Singular genuine rigidity

Luis A. Florit (IMPA) & Felippe Guimarães (IMPA)

March 14, 2018

Definitions

For an *isometric immersion*

$$f\colon M^n \to \mathbb{R}^{n+p}$$

of an n-dimensional connected Riemannian manifold M^n with codimension p into flat Euclidean space, we denote its *second fundamental form* by

$$\alpha^f\colon TM \times TM \to T_f^\perp M, \quad \alpha^f(X,Y) := (\tilde\nabla_X Y)_{T_f^\perp M},$$

and its *normal connection* by

$$\nabla^\perp\colon TM \times T^\perp M \to T_f^\perp M, \quad \nabla^\perp_X \xi = (\tilde\nabla_X \xi)_{T_f^\perp M}.$$

Definitions

For an *isometric immersion*

$$f\colon M^n \to \mathbb{R}^{n+p}$$

of an n-dimensional connected Riemannian manifold M^n with codimension p into flat Euclidean space, we denote its *second fundamental form* by

$$\alpha^f\colon TM \times TM \to T_f^\perp M, \qquad \alpha^f(X,Y) := \left(\tilde\nabla_X Y\right)_{T_f^\perp M},$$

and its *normal connection* by

$$\nabla^\perp\colon TM \times T^\perp M \to T_f^\perp M, \qquad \nabla_X^\perp \xi = \left(\tilde\nabla_X \xi\right)_{T_f^\perp M}.$$

Definitions

For an *isometric immersion*

$$f\colon M^n \to \mathbb{R}^{n+p}$$

of an n-dimensional connected Riemannian manifold M^n with codimension p into flat Euclidean space, we denote its *second fundamental form* by

$$\alpha^f\colon TM \times TM \to \ T_f^{\perp}M, \qquad \alpha^f(X,Y) := \left(\tilde{\nabla}_X Y\right)_{T_f^{\perp}M},$$

and its *normal connection* by

$$\nabla^{\perp} : TM \times T^{\perp}M \to \ T_f^{\perp}M, \qquad \nabla_X^{\perp}\xi = \left(\tilde{\nabla}_X \xi\right)_{T_f^{\perp}M}.$$

General existence

Main existence result:

General existence

Main existence result:

Theorem (Nash, 1956)

Any Riemannian manifold M^n can be isometrically embedded in $\mathbb{R}^{n+p}$, for sufficiently large codimension p.

General existence

Main existence result:

Theorem (Nash, 1956)

Any Riemannian manifold M^n can be isometrically embedded in $\mathbb{R}^{n+p}$, for sufficiently large codimension p.

First key problem:

General existence

Main existence result:

Theorem (Nash, 1956)

Any Riemannian manifold M^n can be isometrically embedded in $\mathbb{R}^{n+p}$, for sufficiently large codimension p.

First key problem: (Only known for a few manifolds)

For a given Riemannian manifold M^n, what is (locally or globally) the lowest possible codimension $p = p(M^n)$ in Nash Theorem?

Uniqueness: rigidity

We say that an isometric immersion

$$f\colon\ M^n \to \mathbb{R}^{n+p}$$

is *(isometrically) rigid*

Uniqueness: rigidity

We say that an isometric immersion

$$f\colon\ M^n \to \mathbb{R}^{n+p}$$

is *(isometrically) rigid* if any other isometric immersion

$$g\colon\ M^n \to \mathbb{R}^{n+p}$$

is congruent to f by a “rigid motion” of the ambient space.

Uniqueness: rigidity

We say that an isometric immersion

$$f\colon\ M^n \to \mathbb{R}^{n+p}$$

is *(isometrically) rigid* if any other isometric immersion

$$g\colon\ M^n \to \mathbb{R}^{n+p}$$

is congruent to f by a "rigid motion" of the ambient space, i.e., when there is an isometry $T\colon\ \mathbb{R}^{n+p} \to \mathbb{R}^{n+p}$ such that

$$g = T \circ f.$$

Uniqueness: rigidity

We say that an isometric immersion

$$f\colon\ M^n \to \mathbb{R}^{n+p}$$

is *(isometrically) rigid* if any other isometric immersion

$$g\colon\ M^n \to \mathbb{R}^{n+p}$$

is congruent to f by a “rigid motion” of the ambient space, i.e., when there is an isometry $T\colon\ \mathbb{R}^{n+p} \to \mathbb{R}^{n+p}$ such that

$$g = T \circ f.$$

(In particular, rigidity implies that $p = p(M^n)$).
If otherwise, we say that f is *(isometrically) deformable*.

Deformations

Basic deformation problem:

Deformations

Basic deformation problem: (too difficult!)

Describe (in some sense) all deformations of

$$f: \; M^n \to \mathbb{R}^{n+p}$$

when $p = p(M^n)$ is the *lowest* possible codimension.

Deformations

Basic deformation problem: (too difficult!)

Describe (in some sense) all deformations of

$$f\colon\ M^n \to \mathbb{R}^{n+p}$$

when $p = p(M^n)$ is the *lowest* possible codimension.

- And if p is not the lowest possible codimension?

Deformations

Basic deformation problem: (too difficult!)

Describe (in some sense) all deformations of

$$f: M^n \to \mathbb{R}^{n+p}$$

when $p = p(M^n)$ is the *lowest* possible codimension.

- And if p is not the lowest possible codimension?

General deformation problem:

Deformations

Basic deformation problem: (too difficult!)

Describe (in some sense) all deformations of

$$f: \ M^n \to \mathbb{R}^{n+p}$$

when $p = p(M^n)$ is the *lowest* possible codimension.

- And if p is not the lowest possible codimension?

General deformation problem: (impossible!)

Describe all isometric immersions of M^n into $\mathbb{R}^{n+q}$ for a given codimension q.

Deformations

Basic deformation problem: (too difficult!)

Describe (in some sense) all deformations of

$$f\colon\ M^n \to \mathbb{R}^{n+p}$$

when $p = p(M^n)$ is the *lowest* possible codimension.

- And if p is not the lowest possible codimension?

General deformation problem: (even more difficult...)

Describe all isometric immersions of M^n into $\mathbb{R}^{n+q}$ for a given codimension q... for 'small' codimension q ($q < n$?).

Local rigidity results: Hypersurfaces ($p = 1$)

The *type number* of $f\colon M^n \to \mathbb{R}^{n+1}$ at $x \in M^n$ is the integer

$$\tau(x) = \operatorname{rank} \alpha^f(x).$$

Local rigidity results: Hypersurfaces ($p = 1$)

The *type number* of $f\colon M^n \to \mathbb{R}^{n+1}$ at $x \in M^n$ is the integer

$$\tau(x) = \operatorname{rank} \alpha^f(x).$$

Theorem (Beez, 1876 & Killing, 1885)

Any immersed hypersurface with $\tau \geq 3$ is rigid.

Local rigidity results: Hypersurfaces ($p = 1$)

The *type number* of $f\colon M^n \to \mathbb{R}^{n+1}$ at $x \in M^n$ is the integer

$$\tau(x) = \operatorname{rank} \alpha^f(x).$$

Theorem (Beez, 1876 & Killing, 1885)

Any immersed hypersurface with $\tau \geq 3$ is rigid.

FACT. By the Gauss equation,

$$\tau(x) \leq 1 \iff K_M(x) = 0.$$

Using the Gauss Parametrization (Sbrana), it is easy to see that the set of (local) flat hypersurfaces of $\mathbb{R}^{n+1}$ can be naturally identified with $C^\infty(\mathbb{R}, \mathbb{R}^{n+1})$: the smooth curves in $\mathbb{R}^{n+1}$.

Sbrana-Cartan Hypersurfaces

Thus, the interesting (local) case is when the type number satisfies

$$\tau \equiv 2.$$

Sbrana-Cartan Hypersurfaces

Thus, the interesting (local) case is when the type number satisfies

$$\tau \equiv 2.$$

The description of the nonflat locally deformable hypersurfaces and their possible deformations is due, independently, to Sbrana (1909) and Cartan (1916), and we call such submanifolds *Sbrana-Cartan hypersurfaces*.

Sbrana-Cartan Hypersurfaces

Thus, the interesting (local) case is when the type number satisfies

$$\tau \equiv 2.$$

The description of the nonflat locally deformable hypersurfaces and their possible deformations is due, independently, to Sbrana (1909) and Cartan (1916), and we call such submanifolds *Sbrana-Cartan hypersurfaces*.

They are divided in 4 types of $(n-2)$-ruled hypersurfaces:

Sbrana-Cartan Hypersurfaces

Thus, the interesting (local) case is when the type number satisfies

$$\tau \equiv 2.$$

The description of the nonflat locally deformable hypersurfaces and their possible deformations is due, independently, to Sbrana (1909) and Cartan (1916), and we call such submanifolds *Sbrana-Cartan hypersurfaces*.

They are divided in 4 types of $(n-2)$-ruled hypersurfaces:

- surface-like (products): deform as surfaces in $\mathbb{R}^3$ or $\mathbb{S}^3$;
- (n-1)-ruled: moduli space of deformations $= C^\infty(\mathbb{R}, \mathbb{R})$;
- continuous type: moduli space of deformations $= \mathbb{R}$;
- discrete type: exactly one more deformation: moduli $\mathbb{Z}_2$.

Local rigidity results: Higher codimensions

With a clever generalization of the concept of type number for higher codimensions, we have:

Local rigidity results: Higher codimensions

With a clever generalization of the concept of type number for higher codimensions, we have:

Theorem (Allendoerfer, 1939)

Any immersed submanifold with $\tau \geq 3$ is rigid.

Local rigidity results: Higher codimensions

With a clever generalization of the concept of type number for higher codimensions, we have:

Theorem (Allendoerfer, 1939)

Any immersed submanifold with $\tau \geq 3$ is rigid.

FACT: $\tau \geq 3$ implies codimension $p \leq n/3$.

Local rigidity results: Higher codimensions

With a clever generalization of the concept of type number for higher codimensions, we have:

Theorem (Allendoerfer, 1939)

Any immersed submanifold with $\tau \geq 3$ is rigid.

FACT: $\tau \geq 3$ implies codimension $p \leq n/3$.

Theorem (E. Berger, R. Bryant & P. Griffiths, 1983)

"Generically", any submanifold with $p \leq n$ and $n \geq 6$ is rigid.

Local rigidity results: Higher codimensions

With a clever generalization of the concept of type number for higher codimensions, we have:

Theorem (Allendoerfer, 1939)

Any immersed submanifold with $\tau \geq 3$ is rigid.

FACT: $\tau \geq 3$ implies codimension $p \leq n/3$.

Theorem (E. Berger, R. Bryant & P. Griffiths, 1983)

"Generically", any submanifold with $p \leq n$ and $n \geq 6$ is rigid.

These are algebraic results, with little geometry...

An observation

To deal with the deformation problem in higher codimension, we introduce a new concept based on the following observation:

An observation

To deal with the deformation problem in higher codimension, we introduce a new concept based on the following observation:

Let $N^{n+1} \subset \mathbb{R}^{n+2}$ be a Sbrana-Cartan (i.e., deformable and classified) hypersurface and

$$M^n \subset N^{n+1} \subset \mathbb{R}^{n+2}$$

any hypersurface.

An observation

To deal with the deformation problem in higher codimension, we introduce a new concept based on the following observation:

Let $N^{n+1} \subset \mathbb{R}^{n+2}$ be a Sbrana-Cartan (i.e., deformable and classified) hypersurface and

$$M^n \subset N^{n+1} \subset \mathbb{R}^{n+2}$$

any hypersurface. Then, in general M^n is a deformable submanifold in $\mathbb{R}^{n+2}$ because deformations of N^{n+1} induce deformations of M^n.

An observation

To deal with the deformation problem in higher codimension, we introduce a new concept based on the following observation:

Let $N^{n+1} \subset \mathbb{R}^{n+2}$ be a Sbrana-Cartan (i.e., deformable and classified) hypersurface and

$$M^n \subset N^{n+1} \subset \mathbb{R}^{n+2}$$

any hypersurface. Then, in general M^n is a deformable submanifold in $\mathbb{R}^{n+2}$ because deformations of N^{n+1} induce deformations of M^n.

This kind of "dishonest" deformations must be discarded since we are reducing the codimension of the problem, and for that purpose we introduce the following general definition:

Isometric extensions

Definition (Isometric extensions)

Isometric extensions

Definition (Isometric extensions)

We say that a pair of isometric immersions

Isometric extensions

Definition (Isometric extensions)

We say that a pair of isometric immersions $f\colon\ M^n \to \mathbb{R}^{n+p}$

$$M^n \xrightarrow{f} \mathbb{R}^{n+p}$$

Isometric extensions

Definition (Isometric extensions)

We say that a pair of isometric immersions $f\colon\ M^n \to \mathbb{R}^{n+p}$ and $g\colon\ M^n \to \mathbb{R}^{n+q}$

$$\begin{tikzcd}
 & \mathbb{R}^{n+p} \\
M^n \arrow[ur, "f"] \arrow[dr, "g"'] & \\
 & \mathbb{R}^{n+q}
\end{tikzcd}$$

Isometric extensions

Definition (Isometric extensions)

We say that a pair of isometric immersions $f\colon M^n \to \mathbb{R}^{n+p}$ and $g\colon M^n \to \mathbb{R}^{n+q}$ *extends isometrically* when there are:

- a Riemannian manifold N^{n+r} with $r \geq 1$,
- an isometric embedding $j\colon M^n \hookrightarrow N^{n+r}$,
- and isometric immersions $F\colon N^{n+r} \to \mathbb{R}^{n+p}$, $G\colon N^{n+r} \to \mathbb{R}^{n+q}$

$$\begin{tikzcd} & \mathbb{R}^{n+p} \\ M^n \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & \mathbb{R}^{n+q} \end{tikzcd}$$

Isometric extensions

Definition (Isometric extensions)

We say that a pair of isometric immersions $f\colon\ M^n \to \mathbb{R}^{n+p}$ and $g\colon\ M^n \to \mathbb{R}^{n+q}$ *extends isometrically* when there are:

- a Riemannian manifold N^{n+r} with $r \geq 1$,
- an isometric embedding $j\colon\ M^n \hookrightarrow N^{n+r}$,
- and isometric immersions $F\colon N^{n+r} \to \mathbb{R}^{n+p}$, $G\colon N^{n+r} \to \mathbb{R}^{n+q}$

$$
\begin{tikzcd}
 & & \mathbb{R}^{n+p} \\
M^n \arrow[urr, "f"] \arrow[drr, "g"'] & N^{n+r} & \\
 & & \mathbb{R}^{n+q}
\end{tikzcd}
$$

Isometric extensions

Definition (Isometric extensions)

We say that a pair of isometric immersions $f\colon M^n \to \mathbb{R}^{n+p}$ and $g\colon M^n \to \mathbb{R}^{n+q}$ *extends isometrically* when there are:

- a Riemannian manifold N^{n+r} with $r \geq 1$,
- an isometric embedding $j\colon M^n \hookrightarrow N^{n+r}$,
- and isometric immersions $F\colon N^{n+r} \to \mathbb{R}^{n+p}$, $G\colon N^{n+r} \to \mathbb{R}^{n+q}$

$$\begin{tikzcd} & & \mathbb{R}^{n+p} \\ M^n \arrow[urr, "f"] \arrow[r, hook, "j"] \arrow[drr, "g"'] & N^{n+r} & \\ & & \mathbb{R}^{n+q} \end{tikzcd}$$

Isometric extensions

Definition (Isometric extensions)

We say that a pair of isometric immersions $f\colon M^n \to \mathbb{R}^{n+p}$ and $g\colon M^n \to \mathbb{R}^{n+q}$ *extends isometrically* when there are:

- a Riemannian manifold N^{n+r} with $r \geq 1$,
- an isometric embedding $j\colon M^n \hookrightarrow N^{n+r}$,
- and isometric immersions $F\colon N^{n+r} \to \mathbb{R}^{n+p}$, $G\colon N^{n+r} \to \mathbb{R}^{n+q}$

$$\begin{tikzcd}
 & & \mathbb{R}^{n+p} \\
M^n \arrow[urr, "f"] \arrow[r, hook, "j"] \arrow[drr, "g"'] & N^{n+r} \arrow[ur, "F"'] & \\
 & & \mathbb{R}^{n+q}
\end{tikzcd}$$

Isometric extensions

Definition (Isometric extensions)

We say that a pair of isometric immersions $f\colon\ M^n \to \mathbb{R}^{n+p}$ and $g\colon\ M^n \to \mathbb{R}^{n+q}$ *extends isometrically* when there are:

- a Riemannian manifold N^{n+r} with $r \geq 1$,
- an isometric embedding $j\colon\ M^n \hookrightarrow N^{n+r}$,
- and isometric immersions $F\colon N^{n+r} \to \mathbb{R}^{n+p}$, $G\colon N^{n+r} \to \mathbb{R}^{n+q}$

$$
\begin{tikzcd}
 & & \mathbb{R}^{n+p} \\
M^n \arrow[urr, "f"] \arrow[r, hook, "j"] \arrow[drr, "g"'] & N^{n+r} \arrow[ur, "F"'] \arrow[dr, "G"] & \\
 & & \mathbb{R}^{n+q}
\end{tikzcd}
$$

Isometric extensions

Definition (Isometric extensions)

We say that a pair of isometric immersions $f\colon\ M^n \to \mathbb{R}^{n+p}$ and $g\colon\ M^n \to \mathbb{R}^{n+q}$ *extends isometrically* when there are:

- a Riemannian manifold N^{n+r} with $r \geq 1$,
- an isometric embedding $j\colon\ M^n \hookrightarrow N^{n+r}$,
- and isometric immersions $F\colon N^{n+r} \to \mathbb{R}^{n+p}$, $G\colon N^{n+r} \to \mathbb{R}^{n+q}$

such that $f = F \circ j$ and $g = G \circ j$.

$$\begin{tikzcd}
 & & \mathbb{R}^{n+p} \\
M^n \arrow[r, hook, "j"] \arrow[urr, "f"] \arrow[drr, "g"'] & N^{n+r} \arrow[ur, "F"'] \arrow[dr, "G"] & \\
 & & \mathbb{R}^{n+q}
\end{tikzcd}$$

commutes!

Genuine deformations

$$
\begin{tikzcd}
 & & \mathbb{R}^{n+p} \\
M^n \arrow[urr, "f"] \arrow[r, hook, "j"] \arrow[drr, "g"'] & N^{n+r} \arrow[ur, "F"'] \arrow[dr, "G"] & \\
 & & \mathbb{R}^{n+q}
\end{tikzcd}
$$

Genuine deformations

$$
\begin{tikzcd}
 & & \mathbb{R}^{n+p} \\
M^n \arrow[urr, "f"] \arrow[r, hook, "j"] \arrow[drr, "g"'] & N^{n+r} \arrow[ur, "F"] \arrow[dr, "G"'] & \\
 & & \mathbb{R}^{n+q}
\end{tikzcd}
$$

NOT a genuine deformation

Genuine deformations

$$
\begin{tikzcd}
& & \mathbb{R}^{n+p} \\
M^n \arrow[urr, "f"] \arrow[r, hook, "j"] \arrow[drr, "g"'] & N^{n+r} \arrow[ur, "F"'] \arrow[dr, "G"] & \\
& & \mathbb{R}^{n+q}
\end{tikzcd}
$$

NOT a genuine deformation

Definition (Genuine deformation)

Given a submanifold $f\colon M^n \to \mathbb{R}^{n+p}$ we say that $g\colon M^n \to \mathbb{R}^{n+q}$ is a *genuine deformation* of f there is no open subset $U \subset M^n$ along which the restrictions $f|_U$ and $g|_U$ extend isometrically.

Genuine deformations

$$
\begin{tikzcd}
 & & \mathbb{R}^{n+p} \\
M^n \arrow[urr, "f"] \arrow[r, hook, "j"] \arrow[drr, "g"'] & N^{n+r} \arrow[ur, "F"'] \arrow[dr, "G"] & \\
 & & \mathbb{R}^{n+q}
\end{tikzcd}
$$

NOT a genuine deformation

Definition (Genuine deformation)

Given a submanifold $f\colon M^n \to \mathbb{R}^{n+p}$ we say that $g\colon M^n \to \mathbb{R}^{n+q}$ is a *genuine deformation* of f there is no open subset $U \subset M^n$ along which the restrictions $f|_U$ and $g|_U$ extend isometrically.

Remark

This is a <u>symmetric</u> concept: for pairs and sets.

Genuine rigidity

$$
\begin{tikzcd}
 & & \mathbb{R}^{n+p} \\
M^n \arrow[urr, "f"] \arrow[r, hook, "j"] \arrow[drr, "g"'] & N^{n+r} \arrow[ur, "F"'] \arrow[dr, "G"] & \\
 & & \mathbb{R}^{n+q}
\end{tikzcd}
$$

Genuine rigidity

$$
\begin{tikzcd}
 & & \mathbb{R}^{n+p} \\
M^n \arrow[urr, "f"] \arrow[drr, "g"'] \arrow[r, hook, "j"] & N^{n+r} \arrow[ur, "F"'] \arrow[dr, "G"] & \\
 & & \mathbb{R}^{n+q}
\end{tikzcd}
$$

genuine rigidity:
only these exist

Genuine rigidity

$$\begin{tikzcd} & & \mathbb{R}^{n+p} \\ M^n \arrow[urr, "f"] \arrow[r, hook, "j"] \arrow[drr, "g"'] & N^{n+r} \arrow[ur, "F"'] \arrow[dr, "G"] & \\ & & \mathbb{R}^{n+q} \end{tikzcd}$$

genuine rigidity: only these exist

Definition (Genuine rigidity)

Given $f\colon M^n \to \mathbb{R}^{n+p}$ and a positive integer q, we say that f is *genuinely rigid* in $\mathbb{R}^{n+q}$ if for any $g\colon M^n \to \mathbb{R}^{n+q}$ there is an open dense subset $U \subset M^n$ such that $f|_U$ and $g|_U$ extend isometrically.

That is, g is <u>nowhere</u> a genuine deformation of f.

Genuine rigidity

$$\begin{tikzcd} & & \mathbb{R}^{n+p} \\ M^n \arrow[urr, "f"] \arrow[r, hook, "j"] \arrow[drr, "g"'] & N^{n+r} \arrow[ur, "F"'] \arrow[dr, "G"] & \\ & & \mathbb{R}^{n+q} \end{tikzcd}$$

genuine rigidity: only these exist

Definition (Genuine rigidity)

Given $f\colon M^n \to \mathbb{R}^{n+p}$ and a positive integer q, we say that f is *genuinely rigid* in $\mathbb{R}^{n+q}$ if for any $g\colon M^n \to \mathbb{R}^{n+q}$ there is an open dense subset $U \subset M^n$ such that $f|_U$ and $g|_U$ extend isometrically.

That is, g is <u>nowhere</u> a genuine deformation of f.

Remark

Genuine rigidity

$$\begin{tikzcd} & & \mathbb{R}^{n+p} \\ M^n \arrow[urr, "f"] \arrow[r, hook, "j"] \arrow[drr, "g"'] & N^{n+r} \arrow[ur, "F"] \arrow[dr, "G"] & \\ & & \mathbb{R}^{n+q} \end{tikzcd}$$

genuine rigidity:
only these exist

Definition (Genuine rigidity)

Given $f\colon M^n \to \mathbb{R}^{n+p}$ and a positive integer q, we say that f is *genuinely rigid* in $\mathbb{R}^{n+q}$ if for any $g\colon M^n \to \mathbb{R}^{n+q}$ there is an open dense subset $U \subset M^n$ such that $f|_U$ and $g|_U$ extend isometrically.
That is, g is nowhere a genuine deformation of f.

Remark

- *If $r = p = q$, we recover the concept of isometric rigidity.*

Genuine rigidity

$$
\begin{tikzcd}
 & & \mathbb{R}^{n+p} \\
M^n \arrow[r, hook, "j"] \arrow[urr, "f"] \arrow[drr, "g"'] & N^{n+r} \arrow[ur, "F"'] \arrow[dr, "G"] & \\
 & & \mathbb{R}^{n+q}
\end{tikzcd}
$$

genuine rigidity:
only these exist

Definition (Genuine rigidity)

Given $f\colon M^n \to \mathbb{R}^{n+p}$ and a positive integer q, we say that f is *genuinely rigid* in $\mathbb{R}^{n+q}$ if for any $g\colon M^n \to \mathbb{R}^{n+q}$ there is an open dense subset $U \subset M^n$ such that $f|_U$ and $g|_U$ extend isometrically.

That is, g is <u>nowhere</u> a genuine deformation of f.

Remark

- *If $r = p = q$, we recover the concept of isometric rigidity.*
- *If $r = p < q$, we recover the concept of compositions.*

The main result

Assume that the submanifolds $f\colon M^n \to \mathbb{R}^{n+p}$ and $g\colon M^n \to \mathbb{R}^{n+q}$ are *mutually D^d-ruled*, that is, D^d is a d-dimensional integrable distribution of M^n whose leaves are mapped by f and g onto open subsets of affine subspaces.

The main result

Assume that the submanifolds $f\colon M^n \to \mathbb{R}^{n+p}$ and $g\colon M^n \to \mathbb{R}^{n+q}$ are *mutually D^d-ruled*, that is, D^d is a d-dimensional integrable distribution of M^n whose leaves are mapped by f and g onto open subsets of affine subspaces.

Set

$$L_D = \mathrm{span}\{\alpha^f(D, TM)\} \quad and \quad L'_D = \mathrm{span}\{\alpha^g(D, TM)\}.$$

The main result

Assume that the submanifolds $f\colon M^n \to \mathbb{R}^{n+p}$ and $g\colon M^n \to \mathbb{R}^{n+q}$ are *mutually D^d-ruled*, that is, D^d is a d-dimensional integrable distribution of M^n whose leaves are mapped by f and g onto open subsets of affine subspaces.
Set

$$L_D = \operatorname{span}\{\alpha^f(D, TM)\} \quad \textit{and} \quad L'_D = \operatorname{span}\{\alpha^g(D, TM)\}.$$

Then, by the Gauss equation, the map $\mathcal{T}_D\colon L_D \to L'_D$ given by

$$\mathcal{T}_D(\alpha^f(Y, X)) = \alpha^g(Y, X), \quad Y \in D^d, X \in TM,$$

is a well defined bundle isometry.

The main result: local in nature

Theorem (Dajczer & F—, 2004)

The main result: local in nature

Theorem (Dajczer & F—, 2004)

Let $g\colon M^n \to \mathbb{R}^{n+q}$ be a genuine deformation of $f\colon M^n \to \mathbb{R}^{n+p}$, with $p + q < n$ and $\min\{p, q\} \leq 5$.

The main result: local in nature

Theorem (Dajczer & F—, 2004)

Let $g\colon M^n \to \mathbb{R}^{n+q}$ be a genuine deformation of $f\colon M^n \to \mathbb{R}^{n+p}$, with $p+q<n$ and $\min\{p,q\} \leq 5$. Then, locally f and g are mutually D^d-ruled, with $d \geq n-p-q+3\dim L_D$.

The main result: local in nature

Theorem (Dajczer & F—, 2004)

Let $g\colon M^n \to \mathbb{R}^{n+q}$ be a genuine deformation of $f\colon M^n \to \mathbb{R}^{n+p}$, with $p+q < n$ and $\min\{p,q\} \leq 5$. Then, locally f and g are mutually D^d-ruled, with $d \geq n-p-q+3\dim L_D$.

Locally $=$ along connected components of an open dense subset.

The main result: local in nature

Theorem (Dajczer & F—, 2004)

Let $g\colon M^n \to \mathbb{R}^{n+q}$ be a genuine deformation of $f\colon M^n \to \mathbb{R}^{n+p}$, with $p+q<n$ and $\min\{p,q\} \leq 5$. Then, locally f and g are mutually D^d-ruled, with $d \geq n-p-q+3\dim L_D$.

Locally $=$ along connected components of an open dense subset.

Remark

The above result generalizes and unifies several rigidity results and all the ones we know about compositions.

$\exists$ genuine deformation $\Rightarrow$ mutually ruled

Thus, for a pair of genuine deformations we have:

$\exists$ genuine deformation $\Rightarrow$ mutually ruled

Thus, for a pair of genuine deformations we have:

$$T_f^\perp M = L_D \oplus L_D^\perp \qquad L_D := \mathrm{span}\,\{\alpha^f(D, TM)\}$$

$$T_g^\perp M = L'_D \oplus L'^\perp_D \qquad L'_D := \mathrm{span}\,\{\alpha^g(D, TM)\}$$

$\exists$ genuine deformation $\Rightarrow$ mutually ruled

Thus, for a pair of genuine deformations we have:

$$L_D := \operatorname{span}\{\alpha^f(D, TM)\}$$

$$\begin{tikzcd}
T_f^{\perp}M = L_D \oplus L_D^{\perp} \arrow[d, "\begin{array}{c}\mathcal{T}_D \\ (\nabla^{\perp})_{L_D} = (\nabla^{\perp})_{L'_D} \\ (\alpha^f)_{L_D} = (\alpha^g)_{L'_D}\end{array}"'] \\
T_g^{\perp}M = L'_D \oplus L'^{\perp}_D
\end{tikzcd}$$

$$L'_D := \operatorname{span}\{\alpha^g(D, TM)\}$$

$\exists$ genuine deformation $\Rightarrow$ mutually ruled

Thus, for a pair of genuine deformations we have:

$$T_f^\perp M = L_D \oplus L_D^\perp \qquad L_D := \operatorname{span}\{\alpha^f(D, TM)\}$$

$$\begin{array}{c} \mathcal{T}_D \\ (\nabla^\perp)_{L_D} = (\nabla^\perp)_{L'_D} \\ (\alpha^f)_{L_D} = (\alpha^g)_{L'_D} \end{array} \Bigg\downarrow \qquad D^d = \ker\left(\alpha^f_{L_D^\perp}\right) \cap \ker\left(\alpha^g_{L'^\perp_D}\right) \text{ rulings}$$

$$T_g^\perp M = L'_D \oplus L'^\perp_D \qquad L'_D := \operatorname{span}\{\alpha^g(D, TM)\}$$

$\exists$ genuine deformation $\Rightarrow$ mutually ruled

Thus, for a pair of genuine deformations we have:

$$T_f^\perp M = L_D \oplus L_D^\perp \qquad\qquad L_D := \operatorname{span}\,\{\alpha^f(D, TM)\}$$

$$\mathcal{T}_D \qquad (\nabla^\perp)_{L_D} = (\nabla^\perp)_{L'_D} \qquad (\alpha^f)_{L_D} = (\alpha^g)_{L'_D}$$

$$D^d = \ker\left(\alpha^f_{L_D^\perp}\right) \cap \ker\left(\alpha^g_{L'^\perp_D}\right) \text{ rulings}$$

$$T_g^\perp M = L'_D \oplus L'^\perp_D \qquad\qquad L'_D := \operatorname{span}\,\{\alpha^g(D, TM)\}$$

Moreover, the estimate $d \geq n - p - q + 3 \dim L_D$ is sharp.

Consequences

Corollary

Consequences

Corollary

Let $f: M^n \to \mathbb{R}^{n+p}$, $q \in \mathbb{N}$ *with* $p+q<n$ *and* $\min\{p,q\} \leq 5$.

Consequences

Corollary

Let $f: M^n \to \mathbb{R}^{n+p}$, $q \in \mathbb{N}$ *with* $p + q < n$ *and* $\min\{p, q\} \leq 5$.

- *Beez-Killing; Allendoerfer (for* $\min\{p, q\} \leq 5$*), etc.*

Consequences

Corollary

Let $f: M^n \to \mathbb{R}^{n+p}$, $q \in \mathbb{N}$ *with* $p + q < n$ *and* $\min\{p, q\} \leq 5$.

- *Beez-Killing; Allendoerfer (for* $\min\{p, q\} \leq 5$*), etc.*
- *If* f *is nowhere* $(n-p-q)$*-ruled,* f *is genuinely rigid in* $\mathbb{R}^{n+q}$.

Consequences

Corollary

Let $f\colon\ M^n \to \mathbb{R}^{n+p}$, $q \in \mathbb{N}$ *with* $p+q<n$ *and* $\min\{p,q\} \leq 5$.

- *Beez-Killing; Allendoerfer (for* $\min\{p,q\} \leq 5$*), etc.*
- *If* f *is nowhere* $(n-p-q)$*-ruled,* f *is genuinely rigid in* $\mathbb{R}^{n+q}$.
- *If* $Ric_M > 0$*, then* f *is genuinely rigid in* $\mathbb{R}^{n+q}$.

Consequences

Corollary

Let $f: M^n \to \mathbb{R}^{n+p}$, $q \in \mathbb{N}$ *with* $p+q<n$ *and* $\min\{p,q\} \leq 5$.

- *Beez-Killing; Allendoerfer (for* $\min\{p,q\} \leq 5$*), etc.*
- *If* f *is nowhere* $(n-p-q)$*-ruled,* f *is genuinely rigid in* $\mathbb{R}^{n+q}$.
- *If* $Ric_M > 0$*, then* f *is genuinely rigid in* $\mathbb{R}^{n+q}$.
- *If* M^n *is compact, then there exists an open subset* $U \subset M^n$ *such that* $f|_U$ *is genuinely rigid in* $\mathbb{R}^{n+q}$.

Consequences

Corollary

Let $f\colon\ M^n \to \mathbb{R}^{n+p}$, $q \in \mathbb{N}$ *with* $p + q < n$ *and* $\min\{p, q\} \leq 5$.

- *Beez-Killing; Allendoerfer (for* $\min\{p, q\} \leq 5$*), etc.*
- *If* f *is nowhere* $(n-p-q)$*-ruled,* f *is genuinely rigid in* $\mathbb{R}^{n+q}$.
- *If* $Ric_M > 0$*, then* f *is genuinely rigid in* $\mathbb{R}^{n+q}$.
- *If* M^n *is compact, then there exists an open subset* $U \subset M^n$ *such that* $f|_U$ *is genuinely rigid in* $\mathbb{R}^{n+q}$.

But: it is very hard to obtain global applications from this our result!

Genuine conformal rigidity

Analogously, given a manifold M^n with a *conformal* structure, we may want to understand its *conformal* immersions $f\colon M^n \to \mathbb{R}^{n+p}$.

Genuine conformal rigidity

Analogously, given a manifold M^n with a *conformal* structure, we may want to understand its *conformal* immersions $f : M^n \to \mathbb{R}^{n+p}$.

Similar rigidity and deformation results to the ones for the isometric case also exist for the conformal realm: Conformal rigidity, (Sbrana-)Cartan hypersurfaces, Conformal compositions, etc.

Genuine conformal rigidity

Analogously, given a manifold M^n with a *conformal* structure, we may want to understand its *conformal* immersions $f: M^n \to \mathbb{R}^{n+p}$.

Similar rigidity and deformation results to the ones for the isometric case also exist for the conformal realm: Conformal rigidity, (Sbrana-)Cartan hypersurfaces, Conformal compositions, etc.

In 2010, together with Ruy Tojeiro (UFSCar), we extended several of these conformal results by means of a genuine *conformal* rigidity framework, giving them a unified character. Actually, we obtained precisely the conformal version of the genuine rigidity theorem above, with its corresponding corollaries.

Global rigidity

Global rigidity results are **much** harder to obtain.

Global rigidity

Global rigidity results are **much** harder to obtain.

The fundamental one is the beautiful Sacksteder's theorem ($n \geq 3$):

Global rigidity

Global rigidity results are **much** harder to obtain.

The fundamental one is the beautiful Sacksteder's theorem ($n \geq 3$):

Theorem (Sacksteder, 1962)

Any compact Euclidean hypersurface is rigid provided its set of totally geodesic points does not disconnect the manifold.

Global rigidity

Global rigidity results are **much** harder to obtain.

The fundamental one is the beautiful Sacksteder's theorem ($n \geq 3$):

Theorem (Sacksteder, 1962)

Any compact Euclidean hypersurface is rigid provided its set of totally geodesic points does not disconnect the manifold.

Sacksteder actually proved this equivalent statement:

Global rigidity

Global rigidity results are **much** harder to obtain.

The fundamental one is the beautiful Sacksteder's theorem ($n \geq 3$):

Theorem (Sacksteder, 1962)

Any compact Euclidean hypersurface is rigid provided its set of totally geodesic points does not disconnect the manifold.

Sacksteder actually proved this equivalent statement:

Theorem (translating Sacksteder...)

Any compact Euclidean hypersurface is genuinely rigid.

Global rigidity

Outside the realm of hypersurfaces, the only known result of this kind is the codimension two version of Sacksteder's theorem:

Global rigidity

Outside the realm of hypersurfaces, the only known result of this kind is the codimension two version of Sacksteder's theorem:

Theorem (Dajczer & Gromoll, 1995)

Any pair of isometric immersions of a compact Riemannian manifold M^n into $\mathbb{R}^{n+2}$, $n \geq 5$, must extend isometrically along each connected component of an open dense subset of M^n

Global rigidity

Outside the realm of hypersurfaces, the only known result of this kind is the codimension two version of Sacksteder's theorem:

Theorem (Dajczer & Gromoll, 1995)

Any pair of isometric immersions of a compact Riemannian manifold M^n into $\mathbb{R}^{n+2}$, $n \geq 5$, must extend isometrically along each connected component of an open dense subset of M^n, **although maybe singularly**.

Global rigidity

Outside the realm of hypersurfaces, the only known result of this kind is the codimension two version of Sacksteder's theorem:

Theorem (Dajczer & Gromoll, 1995)

Any pair of isometric immersions of a compact Riemannian manifold M^n into $\mathbb{R}^{n+2}$, $n \geq 5$, must extend isometrically along each connected component of an open dense subset of M^n, **although maybe singularly**.

Why **"maybe singularly"**??

Global rigidity

Outside the realm of hypersurfaces, the only known result of this kind is the codimension two version of Sacksteder's theorem:

Theorem (Dajczer & Gromoll, 1995)

Any pair of isometric immersions of a compact Riemannian manifold M^n into $\mathbb{R}^{n+2}$, $n \geq 5$, must extend isometrically along each connected component of an open dense subset of M^n, **although maybe singularly**.

Why **"maybe singularly"**?? The introduction of singularities in this result was shown to be a necessary condition in a recent local classification we made with Guilherme Freitas (2017).

Global rigidity

In fact, after all:

Global rigidity

In fact, after all:

We should not try to avoid singularities when working with ruled extensions!

Global rigidity

In fact, after all:

$\boxed{\text{\textit{We should not try to avoid singularities when working with ruled extensions!}}}$

Singularities are not only necessary, but also natural for the problem, deeply simplify the theory, and are easy to deal with for ruled submanifolds, as we do when we classify flat and ruled surfaces in $\mathbb{R}^3$.

Introducing singularities

Our purpose in this joint work with Felippe Guimarães (IMPA) was then to introduce singularities in the genuine rigidity theory, mainly with the double purpose of obtaining new global rigidity results and to unify the known ones.

Introducing singularities

Our purpose in this joint work with Felippe Guimarães (IMPA) was then to introduce singularities in the genuine rigidity theory, mainly with the double purpose of obtaining new global rigidity results and to unify the known ones.

Yet, we got some nice surprises for the local theory...

Definitions

All the definitions for introducing singularities are the natural ones, adapting the regular case:

Definitions

All the definitions for introducing singularities are the natural ones, adapting the regular case:

$$
\begin{tikzcd}
 & & \mathbb{R}^{n+p} \\
M^n \arrow[urr, "f"] \arrow[r, hook, "j"] \arrow[drr, "g"'] & N^{n+r} \arrow[ur, "F"'] \arrow[dr, "G"] & \\
 & & \mathbb{R}^{n+q}
\end{tikzcd}
$$

We say that f and g *singularly extend isometrically* if the diagram commutes, where j is an isometric embedding as before, and F and G are isometric immersions

Definitions

All the definitions for introducing singularities are the natural ones, adapting the regular case:

$$\begin{tikzcd} & & \mathbb{R}^{n+p} \\ M^n \arrow[urr, "f"] \arrow[r, hook, "j"] \arrow[drr, "g"'] & N^{n+r} \arrow[ur, "F"'] \arrow[dr, "G"] & \\ & & \mathbb{R}^{n+q} \end{tikzcd}$$

We say that f and g *singularly extend isometrically* if the diagram commutes, where j is an isometric embedding as before, and F and G are isometric immersions, except maybe on M^n.

Definitions

Analogously, we say that g is a *strong genuine deformation* of f if there is no open subset where they singularly extend isometrically.

Definitions

Analogously, we say that g is a *strong genuine deformation* of f if there is no open subset where they singularly extend isometrically.

Accordingly, we say that f is *singularly genuinely rigid* in $\mathbb{R}^{n+q}$ if it admits no strong genuine deformation in $\mathbb{R}^{n+q}$.

Definitions

Analogously, we say that g is a *strong genuine deformation* of f if there is no open subset where they singularly extend isometrically.

Accordingly, we say that f is *singularly genuinely rigid* in $\mathbb{R}^{n+q}$ if it admits no strong genuine deformation in $\mathbb{R}^{n+q}$.

With these definitions we show:

Global statements

Theorem (F— & Guimarães, preprint, 2018)

Global statements

Theorem (F— & Guimarães, preprint, 2018)

Let $f\colon M^n \to \mathbb{R}^{n+p}$ and $\hat{f}\colon M^n \to \mathbb{R}^{n+q}$ be isometric immersions of a compact Riemannian manifold with $p+q<n$.

Global statements

Theorem (F— & Guimarães, preprint, 2018)

Let $f\colon M^n \to \mathbb{R}^{n+p}$ and $\hat{f}\colon M^n \to \mathbb{R}^{n+q}$ be isometric immersions of a compact Riemannian manifold with $p+q<n$. Then, locally either f and $\hat{f}$ singularly extend isometrically,

Global statements

Theorem (F— & Guimarães, preprint, 2018)

Let $f\colon M^n \to \mathbb{R}^{n+p}$ and $\hat{f}\colon M^n \to \mathbb{R}^{n+q}$ be isometric immersions of a compact Riemannian manifold with $p+q<n$. Then, locally either f and $\hat{f}$ singularly extend isometrically, or f and $\hat{f}$ are mutually D^d-ruled, with $d \geq n-p-q+3$.

Global statements

Theorem (F— & Guimarães, preprint, 2018)

Let $f\colon M^n \to \mathbb{R}^{n+p}$ and $\hat{f}\colon M^n \to \mathbb{R}^{n+q}$ be isometric immersions of a compact Riemannian manifold with $p+q<n$. Then, locally either f and $\hat{f}$ singularly extend isometrically, or f and $\hat{f}$ are mutually D^d-ruled, with $d \geq n-p-q+3$.

The proof actually works untouched for compact (or complete and bounded) submanifolds in hyperbolic space, and with a little bit of extra work we obtained a similar result for complete submanifolds in the sphere.

Global consequences

For $p + q \leq 4$ this unifies Sacksteder and Dajczer-Gromoll theorems above, and states that the only way to isometrically immerse a compact Euclidean hypersurface in codimension 3 is as a composition:

Global consequences

For $p+q \leq 4$ this unifies Sacksteder and Dajczer-Gromoll theorems above, and states that the only way to isometrically immerse a compact Euclidean hypersurface in codimension 3 is as a composition:

Corollary

Any compact (or complete and bounded) submanifold M^n of $\mathbb{R}^{n+p}$ is singularly genuinely rigid in $\mathbb{R}^{n+q}$ for $q \leq 4-p$.

Global consequences

We also obtained a purely topological criteria for singular genuine rigidity without any restriction on the codimensions, answering a question by M. Gromov in this setting:

Global consequences

We also obtained a purely topological criteria for singular genuine rigidity without any restriction on the codimensions, answering a question by M. Gromov in this setting:

Corollary

Let M^n be a compact manifold whose k-th Pontrjagin class satisfies that $[p_k] \neq 0$ for some $k > \frac{3}{4}(p+q-3)$. Then, any analytic immersion $f : M^n \to \mathbb{R}^{n+p}$ (with the induced metric) is singularly genuinely rigid in $\mathbb{R}^{n+q}$ in the C^∞-category.

Local analysis: the form ϕ_τ

Our global results rely on a local analysis of a special bilinear form.

Local analysis: the form ϕ_τ

Our global results rely on a local analysis of a special bilinear form. Let

$$\tau : L^\ell \subset T_f^\perp M \to \hat{L}^\ell \subset T_{\hat{f}}^\perp M$$

be a vector bundle isometry that preserves the second fundamental forms and the normal connections restricted to the rank ℓ vector normal subbundles L and $\hat{L}$. Extend τ as Id on TM: $\bar{\tau} = Id \oplus \tau$.

Local analysis: the form ϕ_τ

Our global results rely on a local analysis of a special bilinear form. Let

$$\tau : L^\ell \subset T_f^\perp M \to \hat{L}^\ell \subset T_{\hat{f}}^\perp M$$

be a vector bundle isometry that preserves the second fundamental forms and the normal connections restricted to the rank ℓ vector normal subbundles L and $\hat{L}$. Extend τ as Id on TM: $\bar{\tau} = Id \oplus \tau$. Let $\phi_\tau : TM \times (TM \oplus L) \to L^\perp \times \hat{L}^\perp$ be the bilinear form

$$\phi_\tau(X, \eta) = \left((\tilde{\nabla}_X \eta)_{L^\perp}, (\tilde{\nabla}_X \bar{\tau}\eta)_{\hat{L}^\perp}\right),$$

Local analysis: the form ϕ_τ

Our global results rely on a local analysis of a special bilinear form. Let

$$\tau : L^\ell \subset T_f^\perp M \to \hat{L}^\ell \subset T_{\hat{f}}^\perp M$$

be a vector bundle isometry that preserves the second fundamental forms and the normal connections restricted to the rank ℓ vector normal subbundles L and $\hat{L}$. Extend τ as Id on TM: $\bar{\tau} = Id \oplus \tau$. Let $\phi_\tau : TM \times (TM \oplus L) \to L^\perp \times \hat{L}^\perp$ be the bilinear form

$$\phi_\tau(X,\eta) = \left((\tilde{\nabla}_X \eta)_{L^\perp}, (\tilde{\nabla}_X \bar{\tau}\eta)_{\hat{L}^\perp}\right),$$

where $\tilde{\nabla}$ stands for the connection in Euclidean space and $L^\perp \times \hat{L}^\perp$ is endowed with the semi-Riemannian metric $\langle\,,\rangle = \langle\,,\rangle|_{L^\perp} - \langle\,,\rangle|_{\hat{L}^\perp}$.

The form ϕ_τ and the main local result

A subset $S \subset L^\perp \oplus \hat{L}^\perp$ is called *null* if $\langle \eta, \xi \rangle = 0$ for all $\eta, \xi \in S$.

The form ϕ_τ and the main local result

A subset $S \subset L^\perp \oplus \hat{L}^\perp$ is called *null* if $\langle \eta, \xi \rangle = 0$ for all $\eta, \xi \in S$. Given a distribution D on M^n, denote by $\mathcal{O}(D)$ the smallest totally geodesic distribution of M^n that contains D.

The form ϕ_τ and the main local result

A subset $S \subset L^\perp \oplus \hat{L}^\perp$ is called *null* if $\langle \eta, \xi \rangle = 0$ for all $\eta, \xi \in S$. Given a distribution D on M^n, denote by $\mathcal{O}(D)$ the smallest totally geodesic distribution of M^n that contains D.

Our main local result applies even to $\tau = 0$:

The form ϕ_τ and the main local result

A subset $S \subset L^\perp \oplus \hat{L}^\perp$ is called *null* if $\langle \eta, \xi \rangle = 0$ for all $\eta, \xi \in S$. Given a distribution D on M^n, denote by $\mathcal{O}(D)$ the smallest totally geodesic distribution of M^n that contains D.

Our main local result applies even to $\tau = 0$:

Theorem

The form ϕ_τ and the main local result

A subset $S \subset L^\perp \oplus \hat{L}^\perp$ is called *null* if $\langle \eta, \xi \rangle = 0$ for all $\eta, \xi \in S$. Given a distribution D on M^n, denote by $\mathcal{O}(D)$ the smallest totally geodesic distribution of M^n that contains D.

Our main local result applies even to $\tau = 0$:

Theorem

Let $\hat{f}$ be a strongly genuine deformation of f and

$$\tau : L \subset T_f^\perp M \to \hat{L} \subset T_{\hat{f}}^\perp M$$

a parallel v. b. isometry that preserves second fundamental forms.

The form ϕ_τ and the main local result

A subset $S \subset L^\perp \oplus \hat{L}^\perp$ is called *null* if $\langle \eta, \xi \rangle = 0$ for all $\eta, \xi \in S$. Given a distribution D on M^n, denote by $\mathcal{O}(D)$ the smallest totally geodesic distribution of M^n that contains D.

Our main local result applies even to $\tau = 0$:

Theorem

Let $\hat{f}$ be a strongly genuine deformation of f and

$$\tau : L \subset T_f^\perp M \to \hat{L} \subset T_{\hat{f}}^\perp M$$

a parallel v. b. isometry that preserves second fundamental forms. Let $D \subset TM \oplus L$ such that $\phi_\tau(TM, D) \subset L^\perp \oplus \hat{L}^\perp$ is a null set.

The form ϕ_τ and the main local result

A subset $S \subset L^\perp \oplus \hat{L}^\perp$ is called *null* if $\langle \eta, \xi \rangle = 0$ for all $\eta, \xi \in S$. Given a distribution D on M^n, denote by $\mathcal{O}(D)$ the smallest totally geodesic distribution of M^n that contains D.

Our main local result applies even to $\tau = 0$:

Theorem

Let $\hat{f}$ be a strongly genuine deformation of f and

$$\tau : L \subset T_f^\perp M \to \hat{L} \subset T_{\hat{f}}^\perp M$$

a parallel v. b. isometry that preserves second fundamental forms. Let $D \subset TM \oplus L$ such that $\phi_\tau(TM, D) \subset L^\perp \oplus \hat{L}^\perp$ is a null set. Then, $D \subset TM$ and locally f and $\hat{f}$ are mutually $\mathcal{O}(D)$-ruled.

The form ϕ_τ and the main local result

A subset $S \subset L^\perp \oplus \hat{L}^\perp$ is called *null* if $\langle \eta, \xi \rangle = 0$ for all $\eta, \xi \in S$. Given a distribution D on M^n, denote by $\mathcal{O}(D)$ the smallest totally geodesic distribution of M^n that contains D.

Our main local result applies even to $\tau = 0$:

Theorem

Let $\hat{f}$ be a strongly genuine deformation of f and

$$\tau : L \subset T_f^\perp M \to \hat{L} \subset T_{\hat{f}}^\perp M$$

a parallel v. b. isometry that preserves second fundamental forms. Let $D \subset TM \oplus L$ such that $\phi_\tau(TM, D) \subset L^\perp \oplus \hat{L}^\perp$ is a null set. Then, $D \subset TM$ and locally f and $\hat{f}$ are mutually $\mathcal{O}(D)$-ruled.

The key advantage here over other local rigidity results: it deals with easily to construct null sets instead of nullity distributions.

The main local tool for actual applications

A good example is the following singular version of the main result in genuine rigidity removing the assumptions on the codimensions.

The main local tool for actual applications

A good example is the following singular version of the main result in genuine rigidity removing the assumptions on the codimensions.

Recall that $Y \in TM$ is a *(left) regular element* of ϕ_τ if

$$\mathrm{rank}(\phi_\tau^Y) = i(\phi_\tau) := \max\{\mathrm{rank}(\phi_\tau^X) \,:\, X \in TM\},$$

where $\phi_\tau^X = \phi_\tau(X, \cdot)$. Denote by $RE(\phi_\tau) \subset TM$ the open dense subset of regular elements of ϕ_τ, and $K_Y = \ker(\phi_\tau^Y)$ for $Y \in RE(\phi_\tau)$.

The main local tool for actual applications

A good example is the following singular version of the main result in genuine rigidity removing the assumptions on the codimensions.

Recall that $Y \in TM$ is a *(left)* *regular element* of ϕ_τ if

$$\mathrm{rank}(\phi_\tau^Y) = i(\phi_\tau) := \max\{\mathrm{rank}(\phi_\tau^X) \, : \, X \in TM\},$$

where $\phi_\tau^X = \phi_\tau(X, \cdot)$. Denote by $RE(\phi_\tau) \subset TM$ the open dense subset of regular elements of ϕ_τ, and $K_Y = \ker(\phi_\tau^Y)$ for $Y \in RE(\phi_\tau)$.

Corollary

The main local tool for actual applications

A good example is the following singular version of the main result in genuine rigidity removing the assumptions on the codimensions.

Recall that $Y \in TM$ is a *(left) regular element* of ϕ_τ if

$$\mathrm{rank}(\phi_\tau^Y) = i(\phi_\tau) := \max\{\mathrm{rank}(\phi_\tau^X) \,:\, X \in TM\},$$

where $\phi_\tau^X = \phi_\tau(X, \cdot)$. Denote by $RE(\phi_\tau) \subset TM$ the open dense subset of regular elements of ϕ_τ, and $K_Y = \ker(\phi_\tau^Y)$ for $Y \in RE(\phi_\tau)$.

Corollary

If $\hat{f}$ is a strongly genuine deformation of f, then locally they are mutually $\mathcal{O}(D_Y^d)$-ruled $\forall Y \in RE(\phi_\tau)$, where $D_Y^d = \ker(\phi_\tau^Y) \subset TM$ and $d = n + \ell - i(\phi_\tau) \geq n - p - q + 3\ell$.

The main local tool for actual applications

By allowing singular extensions we recover all the genuine rigidity corollaries, yet without the restrictions on the codimensions.

The main local tool for actual applications

By allowing singular extensions we recover all the genuine rigidity corollaries, yet without the restrictions on the codimensions.
E.g., the previous global corollary about Pontrjagin classes actually follows from our local result.

The main local tool for actual applications

By allowing singular extensions we recover all the genuine rigidity corollaries, yet without the restrictions on the codimensions.
E.g., the previous global corollary about Pontrjagin classes actually follows from our local result.

Of course, we also obtain local rigidity under any circumstance that excludes ruled submanifolds. For example:

The main local tool for actual applications

By allowing singular extensions we recover all the genuine rigidity corollaries, yet without the restrictions on the codimensions.
E.g., the previous global corollary about Pontrjagin classes actually follows from our local result.

Of course, we also obtain local rigidity under any circumstance that excludes ruled submanifolds. For example:

Corollary

Any immersed submanifold M^n of $\mathbb{R}^{n+p}$ with $Ric > 0$ is singularly genuinely rigid in $\mathbb{R}^{n+q}$ for any $q < n - p$.

All the computations

Let's now carry out the complete (and a little bit long) proofs of our two main theorems.

All the computations

Let's now carry out the complete (and a little bit long) proofs of our two main theorems.

Proof:

All the computations

Let's now carry out the complete (and a little bit long) proofs of our two main theorems.

Proof:........ Just kidding!! :oP ∎

Thanks!!!!

$\mathcal{L}.$

There's nothing here.

Really!

Really really!

Really really reaaaally!

:op