On deformable hypersurfaces in space forms

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Abstract

We first extend the classical Sbrana-Cartan theory of isometrically deformable Euclidean hypersurfaces to the sphere and hyperbolic space. Then we construct and characterize a large family of hypersurfaces which admit a unique deformation. This is used to show, by means of explicit examples, that different types of hypersurfaces in the Sbrana-Cartan classification can be smoothly attached. Finally, among other applications, we discuss the existence of complete deformable hypersurfaces in hyperbolic space.

The classification of all locally isometrically deformable euclidean hypersurfaces due to Sbrana ([**Sb**]) and Cartan ([$\mathbf{Ca}_1$]) has been of crucial importance in several recent developments, among others [$\mathbf{DG}_4$], [$\mathbf{DG}_5$] and [**DF**]. Sbrana, who stated his results in terms of what is now called (see [$\mathbf{DG}_1$]) the Gauss parametrization, was inspired by works due to Schur ([**Sc**]) and Bianchi ([$\mathbf{Bi}_1$]). A few years after Sbrana, Cartan[1] published similar results but in the language of envelopes of hyperplanes.

Deformable hypersurfaces can be divided into four classes. Submanifolds belonging to the two less interesting ones, namely, 'surface-like' and ruled hypersurfaces, are highly deformable. On the contrary, while hypersurfaces in one of the remaining classes admit, precisely, a continuous one-parameter family of isometric deformations, elements belonging to the other class have a unique one.

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[1] For an interesting commentary see §473, p. 550 of [$\mathbf{Bi}_2$].

The main result in the Sbrana-Cartan theory is a parametric classification of all hypersurfaces in the two most important classes. For the first one, the description turns out to be quite satisfactory in the sense that it enables the construction of many explicit examples; cf §22 of [**Ca**$_1$] and §4 of this paper. For the remaining class the situation is quite different. First of all, the parametric description is cumbersome. One has to search for surfaces for which a pair of Christoffel symbols associated to a conjugate system of coordinates satisfies a certain complicated system of second order partial differential equations. Thus, it is not surprising that no example comes out from this result. In fact, it was not clear until now whether hypersurfaces of this type of dimension at least 4 even exist. See [**Ko**] for a claim in the 3-dimensional case.

Our purpose here is threefold. First, to extend the Sbrana-Cartan theory to hypersurfaces in the sphere and hyperbolic space. For that, we follow Sbrana's approach which is more convenient in applications and makes possible to treat the problem in an unified fashion. Nevertheless, we should point out that the one adopted by Cartan has the advantage of naturally extending to the realm of conformal deformations (see [**Ca**$_2$]).

Our second and main goal is to address the existence problem of deformable hypersurfaces of the discrete type. Roughly speaking, we show that intersecting two hypersurfaces with the same constant sectional curvature as that of a ambient space form yields, generically, a hypersurface of discrete type together with its unique isometric deformation. Moreover, we provide a parametric description of all deformable hypersurfaces which can be obtained as intersections. In particular, this enables us to construct explicit examples where deformable hypersurfaces of different types in the Sbrana-Cartan theory are smoothly attached, thus showing the local nature of their results.

We should point out that all of the above examples belong to a class of discretely deformable hypersurfaces which we call of real type. As to now, no example of complex type has been constructed.

Finally, among other applications, we study complete deformable hypersurfaces in hyperbolic space. The euclidean and spherical cases have been analyzed in [**Fe**] (see [**Da**]) and [**DG**$_4$]). First, we fully describe those of a particular simple type, the surface-like ones, which include the family of examples discovered by Mori ([**Mor**$_1$]). Then we discuss existence of examples of the remaining types. Our main accomplishment is a nonexistence result which, to our surprise, implies that deformable hypersurfaces obtained as intersections cannot be complete unless surface-like.

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§1 Surfaces of $1^{\underline{st}}$ and $2^{\underline{nd}}$ species

Deformable hypersurfaces in space forms belonging to the two most interesting classes in the Sbrana-Cartan theory are affine vector bundles build on from the normal bundles of certain spherical surfaces. We introduce these surfaces in this section and then discuss their parametrization.

Let us denote by $\mathbf{O}^{n+1}$ both, the Euclidean space $\mathbf{R}^{n+1}$ or the Lorentzian flat space $\mathbf{L}^{n+1}$. For simplicity, $\mathbf{S}_1^n \subset \mathbf{O}^{n+1}$ will stand for the Riemannian or Lorentzian unit sphere

$$\mathbf{S}_1^n = \{x \in \mathbf{O}^{n+1} \colon \|x\| = 1\}.$$

By a pair $\{h, (u,v)\}$ we mean a spherical surface $h\colon V^2 \to \mathbf{S}_1^n$ endowed with a global coordinate system (u,v). Recall that (u,v) is called a *real conjugate* system of coordinates when the second fundamental form $\alpha_h\colon TV \times TV \to \mathcal{N}$ of h with values in the normal vector bundle satisfies everywhere,

$$\alpha_h(\partial_u, \partial_v) = 0 \tag{1}$$

for the coordinate vector fields $\partial_u = \partial/\partial u,\ \partial_v = \partial/\partial v$. It is called *complex conjugate* when condition (1) holds for the complex coordinate vector fields $\partial_z = \partial_u - i\partial_v,\ \partial_{\bar z} = \partial_u + i\partial_v$, that is,

$$\alpha_h(\partial_z, \partial_{\bar z}) = \alpha_h(\partial_u, \partial_u) + \alpha_h(\partial_v, \partial_v) = 0. \tag{2}$$

For h regarded as an $\mathbf{O}^{n+1}$–valued map, condition (1) takes the form

$$\mathrm{Hess}_h(\partial_u, \partial_v) + \langle \partial_u, \partial_v \rangle h = 0. \tag{3}$$

In other words, $h = (h^1, \ldots, h^{n+1})$ satisfies

$$h^j_{uv} - \Gamma^1 h^j_u - \Gamma^2 h^j_v + \langle \partial_u, \partial_v \rangle h^j = 0, \ \ 1 \le j \le n+1, \tag{4}$$

where Γ^1, Γ^2 are the Christoffel symbols for the induced metric given by

$$\nabla'_{\partial_u} \partial_v = \Gamma^1 \partial_u + \Gamma^2 \partial_v.$$

We look for surfaces $\{h,(u,v)\}$ for which the system of equations

$$\begin{cases} \tau_u = 2\Gamma^2\tau(1-\tau) \\ \tau_v = 2\Gamma^1(1-\tau) \end{cases} \tag{5}$$

has positive solutions other than the trivial one $\tau \equiv 1$. The integrability condition for (5) is

$$(\Gamma^2_v - 2\Gamma^1\Gamma^2)\tau - \Gamma^1_u + 2\Gamma^1\Gamma^2 = 0. \tag{6}$$

Following Sbrana, we say that $\{h,(u,v)\}$ is of *first species* if (6) is trivially satisfied, i.e.,

$$\Gamma^1_u = \Gamma^2_v = 2\Gamma^1\Gamma^2, \tag{7}$$

and, in addition, an everywhere positive solution exists (which is always the case locally; cf. (56)).

We call the surface of *second species* if it is not of first species and

$$\tau = \frac{\Gamma^2_v - 2\Gamma^1\Gamma^2}{\Gamma^1_u - 2\Gamma^1\Gamma^2}$$

is positive and a (necessarily unique) solution of (5).

Proposition 1. *A surface* $h\colon \mathcal{U} \subset \mathbf{R}^2 \to \mathbf{S}^n_1 \subset \mathbf{O}^{n+1}$ *is of first species with real conjugate coordinates* (u,v) *if and only if there exist functions* $U = U(u)$, $V = V(v)$ *and* $F = F(u,v)$ *such that all coordinate functions of* h *satisfy the same differential equation*

$$h^j_{uv} + \frac{V_v}{2(U+V)}h^j_u + \frac{U_u}{2(U+V)}h^j_v + Fh^j = 0. \tag{8}$$

Proof: Solving equations (7), we get

$$\Gamma^1 = \frac{-V_v}{2(U+V)}, \quad \Gamma^2 = \frac{-U_u}{2(U+V)},$$

where $U = U(u)$ and $V = V(v)$, and the direct statement follows from (4). The converse is straightforward. ∎

When $\{h,(u,v)\}$ has complex conjugate coordinates, we define a complex-valued connection function $\Gamma = \Gamma(z,\bar z)$, where $z = u + iv$, by

$$\nabla'_{\partial_z}\partial_{\bar z} = \Gamma\partial_z + \bar\Gamma\partial_{\bar z}.$$

Then condition (2) takes the form

$$\mathrm{Hess}_h(\partial_z, \partial_{\bar z}) + \langle\partial_z, \partial_{\bar z}\rangle h = 0,$$

which can also be written as

$$h^j_{uu} + h^j_{vv} - 2\Gamma^1 h^j_u - 2\Gamma^2 h^j_v + (\langle\partial_u,\partial_u\rangle + \langle\partial_v,\partial_v\rangle)h^j = 0, \quad 1 \le j \le n+1, \tag{9}$$

where $\Gamma = \Gamma^1 + i\Gamma^2$. Consider the differential equation

$$\rho_{\bar z} + \Gamma(\rho - \bar\rho) = 0, \tag{10}$$

where $\rho = \rho(z,\bar z)$ takes values in the unit circle. In this case, we call $\{h,(u,v)\}$ of first species when the integrability condition

$$\mathrm{Im}\ \rho(\Gamma_z - 2\Gamma\bar\Gamma) = 0 \tag{11}$$

of equation (10) is trivially satisfied, i.e.,

$$\Gamma_z (= \bar\Gamma_{\bar z}) = 2\Gamma\bar\Gamma, \tag{12}$$

which is the complex analogue of (7). We say that h is of second species if it is not of first species and (10) has a (necessarily unique) solution determined by (11).

Proposition 1'. *A surface h: $\mathcal{U} \subset \mathbf{R}^2 \to \mathbf{S}^n_1 \subset \mathbf{O}^{n+1}$ with complex conjugate coordinates $\{u,v\}$ is of first species with complex conjugate coordinates $\{u,v\}$ if and only if there exist functions $\phi = \phi(u,v)$ satisfying*

$$\phi_{uu} + \phi_{vv} = 0 \tag{13}$$

and $F = F(u,v)$ such that all coordinate functions of h are solutions of the same differential equation

$$h^j_{uu} + h^j_{vv} + \frac{\phi_u}{\phi}h^j_u + \frac{\phi_v}{\phi}h^j_v + Fh^j = 0. \tag{14}$$

Proof: Solving equation (12), we get

$$\Gamma^1 = -\frac{\phi_u}{2\phi}, \ \ \Gamma^2 = -\frac{\phi_v}{2\phi},$$

where $\phi = \phi(u,v)$ satisfies (13), and the direct statement follows from (9). The converse is straightforward. ∎

Remark 2. As Cartan does, one can look at Propositions 1 and 1′ in a unified way. In fact, replacing (u,v) by $(z,\bar{z})$ and $U(u), V(v)$ by $U(z), V(\bar{z}) = \bar{U}(z)$ in (8), we get (14) for $\phi = \mathrm{Re}\, U(z)$.

§2 The Sbrana-Cartan theory

Following [**DG**$_5$], we call an isometric immersion $f\colon\ M^n \to \mathbf{Q}_c^{n+1},\ n \geq 3$, into a space form a *Sbrana-Cartan hypersurface* if M^n has no points with constant sectional curvature c and f admits a nowhere congruent isometric immersion. In this section, we locally describe in a parametric form all Sbrana-Cartan hypersurfaces, but first we discuss some basic facts and definitions.

For a hypersurface $f\colon\ M^n \to \mathbf{Q}_c^{n+1}$ with a unit normal vector field N and second fundamental form $A^f = A_N^f\colon TM \to TM$, it is a well known fact that the *relative nullity* spaces $\Delta(x) := \ker A^f(x)$ form an integrable distribution along any open subset where the *index of relative nullity* $\nu_f(x) := \dim \Delta(x)$ is constant. Moreover, the leaves of the induced foliation are totally geodesic in $\mathbf{Q}_c^{n+1}$ and N is constant along them.

By the classical Beez–Killing rigidity theorem, a Sbrana-Cartan hypersurface has precisely two nonzero principal curvatures at each point. Therefore, f can be locally parametrized by an inverse to its Gauss map, called in [**DG**$_1$] the *Gauss parametrization*. Let us denote by $\pi\colon U \to V^2$ the quotient space of relative nullity leaves in an open subset $U \subset M^n$. If the ambient space is euclidean, the Gauss image $h\colon\ V^2 \to \mathbf{S}_1^n \subset \mathbf{R}^{n+1}$ is the isometric immersion (with the induced metric) so that

$$h \circ \pi = N,$$

where N: $M^n \to \mathbf{S}_1^n$ is the Gauss map. Then f can be parametrized along the normal bundle $\mathcal{N}$ of h in $\mathbf{S}_1^n$ by Ψ: $\mathcal{N} \to \mathbf{R}^{n+1}$ given by

$$\Psi(x, w) = (\gamma h + \text{grad } \gamma)(x) + w, \tag{15}$$

where γ is the "support function" defined by $\gamma \circ \pi = \langle f, N \rangle$. It is easily seen that Ψ has maximal rank n at (x, w) if and only if the self adjoint operator $\text{Hess}_\gamma + \gamma \text{I} \ - B_w$ is nonsingular. Here Hess_γ stands for the linear operator associated to the Hessian of γ and B_w denotes the tangent valued second fundamental form of h in direction w.

There are similar parametrizations when $c \neq 0$. From now on, we consider $\mathbf{Q}_c^{n+1}$, $c \neq 0$, isometrically embedded into $\mathbf{O}^{n+2}$. For simplicity, we take $c = \pm 1$. Associated to the Gauss map N: $M^n \to \mathbf{S}_1^{n+1} \subset \mathbf{O}^{n+2}$, we have the Gauss image h: $V^2 \to \mathbf{S}_1^{n+1} \subset \mathbf{O}^{n+2}$ satisfying $h \circ \pi = N$. In the spherical case, the Gauss parametrization Ψ: $\mathcal{N}^1 \to \mathbf{S}_1^{n+1}$ is defined on the unit normal bundle $\mathcal{N}^1$ of h and takes the form

$$\Psi(x, w) = w. \tag{16}$$

Here Ψ has rank n at (x, w) if and only if B_w is nonsingular. In hyperbolic space, the Gauss parametrization Ψ: $\mathcal{N}^1 \to \mathbf{H}_{-1}^{n+1}$ is also given by (16), but now the elements of $\mathcal{N}^1$ have length -1. We refer to [$\mathbf{DG}_1$] for further information on the subject.

We say that f: $M^n \to \mathbf{Q}_c^{n+1}$ is *ruled* if M^n admits a foliation by leaves of codimension one which are mapped by f into totally geodesic submanifolds of $\mathbf{Q}_c^{n+1}$.

Theorem 3 ([Sb], [Ca$_1$]). *Let f: $M^n \to \mathbf{Q}_c^{n+1}$, $n \geq 3$, be a Sbrana-Cartan hypersurface. Then, there is an open dense subset $\mathcal{U} \subset M^n$ such that one of the following holds on any connected component U of $\mathcal{U}$:*

(I) i) $c = 0$ and $f(U) \subset L^2 \times \mathbf{R}^{n-2}$ where L^2 is a surface in $\mathbf{R}^3$, or
ii)-a) $c = 0$ and $f(U) \subset CL^2 \times \mathbf{R}^{n-3}$ where $CL^2 \subset \mathbf{R}^4$ is a cone over a surface $L^2 \subset \mathbf{S}^3$, or
ii)-b) $c \neq 0$ and $f(U) \subset CL^2 \times \mathbf{O}^{n-2} \cap \mathbf{Q}_c^{n+1} \subset \mathbf{O}^{n+2}$ where $CL^2 \subset \mathbf{O}^4$ is a cone over a surface $L^2 \subset \mathbf{Q}_{\tilde{c}}^3$ with $\mathbf{Q}_{\tilde{c}}^3 \subset \mathbf{Q}_c^{n+1}$ totally umbilical, and $\mathbf{O}^{n-2}$, $\mathbf{O}^4$ have different signatures when $c < 0$.

(II) f is a ruled hypersurface.

(III) In terms of the Gauss parametrization, f is given by $\{h, (u, v)\}$ of first species and, when $c = 0$, a function γ satisfying the same equation (8) *or* (14) *as any of the coordinate functions of h does.*

(IV) In terms of the Gauss parametrization, f is given by $\{h, (u, v)\}$ of second species and, when $c = 0$, a function γ satisfying the same equation (4) *or* (9) *as any of the coordinate functions of h does.*

Conversely, any simply connected hypersurface which can be described as in (II), (III) or (IV) is Sbrana-Cartan. Moreover, any deformation of a hypersurface of type (I) is given by a deformation of the surface L^2, whereas the set of deformations of a hypersurface of type (II), (III) or (IV) which is not of type (I) is, respectively, parametrized by all smooth functions in an interval, a continuous 1–parameter family or contains only one other immersion. In all cases deformations are always of the same type.

It is easy to verify that hypersurfaces of type (I) and (II) can be smoothly attached; see [$\mathbf{DG}_2$]. In the next section, we construct explicit examples of Sbrana-Cartan hypersurfaces which contain open subsets of type (I), (*III*) and (IV).

We say that a Sbrana-Cartan hypersurface of type (*III*) or (IV) is of *real* or *complex type* according to whether the associated coordinate system on its Gauss image is real or complex conjugate, respectively.

Remarks 4. 1) In Cartan's terminology, a hypersurface in $\mathbf{R}^{n+1}$ with two nonzero principal curvatures is given as an envelope of a two parameter family of hyperplanes,

$$\alpha^1 x_1 + \ldots + \alpha^{n+1} x_{n+1} + \alpha^0 = 0,$$

with $\alpha^j = \alpha^j(u, v),\ 0 \leq j \leq n+1$. In terms of the Gauss parametrization, the hypersurface is determined by

$$h = \frac{1}{\sqrt{\sum_{j=1}^{n+1} (\alpha^j)^2}} (\alpha^1, \ldots, \alpha^{n+1}),\ \ \gamma = \frac{\alpha^0}{\sqrt{\sum_{j=1}^{n+1} (\alpha^j)^2}}.$$

Then the hypersurface is of real type (*III*) if and only if all homogeneous *tangential coordinates* α^j satisfy the same differential equation

$$\alpha^j_{uv} + M\alpha^j = 0,\ \text{where}\ \sum_{j=1}^{n+1} (\alpha^j)^2 = U(u) + V(v). \tag{17}$$

Accordingly, it is of complex type (*III*) if and only if

$$\alpha^j_{uu} + \alpha^j_{vv} + M\alpha^j = 0, \text{ where } \sum_{j=1}^{n+1} (\alpha^j)^2 = \phi(u,v) \ \text{ with } \ \phi_{uu} + \phi_{vv} = 0. \quad (18)$$

To see that Cartan's description is equivalent to the one given in Theorem 3, the key observation is that any solution h^j of (8) (respectively, (14)) gives rise to a solution $\alpha^j = \sqrt{U+V}\, h^j$ (respectively, $\alpha^j = \sqrt{\phi}\, h^j$) of an equation of type (17) (respectively, (18)) and vice-versa.

2) A Sbrana-Cartan hypersurface of type (*III*) or (*IV*) with sectional curvature $K_M \geq 0$ is always of real type.

Before going into the proof of Theorem 3, we review some basic facts and obtain preliminary results. Consider an isometric immersion $f\colon M^n \to \mathbf{Q}_c^{n+1}$ with constant index of relative nullity. Associated to its relative nullity foliation, one defines the *splitting tensor* C which assigns to each $T \in \Delta$ the endomorphism C_T of $\Delta^\perp$ given by

$$C_T X = -(\nabla_X T)_{\Delta^\perp}.$$

A crucial fact is that the splitting tensor is solely determined by the foliation and, in that sense, independent of f. Hence, given two isometric hypersurfaces with the same nullity foliations, as turns out to be the case for two isometric Sbrana-Cartan hypersurfaces, any geometric property derived from the structure of C necessarily holds for both.

In the following statement and the sequel, A^f has to be considered restricted to $\Delta^\perp$.

Lemma 5 ([**DG**$_4$]). *Let $f\colon M^n \to \mathbf{Q}_c^{n+1}$ be an isometric immersion with constant index of relative nullity $\nu_f = k$. Then,*

i) The following differential equations hold:

$$\nabla_T A^f = A^f \circ C_T, \ \ \forall\, T \in \Delta, \quad (19)$$

$$\nabla_{T_1} C_{T_2} = C_{T_2} C_{T_1} + C_{\nabla_{T_1} T_2} + c\langle T_1, T_2\rangle I, \ \ \forall\, T_1, T_2 \in \Delta, \quad (20)$$

$$(\nabla^{\Delta^\perp}_X C_T)Y - (\nabla^{\Delta^\perp}_Y C_T)X = C_{(\nabla_X T)_\Delta} Y - C_{(\nabla_Y T)_\Delta} X, \ \ \forall\, T \in \Delta. \quad (21)$$

ii) The distribution $\Delta^{\perp}$ *is integrable if and only if* C_T *is selfadjoint for all* $T \in \Delta$.

Next, we characterize hypersurfaces with constant index of relative nullity whose splitting tensor has one of two special structures.

Lemma 6. *Let* $f\colon M^n \to \mathbf{Q}_c^{n+1}$ *be an isometric immersion with constant index of relative nullity* $\nu_f = k > 0$*. Then,*

i) C *vanishes identically if and only if* $c = 0$ *and each point has a neighborhood* V *such that* $f(V) \subset L^{n-k} \times \mathbf{R}^k$*, where* L^{n-k} *is a hypersurface of* $\mathbf{R}^{n-k+1}$.

ii) There exists a unit $T \in \Delta$ *such that coker* $C =$ *span*$\{T\}$ *and* $C_T = \mu I$ *,* $\mu \neq 0$*, if and only if one of the following holds:*

a) $c = 0$ *and each point has a neighborhood* V *such that* $f(V) \subset CL^{n-k} \times \mathbf{R}^{k-1}$*, where* $CL^{n-k} \subset \mathbf{R}^{n-k+2}$ *is the cone over a hypersurface* L^{n-k} *of* $\mathbf{S}^{n-k+1}$*, or*

b) $c \neq 0$ *and each point has a neighborhood* V *such that* $f(V) \subset CL^{n-k} \times \mathbf{O}^k \cap \mathbf{Q}_c^{n+1} \subset \mathbf{O}^{n+2}$*, where* $CL^{n-k} \subset \mathbf{O}^{n-k+2}$ *is the cone over a hypersurface* L^{n-k} *in a totally umbilical* $\mathbf{Q}_{\tilde{c}}^{n-k+1} \subset \mathbf{Q}_c^{n+1}$.

Proof: i) If $C = 0$, equation (20) immediately implies that $c = 0$. Moreover, from the definition of C it follows that Δ is parallel in M^n and, therefore, constant in $\mathbf{R}^{n+1}$. The converse is trivial.

ii) We prove the direct statement, the other being easy. By assumption,

$$\langle \nabla_X S, Y\rangle = -\langle C_S X, Y\rangle = -\mu\langle S, T\rangle\langle X, Y\rangle,\ \ \forall\, X, Y \in \Delta^{\perp},\ \forall\, S \in \Delta. \quad (22)$$

On the other hand, we easily obtain from (21) that

$$\langle \nabla_X S, T\rangle = 0,\ \ \forall\, S \in \ker C, \quad (23)$$

and

$$X(\mu) = 0,\ \ \forall\, X \in \Delta^{\perp}. \quad (24)$$

We conclude from (22) and (23) that

$$\widetilde{\nabla}_X S \in \ker C,\ \ \forall\, S \in \ker C, \quad (25)$$

and

$$\tilde{\nabla}_X T = -\mu X, \ \ \forall\, X \in \Delta^\perp, \tag{26}$$

where $\tilde{\nabla}$ denotes the connection in either $\mathbf{R}^{n+1}$ or $\mathbf{O}^{n+2} \supset \mathbf{Q}_c^{n+1}$ according as $c = 0$ or $c \neq 0$, respectively. Equation (20) yields

$$\mu\langle \nabla_R S, T\rangle + c\langle R, S\rangle = 0, \ \forall\, R \in \Delta, S \in \ker C, \tag{27}$$

and

$$T(\mu) = \mu^2 + c, \quad S(\mu) = 0, \ \forall\, S \in \ker C. \tag{28}$$

In particular, we conclude from (27) that

$$\nabla_T T = 0. \tag{29}$$

Assume $c = 0$. It follows from (25) and (27) that the distribution $\ker C$ is constant in $\mathbf{R}^{n+1}$. Hence, each point has a product neighborhood $V = L^{n-k+1} \times V^{k-1}$, $V^{k-1} \subset \mathbf{R}^{k-1}$ open, on which $f = g \times \mathrm{I}$ splits isometrically. It remains to prove that g is a cone. Since $T \in \Delta$, we conclude from (29) that the integral curves of T are straight lines in $\mathbf{R}^{n+1}$. Using (24), (26) and (28) we obtain

$$\tilde{\nabla}_T\left(g + \frac{1}{\mu}T\right) = 0 \ \text{ and } \ \tilde{\nabla}_X\left(g + \frac{1}{\mu}T\right) = 0, \ \ \forall\, X \in \Delta^\perp,$$

which shows that all lines pass through a fixed point and concludes the proof of part a).

Suppose now that $c \neq 0$. By part ii) of Lemma 5, the distribution $\Delta^\perp$ is integrable. For a fixed leaf L^{n-k} of $\Delta^\perp$, equations (25) and (26) imply that $\Omega = \Delta \oplus \mathrm{span}\{f\}$ is a parallel subbundle of the normal bundle of L^{n-k} in $\mathbf{O}^{n+2}$. Moreover, it follows from (22) that

$$(\tilde{\nabla}_X Y)_\Omega = \langle X, Y\rangle(\mu T - cf), \ \ \forall\, X, Y \in \Delta^\perp.$$

Therefore, L^{n-k} is contained in a $(n-k+1)$-dimensional umbilical submanifold $\mathbf{Q}_{\tilde{c}}^{n-k+1}$ of $\mathbf{Q}_c^{n+1}$. To conclude the proof of part b), it suffices to show that the cone over f in $\mathbf{O}^{n+2}$ splits isometrically as $CL^{n-k} \times \mathbf{O}^k$, where $CL^{n-k} \subset \mathbf{O}^{n-k+2}$ is the cone over L^{n-k}. Since the leaves of Δ are totally geodesic in $\mathbf{Q}_c^{n+1}$, the leaves of Ω are $(k+1)$–dimensional subspaces of $\mathbf{O}^{n+2}$.

We claim that the orthogonal complement $\bar{\Omega}$ of $\mu T - cf$ in Ω is constant in $\mathbf{O}^{n+2}$. In fact, we have

$$\bar{\Omega} = \ker C \oplus \mathrm{span}\{T + \mu f\}.$$

By (24) and (26),

$$\widetilde{\nabla}_X (T + \mu f) = 0, \ \ \forall\, X \in \Delta^{\perp}. \tag{30}$$

Equations (28) and (29) yield

$$\widetilde{\nabla}_T (T + \mu f) = \mu (T + \mu f), \tag{31}$$

whereas (27) and (28) give

$$\widetilde{\nabla}_S (T + \mu f) = (\frac{c}{\mu} + \mu) S. \tag{32}$$

On the other hand, for any $S \in \ker C$, we get from (22) and (29) that

$$\widetilde{\nabla}_S S = -\frac{c}{\mu}(T + \mu f) \quad \text{and} \quad \widetilde{\nabla}_T S \in \ker C. \tag{33}$$

The claim follows from (25) and (30) to (33), and this concludes the proof. ∎

Any hypersurface f as in part $ii)-b)$ can be described as a warped product of isometric immersions by the use of the warped product representation of space forms due to Nölker. This will be useful in the last section.

For a fixed point $\bar{x} \in \mathbf{Q}_c^{n+1}$, let

$$T_{\bar{x}} \mathbf{Q}_c^{n+1} = V^k \oplus V^{n-k+1}$$

be an orthogonal decomposition into nontrivial subspaces. Choose $z \in V^k$ and let $\mathbf{Q}_{\tilde{c}}^{n-k+1}$, $\tilde{c} = c + \|z\|^2$, be the umbilical submanifold of $\mathbf{Q}_c^{n+1}$ such that $T_{\bar{x}} \mathbf{Q}_{\tilde{c}}^{n-k+1} = V^{n-k+1}$ and whose mean curvature vector at $\bar{x}$ is z. Let $a = \tilde{c}\bar{x} - z$ be the mean curvature vector of $\mathbf{Q}_{\tilde{c}}^{n-k+1}$ in $\mathbf{O}^{n+2} \supset \mathbf{Q}_c^{n+1}$ at $\bar{x}$ and $\mathbf{Q}_c^k$ the totally geodesic submanifold

$$\mathbf{Q}_c^k = \mathbf{Q}_c^{n+1} \cap \{\mathrm{span}\{\bar{x}\} \oplus V^k\}.$$

Set $N^k = \mathbf{Q}_c^k$ if $\|a\|^2 \leq 0$ and $N^k = \mathbf{Q}_c^k \cap \{x : \langle a, x \rangle > 0\}$ otherwise, and consider the warped product $N^k \times_\sigma \mathbf{Q}_{\tilde{c}}^{n-k+1}$ with warping function $\sigma(x) = \langle a, x \rangle$. The map $\psi\colon N^k \times_\sigma \mathbf{Q}_{\tilde{c}}^{n-k+1} \to \mathbf{Q}_c^{n+1}$ given by

$$\psi(x, y) = x + \sigma(x)(y - \bar{x})$$

is an isometry onto either $\mathbf{Q}_c^{n+1}$ or $\mathbf{Q}_c^{n+1} \setminus \{\mathrm{span}\{a\} \oplus V^{n-k+1}\}^\perp$, according as $\|a\|^2 \leq 0$ or $\|a\|^2 > 0$, respectively.

Then, f is the warped product of the hypersurface $f_1\colon L^{n-k} \to \mathbf{Q}_{\tilde{c}}^{n-k+1}$ with the identity map on N^k given by

$$f = \psi \circ (\mathrm{id} \times f_1)\colon N^k \times_\sigma L^{n-k} \to \mathbf{Q}_c^{n+1}. \tag{34}$$

In fact, it is easily seen that the cone in $\mathbf{O}^{n+2}$ over the hypersurface defined by (34) factors as $CL^{n-k} \times \mathbf{O}^k$, where $CL^{n-k} \subset \mathbf{O}^{n-k+2}$ is the cone over L^{n-k}.

Proof of Theorem 3: Let $g\colon M^n \to \mathbf{Q}_c^{n+1}$ be a nowhere congruent isometric deformation of f. For each point $x \in M^n$, let

$$W(x) = T_{f(x)}^\perp M \oplus T_{g(x)}^\perp M$$

be endowed with the natural inner product $\langle\langle\ ,\ \rangle\rangle$ of type $(1,1)$, and set

$$\beta = \alpha_f \oplus \alpha_g\colon T_xM \times T_xM \to W(x).$$

The Gauss equations for f and g imply that β is *flat*, that is,

$$\langle\langle\beta((X,Y),\beta(Z,W)\rangle\rangle - \langle\langle\beta((X,W),\beta(Z,Y)\rangle\rangle = 0,\ \ \forall\, X,Y,Z,W \in TM.$$

Being g nowhere congruent to f, the subset $\mathcal{V}_0 \subset M^n$ where β is *null*, i.e., β satisfies

$$\langle\langle\beta((X,Y),\beta(Z,W)\rangle\rangle = 0,\ \ \forall\, X,Y,Z,W \in TM,$$

has empty interior. Moreover, it follows from Corollary 2 of [**Mo**] and the assumption that M^n has no points with constant sectional curvature c that

$$\Delta_f = \Delta_g \text{ and } \nu_f = \nu_g = n-2. \tag{35}$$

Lemma 7. *The endomorphism $D := (A^f)^{-1} \circ A^g\colon \Delta^\perp \to \Delta^\perp$ satisfies:*

i) $\det D = 1$,

ii) $[D, C_T] = 0,\ \ \forall\, T \in \Delta$,

iii) $\nabla_T D = 0,\ \ \forall\, T \in \Delta$.

Proof: i) It is immediate from the Gauss equations for f and g.
ii) Since the term on the left hand side of (19) is symmetric, we have

$$A^f \circ C_T = C_T^* \circ A^f, \tag{36}$$

where C_T^* denotes the adjoint operator of C_T. A similar equation also holds for $A^g = A^f \circ D$, thus

$$A^f D C_T = A^g C_T = C_T^* A^g = C_T^* A^f D = A^f C_T D,$$

and the proof follows.
iii) Equation (19) for A^f and A^g yields, respectively,

$$A^f C_T D = (\nabla_T A^f) D$$

and

$$A^f D C_T = A^g C_T = \nabla_T A^g = \nabla_T (A^f D).$$

Hence,

$$A^f [D, C_T] = A^f (\nabla_T D),$$

and the proof follows from ii). ∎

Lemma 8. *Let $f \colon M^n \to \mathbf{Q}_c^{n+1}$ be a Sbrana-Cartan hypersurface. Then,*

$$\dim \textit{coker}\ C \leq 2.$$

Moreover, when equality holds, then either C_T is symmetric for all $T \in \Delta$ or there exists $S \in$ coker C such that $C_S = \mu I$.

Proof: The first assertion follows immediately from ii) of Lemma 7. When $\dim \text{coker}\ C = 2$, by dimension reasons there exists $\bar{S} \in \text{coker}\ C$ such that $C_{\bar{S}}$ is symmetric. The last assertion then follows easily using again part ii) of Lemma 7. ∎

Let $M_0, M_1 \subset \mathcal{V} = M^n \backslash \mathcal{V}_0$ be the interiors of the subsets where $C = 0$ and where C satisfies the conditions of part ii) in Lemma 6, respectively. It follows from Lemma 6 that f is as in part $(I) - i$) or ii) along any connected component of M_0 or M_1, respectively.

Let $M_2 \subset \mathcal{V}$ be the interior of the set where $\dim \ker C$ is locally constant and there exists $S \in \text{coker}\ C$ so that C_S has one eigenvalue of multiplicity two without being symmetric. By Lemma 8, there exist $S \in \operatorname{coker} C$ smooth and a unique, up to signs, orthonormal frame $\{X, Y\}$ in $\Delta^\perp$ for which

$$C_S = \begin{bmatrix} a & 0 \\ b & a \end{bmatrix}, \quad b \neq 0. \tag{37}$$

Being $A^f C_S$ symmetric by (19), we have

$$a\langle A^f Y, X\rangle = \langle A^f C_S Y, X\rangle = \langle Y, A^f C_S X\rangle = a\langle A^f Y, X\rangle + b\langle A^f Y, Y\rangle,$$

which implies that Y is asymptotic for f, i.e.,

$$\langle A^f Y, Y\rangle = 0.$$

Since the same holds for g, we conclude from the Gauss equations that there exist smooth functions λ, μ and θ such that

$$A^f = \begin{bmatrix} \lambda & \mu \\ \mu & 0 \end{bmatrix}, \quad A^g = \begin{bmatrix} \lambda+\theta & \mu \\ \mu & 0 \end{bmatrix} \tag{38}$$

with respect to the frame $\{X, Y\}$. In particular,

$$DY = Y. \tag{39}$$

We claim that the distribution

$$x \mapsto \operatorname{span}\{Y(x)\} \oplus \Delta(x)$$

is integrable and totally geodesic. In fact, by (37) and Lemma (8),

$$\langle \nabla_Y T, X\rangle = -\langle C_T Y, X\rangle = 0, \ \ \forall\, T \in \Delta. \tag{40}$$

Moreover, from *iii*) of Lemma 7 and (39), we get $D\nabla_T Y = \nabla_T Y$, which implies that

$$\nabla_T Y = 0, \ \ \forall\, T \in \Delta. \tag{41}$$

On the other hand, an easy computation shows that A^f and A^g satisfy simultaneously the Codazzi equations if and only if

$$\langle \nabla_Y Y, X\rangle = 0, \tag{42}$$

$$Y(\theta) = \langle \nabla_X X, Y\rangle \theta \;\text{ and }\; T(\theta) = \langle \nabla_X X, T\rangle \theta, \;\; \forall\, T \in \Delta. \tag{43}$$

The claim follows from (40), (41), (42) and the fact that Δ is totally geodesic. We conclude that f is locally ruled on M_2.

Consider the open subset $M_3 \subset \mathcal{V}$ where there exists $S \in \Delta$ such that C_S has two distinct real eigenvalues. By *ii*) of Lemma 7, there exists a unique, up to signs, frame $\{Y_1, Y_2\}$ of unit eigenvectors of C_T for all $T \in \Delta$, with respect to which D has the form

$$D = \begin{bmatrix} \theta & 0 \\ 0 & 1/\theta \end{bmatrix}, \quad \theta \neq \pm 1. \tag{44}$$

Then, equation *iii*) of Lemma 7 is equivalent to

$$T(\theta) = 0, \quad \forall\, T \in \Delta, \tag{45}$$

and

$$\nabla_T Y_j = 0, \quad \forall\, T \in \Delta, \;\; 1 \leq j \leq 2. \tag{46}$$

Equation (45) says that θ is a function on the Gauss image V^2. We claim that there exist smooth functions μ_1 and μ_2 and a coordinate system (u, v) on V^2 such that the frame $\{X_1, X_2\}$ defined by $X_j = \mu_j Y_j$, $1 \leq j \leq 2$, satisfies

$$\partial_u \circ \pi = \pi_* X_1, \quad \partial_v \circ \pi = \pi_* X_2. \tag{47}$$

It suffices to have

$$\widetilde{\nabla}_T N_* X_j = 0 \tag{48}$$

and

$$[X_1, X_2] \in \Delta. \tag{49}$$

By the Codazzi equation,

$$\widetilde{\nabla}_T N_* X_j = -\widetilde{\nabla}_T A X_j = -\nabla_T A X_j = A[X_j, T].$$

Hence, (48) is equivalent to

$$[X_j, T] \in \Delta, \;\; \forall\, T \in \Delta, \;\; 1 \leq j \leq 2. \tag{50}$$

A straightforward calculation shows that each μ_j can be arbitrarily prescribed along an integral curve γ of Y_j and then extended along each integral curve of Y_i, $i \neq j$, and each geodesic of Δ through γ, as a solution of the linear first order differential equations

$$T(\mu_j) + b_j\mu_j = 0, \quad Y_i(\mu_j) + r_j\mu_j = 0,$$

where

$$C_T Y_j = b_j Y_j \quad \text{and} \quad [Y_1, Y_2] + r_1 Y_1 - r_2 Y_2 \in \Delta,$$

which proves the claim.

The Codazzi equation for $A^g = A^f \circ D$ and (49) yield

$$\nabla_{X_1}(A^f DX_2) = \nabla_{X_2}(A^f DX_1). \tag{51}$$

We have,

$$\begin{aligned}
\nabla_{X_2}(A^f DX_1) &= \nabla_{X_2}(A^f\theta X_1) = \widetilde{\nabla}_{X_2}(A^f\theta X_1) - \theta\langle A^f X_1, A^f X_2\rangle N \\
&= -\widetilde{\nabla}_{X_2} N_*\theta X_1 - \theta\langle N_* X_1, N_* X_2\rangle N = -\widetilde{\nabla}_{\partial_v}\theta\partial_u - \theta\langle\partial_u, \partial_v\rangle h \\
&= -\theta_v\partial_u - \theta\left(\nabla'_{\partial_v}\partial_u + \alpha_h(\partial_u, \partial_v)\right).
\end{aligned}$$

Similarly,

$$\nabla_{X_1}(A^f DX_2) = \frac{\theta_u}{\theta^2}\partial_v - \frac{1}{\theta}\left(\nabla'_{\partial_u}\partial_v + \alpha_h(\partial_u, \partial_v)\right).$$

Setting

$$\tau = \theta^2, \tag{52}$$

we easily see that (51) is equivalent to conditions (1) and (5). We conclude that V^2 is a surface of first or second species with real conjugate coordinates.

On the other hand, we have

$$\langle A^g X_1, X_2\rangle = \langle A^f DX_1, X_2\rangle = \theta\langle A^f X_1, X_2\rangle = \theta^2\langle A^g X_1, X_2\rangle,$$

hence X_1, X_2 are conjugate vectors, i.e.,

$$\langle A^f X_1, X_2\rangle = 0. \tag{53}$$

Moreover, by Proposition 1.8 in [$\mathbf{DG}_1$], we have for f in terms of the Gauss parametrization (15) that

$$A^f(w) = -(\mathrm{Hess}_\gamma + \gamma \mathrm{I} - B_w)^{-1}. \tag{54}$$

It follows from (1), (53) and (54) that γ satisfies (4). Of course, there is no support function when $c \neq 0$. We conclude that f is of real type (*III*) or (*IV*).

Finally, let $M_4 \subset \mathcal{V}$ be the open subset where there exists $S \in \Delta$ such that C_S has complex conjugate eigenvalues. Then D takes the form

$$D = \begin{bmatrix} \rho & 0 \\ 0 & \bar{\rho} \end{bmatrix}, \quad \|\rho\| = 1, \ \ \rho^2 \neq 1, \tag{55}$$

in the basis $Z = X_1 - iX_2, \bar{Z} = X_1 + iX_2$ of eigenvectors. A computation similar to the previous one shows that $Z, \bar{Z}$ induce conjugate coordinates $(z, \bar{z})$ on the Gauss image V^2 and that the function ρ defines a function on V^2 satisfying (10). Thus, V^2 is a surface of first or second species in the sphere with complex conjugate coordinates. Moreover, as in the real case, we conclude that the support function γ is a solution of equation (9), hence f is of complex type (*III*) or (*IV*).

We now prove the converse. First observe that isometric Sbrana-Cartan hypersurfaces have the same relative nullity foliations by (35), hence the same splitting tensor C. It follows from the proof of the direct statement that they are necessarily of the same type. The assertion on hypersurfaces of type (*I*) is now clear.

Suppose first that $f\colon M^n \to \mathbf{Q}_c^{n+1}$ is ruled. Let $\{X, Y\}$ be an orthonormal frame of $\Delta^\perp$ as before with X orthogonal to the rulings. Then (40), (41) and (42) hold, and there exist smooth functions λ and μ such that A^f is given as in (38). Thus, any solution θ of (43) defines a tensor A^g as in (38) satisfying the Gauss and Codazzi equations for an isometric immersion into $\mathbf{Q}_c^{n+1}$, hence gives rise to an isometric deformation of f. By the previous observation, any deformation arises this way. Notice that each such function is completely determined once an initial condition is chosen along a fixed orthogonal trajectory to the rulings.

Assume now that f is of real type (*III*) or (*IV*) and let $\{h, (u, v)\}$ be its Gauss image. By (54), the coordinate vector fields of h induce a frame of conjugate vector fields for f. Defining D in this frame by (44) with θ given by (52) in terms of a positive solution $\tau \neq 1$ of (5), the computations in the proof of the direct statement show that $A^f \circ D$ satisfies the Gauss and Codazzi equations. Therefore f is a Sbrana-Cartan hypersurface and the set of deformations is in correspondence with the set of positive solutions of (5). When V^2 is of first species, the general solution of (5) is given in terms of the functions $U(u), V(v)$ of Proposition 1 by

$$\tau(u,v) = \frac{c - V(v)}{c + U(u)}, \quad c \in \mathbf{R}, \tag{56}$$

and, therefore, f admits a one-parameter family of deformations. Notice that the constant c has to be chosen do that τ is positive. When V^2 is of second species, (5) has only one positive solution $\tau \neq 1$, and hence f admits a unique isometric deformation.

The proof that any simply connected hypersurface of complex type (*III*) or (IV) admits, respectively, a one-parameter family or a unique isometric deformation is completely similar to the one for hypersurfaces of real type. Now, however, one has to make use of the fact that, when V^2 is of first species, the general solution of (10) is given in terms of the function ϕ of Proposition 1' by

$$\rho = e^{i\theta}, \;\; \cot\theta = \frac{\mu + \lambda}{\phi}, \tag{57}$$

where μ is any particular solution of $\mu_v = \phi_u$ and $\lambda = \lambda(u)$ is determined, up to a constant, by

$$\lambda' + \mu_u + \phi_v = 0. \tag{58}$$

This concludes the proof. ∎

The works by Sbrana and Cartan (see also [**Bo**$_1$] and [**Bo**$_2$]) contain interesting additional information. One result in Sbrana's paper is particularly remarkable. Given a set of $n+1$ solutions $h^1, \ldots, h^{n+1}$ of the equation

$$\frac{\partial^2 \psi}{\partial u \partial v} = M\psi$$

satisfying the condition $\sum_{j=1}^{n+1}(h^j)^2 = 1$, Bianchi ([**Bi**$_1$]) had found an analytic transformation which generates a n–parameter family of new sets of solutions satisfying the same quadratic condition. The transformation reduces to solving a first order completely integrable system of differential equations. In view of the discussion in Remark 4, this transformation can be used to generate a n–parameter family of new surfaces of first species from a given one. In addition, there is a permutability formula which allows, given any two transforms of a surface of first species, to generate a third surface just by an algebraic procedure.

§3 Intersections

The main purpose of this section is to construct and characterize a large family of deformable hypersurfaces of class (IV). We also show, by means of explicit examples, that different types of hypersurfaces in the Sbrana-Cartan classification can be smoothly attached. For simplicity, statements and proofs are given for euclidean ambient space. Extensions to the sphere and hyperbolic space are straightforward.

Let $F_i\colon U_i \subset \mathbf{R}^{n+1} \to \mathbf{R}^{n+2}$ be two isometric embeddings of rank 1, i.e., free of totally geodesic points, whose Gauss maps $\eta_i\colon U_i \to \mathbf{S}_1^{n+1} \subset \mathbf{R}^{n+2}$ satisfy

$$0 < \langle \eta_1, \eta_2 \rangle < 1 \tag{59}$$

along $M^n = F_1(U_1) \cap F_2(U_2)$. Notice that (59) is just a condition on two curves. In fact, any flat hypersurface $F\colon U \subset \mathbf{R}^m \to \mathbf{R}^{m+1}$ of rank 1 can be locally parametrized (Gauss parametrization) by a map $\Psi\colon \mathcal{N} \to \mathbf{R}^{m+1}$ defined on the normal bundle of a unit speed curve $c\colon I \subset \mathbf{R} \to \mathbf{S}_1^m$, and given by

$$\Psi(s,w) = (rc + r'c')(s) + w,$$

where $r \in C^\infty(I)$. In particular, the Gauss map $\eta\colon U \to \mathbf{S}_1^m \subset \mathbf{R}^{m+1}$ satisfies $\eta(s,w) = c(s)$.

We say that F_1, F_2 satisfying the above conditions are in *general position* when, in addition, their relative nullity spaces $\Delta_{F_1}, \Delta_{F_2}$ are transversal at any point of M^n.

Theorem 9. *Given F_1, F_2 in general position, define $f_1, f_2\colon M^n \to \mathbf{R}^{n+1}$ by*

$$f_i = (F_i|_{F_i^{-1}(M^n)})^{-1}\colon M^n \to U_i \subset \mathbf{R}^{n+1}.$$

Then f_1, f_2 are isometric Sbrana-Cartan hypersurfaces. In terms of the Gauss parametrization, each hypersurface can be locally described by a pair $\{h, \gamma\}$ of the form

$$h(u,v) = \frac{1}{\sqrt{1-\tau}} \left(\alpha(u) + \int_0^v \sqrt{\tau(u,s)}\beta(s)ds \right) \tag{60}$$

and

$$\gamma(u,v)=\frac{1}{\sqrt{1-\tau}}\left(a(u)+\int_0^v\sqrt{\tau(u,s)}b(s)ds\right), \tag{61}$$

given in terms of two smooth curves $\alpha(u)$ and $\beta(v)$ in $\mathbf{R}^{n+1}$ and three smooth functions $a(u)$, $b(v)$ and $\tau(u,v)$.

Conversely, given two smooth curves $\alpha(u)$, $\beta(v)$ in $\mathbf{R}^{n+1}$ with $\|\alpha(u)\|<1$, there exists locally a unique smooth function $\tau(u,v)\in(0,1)$ so that $h(u,v)$ has unit length. Then, given arbitrary smooth functions $a(u)$ and $b(v)$, the pair $\{h,\gamma\}$ defines locally at regular points a Sbrana-Cartan hypersurface which can be obtained as an intersection as above.

Let us first prove the following basic fact.

Lemma 10. *A surface $h\colon V^2\to\mathbf{S}^n_1$ admits a parametric description (60) if and only if (u,v) are real conjugate coordinates for h, τ is a solution of (5) and the Christoffel symbols satisfy the additional condition*

$$\Gamma^1_u-\Gamma^1\Gamma^2+F=0,\quad F=\langle\partial_u,\partial_v\rangle. \tag{62}$$

Proof: That a surface $h\colon V^2\to\mathbf{S}^n_1$ admits a parametric description (60) is clearly equivalent to

$$\left(\frac{(\sqrt{1-\tau}\,h)_v}{\sqrt{\tau}}\right)_u=0. \tag{63}$$

A straightforward computation shows that (63) holds if and only if (u,v) are real conjugate coordinates for h, τ is a solution of (5) and condition (62) is satisfied. ∎

Proof of Theorem 9: We denote by $B_i=B_{\eta_i}$ the second fundamental form of F_i and by $A_i=A_{\widetilde{N}_i}$ the second fundamental form of f_i associated to the Gauss map $\widetilde{N}_i\colon M^n\to\mathbf{S}^n_1\subset\mathbf{R}^{n+1}$. Along M^n, we have from

$$F_1\circ f_1=F_2\circ f_2 \tag{64}$$

that

$$\langle B_1X,Y\rangle\eta_1+\langle A_1X,Y\rangle N_1=\langle B_2X,Y\rangle\eta_2+\langle A_2X,Y\rangle N_2,\ \ \forall X,Y\in TM, \tag{65}$$

where $N_i=F_{i*}\widetilde{N}_i$. Clearly, we may write

$$\eta_2=\cos\alpha\,\eta_1+\sin\alpha\,N_1,\quad N_2=-\sin\alpha\,\eta_1+\cos\alpha\,N_1, \tag{66}$$

where the function $\alpha\colon M^n \to (0,\pi)$ denotes the angle between both hypersurfaces.

Define $\bar{B}_i\colon TM \to TM$ by

$$\langle \bar{B}_i X, Y\rangle = \langle B_i X, Y\rangle. \tag{67}$$

Then (65) is equivalent to

$$A_1 - \cos\alpha\, A_2 = \sin\alpha\, \bar{B}_2 \tag{68}$$

and

$$\sin\alpha\, A_2 = \cos\alpha\, \bar{B}_2 - \bar{B}_1. \tag{69}$$

Clearly,

$$\ker \bar{B}_i = \Delta_{F_i} \cap TM. \tag{70}$$

Our assumption that Δ_{F_1} and Δ_{F_2} are everywhere transversal implies that $\ker \bar{B}_1 \cap \ker \bar{B}_2 = \Delta_{F_1} \cap \Delta_{F_2} \cap TM$ has always dimension $n-2$. Hence $\bar{B}_1, \bar{B}_2$ are linearly independent of rank one. It follows from (59) and (69) that

$$\Delta = \ker \bar{B}_1 \cap \ker \bar{B}_2. \tag{71}$$

Hence, M^n does not have flat points. Consider vector fields X_1, X_2 so that

$$\operatorname{span}\{X_i\} = \ker \bar{B}_i \cap \Delta^\perp, \quad 1 \le i \le 2.$$

Set $\tau = \cos^2\alpha$. Then (68) and (69) yield

$$\bar{B}_i X_j = (-1)^i \sqrt{\tau^{-1} - 1}\, A_i X_j, \quad 1 \le i \ne j \le 2, \tag{72}$$

and

$$A_2 = A_1 D, \qquad D = \begin{bmatrix} \sqrt{\tau} & 0 \\ 0 & \sqrt{\tau^{-1}} \end{bmatrix}, \tag{73}$$

when A_1, A_2 are restricted do $\Delta^\perp$. Therefore, f_1, f_2 are nowhere congruent of real type with conjugate directions X_1, X_2, which we choose to satisfy (47).

Set

$$\lambda_i = \langle B_i X_j, \widetilde{N}_i\rangle, \quad 1 \le i \ne j \le 2.$$

Taking derivatives and using Codazzi's equation for F_i yields

$$T(\lambda_i) = \langle \widetilde{\nabla}_{X_j} B_i T - B_i[X_j, T], \widetilde{N}_i\rangle - \langle B_i X_j, B_i T\rangle. \tag{74}$$

It follows from (50), (71) and (74) that

$$T(\lambda_i) = 0, \quad \forall\, T \in \Delta,$$

hence λ_1, λ_2 can be viewed as functions on V^2. We have from (49), (70), (71) and the Codazzi equation that

$$\begin{aligned} 0 \;=\; \widetilde{\nabla}_{X_i}(B_i X_j) = \Big(X_i(\lambda_i) + (-1)^i \sqrt{\tau^{-1} - 1}\, \langle A_i X_i, A_i X_j \rangle \Big)\, \widetilde{N}_i - \lambda_i\, A_i X_i \\ +\, (-1)^i X_i \left(\sqrt{\tau^{-1} - 1} \right) A_i X_j + (-1)^i \sqrt{\tau^{-1} - 1}\, \nabla_{X_i} A_i X_j. \end{aligned}$$

Now consider on V^2 the metric induced by h_1 and the real conjugate coordinates (u, v) satisfying (47) for suitably normalized X_1, X_2. The above equation can be replaced by

$$X_i(\lambda_i) + (-1)^i \sqrt{\tau^{-1} - 1} \langle A_i X_i, A_i X_j \rangle = 0, \tag{75}$$

$$\lambda_1 \partial_u + \left(\sqrt{\tau^{-1} - 1} \right)_u \partial_v + \sqrt{\tau^{-1} - 1}\, \nabla'_{\partial_u} \partial_v = 0, \tag{76}$$

and

$$\frac{\lambda_2}{\sqrt{\tau}} \partial_v - \left(\sqrt{1 - \tau} \right)_v \partial_u - \sqrt{1 - \tau}\; \nabla'_{\partial_u} \partial_v = 0, \tag{77}$$

where (76) and (77) are equivalent to (5) and

$$\begin{cases} \lambda_1 + \dfrac{1}{\sqrt{\tau}} \sqrt{1 - \tau}\, \Gamma^1 = 0 \\[2ex] \lambda_2 - \sqrt{\tau} \sqrt{1 - \tau}\, \Gamma^2 = 0. \end{cases} \tag{78}$$

Using (5) and (78), we easily see that (75) reduces to (6) and the additional condition (62) on V^2. It follows from Lemma 10 that h admits a parametric description (60). Moreover, an easy computation shows that, under condition (62), equation (3) for the support function holds if and only if

$$\left(\frac{(\sqrt{1 - \tau}\, \gamma)_v}{\sqrt{\tau}} \right)_u = 0. \tag{79}$$

This is clearly equivalent to γ being given by (61) and concludes the proof of the direct statement.

For the converse, we first show the existence of a unique smooth function $\tau(u,v)$ so that $h(u,v)$ has unit length. But this is equivalent to

$$\theta^2(u,v) = 1 - \|\alpha(u)\|^2 - 2\int_0^v \langle\alpha(u), \beta(s)\rangle\theta(u,s)ds$$
$$- \int_0^v \int_0^v \langle\beta(s), \beta(t)\rangle\theta(u,s)\theta(u,t)dtds,$$

where $\theta^2 = \tau$. Taking derivatives with respect to v yields

$$\theta_v(u,v) = -\langle\alpha(u), \beta(v)\rangle - \int_0^v \langle\beta(v), \beta(t)\rangle\theta(u,t)dt.$$

Using

$$\theta(u,t) = \theta(u,0) + \int_0^t \theta_v(u,s)ds, \quad \theta(u,0) = \sqrt{1 - \|\alpha(u)\|^2}$$

and Fubini's theorem, we get

$$\theta_v(u,v) = -\langle\alpha(u), \beta(v)\rangle - \left(\int_0^v \langle\beta(v), \beta(t)\rangle dt\right)\theta(u,0)$$
$$- \int_0^v \left(\int_s^v \langle\beta(v), \beta(t)\rangle dt\right)\theta_v(u,s)ds.$$

The above is an integral equation of Volterra type which has a unique (smooth) solution $\theta_v(u,v)$. Finally, notice that $0 < \theta(u,v) < 1$ for sufficiently small values of the variables.

Now suppose that M^n is simply connected and that $f\colon M^n \to \mathbf{R}^{n+1}$ is given by a pair $\{h, \gamma\}$ as in (60) and (61). By Lemma 10, h satisfies condition (62) and τ is a solution of (5). Let $f_2\colon M^n \to \mathbf{R}^{n+1}$ be the isometric deformation of $f_1 = f$ determined by τ, and let $h_2\colon V^2 \to \mathbf{S}_1^n$ be its Gauss image. We claim that h_2 satisfies (62) for (v,u). The metrics induced on V^2 by h_2 and $h_1 = h$ are related by

$$\langle\ ,\ \rangle_{h_2} = \langle D, D\rangle_{h_1}.$$

Using (5), this yields for the Christoffel symbols $\tilde{\Gamma}^i$ of h_2,

$$\tilde{\Gamma}^1 = \frac{1}{\tau}\Gamma^1, \quad \tilde{\Gamma}^2 = \tau\Gamma^2. \tag{80}$$

It follows from (5), (6) and (80) that

$$\widetilde{\Gamma}^1_u = \Gamma^2_v, \quad \widetilde{\Gamma}^2_v = \Gamma^1_u, \tag{81}$$

which proves the claim. By Lemma 10, h_2 can be parametrized as

$$h_2(u,v) = \frac{1}{\sqrt{1-\tau}}\left(\alpha_2(v) + \int_0^u \sqrt{\tau(s,v)}\beta_2(s)ds\right)$$

for smooth curves $\alpha_2(v), \beta_2(u)$ and the same function $\tau(u,v)$. In particular,

$$\beta_1(v) = \beta(v) = \frac{1}{\sqrt{\tau}}\left(\sqrt{1-\tau}\, h_1\right)_v, \quad \beta_2(u) = \frac{1}{\sqrt{\tau}}\left(\sqrt{1-\tau}\, h_2\right)_u.$$

Therefore,

$$\beta_1(v) \in \operatorname{span}\{h_1, h_{1*}\partial_v\}, \quad \beta_2(u) \in \operatorname{span}\{h_2, h_{2*}\partial_u\}.$$

Take a smooth unit vector field $\delta_i = \delta_i(u,v)$, $1 \leq i \leq 2$, orthogonal to β_i in each of the above plane bundles. Observe that the δ_i's are nowhere tangent to M^n. Let $\psi^i\colon M^n \times (-\epsilon,\epsilon) \to U_i \subset \mathbf{R}^{n+1}$ be a parametrization of a tubular neighborhood of $f_i(M)$ defined as

$$\psi^i(x,t) = f_i(x) + t\delta_i(\pi(x)), \quad i = 1,2.$$

Now define rank one endomorphisms B_i along ψ_i by

$$B_1(\Delta) = 0, \;\; B_1\psi^1_u = 0, \;\; B_1\psi^1_t = 0, \;\; B_1\psi^1_v = \beta_1(v),$$

and

$$B_2(\Delta) = 0, \;\; B_2\psi^2_v = 0, \;\; B_2\psi^1_t = 0, \;\; B_2\psi^2_u = -\beta_2(u).$$

One can easily check that the B_i's are symmetric tensors. Moreover, the B_i's trivially satisfy the Codazzi equations. Therefore, there exist isometric immersions $F_i\colon U_i \subset \mathbf{R}^{n+1} \to \mathbf{R}^{n+2}$ with second fundamental forms B_i.

We have for the $\bar{B}_i$'s defined by (67) that

$$\bar{B}_1 X_2 = (\beta_1(v))_{TM} = \sqrt{\tau^{-1}-1}\, h_{1*}\partial_v = -\sqrt{\tau^{-1}-1}\, A_1 X_2 \tag{82}$$

and, similarly,

$$\bar{B}_2 X_1 = \sqrt{\tau^{-1}-1}\, A_2 X_1. \tag{83}$$

For $\widetilde{N}_i, N_i$ and η_i as before, define an isometry $\Upsilon\colon T^{\perp}_{F_2\circ f_2}M \to T^{\perp}_{F_1\circ f_1}M$ between the normal bundles by

$$\Upsilon(\eta_2) = \sqrt{\tau}\,\eta_1 + \sqrt{1-\tau}\,N_1,\;\; \Upsilon(N_2) = -\sqrt{1-\tau}\,\eta_1 + \sqrt{\tau}\,N_1.$$

It follows easily from (82) and (83) that Υ preserves the second fundamental forms. We claim that Υ is parallel with respect to the normal connections. In fact, we have

$$\Upsilon\nabla^{\perp}_{X_1}\eta_2 = -\Upsilon(B_2X_1)^{\perp} = \frac{(\sqrt{1-\tau})_u}{\sqrt{\tau}}\Upsilon N_2.$$

Thus,

$$\nabla^{\perp}_{X_1}\Upsilon\eta_2 = \nabla^{\perp}_{X_1}(\sqrt{\tau}\,\eta_1 + \sqrt{1-\tau}N_1) = (\sqrt{\tau})_u\eta_1 + (\sqrt{1-\tau})_uN_1 = \Upsilon\nabla^{\perp}_{X_1}\eta_2.$$

Similarly,

$$\Upsilon\nabla^{\perp}_{X_2}\eta_2 = -\Upsilon(B_2X_2)^{\perp} = 0$$

and

$$\nabla^{\perp}_{X_2}\Upsilon\eta_2 = (\sqrt{\tau})_v\eta_1 - \sqrt{\tau}(B_1X_2)^{\perp} + (\sqrt{1-\tau})_vN_1 + \sqrt{1-\tau}\langle N_1, B_1X_2\rangle\eta_1 = 0$$

which proves the claim. We conclude from the fundamental theorem of submanifolds that (64) holds up to a rigid motion. ∎

We show below that Sbrana-Cartan hypersurfaces obtained as intersections are generically of type (IV). Nevertheless, there exist deformable hypersurfaces of type (I) or (III) which can be obtained this way. In fact, it is easy to see that the flat hypersurfaces which yield the ones of type (I) are precisely as follows:

1) $F_i = F_i^0 \times \mathrm{I}$ splits, where $F_i^0\colon U_i^0 \subset \mathbf{R}^3 \to \mathbf{R}^4$ and the euclidean factors $\mathbf{R}^{n-2}$ are parallel, or

2) $F_i = F_i^0 \times \mathrm{I}$ splits, where the $F_i^0\colon U_i^0 \subset \mathbf{R}^4 \to \mathbf{R}^5$ are cones over hypersurfaces of rank 1 in $\mathbf{S}^4_1 \subset \mathbf{R}^5$ and the euclidean factors $\mathbf{R}^{n-3}$ are parallel.

It is also not difficult to characterize the above hypersurfaces of type (I) in terms of the Gauss parametrization. They are given by a pair $\{h,\gamma\}$ as in (60), (61), satisfying: 1) h is contained in a totally geodesic $\mathbf{S}_1^2 \subset \mathbf{S}_1^n$, or 2) h is contained in a totally geodesic $\mathbf{S}_1^3 \subset \mathbf{S}_1^n$ and γ is a height function, i.e., $a = \langle \alpha, v_0\rangle$ and $b = \langle \beta, v_0\rangle$ for $v_0 \in \mathbf{R}^{n+1}$.

Hypersurfaces of type (III) which can be obtained as intersections admit a simpler parametrization determined by

$$h = \frac{(\alpha_1(u), \alpha_2(v))}{\sqrt{\|\alpha_1(u)\|^2 + \|\alpha_2(v)\|^2}} \quad \text{and} \quad \gamma = \frac{a_1(u) + a_2(v)}{\sqrt{\|\alpha_1(u)\|^2 + \|\alpha_2(v)\|^2}},$$

where the smooth curves $\alpha_j\colon I_j \to E_j$ lie in affine orthogonal subspaces $E_j \subset \mathbf{R}^{n+1}$ and $a_1(u), a_2(v)$ are arbitrary smooth functions. This class was already obtained by Cartan ([**Ca**$_1$]) as a result of a completely different approach. Notice that in this case condition (62) reduces to $\Gamma^1\Gamma^2 + F = 0$. See [**DF**] for details.

Theorem 11. *A Sbrana-Cartan hypersurface obtained as an intersection can only be of type (I), (III) or (IV). Moreover, if nowhere of type (I), it is of type (III) if and only if one of the following equivalent conditions hold:*

1) The curves $\alpha(u)$, $\beta(v)$ in (60) satisfy

$$\alpha(u) = \alpha_0(u) + \sqrt{1 - \|\alpha_0(u)\|^2}\, v_0$$

with $v_0 \in \mathbf{R}^{n+1}$ constant and $\langle \alpha_0(u), v_0\rangle = \langle \alpha_0(u), \beta(v)\rangle = 0$.

2) $\dim\left(\mathit{span}\{\eta_1\} \cap \mathit{span}\{\eta_2\}\right) = 1$.

Proof: For a pair of isometric Sbrana-Cartan hypersurfaces obtained as an intersection, the tensor $D = A_1^{-1} \circ A_2$ has two real distinct eigenvalues. On the other hand, the tensor D associated to a pair of isometric hypersurfaces of type (II) has 1 as an eigenvalue of multiplicity two. This proves the first assertion.

For a surface $\{h,(u,v)\}$ of first or second species and a solution τ of (5), we have

$$\left(\frac{\tau_u}{\tau}\right)_v = 2(1-\tau)(\Gamma^2_v - 2\Gamma^1\Gamma^2).$$

Therefore, h is of first species if and only if there exist functions $V = V(v) < 0$ and $U = U(u) > 0$ such that

$$\tau(u,v) = \frac{-V(v)}{U(u)}. \tag{84}$$

1) Suppose that h is of first species. Replacing (84) in (60) yields

$$U(u) + V(v) = \left\| \sqrt{U(u)}\,\alpha(u) + \int_0^v \sqrt{-V(s)}\,\beta(s)\,ds \right\|^2. \tag{85}$$

It follows easily that

$$\left\langle \left(\sqrt{U(u)}\,\alpha(u)\right)_u, \beta(v) \right\rangle = 0.$$

Hence, we have that

$$\alpha(u) = \alpha_0(u) + \frac{1}{\sqrt{U(u)}}\,w_0, \tag{86}$$

where $\langle \alpha_0(u), w_0\rangle = \langle \alpha_0(u), \beta(v)\rangle = 0$. From (85) and (86), we obtain

$$U(u)(1 - \|\alpha_0(u)\|^2) + V(v) - \|w_0 + \int_0^v \sqrt{-V(s)}\,\beta(s)\,ds\|^2 = 0.$$

Therefore, there exists a constant $k_0 > 0$ such that

$$U(u) = \frac{k_0}{1 - \|\alpha_0(u)\|^2}, \quad V(v) = -k_0 + \|w_0 + \int_0^v \sqrt{-V(s)}\,\beta(s)\,ds\|^2, \tag{87}$$

and the proof of the direct statement follows by setting $v_0 = w_0/\sqrt{k_0}$.

To prove the converse, we first show that there exists a function $V(v)$ satisfying the second equation in (87). Set $V(v) = -\psi^2(v)$. Differentiation yields

$$\psi'(v) = \langle \beta(v), v_0\rangle - \int_0^v \langle \beta(v), \beta(s)\rangle \psi(s)\,ds.$$

By a similar procedure as before, this equation can be transformed into an integral equation of Volterra type, hence has a unique solution. Going backwards in the above calculations shows that the function $\bar\tau(u,v) = -V(v)/U(u)$, with $V(v)$ and $U(u)$ as in (87), has the property that

$$h(u,v) = \frac{1}{\sqrt{1-\bar\tau}} \left(\alpha_0(u) + \sqrt{1 - \|\alpha_0(u)\|^2}\; v_0 + \int_0^v \sqrt{\bar\tau(u,s)}\beta(s)ds \right)$$

has unit length. By the uniqueness established in the proof of the converse of Theorem 9, we have $\bar{\tau} = \tau$ and the proof of 1) is completed.

2) Since $\sqrt{\tau} = \langle \eta_1, \eta_2 \rangle$ for a hypersurface obtained as an intersection, we have that (84) holds if and only if

$$\langle (\eta_1, \frac{-1}{\sqrt{U}}), (\eta_2, \sqrt{-V}) \rangle = 0$$

in $\mathbf{R}^{n+3}$. This is clearly equivalent to the subspaces $\mathrm{span}\{\eta_1\} \cap \mathrm{span}\{\eta_2\}$ having dimension 1. ∎

Our last result has the following important consequence.

Corollary 12. *There exist connected Sbrana-Cartan hypersurfaces which are of type (III) and (IV) but not of type (I) on some open subsets, and of type (I) along some other open subsets.*

Examples 13. Examples of Sbrana-Cartan hypersurfaces as in Corollary 12 can be explicitly constructed. Take $\beta(v) = e$ constant with $\|e\| = 1$, and $\alpha \colon I \to \mathbf{R}^{n+1}$ any curve so that $\|\alpha(u)\|^2 = k < 1$ is constant. Set

$$\sqrt{\tau} = -\langle \alpha(u), e \rangle \sin v + \sqrt{1-k} \cos v \tag{88}$$

and let V^2 be any connected component of $I \times \mathbf{R}$ where $0 < \sqrt{\tau} < 1$. Then $h \colon V^2 \to \mathbf{R}^{n+1}$ given by

$$h(u,v) = \frac{1}{\sqrt{1-\tau}} \Big(\alpha(u) + \int_0^v \sqrt{\tau(u,s)} ds\, e \Big) \tag{89}$$

has unit length. We have everywhere on V^2 that

$$W := \mathrm{span}\{h_u, h_v, h\} = \mathrm{span}\{\alpha(u), \alpha'(u), e\}.$$

Hence, the singular points of h occur precisely along the v-coordinate curves for which $e \in \mathrm{span}\{\alpha(u), \alpha'(u)\}$. Since $h_{uv}, h_{vv} \in W$, we conclude that (u, v) are real conjugate coordinates and that h_v belongs to the relative nullity of h. In particular, V^2 has constant Gauss curvature 1 at its regular points. Moreover, $h_{uu} \in W$, i.e., h is totally geodesic, if and only if $\alpha''(u) \in W$, that is,

$$\alpha(I) \subset \mathbf{R}^3 \ni e. \tag{90}$$

By part 1) of Theorem 9, we have that h is of first species (with $\Gamma^1 = \Gamma^1(v)$ and $\Gamma^2 = 0$) on any open subset of V^2 where $\langle \alpha(u), e\rangle$ is constant, and of second species on an open subset where $\langle \alpha'(u), e\rangle \neq 0$. In the first case, the singular points of h occur along the v-coordinate curves for which e and $\alpha(u)$ are colinear.

A straightforward computation for τ as in (88) and γ as in (61) shows that

$$\langle (\mathrm{Hess}_\gamma + \gamma \mathrm{I}\ )h_v, h_v\rangle = 0 \iff b'(v) = 0.$$

On the other hand, where h is not totally geodesic, let $\eta, \xi_1, \ldots, \xi_{n-3}$ be an orthonormal frame of $T^\perp V$ such that η spans the first normal space of h. For $x \in V^2$ and $w = t\eta + \sum_1^{n-3} t_i \xi_i \in T_x^\perp V$, the points $(x, w) \in T^\perp V$ where

$$\langle (\mathrm{Hess}_\gamma + \gamma \mathrm{I}\ - B_w)h_u, h_u\rangle = 0$$

can only occur along the hypersurface $\mathcal{S}$ given by

$$\mathcal{S} = \left\{ (x, w) \in T^\perp V \colon t = \frac{\langle (\mathrm{Hess}_\gamma + \gamma \mathrm{I}\ )h_u, h_u\rangle}{\langle B_\eta h_u, h_u\rangle} \right\}.$$

Therefore, a hypersurface generated by $\{h, \gamma\}$ is regular outside $\mathcal{S}$ if $b(v)$ is chosen so that $b'(v) \neq 0$ everywhere. Moreover, it is of type (III) on any open subset where $\langle \alpha(u), e\rangle$ is constant, of type (IV) on an open subset where $\langle \alpha'(u), e\rangle \neq 0$ and of type $(I)\,i)$ on an open subset where $\alpha(u)$ satisfies (90). Notice that it cannot be of type $(I)\,ii)$ because in that case $b'(v) = 0$.

On any open subset of type (III), an easy computation shows that the one parameter family of local deformations is given by

$$\bar\tau = c + (1 - c)\tau, \quad c \in \mathbf{R}.$$

Notice also that the sectional curvature changes signs at opposite sides of $\mathcal{S}$.

§4 Further examples

In this section we construct large families of surfaces in Riemannian and Lorentzian sphere which are of first species with respect to real conjugate coordinates. They do not satisfy condition (62), thus they yield Sbrana-Cartan hypersurfaces of real type (III) which cannot be obtained as intersections. Verifications are left to the reader.

Proposition 14. *Given two curves $c_j\colon I_j \to \mathbf{R}^{n_j}$, $c_j = c_j(u_j)$, with Frenet frames $c_j' = e_1^j, \dots, e_{n_j}^j$ and first curvature functions $k_j^1(u_j) > 0$ for all $u_j \in I_j$, the map $h\colon I_1 \times I_2 \to \mathbf{S}_1^n \subset \mathbf{R}^{n+1} = \mathbf{R}^{n_1} \oplus \mathbf{R}^{n_2}$ given by*

$$h = \frac{1}{\sqrt{\cos^2\theta_1 + \cos^2\theta_2}} \left(\cos\theta_2\, e_2^1, \cos\theta_1\, e_2^2\right),$$

where

$$\frac{d}{du_j}\theta_j = k_j^1,\ 1 \leq j \leq 2,$$

is a surface of first species with real conjugate coordinates (u_1, u_2).

Proposition 15. *Consider curves $c_1\colon I_1 \to \mathbf{L}^{n_1}$, $c_1 = c_1(u_1)$, with Frenet frame $c_1' = e_1^1, \dots, e_{n_1}^1$, $\|e_2^1\| = -1$, and $c_2\colon I_2 \to \mathbf{R}^{n_2}$, $c_2 = c_2(u_2)$, with Frenet frame $c_2' = e_1^2, \dots, e_{n_2}^2$. Assume that the first and second Frenet curvatures k_j^1, k_j^2 of c_j, $1 \leq j \leq 2$, satisfy the following conditions:*

i) $k_j^1(u_j) > 0,\ \ \forall\, u_j \in I_j,$

ii) $\theta_1(u_1) > \theta_2(u_2),\ \ \forall\, u_j \in I_j,$ *where* $\dfrac{d}{du_j}\theta_j = k_j^1,$

iii) $1 + \left(\dfrac{k_j^2}{k_j^1}\right)^2 > \dfrac{\sinh^2\theta_j}{\cosh^2\theta_1 - \cosh^2\theta_2}.$

Then, the map $h\colon I_1 \times I_2 \to \mathbf{S}_1^n \subset \mathbf{L}^{n+1} = \mathbf{L}^{n_1} \oplus \mathbf{R}^{n_2}$ given by

$$h = \frac{1}{\sqrt{\cosh^2\theta_1 - \cosh^2\theta_2}} \left(\cosh\theta_2\, e_2^1, \cosh\theta_1\, e_2^2\right)$$

is a surface of first species with real conjugate coordinates (u_1, u_2).

Remark 16. The examples above are totally geodesic for $n = 3$, but for higher dimensions this, in general, is not the case.

It is known that simply-connected minimal hypersurfaces of rank 2 are Sbrana-Cartan hypersurfaces; cf. [$\mathbf{DG}_1$]. In terms of the Gauss parametrization, they are given by a minimal surface in the sphere and, in case the ambient space is euclidean, a function γ on the surface satisfying

$$\Delta\gamma + 2\gamma = 0. \tag{91}$$

Notice that (91) is equation (9) with respect to isothermal coordinates. In our context, this result can be proved using the following.

Proposition 17. *Minimal surfaces endowed with isothermal coordinates are surfaces of first species with complex conjugate coordinates.*

Proof: They correspond to solutions of (14) for constant ϕ. ∎

§5 Some applications

Theorem 3 yields two applications. The first result deals with the rigidity question for real Kaehler hypersurfaces in euclidean space.

Theorem 18. *Let* $f\colon M^{2n} \to \mathbf{R}^{2n+1}$, $n \geq 2$, *be an isometric immersion of a locally irreducible Kaehler manifold. Then* f *is isometrically deformable if and only if it is minimal.*

Proof: Suppose that the Gauss image $h\colon V^2 \to \mathbf{S}_1^n$ of a locally irreducible hypersurface $g\colon M^n \to \mathbf{R}^{n+1}$, $n \geq 4$, is a minimal surface. By Theorem 3, we have that g is deformable if and only if it is either minimal or ruled. We argue that it cannot be ruled. In fact, if otherwise, g has everywhere an asymptotic direction. It follows from (54) that also h must have everywhere an asymptotic direction. Being minimal, either h is totally geodesic or the dimension of its first normal space is equal to 1 except possibly at isolated points. Hence, the substantial codimension is at most 1, which is in contradiction with our assumption that g is irreducible. To conclude the proof observe that, by Theorem 2.5 of [**DG**$_2$], the Gauss image of f in the statement has to be a pseudoholomorphic surface, hence minimal. ∎

Next we provide a short proof of part of the result in [**DG**$_3$].

Theorem 19 ([DG$_3$]). *Let* $f\colon M^n \to \mathbf{R}^{n+1}$ *be a Sbrana-Cartan hypersurface and* $g\colon M^n \to \mathbf{R}^{n+1}$ *an isometric deformation. If* f, g *have isometric Gauss maps, then* f *is minimal and* g *belongs to its associated family.*

Proof: Having isometric Gauss maps means that

$$(A^f)^2 = (A^g)^2 = (A^f D)^2.$$

This easily implies the statement for hypersurfaces of type (I), and excludes ruled and hypersurfaces of real type (III) and (IV) .

If f is of complex type (III) or (IV), we have for the complex coordinate vector $\partial_z = \partial_u - i\partial_v$ induced on the Gauss image by the complex eigenfield Z of D that

$$\langle \partial_z, \partial_z \rangle = \langle A^f Z, A^f Z \rangle = \langle A^f DZ, A^f DZ \rangle = \rho^2 \langle A^f Z, A^f Z \rangle = \rho^2 \langle \partial_z, \partial_z \rangle,$$

hence $\langle \partial_z, \partial_z \rangle = 0$. This is equivalent to asking the coordinates (u, v) to be isothermal. We conclude from (2) that the Gauss image of f is minimal. Moreover, since the solutions of (13) associated to minimal surfaces are the constant ones, the corresponding solutions of (10) given by (57) are $\rho = e^{i\theta}$, $\theta \in \mathbf{R}$. Therefore, the 1-parameter family of deformations of a minimal hypersurface coincides with its associated family. ∎

§6 Global results

In this section we study complete Sbrana-Cartan hypersurfaces in hyperbolic space. We first characterize those of type (I).

Theorem 20. *Let $f\colon M^n \to \mathbf{H}^{n+1}_{-1}$ be a Sbrana-Cartan hypersurface of type (I) determined by a surface L^2 in $\mathbf{Q}^3_{\tilde{c}}$. Then M^n is complete if and only if $\tilde{c} \leq 0$ and L^2 is complete.*

Proof: Let a denote the mean curvature vector of $\mathbf{Q}^3_{\tilde{c}}$ in $\mathbf{L}^{n+2} \supset \mathbf{H}^{n+1}_{-1}$. By the discussion following Lemma 6, M^n is isometric to $N^{n-2} \times_\sigma L^2$, where $N^{n-2} = \mathbf{H}^{n-2}_{-1}$ if $\tilde{c} \leq 0$ and $N^{n-2} = \mathbf{H}^{n-2}_{-1} \cap \{x : \langle a, x \rangle > 0\}$ otherwise, the warping function $\sigma\colon N^{n-2} \to \mathbf{R}$ being $\sigma(x) = \langle a, x \rangle$. The conclusion follows from the standard characterization of whether a warped product is complete. ∎

Remark 21. The examples of complete deformable hypersurfaces in hyperbolic space in [$\mathbf{Mor}_1$] are rather simple. They are of type (I) where L^2 is a complete ruled surface in a totally geodesic $\mathbf{H}^3_{-1} \subset \mathbf{H}^{n+1}_{-1}$. Moreover, being L^2 ruled, they also turn out to be ruled, that is, of type (II).

There exist many complete ruled hypersurfaces in hyperbolic space of any dimension which are not of type (I). To generate simple examples one starts with a substantial curve $d\colon \mathbf{R} \to \mathbf{H}^{n+1}$, $d = d(s)$, $n \geq 4$, with Frenet frame $d' = e_1, \ldots, e_{n+1}$, and consider the parametrized hypersurface

$$F(s, t_1, \ldots, t_{n-1}) = \exp_{d(s)} \left(\sum_{j=1}^{n-1} t_j e_{j+2}(s) \right).$$

Next we prove a non-existence result for complete hypersurfaces of real type (*III*) and (*IV*).

Theorem 22. *No hypersurface $f\colon M^n \to \mathbf{H}^{n+1}_{-1}$, $n \geq 4$, can be Sbrana-Cartan of real type (III) or (IV) on an open subset with complete relative nullity leaves unless it is also of type (I) on that subset.*

Proof: Suppose that the hypersurface is of real type (*III*) or (*IV*) on an open subset U as in the statement. Then, it can be described along U by the Gauss parametrization (16) defined on the normal bundle of a surface $h\colon V^2 \to \mathbf{S}^n_1$ of first or second species, respectively. Being the leaves of relative nullity complete on U, Ψ must have maximal rank everywhere on $T^\perp V$. As already pointed out in §2, this is equivalent to B_w being nonsingular for any normal vector w of length -1 on V^2. We claim that this is the case if and only if the first normal spaces N^h_1 form a rank-1 parallel subbundle of the normal bundle.

We first show that, at any point $x \in V^2$ where N^h_1 has the maximal dimension 2, there exists a vector $w \in T^\perp_x V$ such that $\|w\| = -1$ and B_w is singular. The assertion is trivial if $N^h_1(x)$ is Riemannian. If it is degenerate, we may assume that $\|\alpha_h(\partial_u, \partial_u)\| \neq 0$ and let $w_0 \in N^h_1(x) \cap N^h_1(x)^\perp$. Now take $\hat{w}_0$ such that $\|\hat{w}_0\| = 0$, $\langle w_0, \hat{w}_0 \rangle = -1/2$, $\langle \hat{w}_0, \alpha_h(\partial_u, \partial_u) \rangle = 0$, and set $w = w_0 + \hat{w}_0$. When $N^h_1(x)$ is Lorentzian and either $\alpha_h(\partial_u, \partial_u)$ or $\alpha_h(\partial_v, \partial_v)$, say, the former, is space-like, simply choose $w \in N^h_1(x)$ orthogonal to $\alpha_h(\partial_u, \partial_u)$. Finally, if both $\alpha_h(\partial_u, \partial_u)$ and $\alpha_h(\partial_v, \partial_v)$ are time-like, let ξ, δ be an orthonormal basis of $N^h_1(x)$ with ξ colinear with $\alpha_h(\partial_u, \partial_u)$. Then set $w = \cosh\theta\, \xi + \sinh\theta\, \delta$, where $\tanh\theta = -\langle \alpha_h(\partial_v, \partial_v), \xi \rangle / \langle \alpha_h(\partial_v, \partial_v), \delta \rangle$.

Now assume that $N^h_1(x) = \operatorname{span}\{w\}$ and there exist $\zeta \in T^\perp_x V$ orthogonal to w and $X \in T_x V$ such that $\langle \nabla^\perp_X w, \zeta \rangle \neq 0$. The Codazzi equation for B_ζ easily implies that $B_w Y = 0$ for any $Y \in T_x V$ such that $\langle \nabla^\perp_Y w, \zeta \rangle = 0$. Hence, B_w is singular and the claim is proved.

We conclude that h reduces codimension to one, that is, there exists a totally geodesic $\mathbf{S}^3_1 \subset \mathbf{S}^n_1$ such that $h(V^2) \subset \mathbf{S}^3_1$. Then, it is easily seen that

$f|_U$ is of type (I), where the surface $L^2 \in \mathbf{H}^3_{-1}$ in the statement of Theorem 3 is the polar surface of h. ∎

Although we do not have examples of complete hypersurfaces in hyperbolic space of complex type (*III*), we believe that there exists an abundance. A straightforward computation shows that, for any surface of first species in Lorentzian sphere with complex conjugate coordinates whose first normal space is nowhere Riemannian, the Gauss parametrization (16) is everywhere an immersion. Hence, a natural way to produce examples would be to consider hypersurfaces given by (16) for compact surfaces of first species, perhaps minimal ones, with nowhere Riemannian first normal spaces.

Remark 23. We point out that the claim in [$\mathbf{Mor}_2$] is not correct. In fact, the three dimensional complete deformable euclidean hypersurfaces constructed there are of type $(I)\, i)$ in Theorem 3 and not of type (*III*).

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