Singular genuine rigidity

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Definitions

For an *isometric immersion*

$$f\colon M^n \to \mathbb{R}^{n+p}$$

of an n-dimensional connected Riemannian manifold M^n with codimension p into flat Euclidean space, we denote its *second fundamental form* by

$$\alpha^f\colon TM \times TM \to T_f^\perp M, \quad \alpha^f(X,Y) := (\tilde\nabla_X Y)_{T_f^\perp M},$$

and its *normal connection* by

$$\nabla^\perp\colon TM \times T^\perp M \to T_f^\perp M, \quad \nabla_X^\perp \xi = (\tilde\nabla_X \xi)_{T_f^\perp M}.$$

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First key problem: (Only known for a few manifolds)

For a given Riemannian manifold M^n, what is (locally or globally) the lowest possible codimension $p = p(M^n)$ in Nash Theorem?

Uniqueness: rigidity

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(In particular, rigidity implies that $p = p(M^n)$).
If otherwise, we say that f is *(isometrically) deformable*.

Deformations

Basic deformation problem:

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Basic deformation problem: (too difficult!)

Describe (in some sense) all deformations of

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when $p = p(M^n)$ is the *lowest* possible codimension.

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General deformation problem: (even more difficult...)

Describe all isometric immersions of M^n into $\mathbb{R}^{n+q}$ for a given codimension q... for 'small' codimension q ($q < n$?).

Local rigidity results: Hypersurfaces ($p = 1$)

The *type number* of $f\colon M^n \to \mathbb{R}^{n+1}$ at $x \in M^n$ is the integer

$$\tau(x) = \operatorname{rank} \alpha^f(x).$$

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FACT. By the Gauss equation,

$$\tau(x) \leq 1 \iff K_M(x) = 0.$$

Using the Gauss Parametrization (Sbrana), it is easy to see that the set of (local) flat hypersurfaces of $\mathbb{R}^{n+1}$ can be naturally identified with $C^\infty(\mathbb{R}, \mathbb{R}^{n+1})$: the smooth curves in $\mathbb{R}^{n+1}$.

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They are divided in 4 types of $(n-2)$-ruled hypersurfaces:

- surface-like (products): deform as surfaces in $\mathbb{R}^3$ or $\mathbb{S}^3$;
- (n-1)-ruled: moduli space of deformations $= C^\infty(\mathbb{R}, \mathbb{R})$;
- continuous type: moduli space of deformations $= \mathbb{R}$;
- discrete type: exactly one more deformation: moduli $\mathbb{Z}_2$.

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These are algebraic results, with little geometry...

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Let $N^{n+1} \subset \mathbb{R}^{n+2}$ be a Sbrana-Cartan (i.e., deformable and classified) hypersurface and

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This kind of "dishonest" deformations must be discarded since we are reducing the codimension of the problem, and for that purpose we introduce the following general definition:

Isometric extensions

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We say that a pair of isometric immersions $f\colon\ M^n \to \mathbb{R}^{n+p}$ and $g\colon\ M^n \to \mathbb{R}^{n+q}$ *extends isometrically* when there are:

- a Riemannian manifold N^{n+r} with $r \geq 1$,
- an isometric embedding $j\colon\ M^n \hookrightarrow N^{n+r}$,
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$$\text{such that } f = F \circ j \text{ and } g = G \circ j.$$

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Given a submanifold $f\colon M^n \to \mathbb{R}^{n+p}$ we say that $g\colon M^n \to \mathbb{R}^{n+q}$ is a *genuine deformation* of f there is no open subset $U \subset M^n$ along which the restrictions $f|_U$ and $g|_U$ extend isometrically.

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Remark

This is a symmetric concept: for pairs and sets.

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Definition (Genuine rigidity)

Given $f\colon M^n \to \mathbb{R}^{n+p}$ and a positive integer q, we say that f is *genuinely rigid* in $\mathbb{R}^{n+q}$ if for any $g\colon M^n \to \mathbb{R}^{n+q}$ there is an open dense subset $U \subset M^n$ such that $f|_U$ and $g|_U$ extend isometrically.
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Remark

- *If $r = p = q$, we recover the concept of isometric rigidity.*

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Remark

- *If $r = p = q$, we recover the concept of isometric rigidity.*
- *If $r = p < q$, we recover the concept of compositions.*

The main result

Assume that the submanifolds $f\colon M^n \to \mathbb{R}^{n+p}$ and $g\colon M^n \to \mathbb{R}^{n+q}$ are *mutually D^d-ruled*, that is, D^d is a d-dimensional integrable distribution of M^n whose leaves are mapped by f and g onto open subsets of affine subspaces.

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Set

$$L_D = \text{span}\{\alpha^f(D, TM)\} \quad and \quad L'_D = \text{span}\{\alpha^g(D, TM)\}.$$

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Set

$$L_D = \mathrm{span}\{\alpha^f(D, TM)\} \quad and \quad L'_D = \mathrm{span}\{\alpha^g(D, TM)\}.$$

Then, by the Gauss equation, the map $\mathcal{T}_D\colon\ L_D \to L'_D$ given by

$$\mathcal{T}_D(\alpha^f(Y, X)) = \alpha^g(Y, X), \quad Y \in D^d, X \in TM,$$

is a well defined bundle isometry.

The main result: local in nature

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Let $g\colon M^n \to \mathbb{R}^{n+q}$ be a genuine deformation of $f\colon M^n \to \mathbb{R}^{n+p}$, with $p + q < n$ and $\min\{p, q\} \leq 5$.

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Let $g\colon M^n \to \mathbb{R}^{n+q}$ be a genuine deformation of $f\colon M^n \to \mathbb{R}^{n+p}$, with $p+q<n$ and $\min\{p,q\} \leq 5$. Then, locally f and g are mutually D^d-ruled, with $d \geq n-p-q+3\dim L_D$.

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Let $g\colon M^n \to \mathbb{R}^{n+q}$ *be a genuine deformation of* $f\colon M^n \to \mathbb{R}^{n+p}$, *with* $p+q < n$ *and* $\min\{p,q\} \leq 5$. *Then, locally* f *and* g *are mutually* D^d*-ruled, with* $d \geq n-p-q+3\dim L_D$.

Locally $=$ along connected components of an open dense subset.

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Locally = along connected components of an open dense subset.

Remark

The above result generalizes and unifies several rigidity results and all the ones we know about compositions.

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Moreover, the estimate $d \geq n - p - q + 3 \dim L_D$ is sharp.

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But: it is very hard to obtain global applications from this result!

Genuine conformal rigidity

Analogously, given a manifold M^n with a *conformal* structure, we may want to understand its *conformal* immersions $f\colon M^n \to \mathbb{R}^{n+p}$.

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In 2010, together with Ruy Tojeiro (UFSCar), we extended several of these conformal results by means of a genuine *conformal* rigidity framework, giving them a unified character. Actually, we obtained precisely the conformal version of the genuine rigidity theorem above, with its corresponding corollaries.

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Theorem (translating Sacksteder's Theorem...)

Any compact Euclidean hypersurface is genuinely rigid.

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Outside the realm of hypersurfaces, the only known result of this kind is the codimension two version of Sacksteder's theorem:

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Why **"maybe singularly"**?? The introduction of singularities in this result was shown to be a necessary condition in a recent local classification we made with Guilherme Freitas (2017).

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We should not try to avoid singularities when working with ruled extensions!

Singularities are not only necessary, but also natural for the problem, deeply simplify the theory, and are easy to deal with for ruled submanifolds, as we do when we classify flat and ruled surfaces in $\mathbb{R}^3$.

Introducing singularities

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Yet, we got some nice surprises for the local theory...

Definitions

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M^n \arrow[urr, "f"] \arrow[r, hook, "j"] \arrow[drr, "g"'] & N^{n+r} \arrow[ur, "F"] \arrow[dr, "G"] & \\
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We say that f and g *singularly extend isometrically* if the diagram commutes, where j is an isometric embedding as before, and F and G are isometric immersions, except maybe on M^n.

Definitions

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With these definitions we show:

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The proof actually works untouched for compact (or complete and bounded) submanifolds in hyperbolic space, and with a little bit of extra work we obtained a similar result for complete submanifolds in the sphere.

Global consequences

For $p+q \leq 4$ this unifies Sacksteder and Dajczer-Gromoll theorems above, and states that the only way to isometrically immerse a compact Euclidean hypersurface in codimension 3 is as a composition:

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Corollary

Any compact (or complete and bounded) submanifold M^n of $\mathbb{R}^{n+p}$, $n \geq 5$, is singularly genuinely rigid in $\mathbb{R}^{n+q}$ for $q \leq 4 - p$.

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Corollary

Let M^n be a compact manifold whose k-th Pontrjagin class satisfies that $[p_k] \neq 0$ for some $k > \frac{3}{4}(p+q-3)$. Then, any analytic immersion $f : M^n \to \mathbb{R}^{n+p}$ (with the induced metric) is singularly genuinely rigid in $\mathbb{R}^{n+q}$ in the C^∞-category.

Local analysis: the form ϕ_τ

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be a vector bundle isometry that preserves the second fundamental forms and the normal connections restricted to the rank ℓ vector normal subbundles L and $\hat{L}$. Extend τ as Id on TM: $\bar{\tau} = Id \oplus \tau$.

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$$\phi_\tau(X, \eta) = \left((\tilde{\nabla}_X \eta)_{L^\perp}, (\tilde{\nabla}_X \bar{\tau}\eta)_{\hat{L}^\perp}\right),$$

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$$\phi_\tau(X, \eta) = \left((\tilde{\nabla}_X \eta)_{L^\perp}, (\tilde{\nabla}_X \bar{\tau}\eta)_{\hat{L}^\perp}\right),$$

where $\tilde{\nabla}$ stands for the connection in Euclidean space and $L^\perp \times \hat{L}^\perp$ is endowed with the semi-Riemannian metric $\langle\,,\rangle = \langle\,,\rangle|_{L^\perp} - \langle\,,\rangle|_{\hat{L}^\perp}$.

The form ϕ_τ and the main local result

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The key advantage here over other local rigidity results: it deals with easily to construct null sets instead of nullity distributions.

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Recall that $Y \in TM$ is a *(left) regular element* of ϕ_τ if

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where $\phi_\tau^X = \phi_\tau(X, \cdot)$. Denote by $RE(\phi_\tau) \subset TM$ the open dense subset of regular elements of ϕ_τ, and $K_Y = \ker(\phi_\tau^Y)$ for $Y \in RE(\phi_\tau)$.

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Corollary

If $\hat{f}$ is a strongly genuine deformation of f, then locally they are mutually $\mathcal{O}(D_Y^d)$-ruled $\forall Y \in RE(\phi_\tau)$, where $D_Y^d = \ker(\phi_\tau^Y) \subset TM$ and $d = n + \ell - i(\phi_\tau) \geq n - p - q + 3\ell$.

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Of course, we also obtain local rigidity under any circumstance that excludes ruled submanifolds. For example:

Corollary

Any immersed submanifold M^n of $\mathbb{R}^{n+p}$ with $Ric > 0$ is singularly genuinely rigid in $\mathbb{R}^{n+q}$ for any $q < n - p$.

All the computations

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Proof:

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Proof:........ Just kidding!! :oP ∎

Thanks!!!!

$\mathcal{L}.$

There's nothing here.

Really!

Really really!

Really really reaaaally!

:op