vector bundle isometry that preserves the second fundamental forms and the normal connections restricted to the rank  $\ell$  vector normal subbundles  $L$  and  $\hat{L}$ . Extend  $\tau$  as  $Id$  on  $TM$ :  $\bar{\tau} = Id \oplus \tau$ .

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Let  $\phi_\tau : TM \times (TM \oplus L) \rightarrow L^\perp \times \hat{L}^\perp$  be the bilinear form

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where  $\tilde{\nabla}$  stands for the connection in Euclidean space and  $L^\perp \times \hat{L}^\perp$  is endowed with the semi-Riemannian metric  $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^\perp} - \langle \cdot, \cdot \rangle_{\hat{L}^\perp}$ .



# The form $\phi_T$ and the main local result

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The key advantage here over other local rigidity results: it deals with [easily to construct null sets](#) instead of nullity distributions.



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where  $\phi_\tau^X = \phi_\tau(X, \cdot)$ . Denote by  $RE(\phi_\tau) \subset TM$  the open dense subset of regular elements of  $\phi_\tau$ , and  $K_Y = \ker(\phi_\tau^Y)$  for  $Y \in RE(\phi_\tau)$ .

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Recall that  $Y \in TM$  is a (left) *regular element* of  $\phi_\tau$  if

$$\text{rank}(\phi_\tau^Y) = i(\phi_\tau) := \max\{\text{rank}(\phi_\tau^X) : X \in TM\},$$

where  $\phi_\tau^X = \phi_\tau(X, \cdot)$ . Denote by  $RE(\phi_\tau) \subset TM$  the open dense subset of regular elements of  $\phi_\tau$ , and  $K_Y = \ker(\phi_\tau^Y)$  for  $Y \in RE(\phi_\tau)$ .

## Corollary

If  $\hat{f}$  is a strongly genuine deformation of  $f$ , then locally they are *mutually  $\mathcal{O}(D_Y^d)$ -ruled*  $\forall Y \in RE(\phi_\tau)$ , where  $D_Y^d = \ker(\phi_\tau^Y) \subset TM$  and  $d = n + \ell - i(\phi_\tau) \geq n - p - q + 3\ell$ .

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Of course, we also obtain local rigidity under any circumstance that excludes ruled submanifolds. For example:

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## Corollary

*Any immersed submanifold  $M^n$  of  $\mathbb{R}^{n+p}$  with  $\text{Ric} > 0$  is singularly genuinely rigid in  $\mathbb{R}^{n+q}$  for any  $q < n - p$ .*



# All the computations

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*Proof:*

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Let's now carry out the complete (and a little bit long) proofs of our two main theorems.

*Proof:..... Just kidding!! :oP* ■

*Thanks!!!!*

*L.*

There's nothing here.

Really!

Really really!

Really really reaaaally!

:op



Outline

Local rigidity

○○○○○○○

Genuine Rigidity

○○○○○○○○○

Global rigidity

○○○

Singular genuine rigidity

○○○○○○○○○○○

The proofs

○○○○○○●○○○







