The vectorial Ribaucour transformation for submanifolds and applications*

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Abstract

In this paper we develop the vectorial Ribaucour transformation for Euclidean submanifolds. We prove a general decomposition theorem showing that under appropriate conditions the composition of two or more vectorial Ribaucour transformations is again a vectorial Ribaucour transformation. An immediate consequence of this result is the classical permutability of Ribaucour transformations. Our main application is to provide an explicit local construction of an arbitrary Euclidean n-dimensional submanifold with flat normal bundle and codimension m by means of a commuting family of m Hessian matrices on an open subset of Euclidean space $\mathbb{R}^n$. Actually, this is a particular case of a more general result. Namely, we obtain a similar local construction of all Euclidean submanifolds carrying a parallel flat normal subbundle, in particular of all those that carry a parallel normal vector field. Finally, we describe all submanifolds carrying a Dupin principal curvature normal vector field with integrable conullity, a concept that has proven to be crucial in the study of reducibility of Dupin submanifolds.

An explicit construction of all submanifolds with flat normal bundle of the Euclidean sphere carrying a holonomic net of curvature lines, that is, admitting principal coordinate systems, was given by Ferapontov in [8]. The author points out that his construction "resembles" the vectorial Ribaucour transformation for orthogonal systems developed in [13]. The latter provides a convenient framework for understanding the permutability properties of the classical Ribaucour transformation.

This paper grew out as an attempt to better understand the connection between those two subjects, as a means of unravelling the geometry behind Ferapontov's construction. This has led us to develop a vectorial Ribaucour transformation for Euclidean submanifolds, extending the transformation in [13] for orthogonal coordinate systems. It turns out that any n-dimensional submanifold with flat normal bundle of $\mathbb{R}^{n+m}$ can be locally transformed by a suitable vectorial Ribaucour transformation to the inclusion map of an open subset of an n-dimensional subspace of $\mathbb{R}^{n+m}$. Inverting such a transformation yields

*Mathematics Subject Classification 2000. Primary 53B25, 58J72.

the following explicit local construction of an arbitrary n-dimensional submanifold with flat normal bundle of $\mathbb{R}^{n+m}$ by means of a commuting family of m Hessian matrices on an open subset of $\mathbb{R}^n$. Notice that carrying a principal coordinate system is not required.

Theorem 1. *Let $\varphi_1, \ldots, \varphi_m$ be smooth real functions on an open simply connected subset $U \subset \mathbb{R}^n$ satisfying*

$$[\operatorname{Hess} \varphi_i, \operatorname{Hess} \varphi_j] = 0, \quad 1 \leq i, j \leq m,$$

and let $\mathcal{G}\colon U \to M_{n\times m}(\mathbb{R})$ be defined by $\mathcal{G} = (\nabla\varphi_1, \ldots, \nabla\varphi_m)$. Then for any $x \in U$ there exists a smooth map $\Omega\colon V \to Gl(\mathbb{R}^m)$ on an open subset $V \subset U$ containing x such that $d\Omega = \mathcal{G}^t\, d\mathcal{G}$ and $\Omega + \Omega^t = \mathcal{G}^t\mathcal{G} + I$. Moreover, the map

$$f(u) = \begin{pmatrix} u^t + \mathcal{G}\Omega^{-1}\varphi(u) \\ \Omega^{-1}\varphi(u) \end{pmatrix}$$

with $u = (u_1, \ldots, u_n) \in V$ and $\varphi(u) = (\varphi_1(u), \ldots, \varphi_m(u))^t$ defines, at regular points, an immersion $f\colon V \to \mathbb{R}^{n+m}$ with flat normal bundle.

Conversely, any isometric immersion $f\colon M^n \to \mathbb{R}^{n+m}$ with flat normal bundle can be locally constructed in this way.

The case of submanifolds of the sphere can be easily derived from the preceding result and the observation that any such submanifold arises as the image of a unit parallel normal vector field to a submanifold with flat normal bundle of Euclidean space (see Corollary 19). In this way we recover Ferapontov's result for the holonomic case (see Theorem 20), thus proving his guess correct.

Theorem 1 is actually a particular case of a more general result. In fact, we obtain a similar local explicit construction (see Theorem 18) of all isometric immersions $\tilde{f}\colon \tilde{M}^{n+m} \to \mathbb{R}^{n+m+p}$ carrying a parallel flat normal subbundle of rank m, in particular of all those that carry a parallel normal vector field, starting with an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ and a set of Codazzi tensors $\Phi_1, \ldots, \Phi_m$ on M^n that commute one with each other and with the second fundamental form of f. We refer the reader to [1] for results of a global nature on such isometric immersions, with strong implications for the submanifold geometry of orbits of orthogonal representations.

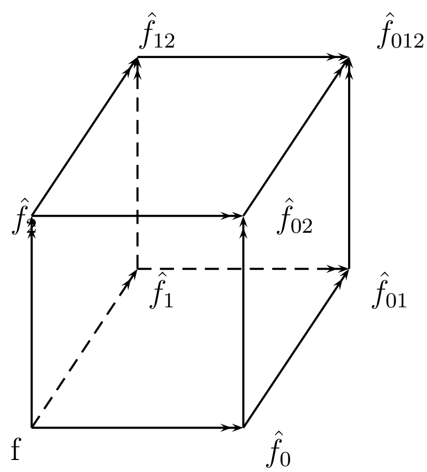
By putting together the preceding result with Theorem 8 of [6], we obtain an explicit construction (see Theorem 22) in terms of the vectorial Ribaucour transformation of all Euclidean submanifolds that carry a Dupin principal curvature normal vector field with integrable conullity (see Section 7 for the precise definitions), a concept that has proven to be crucial in the study of reducibility of Dupin submanifolds (see [6]).

A key feature of the Ribaucour transformation for submanifolds (in particular, orthogonal systems) is its permutability property. Namely, given two Ribaucour transforms of a submanifold, there is, generically, a fourth submanifold that is a simultaneous Ribaucour transform of the first two, giving rise to a *Bianchi quadrilateral.*

More generally, for any integer $k \geq 2$ we define a *Bianchi k-cube* as a $(k+1)$-tuple $(\mathcal{C}_0, \ldots, \mathcal{C}_k)$, where each $\mathcal{C}_i$, $0 \leq i \leq k$, is a family of submanifolds with exactly $\binom{k}{i}$ elements, such that every element of $\mathcal{C}_1$ is a Ribaucour transform of the unique element of $\mathcal{C}_0$ and such that, for every $\hat{f} \in \mathcal{C}_{s+1}$, $1 \leq s \leq k-1$, there exist unique elements $\hat{f}_1, \ldots, \hat{f}_{s+1} \in \mathcal{C}_s$ satisfying the following conditions:

(*i*) $\hat{f}$ is a Ribaucour transform of $\hat{f}_1, \ldots, \hat{f}_{s+1}$.

(*ii*) For each pair of indices $1 \leq i \neq j \leq s+1$ there exists a unique element $\hat{f}_{ij} \in \mathcal{C}_{s-1}$ such that $\{\hat{f}_{ij}, \hat{f}_i, \hat{f}_j, \hat{f}\}$ is a Bianchi quadrilateral.



The following Bianchi k-cube theorem was proved in [11] for $k = 3$ in the context of triply orthogonal systems of Euclidean space. A nice proof in the setup of Lie sphere geometry was recently given in [2], where also an indication was provided of how the general case can be settled by using results of [12] for discrete orthogonal nets together with an induction argument.

Theorem 2. *Let $f\colon M^n \to \mathbb{R}^N$ be an isometric immersion and let $f_1, \ldots, f_k$ be independent Ribaucour transforms of f. Then, for a generic choice of simultaneous Ribaucour transforms f_{ij} of f_i and f_j such that $\{f_{ij}, f_i, f_j, f\}$ is a Bianchi quadrilateral for all pairs $\{i,j\} \subset \{1, \ldots, k\}$ with $i \neq j$, there exists a unique Bianchi k-cube $(\mathcal{C}_0, \ldots, \mathcal{C}_k)$ such that $\mathcal{C}_0 = \{f\}$, $\mathcal{C}_1 = \{f_1, \ldots, f_k\}$ and $\mathcal{C}_2 = \{f_{ij}\}_{1 \leq i \neq j \leq k}$.*

We give a simple and direct proof of Theorem 2 in Section 5, where the precise meanings of *independent* and *generic* are explained. The proof relies on a general decomposition theorem for the vectorial Ribaucour transformation for submanifolds (Theorem 14), according to which the composition of two or more vectorial Ribaucour transformations with appropriate conditions is again a vectorial Ribaucour transformation. The latter extends a similar result of [13] for the case of orthogonal systems and implies, in particular, the classical permutability of Ribaucour transformations for surfaces and, more generally, the permutability of vectorial Ribaucour transformations for submanifolds.

Acknowledgment. We are grateful to the referee for useful remarks and for bringing to our attention the papers [9] and [10]. As pointed out by him, in those articles a canonical correspondence between Hamiltonian systems of hydrodynamic type and hypersurfaces of a (pseudo) Euclidean space is provided, and that construction coincides when $m = 1$ with (in fact it is the "Legendre dual" of) the one given by Theorem 1. He also remarks that commuting families of Hessian matrices arise in the theory of Hamiltonian systems of hydrodynamic type by Dubrovin, Novikov and Tsarev among others.

§1 Preliminaries.

Let M^n be an n-dimensional Riemannian manifold and let ξ be a Riemannian vector bundle over M^n endowed with a compatible connection ∇^ξ. We denote by $\Gamma(\xi)$ the space of smooth sections of ξ and by R^ξ its curvature tensor. If $\zeta = \xi^* \otimes \eta = \mathrm{Hom}(\xi, \eta)$ is the tensor product of the vector bundles ξ^* and η, where ξ^* stands for the dual vector bundle of ξ and η is a Riemannian vector bundle over M^n, then the covariant derivative $\nabla Z \in \Gamma(T^*M \otimes \zeta)$ of $Z \in \Gamma(\zeta)$ is given by

$$(\nabla^\zeta_X Z)(v) = \nabla^\eta_X Z(v) - Z(\nabla^\xi_X v)$$

for any $X \in \Gamma(TM)$ and $v \in \Gamma(\xi)$. In particular, if $\omega \in \Gamma(T^*M \otimes \xi)$ is a smooth one-form on M^n with values in ξ, then $\nabla\omega \in \Gamma(T^*M \otimes T^*M \otimes \xi)$ is given by

$$\nabla\omega(X, Y) := (\nabla^{T^*M\otimes\xi}_X \omega)(Y) = \nabla^\xi_X \omega(Y) - \omega(\nabla_X Y),$$

where in the right hand side ∇ denotes the Levi-Civita connection of M^n. The exterior derivative $d\omega \in \Gamma(\Lambda^2 T^*M \otimes \xi)$ of ω is related to $\nabla\omega$ by

$$d\omega(X, Y) = \nabla\omega(X, Y) - \nabla\omega(Y, X).$$

The one-form ω is *closed* if $d\omega = 0$. If $Z \in \Gamma(\xi)$, then $\nabla Z = dZ \in \Gamma(T^*M \otimes \xi)$ is the one-form given by $\nabla Z(X) = \nabla^\xi_X Z$. In case $\xi = M \times V$ is a trivial vector bundle over M^n, with V an Euclidean vector space, that is, a vector space endowed with an inner product, then $\Gamma(T^*M \otimes \xi)$ is identified with the space of smooth one-forms with values in V. We use the same notation for the vector space V and the trivial vector bundle $\xi = M \times V$ over M^n.

Given $Z_1 \in \Gamma(\xi^* \otimes \eta)$ and $Z_2 \in \Gamma(\eta^* \otimes \gamma)$, we define $Z_2 Z_1 \in \Gamma(\xi^* \otimes \gamma)$ by

$$Z_2 Z_1(v) = Z_2(Z_1(v)), \ \ v \in \Gamma(\xi).$$

For $Z \in \Gamma(\xi^* \otimes \eta)$, we define $Z^t \in \Gamma(\eta^* \otimes \xi)$ by

$$\langle Z^t(u), v\rangle = \langle u, Z(v)\rangle, \ \ u \in \Gamma(\eta) \ \text{ and } \ v \in \Gamma(\xi).$$

For later use, we summarize in the following lemma a few elementary properties of covariant and exterior derivatives, which follow by straightforward computations.

Lemma 3. *The following facts hold:*

(i) If $Z_1 \in \Gamma(\xi^* \otimes \eta)$ and $Z_2 \in \Gamma(\eta^* \otimes \gamma)$, then $d(Z_2 Z_1) = (dZ_2)Z_1 + Z_2(dZ_1)$.

(ii) If $Z \in \Gamma(\xi^* \otimes \eta)$ then $dZ^t = (dZ)^t$.

(iii) If $Z \in \Gamma(\xi)$ then $d^2 Z(X,Y) = R^\xi(X,Y)Z$.

(iv) If $\zeta = \xi^* \otimes \eta$ and $Z \in \Gamma(\zeta)$ then $(R^\zeta(X,Y)Z)(v) = R^\eta(X,Y)Z(v) - Z(R^\xi(X,Y)v)$.

We also need the following result.

Proposition 4. *Let ξ, η be Riemannian vector bundles over M^n and $\omega \in \Gamma(T^*M \otimes \xi)$. Set $\zeta = \eta^* \otimes TM$ and $\gamma = \eta^* \otimes \xi$. Let $\Phi \in \Gamma(T^*M \otimes \zeta)$ be a closed one-form such that*

$$\nabla\omega(X, \Phi_u Y) = \nabla\omega(Y, \Phi_u X) \quad \textit{for all} \;\; u \in \Gamma(\eta), \tag{1}$$

where we write $\Phi_u X = \Phi(X)(u)$. Then the one-form $\rho = \rho(\omega, \Phi) \in \Gamma(T^*M \otimes \gamma)$ defined by $\rho(X)(u) = \omega(\Phi_u X)$ is also closed.

Proof: We have

$$\begin{aligned} \nabla\rho(X,Y)(u) &= \nabla^\xi_X \rho(Y)(u) - \rho(Y)(\nabla^\eta_X u) - \rho(\nabla_X Y)(u) \\ &= \nabla^\xi_X \omega(\Phi_u Y) - \omega(\Phi_{\nabla^\eta_X u} Y) - \omega(\Phi_u \nabla_X Y) \\ &= \nabla\omega(X, \Phi_u Y) + \omega(\nabla\Phi(X,Y)u). \quad \blacksquare \end{aligned} \tag{2}$$

The following consequence of Proposition 4 will be used throughout the paper.

Corollary 5. *Under the assumptions of Proposition 4, assume further that M^n is simply-connected and that ξ and η are flat. Then there exists $\Omega(\omega, \Phi) \in \Gamma(\eta^* \otimes \xi)$ such that*

$$d\Omega(\omega, \Phi)(X)(u) = \omega(\Phi_u X) \;\; \textit{for all} \;\; X \in TM \;\; \textit{and} \;\; u \in \eta.$$

Proof: Since ξ and η are flat, the same holds for $\gamma = \eta^* \otimes \xi$ by Lemma 3. The manifold M^n being simply-connected, a one-form $\rho \in \Gamma(T^*M \otimes \gamma)$ is exact if and only if it is closed. $\blacksquare$

§2 The Combescure transformation.

In this section we introduce a vectorial version of the Combescure transformation for submanifolds and derive a few properties of it that will be needed later.

Proposition 6. *Let $f \colon M^n \to \mathbb{R}^N$ be an isometric immersion of a simply connected Riemannian manifold, let V be an Euclidean vector space and let $\Phi \in \Gamma(T^*M \otimes V^* \otimes TM)$. Then there exists $\mathcal{F} \in \Gamma(V^* \otimes f^*T\mathbb{R}^N)$ such that*

$$d\mathcal{F}(X)(v) = f_* \Phi_v X \;\; \textit{for all} \;\; X \in TM \; \textit{and} \;\; v \in V \tag{3}$$

if and only if Φ is closed and satisfies

$$\alpha(X,\Phi_vY)=\alpha(Y,\Phi_vX)\ \ \textit{for all } v\in\Gamma(V), \tag{4}$$

where $\alpha\colon TM\times TM\to T^\perp M$ is the second fundamental form of f.

Proof: Applying (2) for $\omega=f_*\in\Gamma(T^*M\otimes f^*T\mathbb{R}^N)$ and $\Phi\in\Gamma(T^*M\otimes V^*\otimes TM)$ we obtain that the one-form $\rho=\rho(f_*,\Phi)\in\Gamma(T^*M\otimes V^*\otimes f^*T\mathbb{R}^N)$ satisfies

$$\nabla\rho(X,Y)u=\alpha(X,\Phi_uY)+f_*(\nabla\Phi(X,Y)u).$$

Therefore, Φ being closed and (4) are both necessary and sufficient conditions for ρ to be closed. Since V and $f^*T\mathbb{R}^N$ are flat, the result follows from Corollary 5. ∎

We call $\mathcal{F}$ a *Combescure transform* of f determined by Φ if, in addition,

$$\langle\Phi_vX,Y\rangle=\langle X,\Phi_vY\rangle\ \ \text{for all } v\in\Gamma(V). \tag{5}$$

Observe that $\mathcal{F}$ is determined up to a parallel element in $\Gamma(V^*\oplus f^*T\mathbb{R}^N))$. Notice also that for each fixed vector $v\in V$, regarded as a parallel section of the trivial vector bundle V, we have that $\mathcal{F}(v)\in\Gamma(f^*T\mathbb{R}^N)$ satisfies $d\mathcal{F}(v)(X)=f_*\Phi_v(X)$, and hence $\mathcal{F}(v)$ is a Combescure transform of f in the sense of [5] determined by the Codazzi tensor Φ_v.

Proposition 7. *Let $f\colon M^n\to\mathbb{R}^N$ be an isometric immersion of a simply connected Riemannian manifold, let V be an Euclidean vector space and let $\Phi\in\Gamma(T^*M\otimes V^*\otimes TM)$ be closed and satisfy (4). For $\mathcal{F}\in\Gamma(V^*\otimes f^*T\mathbb{R}^N)$ satisfying (3) write*

$$\mathcal{F}=f_*\omega^t+\beta, \tag{6}$$

where $\omega\in\Gamma(T^*M\otimes V)$ and $\beta\in\Gamma(V^*\otimes T^\perp M)$. Then

$$\alpha(X,\omega^t(v))+(\nabla_X^{V^*\otimes T^\perp M}\beta)v=0\ \ \textit{for all } v\in\Gamma(V), \tag{7}$$

and Φ is given by

$$\Phi_vX=(\nabla_X^{V^*\otimes TM}\omega^t)v-A_{\beta(v)}X. \tag{8}$$

Conversely, if $\omega\in\Gamma(T^*M\otimes V)$ and $\beta\in\Gamma(V^*\otimes T^\perp M)$ satisfy (7), then (3) holds for $\mathcal{F}=\mathcal{F}(\omega,\beta)$ and $\Phi=\Phi(\omega,\beta)$ given by (6) and (8), respectively. In particular, Φ is closed and (4) holds. Moreover, Φ satisfies (5) if and only if $\omega=d\varphi$ for some $\varphi\in\Gamma(V)$.

Proof: Denote by ∇^* the covariant derivative of $V^*\otimes f^*T\mathbb{R}^N$. Then,

$$\begin{aligned} d\mathcal{F}(X)(v)&=(\nabla^*_Xf_*\omega^t)v+(\nabla^*_X\beta)v=\nabla_X^{f^*T\mathbb{R}^N}f_*\omega^t(v)-f_*\omega^t(\nabla^V_Xv)+(\nabla^*_X\beta)v\\ &=f_*\nabla_X\omega^t(v)+\alpha(X,\omega^t(v))-f_*\omega^t(\nabla^V_Xv)+(\nabla_X^{V^*\otimes T^\perp M}\beta)v-f_*A_{\beta(v)}X.\end{aligned}$$

Since, on the other hand, $\mathcal{F}$ satisfies (3), then (7) and (8) follow.

Conversely, if ω and β satisfy (7), then the preceding computation yields (3) with Φ given by (8). Finally, taking the inner product of (8) with $Y \in \Gamma(TM)$ gives

$$\langle \Phi_v X, Y\rangle = \langle v, \nabla\omega(X,Y)\rangle - \langle A_{\beta(v)}X, Y\rangle,$$

thus the symmetry of $\nabla\omega$ is equivalent to (5). ∎

Proposition 8. *Let $f\colon M^n \to \mathbb{R}^N$ be an isometric immersion of a simply connected Riemannian manifold. Let V_i, $1 \le i \le 2$, be Euclidean vector spaces, and assume that $\omega_i \in \Gamma(T^*M \otimes V_i)$ and $\beta_i \in \Gamma(V_i^* \otimes T^\perp M)$ satisfy*

$$\alpha(X, \omega_i^t(v_i)) + (\nabla_X^{V_i^*\otimes T^\perp M}\beta_i)v_i = 0 \quad \textit{for all } v_i \in \Gamma(V_i). \tag{9}$$

Set $\mathcal{F}_i = f_*\omega_i^t + \beta_i$ and $\Phi^i_{v_i}X = (\nabla_X^{V_i^*\otimes TM}\omega_i^t)v_i - A_{\beta_i(v_i)}X$. Then,

$$\nabla\omega_i(X, \Phi^j_{v_j}Y) = \nabla\omega_i(Y, \Phi^j_{v_j}X) \quad \textit{for all } v_j \in \Gamma(V_j) \tag{10}$$

if and only if

$$\langle \Phi^i_{v_i}X, \Phi^j_{v_j}Y\rangle = \langle \Phi^i_{v_i}Y, \Phi^j_{v_j}X\rangle \quad \textit{for all } v_i \in \Gamma(V_i) \textit{ and } v_j \in \Gamma(V_j). \tag{11}$$

When this is the case, there exists $\Omega_{ij} = \Omega(\omega_i, \Phi^j) \in \Gamma(V_j^* \otimes V_i)$ satisfying

$$d\Omega_{ij}(X)(v_j) = \omega_i(\Phi^j_{v_j}X) \quad \textit{for all } v_j \in \Gamma(V_j). \tag{12}$$

In particular,

$$d\Omega_{ij} = \mathcal{F}_i^t\, d\mathcal{F}_j \tag{13}$$

and

$$\Omega_{ij} + \Omega_{ji}^t = \mathcal{F}_i^t\mathcal{F}_j = \omega_i\omega_j^t + \beta_i^t\beta_j, \tag{14}$$

up to a parallel element in $\Gamma(V_j^* \otimes V_i)$.

Proof: Since ω_j and β_j satisfy (9), we have $\alpha(X, \Phi^j_{v_j}Y) = \alpha(Y, \Phi^j_{v_j}X)$ by Proposition 7. Thus, it follows from

$$\langle \Phi^i_{v_i}X, \Phi^j_{v_j}Y\rangle = \langle v_i, \nabla\omega_i(X, \Phi^j_{v_j}Y)\rangle - \langle \alpha(X, \Phi^j_{v_j}Y), \beta_i(v_i)\rangle$$

that conditions (10) and (11) are equivalent. If (10) holds, then by Corollary 5 there exists $\Omega_{ij} \in \Gamma(V_j^* \otimes V_i)$ satisfying (12). On the other hand,

$$\mathcal{F}_i^t\, d\mathcal{F}_j(X)v_j = \mathcal{F}_i^t f_*\Phi^j_{v_j}X = \omega_i(\Phi^j_{v_j}X) \quad \text{for all } v_j \in \Gamma(V_j),$$

and (13) follows. Finally, (13) implies that the exterior derivatives of both sides in the first equality of (14) coincide. ∎

§3 The vectorial Ribaucour transformation.

We now introduce the main concept of this paper.

Definition 9. Let $f\colon M^n \to \mathbb{R}^N$ be an isometric immersion of a simply connected Riemannian manifold, and let V be an Euclidean vector space. Let $\varphi \in \Gamma(V)$ and $\beta \in \Gamma(V^* \otimes T^\perp M)$ satisfy (7) with $\omega = d\varphi$, and let $\Omega \in \Gamma(Gl(V))$ be a solution of the completely integrable first order system

$$d\Omega = \mathcal{F}^t\, d\mathcal{F} \tag{15}$$

such that

$$\Omega + \Omega^t = \mathcal{F}^t \mathcal{F}, \tag{16}$$

where $\mathcal{F} = f_* \omega^t + \beta$. If the map $\tilde{f}\colon M^n \to \mathbb{R}^N$ given by

$$\tilde{f} = f - \mathcal{F}\Omega^{-1}\varphi \tag{17}$$

is an immersion, then the isometric immersion $\tilde{f}\colon \tilde{M}^n \to \mathbb{R}^N$, where $\tilde{M}^n$ stands for M^n with the metric induced by $\tilde{f}$, is called a *vectorial Ribaucour transform* of f determined by (φ, β, Ω), and it is denoted by $\mathcal{R}_{\varphi,\beta,\Omega}(f)$.

Remark 10. If $\dim V = 1$, after identifying $V^* \otimes T_f^\perp M$ with $T_f^\perp M$ then φ and β become elements of $C^\infty(M)$ and $\Gamma(T_f^\perp M)$, respectively, and (7) reduces to

$$\alpha(X, \nabla\varphi) + \nabla_X^\perp \beta = 0.$$

Moreover, $\Omega = (1/2)\langle \mathcal{F}, \mathcal{F}\rangle$ for $\mathcal{F} = f_*\nabla\varphi + \beta$, and (17) reduces to the parameterization of a *scalar* Ribaucour transform of f obtained in Theorem 17 of [5]. In this case, since Ω is determined by φ and β we write $\tilde{f} = \mathcal{R}_{\varphi,\beta}(f)$ instead of $\tilde{f} = \mathcal{R}_{\varphi,\beta,\Omega}(f)$.

Next we derive several basic properties of the vectorial Ribaucour transformation.

Proposition 11. *The bundle map* $\mathcal{P} \in \Gamma((f^*T\mathbb{R}^N)^* \otimes \tilde{f}^*T\mathbb{R}^N)$ *given by*

$$\mathcal{P} = I - \mathcal{F}\Omega^{-1}\mathcal{F}^t \tag{18}$$

is a vector bundle isometry and

$$\tilde{f}_* = \mathcal{P} f_* D, \tag{19}$$

where $D = I - \Phi_{\Omega^{-1}\varphi} \in \Gamma(T^*M \otimes TM)$. *In particular,* $\tilde{f}$ *has the metric* $\langle\ ,\ \rangle^\sim = D^*\langle\ ,\ \rangle$.

Proof: We have

$$\begin{aligned}\mathcal{P}^t\mathcal{P} &= (I - \mathcal{F}(\Omega^{-1})^t\mathcal{F}^t)(I - \mathcal{F}\Omega^{-1}\mathcal{F}^t)\\ &= I - \mathcal{F}\Omega^{-1}\mathcal{F}^t - \mathcal{F}(\Omega^{-1})^t\mathcal{F}^t + \mathcal{F}(\Omega^{-1})^t\mathcal{F}^t\mathcal{F}\Omega^{-1}\mathcal{F}^t.\end{aligned}$$

Using (16) in the last term implies that the three last terms cancel out. Thus $\mathcal{P}$ is an isometry. Now, using (3) and (15) we obtain

$$\begin{aligned}\tilde{f}_* &= f_* - d\mathcal{F}\Omega^{-1}\varphi + \mathcal{F}\Omega^{-1}d\Omega\Omega^{-1}\varphi - \mathcal{F}\Omega^{-1}\omega\\ &= f_* - f_*\Phi_{\Omega^{-1}\varphi} + \mathcal{F}\Omega^{-1}\mathcal{F}^t\, d\mathcal{F}\Omega^{-1}\varphi - \mathcal{F}\Omega^{-1}\mathcal{F}^t f_*\\ &= f_*(I - \Phi_{\Omega^{-1}\varphi}) - \mathcal{F}\Omega^{-1}\mathcal{F}^t f_*(I - \Phi_{\Omega^{-1}\varphi}) = \mathcal{P} f_* D. \quad \blacksquare\end{aligned}$$

Proposition 12. *The normal connections and second fundamental forms of f and $\tilde{f}$ are related by*

$$\tilde{\nabla}^\perp_X \xi = \mathcal{P}\nabla^\perp_X \xi \tag{20}$$

and

$$\tilde{A}_{\mathcal{P}\xi} = D^{-1}(A_\xi + \Phi_{\Omega^{-1}\beta^t\xi}), \tag{21}$$

or equivalently,

$$\tilde{\alpha}(X,Y) = \mathcal{P}(\alpha(X,DY) + \beta(\Omega^{-1})^t\Phi(X)^t DY). \tag{22}$$

Proof: Let $\bar{\nabla}$ denote the connection of $\tilde{f}^*T\mathbb{R}^N$. Observing that $d\mathcal{F}^t(X)$ vanishes on $T^\perp M$, for $\langle d\mathcal{F}^t(X)\xi, v\rangle = \langle \xi, d\mathcal{F}(X)v\rangle = \langle \xi, f_*\Phi_v(X)\rangle = 0$, and using (3), (15) and (19), we get

$$\begin{aligned}
&-\tilde{f}_*\tilde{A}_{\mathcal{P}\xi}X + \tilde{\nabla}^\perp_X\xi = \bar{\nabla}_X\mathcal{P}\xi = \bar{\nabla}_X(\xi - \mathcal{F}\Omega^{-1}\mathcal{F}^t\xi)\\
&= -f_*A_\xi X + \nabla^\perp_X\xi - d\mathcal{F}(X)\Omega^{-1}\mathcal{F}^t\xi + \mathcal{F}\Omega^{-1}d\Omega(X)\Omega^{-1}\mathcal{F}^t\xi + \mathcal{F}\Omega^{-1}\mathcal{F}^t(f_*A_\xi X - \nabla^\perp_X\xi)\\
&= -\mathcal{P}f_*A_\xi X + \mathcal{P}\nabla^\perp_X\xi - \mathcal{P}f_*\Phi(X)\Omega^{-1}\mathcal{F}^t\xi,
\end{aligned}$$

which gives (20) and (21). ∎

Proposition 13. *The triple $(\tilde{\varphi}, \tilde{\beta}, \tilde{\Omega}) = (\Omega^{-1}\varphi, \mathcal{P}\beta(\Omega^{-1})^t, \Omega^{-1})$ satisfies the conditions of Definition 9 with respect to $\tilde{f}$, and $f = \mathcal{R}_{\tilde{\varphi},\tilde{\beta},\tilde{\Omega}}(\tilde{f})$. Moreover, $\tilde{\mathcal{F}} = \tilde{f}_*(d\tilde{\varphi})^t + \tilde{\beta}$ and $\tilde{\Phi} = \Phi(d\tilde{\varphi}, \tilde{\beta})$ are given, respectively, by*

$$\tilde{\mathcal{F}} = -\mathcal{F}\Omega^{-1} \quad \textit{and} \quad D\tilde{\Phi}_v = -\Phi_{\Omega^{-1}v}. \tag{23}$$

Proof: Since $\tilde{\Omega} = d\tilde{\varphi} = -\Omega^{-1}\omega\Phi_{\Omega^{-1}\varphi} + \Omega^{-1}\omega = \Omega^{-1}\omega D$, we have

$$\langle \tilde{\Omega}^t(v), X\rangle^\sim = \langle v, \Omega^{-1}\omega(DX)\rangle = \langle D\omega^t(\Omega^{-1})^t v, X\rangle = \langle \omega^t(\Omega^{-1})^t v, D^{-1}X\rangle^\sim,$$

thus

$$D\tilde{\Omega}^t = \omega^t(\Omega^{-1})^t, \tag{24}$$

where we have used that D^{-1} is symmetric with respect to $\langle\ ,\ \rangle^\sim$. We now prove that

$$\tilde{\alpha}(X, \tilde{\Omega}^t(v)) + (\nabla_X^{V^*\otimes T^\perp_{\tilde{f}}\tilde{M}}\tilde{\beta})v = 0 \quad \text{for all} \quad v \in \Gamma(V). \tag{25}$$

Equations (22) and (24) yield

$$\tilde{\alpha}(X, \tilde{\Omega}^t(v)) = \mathcal{P}(\alpha(X, \omega^t(\Omega^{-1})^t v) + \beta(\Omega^{-1})^t\Phi(X)^t\omega^t(\Omega^{-1})^t v), \tag{26}$$

whereas (20) gives

$$\begin{aligned}
(\nabla_X^{V^*\otimes T^\perp_{\tilde{f}}\tilde{M}}\tilde{\beta})v &= \tilde{\nabla}^\perp_X\tilde{\beta}(v) - \tilde{\beta}(\nabla^V_X v) = \mathcal{P}(\nabla^\perp_X\beta(\Omega^{-1})^t v - \beta(\Omega^{-1})^t\nabla^V_X v)\\
&= \mathcal{P}((\nabla_X^{V^*\otimes T^\perp M}\beta)(\Omega^{-1})^t v - \beta(\Omega^{-1})^t\Phi(X)^t\omega^t(\Omega^{-1})^t v).
\end{aligned} \tag{27}$$

It follows from (7), (26) and (27) that (25) holds.

We now compute $\tilde{\mathcal{F}} = \tilde{f}_*\tilde{\Omega}^t + \tilde{\beta}$. Using (19) in the first equality below, (24) in the second and (16) in the last one, we obtain

$$\tilde{\mathcal{F}} = \mathcal{P}f_*D\tilde{\Omega}^t + \tilde{\beta} = \mathcal{P}(f_*\omega^t(\Omega^{-1})^t + \beta(\Omega^{-1})^t) = (I - \mathcal{F}\Omega^{-1}\mathcal{F}^t)\mathcal{F}(\Omega^{-1})^t = -\mathcal{F}\Omega^{-1}. \quad (28)$$

Then, it follows from (15), (16) and (28) that

$$\tilde{\mathcal{F}}^t\, d\tilde{\mathcal{F}} = (\Omega^{-1})^t\mathcal{F}^t\, d\mathcal{F}\Omega^{-1} - (\Omega^{-1})^t\mathcal{F}^t\mathcal{F}\Omega^{-1}d\Omega\Omega^{-1} = d\tilde{\Omega},$$

and

$$\tilde{\mathcal{F}}^t\tilde{\mathcal{F}} = (\Omega^{-1})^t\mathcal{F}^t\mathcal{F}\Omega^{-1} = \tilde{\Omega} + \tilde{\Omega}^t.$$

Therefore,

$$\mathcal{R}_{\tilde{\varphi},\tilde{\beta},\tilde{\Omega}}(\tilde{f}) = \tilde{f} - \tilde{\mathcal{F}}\tilde{\Omega}^{-1}\tilde{\varphi} = f - \mathcal{F}\Omega^{-1}\varphi - (-\mathcal{F}\Omega^{-1})\Omega\Omega^{-1}\varphi = f.$$

Finally, the second formula in (23) follows from

$$\begin{aligned}\tilde{f}_*\tilde{\Phi}_v(X) &= d\tilde{\mathcal{F}}(X)v = -d\mathcal{F}(X)\Omega^{-1}v + \mathcal{F}\Omega^{-1}d\Omega(X)\Omega^{-1}v\\ &= -f_*\Phi_{\Omega^{-1}v}(X) + \mathcal{F}\Omega^{-1}\omega\Phi_{\Omega^{-1}v}(X) = -\mathcal{P}f_*\Phi_{\Omega^{-1}v}(X)\\ &= -\tilde{f}_*D^{-1}\Phi_{\Omega^{-1}v}(X). \qquad \blacksquare\end{aligned}$$

§4 The decomposition theorem.

A fundamental feature of the vectorial Ribaucour transformation for submanifolds is the following decomposition property, first proved in [13] in the context of orthogonal systems.

Theorem 14. *Let $\mathcal{R}_{\varphi,\beta,\Omega}(f)\colon \tilde{M^n} \to \mathbb{R}^N$ be a vectorial Ribaucour transform of an isometric immersion $f\colon M^n \to \mathbb{R}^N$. For an orthogonal decomposition $V = V_1 \oplus V_2$ define*

$$\varphi_j = \pi_{V_j}\circ\varphi, \quad \beta_j = \beta|_{V_j} \quad \text{and} \quad \Omega_{ij} = \pi_{V_i}\circ\Omega|_{V_j} \in \Gamma(V_j^*\otimes V_i), \quad 1 \le i,j \le 2. \quad (29)$$

Assume that Ω_{jj} is invertible and, for $i \neq j$, define $\mathcal{R}_{\varphi,\beta,\Omega}(\varphi_i,\beta_i,\Omega_{ii}) = (\bar{\varphi}_i,\bar{\beta}_i,\bar{\Omega}_{ii})$ by

$$\bar{\varphi}_i = \varphi_i - \Omega_{ij}\Omega_{jj}^{-1}\varphi_j, \quad \bar{\beta}_i = \mathcal{P}_j(\beta_i - \beta_j(\Omega_{jj}^{-1})^t\Omega_{ij}^t) \quad \text{and} \quad \bar{\Omega}_{ii} = \Omega_{ii} - \Omega_{ij}\Omega_{jj}^{-1}\Omega_{ji},$$

where $\mathcal{P}_j = I - \mathcal{F}_j\Omega_{jj}^{-1}\mathcal{F}_j^t$. Then the triples $(\varphi_j,\beta_j,\Omega_{jj})$ and $(\bar{\varphi}_i,\bar{\beta}_i,\bar{\Omega}_{ii})$ satisfy the conditions of Definition 9 with respect to f and f_j, respectively, and we have

$$\mathcal{R}_{\varphi,\beta,\Omega}(f) = \mathcal{R}_{\bar{\varphi}_i,\bar{\beta}_i,\bar{\Omega}_{ii}}(\mathcal{R}_{\varphi_j,\beta_j,\Omega_{jj}}(f)).$$

Proof: That $(\varphi_j, \beta_j, \Omega_{jj})$, $1 \le j \le 2$, satisfies the conditions of Definition 9 with respect to f is clear. In order to prove that $(\bar\varphi_i, \bar\beta_i, \bar\Omega_{ii})$ satisfies the conditions of Definition 9 with respect to f_j for $i \neq j$ we first compute $\bar\omega_i = d\bar\varphi_i$. We have

$$\begin{aligned}\bar\omega_i(X) &= \omega_i(X) + -d\Omega_{ij}(X)\Omega_{jj}^{-1}\varphi_j + \Omega_{ij}\Omega_{jj}^{-1}d\Omega_{jj}(X)\Omega_{jj}^{-1}\varphi_j - \Omega_{ij}\Omega_{jj}^{-1}\omega_j(X)\\ &= \omega_i(X) - \omega_i(\Phi^j(X)\Omega_{jj}^{-1}\varphi_j) + \Omega_{ij}\Omega_{jj}^{-1}\omega_j(\Phi^j(X)\Omega_{jj}^{-1}\varphi_j) - \Omega_{ij}\Omega_{jj}^{-1}\omega_j(X)\\ &= \omega_i(D_jX) - \Omega_{ij}\Omega_{jj}^{-1}\omega_j(D_jX),\end{aligned}$$

where $D_j = I - \Phi^j_{\Omega_{jj}^{-1}\varphi_j}$. Then

$$\begin{aligned}\langle\bar\omega_i^t(v_i), X\rangle_j &= \langle v_i, \omega_i(D_jX) - \Omega_{ij}\Omega_{jj}^{-1}\omega_j(D_jX)\rangle\\ &= \langle D_j\omega_i^t(v_i) - D_j\omega_j^t(\Omega_{jj}^{-1})^t\Omega_{ij}^t(v_i), X\rangle\\ &= \langle\omega_i^t(v_i) - \omega_j^t(\Omega_{jj}^{-1})^t\Omega_{ij}^t(v_i), D_j^{-1}X\rangle_j,\end{aligned}$$

where $\langle\ ,\ \rangle_j$ denotes the metric induced by f_j. Using that D_j^{-1} is symmetric with respect to $\langle\ ,\ \rangle_j$, we obtain that

$$D_j\bar\omega_i^t = \omega_i^t - \omega_j^t(\Omega_{jj}^{-1})^t\Omega_{ij}^t.$$

It follows from (22) that

$$\begin{aligned}\alpha_j(X, \bar\omega_i^t(v_i)) = \mathcal{P}_j(&\alpha(X, \omega_i^t(v_i)) - \alpha(X, \omega_j^t(\Omega_{jj}^{-1})^t\Omega_{ij}^t(v_i)) + \beta_j(\Omega_{jj}^{-1})^t\Phi^j(X)^t\omega_i^t(v_i)\\ &+\beta_j(\Omega_{jj}^{-1})^t\Phi^j(X)^t\omega_j^t(\Omega_{jj}^{-1})^t\Omega_{ij}^t(v_i))\end{aligned} \tag{30}$$

where α_j is the second fundamental form of f_j. On the other hand, we obtain from

$$\begin{aligned}(\nabla_X\bar\beta_i)(v_i) &= \nabla_X^\perp\bar\beta_i(v_i) - \bar\beta_i(\nabla_X^V v_i)\\ &= \mathcal{P}_j(\nabla_X^\perp\beta_i(v_i) - \nabla_X^\perp\beta_j(\Omega_{jj}^{-1})^t\Omega_{ij}^t(v_i) - \beta_i(\nabla_X^V v_i) + \beta_j(\Omega_{jj}^{-1})^t\Omega_{ij}^t(\nabla_X^V v_i))\end{aligned}$$

and

$$\begin{aligned}&-\nabla_X^\perp\beta_j(\Omega_{jj}^{-1})^t\Omega_{ij}^t(v_i) + \beta_j(\Omega_{jj}^{-1})^t\Omega_{ij}^t(\nabla_X v_i)\\ &\quad= -(\nabla_X\beta_j)((\Omega_{jj}^{-1})^t\Omega_{ij}^t(v_i)) - \beta_j(d(\Omega_{jj}^{-1})^t(X)\Omega_{ij}^t(v_i)) - \beta_j(\Omega_{jj}^{-1})^t\, d\Omega_{ij}^t(X)(v_i)\end{aligned}$$

that

$$\begin{aligned}(\nabla_X\bar\beta_i)(v_i) = \mathcal{P}_j(&(\nabla_X\beta_i)(v_i) - (\nabla_X\beta_j)((\Omega_{jj}^{-1})^t\Omega_{ij}^t(v_i)) - \beta_j(\Omega_{jj}^{-1})^t\Phi^j(X)^t\omega_i^t(v_i)\\ &+\beta_j(\Omega_{jj}^{-1})^t\Phi^j(X)^t\omega_j^t(\Omega_{jj}^{-1})^t\Omega_{ij}^t(v_i)),\end{aligned} \tag{31}$$

where we used $d\Omega_{ij}^t(X) = \Phi^j(X)^t\omega_i^t$. It follows from (30) and (31) that

$$\alpha_j(X, \bar\omega_i^t(v_i)) + (\nabla_X\bar\beta_i)(v_i) = 0.$$

Now we have

$$
\begin{aligned}
\bar{\mathcal{F}}_i &= f_{j*}\bar{\omega}_i^t + \bar{\beta}_i = \mathcal{P}_j f_* D_j \bar{\omega}_i^t + \mathcal{P}_j(\beta_i - \beta_j(\Omega_{jj}^{-1})^t\Omega_{ij}^t)\\
&= \mathcal{P}_j(f_*\omega_i^t + \beta_i - f_*\omega_j^t(\Omega_{jj}^{-1})^t\Omega_{ij}^t - \beta_j(\Omega_{jj}^{-1})^t\Omega_{ij}^t = \mathcal{P}_j(\mathcal{F}_i - \mathcal{F}_j(\Omega_{jj}^{-1})^t\Omega_{ij}^t)\\
&= \mathcal{F}_i - \mathcal{F}_j(\Omega_{jj}^{-1})^t\Omega_{ij}^t - \mathcal{F}_j\Omega_{jj}^{-1}\mathcal{F}_j^t\mathcal{F}_i + \mathcal{F}_j\Omega_{jj}^{-1}\mathcal{F}_j^t\mathcal{F}_j(\Omega_{jj}^{-1})^t\Omega_{ij}^t\\
&= \mathcal{F}_i - \mathcal{F}_j\Omega_{jj}^{-1}\Omega_{ji},
\end{aligned}
$$

where we used that $\mathcal{F}_j^t\mathcal{F}_i = \Omega_{ji} + \Omega_{ij}^t$. Then,

$$
\begin{aligned}
\bar{\mathcal{F}}_i^t\, d\bar{\mathcal{F}}_i &= \bar{\mathcal{F}}_i^t(d\mathcal{F}_i - d\mathcal{F}_j\Omega_{jj}^{-1}\Omega_{ji} + \mathcal{F}_j\Omega_{jj}^{-1}d\Omega_{jj}\Omega_{jj}^{-1}\Omega_{ji} - \mathcal{F}_j\Omega_{jj}^{-1}d\Omega_{ji})\\
&= \mathcal{F}_i^t d\mathcal{F}_i - \mathcal{F}_i^t d\mathcal{F}_j\Omega_{jj}^{-1}\Omega_{ji} + \mathcal{F}_i^t\mathcal{F}_j\Omega_{jj}^{-1}d\Omega_{jj}\Omega_{jj}^{-1}\Omega_{ji} - \mathcal{F}_i^t\mathcal{F}_j\Omega_{jj}^{-1}d\Omega_{ji} - \Omega_{ji}^t(\Omega_{jj}^{-1})^t\mathcal{F}_j^t\, d\mathcal{F}_i\\
&\quad + \Omega_{ji}^t(\Omega_{jj}^{-1})^t\mathcal{F}_j^t\, d\mathcal{F}_j\Omega_{jj}^{-1}\Omega_{ji} - \Omega_{ji}^t(\Omega_{jj}^{-1})^t\mathcal{F}_j^t\mathcal{F}_j\Omega_{jj}^{-1}d\Omega_{jj}\Omega_{jj}^{-1}\Omega_{ji} + \Omega_{ji}^t(\Omega_{jj}^{-1})^t\mathcal{F}_j^t\mathcal{F}_j\Omega_{jj}^{-1}d\Omega_{ji}.
\end{aligned}
$$

Using that $d\Omega_{ji} = \mathcal{F}_j^t\, d\mathcal{F}_i$ and $d\Omega_{jj} = \mathcal{F}_j^t\, d\mathcal{F}_j$, we obtain

$$
\bar{\mathcal{F}}_i^t\, d\bar{\mathcal{F}}_i = d\Omega_{ii} - d\Omega_{ij}\Omega_{jj}^{-1}\Omega_{ji} + \Omega_{ij}\Omega_{jj}^{-1}d\Omega_{jj}\Omega_{jj}^{-1}\Omega_{ji} - \Omega_{ij}\Omega_{jj}^{-1}d\Omega_{ji} = d\bar{\Omega}_{ii}.
$$

Moreover,

$$
\begin{aligned}
\bar{\mathcal{F}}_i^t\bar{\mathcal{F}}_i &= (\mathcal{F}_i^t - \Omega_{ji}^t(\Omega_{jj}^{-1})^t\mathcal{F}_j^t)(\mathcal{F}_i - \mathcal{F}_j\Omega_{jj}^{-1}\Omega_{ji})\\
&= \mathcal{F}_i^t\mathcal{F}_i - \mathcal{F}_i^t\mathcal{F}_j\Omega_{jj}^{-1}\Omega_{ji} - \Omega_{ji}^t(\Omega_{jj}^{-1})^t\mathcal{F}_j^t\mathcal{F}_i + \Omega_{ji}^t(\Omega_{jj}^{-1})^t\mathcal{F}_j^t\mathcal{F}_j\Omega_{jj}^{-1}\Omega_{ji}\\
&= \Omega_{ii} - \Omega_{ij}\Omega_{jj}^{-1}\Omega_{ji} + \Omega_{ii}^t - \Omega_{ji}^t(\Omega_{jj}^{-1})^t\Omega_{ij}^t = \bar{\Omega}_{ii} + \bar{\Omega}_{ii}^t,
\end{aligned}
$$

which completes the proof that $(\bar{\varphi}_i, \bar{\beta}_i, \bar{\Omega}_{ii})$ satisfies the required conditions.

Now write Ω in matrix notation as

$$
\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12}\\ \Omega_{21} & \Omega_{22} \end{pmatrix}.
$$

Since Ω and Ω_{ii} are invertible, then $\bar{\Omega}_{ii}$ is invertible for $1 \le i \le 2$ and

$$
\Omega^{-1} = \begin{pmatrix} \bar{\Omega}_{11}^{-1} & -\bar{\Omega}_{11}^{-1}\Omega_{12}\Omega_{22}^{-1}\\ -\bar{\Omega}_{22}^{-1}\Omega_{21}\Omega_{11}^{-1} & \bar{\Omega}_{22}^{-1} \end{pmatrix}.
$$

In particular,

$$
\bar{\Omega}_{ii}^{-1} = \Omega_{ii}^{-1} + \Omega_{ii}^{-1}\Omega_{ij}\bar{\Omega}_{jj}^{-1}\Omega_{ji}\Omega_{ii}^{-1} \quad \text{and} \quad \bar{\Omega}_{ii}^{-1}\Omega_{ij}\Omega_{jj}^{-1} = \Omega_{ii}^{-1}\Omega_{ij}\bar{\Omega}_{jj}^{-1} \tag{32}
$$

for $1 \le i \ne j \le 2$. Then,

$$
\begin{aligned}
\mathcal{R}_{\varphi,\beta,\Omega}(f) &= f - \mathcal{F}\Omega^{-1}\varphi\\
&= f - \mathcal{F}(\bar{\Omega}_{11}^{-1}(\varphi_1 - \Omega_{12}\Omega_{22}^{-1}\varphi_2) + \bar{\Omega}_{22}^{-1}(\varphi_2 - \Omega_{21}(\Omega_{11})^{-1}\varphi_1))\\
&= f - \mathcal{F}_1\bar{\Omega}_{11}^{-1}(\varphi_1 - \Omega_{12}\Omega_{22}^{-1}\varphi_2) - \mathcal{F}_2\bar{\Omega}_{22}^{-1}(\varphi_2 - \Omega_{21}\Omega_{11}^{-1}\varphi_1).
\end{aligned}
$$

On the other hand, by (32) and (32) we have

$$\begin{aligned}\mathcal{R}_{\bar\varphi_i,\bar\beta_i,\bar\Omega_{ii}}(f_j) &= f_j - \bar{\mathcal{F}}_i\bar\Omega_{ii}\bar\varphi_i\\ &= f - \mathcal{F}_j\Omega_{jj}^{-1}\varphi_j - (\mathcal{F}_i - \mathcal{F}_j\Omega_{jj}^{-1}\Omega_{ji})\bar\Omega_{ii}^{-1}(\varphi_i - \Omega_{ij}\Omega_{jj}^{-1}\varphi_j)\\ &= f - (\mathcal{F}_j\bar\Omega_{jj}^{-1} - \mathcal{F}_i\bar\Omega_{ii}^{-1}\Omega_{ij}\Omega_{jj}^{-1})\varphi_j - (\mathcal{F}_i\bar\Omega_{ii}^{-1} - \mathcal{F}_j\bar\Omega_{jj}^{-1}\Omega_{ji}\Omega_{ii}^{-1})\varphi_i.\end{aligned}$$

We conclude that $\mathcal{R}_{\varphi,\beta,\Omega}(f) = \mathcal{R}_{\bar\varphi_i,\bar\beta_i,\bar\Omega_{ii}}(f_j)$ for $1 \le i \ne j \le 2$. ∎

Remark 15. It follows from Theorem 14 that a vectorial Ribaucour transformation whose associated data (φ, β, Ω) are defined on a vector space V can be regarded as the iteration of $k = \dim V$ scalar Ribaucour transformations.

In applying Theorem 14, it is often more convenient to use one of its two following consequences.

Corollary 16. *Let $f_i = \mathcal{R}_{\varphi_i,\beta_i,\Omega_{ii}}(f)\colon M_i^n \to \mathbb{R}^N$, $1 \le i \le 2$, be two vectorial Ribaucour transforms of $f\colon M^n \to \mathbb{R}^N$. Assume that the tensors $\Phi^i = \Phi(d\varphi_i, \beta_i)$ satisfy*

$$[\Phi^i_{v_i}, \Phi^j_{v_j}] = 0 \quad \textit{for all } v_i \in V_i \textit{ and } v_j \in V_j,\ \ 1 \le i \ne j \le 2.$$

Set $\mathcal{F}_i = f_*(d\varphi_i)^t + \beta_i$. Then there exists $\Omega_{ij} \in \Gamma(V_j^* \otimes V_i)$, such that

$$d\Omega_{ij} = \mathcal{F}_i^t\, d\mathcal{F}_j \quad \textit{and} \quad \mathcal{F}_i^t\mathcal{F}_j = \Omega_{ij} + \Omega_{ji}^t, \tag{33}$$

and such that $\varphi \in \Gamma(V), \beta \in \Gamma(V^* \otimes T^\perp M)$ and $\Omega \in \Gamma(V^* \otimes V)$ defined by (29) for $V = V_1 \oplus V_2$ satisfy the conditions of Definition 9 (and therefore the remaining of the conclusions of Theorem 14 hold).

Proof: The first assertion is a consequence of Proposition 8. It is now easily seen that $\varphi \in \Gamma(V), \beta \in \Gamma(V^* \otimes T^\perp M)$, $\Omega \in \Gamma(V^* \otimes V)$ defined by (29) for $V = V_1 \oplus V_2$ satisfy the conditions of Definition 9 with respect to f if and only if the same holds for $(\varphi_i, \beta_i, \Omega_{ii})$ and, in addition, (33) holds. ∎

Corollary 17. *Let $f_1 = \mathcal{R}_{\varphi_1,\beta_1,\Omega_{11}}(f)\colon \tilde M^n \to \mathbb{R}^N$ be a vectorial Ribaucour transform of $f\colon M^n \to \mathbb{R}^N$. Let $(\bar\varphi_2, \bar\beta_2, \bar\Omega_{22})$ satisfy the conditions of Definition 9 with respect to f_1. Assume further that $\bar\Phi^2 = \Phi(d\bar\varphi_2, \bar\beta_2)$ satisfies*

$$[\bar\Phi^2_{v_2}, \bar\Phi^1_{v_1}] = 0, \ \textit{for all } v_1 \in V_1,\ v_2 \in V_2,$$

where $D_1\bar\Phi^1_{v_1} = -\Phi^1_{\Omega_{11}^{-1}v_1}$ for $\Phi^1 = \Phi(d\varphi_1, \beta_1)$. Then there exist $\bar\Omega_{ij} \in \Gamma(V_j^* \otimes V_i)$, $i \ne j$, such that

$$d\bar\Omega_{ij} = \bar{\mathcal{F}}_i^t\, d\bar{\mathcal{F}}_j \quad \textit{and} \quad \bar{\mathcal{F}}_i^t\bar{\mathcal{F}}_j = \bar\Omega_{ij} + \bar\Omega_{ji}^t, \tag{34}$$

where $\bar{\mathcal{F}}_1 = -\mathcal{F}_1\Omega_{11}^{-1}$ *and* $\bar{\mathcal{F}}_2 = (f_1)_*(d\bar{\varphi}_2)^t + \bar{\beta}_2$. *Now define*

$$(\varphi_2, \beta_2, \Omega_{22}) = (\bar{\varphi}_2 - \bar{\Omega}_{21}\varphi_1, \mathcal{P}_1^{-1}\bar{\beta}_2 - \beta_1\bar{\Omega}_{21}^t, \bar{\Omega}_{22} - \bar{\Omega}_{21}\Omega_{11}\bar{\Omega}_{12}),$$

$$\Omega_{12} = \Omega_{11}\bar{\Omega}_{12} \quad and \quad \Omega_{21} = -\bar{\Omega}_{21}\Omega_{11}.$$

Then $\varphi \in \Gamma(V), \beta \in \Gamma(V^* \otimes T^\perp M)$ *and* $\Omega \in \Gamma(V^* \otimes V)$ *defined by (29) for* $V = V_1 \oplus V_2$ *satisfy the conditions of Definition 9 and* $\mathcal{R}_{\varphi,\beta,\Omega}(f) = \mathcal{R}_{\bar{\varphi}_2,\bar{\beta}_2,\bar{\Omega}_{22}}(\mathcal{R}_{\varphi_1,\beta_1,\Omega_{11}}(f))$.

Proof: By Proposition 13, we have $f = \mathcal{R}_{\bar{\varphi}_1,\bar{\beta}_1,\bar{\Omega}_{11}}(f_1)$ where

$$(\bar{\varphi}_1, \bar{\beta}_1, \bar{\Omega}_{11}) = (\Omega_{11}^{-1}\varphi_1, \mathcal{P}_1\beta_1(\Omega_{11}^{-1})^t, \Omega_{11}^{-1}).$$

Moreover, $\bar{\mathcal{F}}_1 = (f_1)_*(d\bar{\varphi}_1)^t + \bar{\beta}_1$ and $\bar{\Phi}_1 = \Phi(d\bar{\varphi}_1, \bar{\beta}_1)$ are given by

$$\bar{\mathcal{F}}_1 = -\mathcal{F}_1\Omega_{11}^{-1} \quad \text{and} \quad D_1\bar{\Phi}^1_{v_1} = -\Phi^1_{\Omega_{11}^{-1}v_1}.$$

Thus, the existence of $\bar{\Omega}_{ij} \in \Gamma(V_j^* \otimes V_i)$, $i \neq j$, satisfying condition (34) follows from Proposition 8 applied to f_1 and the triples $(\bar{\varphi}_i, \bar{\beta}_i, \bar{\Omega}_{ii})$, $1 \leq i \leq 2$. Now observe that $(\varphi_2, \beta_2, \Omega_{22}) = \mathcal{R}_{\bar{\varphi},\bar{\beta},\bar{\Omega}}(\bar{\varphi}_2, \bar{\beta}_2, \bar{\Omega}_2)$, and hence $(\varphi_2, \beta_2, \Omega_{22})$ satisfies (7) with respect to f_1 and $d\Omega_{22} = \mathcal{F}_2^t\, d\mathcal{F}_2$ by Theorem 14.

It remains to check that $d\Omega_{ij} = \mathcal{F}_i^t\, d\mathcal{F}_j$ and $\Omega_{ij} + \Omega_{ji}^t = \mathcal{F}_i^t\mathcal{F}_j$, $\leq i \neq j$. From the proof of Theorem 14 (see (32)) we have $\mathcal{F}_2 = \bar{\mathcal{F}}_2 - \bar{\mathcal{F}}_1\bar{\Omega}_{11}^{-1}\bar{\Omega}_{12} = \bar{\mathcal{F}}_2 - \bar{\mathcal{F}}_1\Omega_{11}\bar{\Omega}_{12}$. Then,

$$\begin{aligned}\mathcal{F}_2^t\, d\mathcal{F}_1 &= -(\bar{\mathcal{F}}_2^t - \bar{\Omega}_{12}^t\Omega_{11}^t\bar{\mathcal{F}}_1^t\, d(\bar{\mathcal{F}}_1\Omega_{11}) = -\bar{\mathcal{F}}_2^t\, d\bar{\mathcal{F}}_1\Omega_{11} - \bar{\mathcal{F}}_2^t\bar{\mathcal{F}}_1\, d\Omega_{11} + \bar{\Omega}_{12}^t\mathcal{F}_1^t\, d\mathcal{F}_1\\ &= -d\bar{\Omega}_{21}\Omega_{11} - (\bar{\Omega}_{21} + \bar{\Omega}_{12}^t)\, d\Omega_{11} + \bar{\Omega}_{12}^t\, d\Omega_{11} - d(\bar{\Omega}_{21}\Omega_{11})\\ &= d\Omega_{21}.\end{aligned}$$

A similar computation shows that $\mathcal{F}_1^t\, d\mathcal{F}_2 = \Omega_{12}$.

Finally, we have

$$\begin{aligned}\mathcal{F}_1^t\mathcal{F}_2 &= -\Omega_{11}^t\bar{\mathcal{F}}_1^t(\bar{\mathcal{F}}_2 - \bar{\mathcal{F}}_1\bar{\Omega}_{11}^{-1}\bar{\Omega}_{12}) = -\Omega_{11}^t(\bar{\Omega}_{12} + \bar{\Omega}_{21}^t - (\bar{\Omega}_{11} + \bar{\Omega}_{11}^t)\bar{\Omega}_{11}^{-1}\bar{\Omega}_{12})\\ &= -\Omega_{11}^t(\bar{\Omega}_{21}^t - \bar{\Omega}_{11}^t\Omega_{11}\bar{\Omega}_{12}) = -\Omega_{11}^t\bar{\Omega}_{21}^t + \Omega_{11}\bar{\Omega}_{12} = \Omega_{11}\bar{\Omega}_{12} - (\bar{\Omega}_{21}\Omega_{11})^t\\ &= \Omega_{12} + \Omega_{21}^t,\end{aligned}$$

and similarly one checks that $\mathcal{F}_2^t\mathcal{F}_1 = \Omega_{21} + \Omega_{12}^t$. ∎

Given four submanifolds $f_i\colon M_i^n \to \mathbb{R}^N$, $1 \leq i \leq 4$, we say that they form a *Bianchi quadrilateral* if for each of them both the preceding and subsequent ones (thought of as points on an oriented circle) are Ribaucour transforms of it, and the Codazzi tensors associated to the transformations commute.

Proof of Theorem 2: We first prove existence. Write $f_i = \mathcal{R}_{\varphi_i,\beta_i}(f)$, $1 \leq i \leq k$. For each pair $\{i,j\} \subset \{1,\ldots,k\}$ with $i < j$ define $\varphi^{ij} \in \Gamma(\mathbb{R}^2)$ and $\beta^{ij} \in \Gamma((\mathbb{R}^2)^* \otimes T^\perp M)$ by

$$\varphi^{ij} = (\varphi_i, \varphi_j) \quad \text{and} \quad \beta^{ij} = dx_1 \otimes \beta_i + dx_2 \otimes \beta_j.$$

By the assumption that $\{f_{ij}, f_i, f_j, f\}$ is a Bianchi quadrilateral, there $\Omega^{ij} \in Gl(\mathbb{R}^2)$ with

$$\Omega^{ij}(e_1) = \Omega_{ii} = (1/2)\langle \mathcal{F}_i, \mathcal{F}_i\rangle \quad \text{and} \quad \Omega^{ij}(e_2) = \Omega_{jj} = (1/2)\langle \mathcal{F}_j, \mathcal{F}_j\rangle,$$

where $\mathcal{F}_r = f_*\nabla\varphi_r + \beta_r$, $r \in \{i, j\}$, such that $(\varphi^{ij}, \beta^{ij}, \Omega^{ij})$ satisfies the conditions of Definition 9 with respect to f and such that $f_{ij} = \mathcal{R}_{\varphi^{ij},\beta^{ij},\Omega^{ij}}(f)$. Define $\varphi \in \Gamma(\mathbb{R}^k)$, $\beta \in \Gamma((\mathbb{R}^k)^* \otimes T^\perp M)$ and $\Omega \in \Gamma((\mathbb{R}^k)^* \otimes \mathbb{R}^k)$ by

$$\varphi = (\varphi_1, \ldots, \varphi_k), \quad \beta = \sum_{i=1}^{k} dx_i \otimes \beta_i$$

and

$$\Omega = \sum_{i=1}^{k} \Omega_{ii} dx_i \otimes e_i + \sum_{i<j} (\langle \Omega^{ij}(e_1), e_2\rangle dx_i \otimes e_j + \langle \Omega^{ij}(e_2), e_1\rangle dx_j \otimes e_i).$$

It is easy to check that (φ, β, Ω) satisfies the conditions of Definition 9 with respect to f.

We now make precise the "generic" assumption on the statement of Theorem 2. Namely, we require that no principal minor of Ω vanishes, where Ω is regarded as a square $(k \times k)$-matrix. That is, for any multi-index $\alpha = \{i_1 < \ldots < i_r\} \subset \{1, \ldots, k\}$, the sub-matrix Ω_α of Ω, formed by those elements of Ω that belong to the rows and columns with indexes in α, has nonzero determinant. Now, for any such α set

$$\varphi^\alpha = (\varphi_{i_1}, \ldots, \varphi_{i_r}), \quad \beta^\alpha = \sum_{j=1}^{r} dx_{i_j} \otimes \beta_{i_j} \quad \text{and} \quad \Omega^\alpha = \Omega_\alpha.$$

We define $\mathcal{C}_r$ as the family of $\binom{k}{r}$ elements formed by the vectorial Ribaucour transforms $\mathcal{R}_{\varphi^\alpha,\beta^\alpha,\Omega^\alpha}(f)$, where α ranges on the set of multi-indexes $\alpha = \{i_1 < \ldots < i_r\} \subset \{1, \ldots, k\}$ with r elements. Given

$$\hat{f} = \mathcal{R}_{\varphi^\alpha,\beta^\alpha,\Omega^\alpha}(f) \in \mathcal{C}_{s+1}, \quad 1 \le s \le k-1 \;\text{ and }\; \alpha = \{i_1 < \ldots < i_{s+1}\} \subset \{1, \ldots, k\},$$

let $\alpha_1, \ldots, \alpha_{s+1}$ be the $(s+1)$ multi-indexes with s elements that are contained in α. For each $j = 1, \ldots, s+1$ write $\alpha = \alpha_j \cup \{i_j\}$. Then,

$$\hat{f}_j := \mathcal{R}_{\varphi^{\alpha_j},\beta^{\alpha_j},\Omega^{\alpha_j}}(f) \in \mathcal{C}_s \quad \text{and} \quad \hat{f} = \mathcal{R}_{\bar{\varphi}_{i_j},\bar{\beta}_{i_j}}(\hat{f}_j)$$

by Theorem 14. Therefore $\hat{f}$ is a Ribaucour transform of $\hat{f}_1, \ldots, \hat{f}_{s+1}$. Moreover, for each pair $\{\alpha_i, \alpha_j\}$, set $\alpha_{ij} = \alpha_i \cap \alpha_j$, and let $\hat{f}_{ij} = \mathcal{R}_{\varphi^{\alpha_{ij}},\beta^{\alpha_{ij}},\Omega^{\alpha_{ij}}}(f)$. Then $\hat{f}_{ij} \in \mathcal{C}_{s-1}$ and $\{\hat{f}_{ij}, \hat{f}_i, \hat{f}_j, \hat{f}\}$ is a Bianchi quadrilateral

Next we argue for the uniqueness. We first make precise the meaning of $f_1, \ldots, f_k$ being *independent* Ribaucour transforms of f. Namely, if f_i is determined by the pair (φ_i, β_i) with $\varphi_i \in C^\infty(M)$ and $\beta_i \in \Gamma(T_f^\perp M)$, $1 \le i \le k$, we require that the image of the map $\varphi = (\varphi_1, \ldots, \varphi_k)\colon\ M \to \mathbb{R}^k$ spans $\mathbb{R}^k$ and, in addition, that the linear map $\mathcal{F}\colon\ \mathbb{R}^k \to f^*T\mathbb{R}^N$ given by $\mathcal{F} = \sum_{i=1}^k dx_i \otimes \mathcal{F}_i$, with $\mathcal{F}_i = f_*\nabla\varphi_i + \beta_i$, is injective.

It is easily seen that all uniqueness assertions follow from the uniqueness for $k = 3$. For this case, the independence assumption is equivalent to the condition that neither of f_1, f_2 or f_3 belong to the *associated family* determined by the other two. Then, uniqueness was proved in [11] by using a nice elementary argument relying on the version of Miquel's Theorem for four circumferences. ∎

§5 Submanifolds carrying a parallel flat normal subbundle.

In this section we give an explicit local construction of all submanifolds of Euclidean space that carry a parallel flat normal subbundle, from which Theorem 1 in the introduction follows as a special case.

Theorem 18. *Let $f\colon\ M^n \to \mathbb{R}^{n+p}$ be an isometric immersion of a simply connected Riemannian manifold and let $\varphi_i \in C^\infty(M)$ and $\beta_i \in \Gamma(T^\perp M)$, $1 \le i \le m$, satisfy*

$$\alpha(X, \nabla\varphi_i) + \nabla^\perp_X \beta_i = 0 \tag{35}$$

and

$$[\Phi_i, \Phi_j] = 0, \quad 1 \le i, j \le m, \tag{36}$$

where $\Phi_i = \operatorname{Hess}\varphi_i - A_{\beta_i}$. Define $\mathcal{G}\colon\ M^n \to M_{(n+p)\times m}(\mathbb{R})$ by

$$\mathcal{G} = (f_*\nabla\varphi_1 + \beta_1, \ldots, f_*\nabla\varphi_m + \beta_m).$$

Then there exists a smooth map $\Omega\colon\ U \to GL(\mathbb{R}^m)$ on an open subset $U \subset M^n$ such that

$$d\Omega = \mathcal{G}^t\, d\mathcal{G} \quad \text{and} \quad \Omega + \Omega^t = \mathcal{G}^t\mathcal{G} + I. \tag{37}$$

Moreover, the map $\tilde f\colon\ \tilde M^n \to \mathbb{R}^{n+p+m}$ given by

$$\tilde f(p) = \begin{pmatrix} f(p)^t - \mathcal{G}\Omega^{-1}\varphi(p) \\ -\Omega^{-1}\varphi(p) \end{pmatrix} \tag{38}$$

where $\varphi(p) = (\varphi_1(p), \ldots, \varphi_m(p))^t$, defines, on an open subset $\tilde M^n \subset U$ of regular points, an immersion carrying a parallel flat normal subbundle of rank m.

Conversely, any isometric immersion carrying a parallel flat normal subbundle of rank m can be locally constructed in this way.

Proof: Set $V = \mathbb{R}^m$ and define $\beta \in \Gamma(V^* \oplus T^\perp M)$ by $\beta(e_i) = \beta_i + e_i$, where $\{e_i\}_{1\le i\le m}$ is the canonical basis of $\mathbb{R}^m$ regarded as the orthogonal complement of $\mathbb{R}^{n+p}$ in $\mathbb{R}^{n+p+m}$.

Then $\omega = d\varphi$ and β satisfy (7) in view of (35). Moreover, $\mathcal{F} \in \Gamma(V^* \oplus f^*T\mathbb{R}^{n+m+p})$ given by $\mathcal{F}(v) = f_*\omega^t(v) + \beta(v)$ satisfies

$$\mathcal{F} = \begin{pmatrix} \mathcal{G} \\ I_m \end{pmatrix},$$

where I_m denotes the $m \times m$ identity matrix. Thus

$$\mathcal{F}^t \, d\mathcal{F} = \mathcal{G}^t d\mathcal{G} \quad \text{and} \quad \mathcal{F}^t\mathcal{F} = \mathcal{G}^t\mathcal{G} + I_m,$$

and the existence of Ω satisfying (37) follows from Proposition 8 by using (36). Moreover, comparing (17) and (38) we have that $\tilde{f} = \mathcal{R}_{\varphi,\beta,\Omega}(f)$. Since $\mathbb{R}^m$ is a parallel flat normal vector subbundle of $T^\perp M$ (where f is regarded as an immersion into $\mathbb{R}^{n+p+m}$) and $\mathcal{P} \in \Gamma((f^*T\mathbb{R}^N)^* \otimes \tilde{f}^*T\mathbb{R}^N)$ given by (18) is a parallel vector bundle isometry by virtue of (20), it follows that $\mathcal{P}(\mathbb{R}^m)$ is a parallel flat normal vector subbundle of $T_{\tilde{f}}M$ of rank m.

In order to prove the converse, it suffices to show that, given an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p+m}$ carrying a parallel flat normal subbundle E of rank m, there exist locally an immersion $\tilde{f}\colon M^n \to \mathbb{R}^{n+p} \subset \mathbb{R}^{n+p+m}$ and $\tilde{\varphi} \in \Gamma(V := \mathbb{R}^m)$, $\tilde{\beta} \in \Gamma(V^* \oplus T_{\tilde{f}}^\perp M)$ and $\tilde{\Omega} \in \Gamma(Gl(V))$ satisfying the conditions of Definition 9 such that $(\tilde{\beta}(e_i))_{\mathbb{R}^m} = e_i$, $1 \leq i \leq m$, and that $f = \mathcal{R}_{\tilde{\varphi},\tilde{\beta},\tilde{\Omega}}(\tilde{f})$.

Let $\xi_1, \ldots, \xi_m$ be an orthonormal parallel frame of E. Let $V = \mathbb{R}^m$ be identified with a subspace of $\mathbb{R}^N$ and let $e_1, \ldots, e_m$ be the canonical basis of $\mathbb{R}^m$. Define $\varphi \in \Gamma(V)$ and $\beta \in \Gamma(V^* \otimes T^\perp M)$ by

$$\varphi = -\sum_{i=1}^m \langle f, e_i \rangle e_i \quad \text{and} \quad \beta(v) = \sum_{i=1}^m x_i(\xi_i - e_i^\perp),$$

for $v = (x_1, \ldots, x_m)$, where $e_i^\perp$ denotes the normal vector field obtained by orthogonally projecting e_i pointwise onto $T^\perp M$. Then

$$\omega^t(v) = -\sum_{i=1}^m x_i f_*^t e_i \quad \text{and} \quad \mathcal{F}(v) = \sum_{i=1}^m x_i(\xi_i - e_i). \tag{39}$$

Therefore,

$$\begin{aligned} \alpha(X, \omega^t(v)) + (\nabla_X^{V^*\otimes T^\perp M}\beta)v &= -\sum_{i=1}^m x_i \alpha(X, f_*^t e_i) + \sum_{i=1}^m x_i \nabla_X^\perp(\xi_i - e_i^\perp) \\ &= -\sum_{i=1}^m x_i(\tilde{\nabla}_X(f_*^t e_i + e_i^\perp))^\perp = \sum_{i=1}^m x_i(\tilde{\nabla}_X e_i)^\perp = 0, \end{aligned}$$

where $\tilde{\nabla}$ denotes the Euclidean connection, and hence (7) is satisfied.

It also follows from (39) that

$$\langle \mathcal{F}^t\xi_j, v\rangle = \langle \xi_j, \mathcal{F}(v)\rangle = x_j - \sum_{i=1}^{m} x_i\langle \xi_j, e_i\rangle,$$

thus $\mathcal{F}^t\xi_j = e_j - \sum_{i=1}^m \langle \xi_j, e_i\rangle e_i$. Similarly, $\mathcal{F}^t e_j = -e_j + \sum_{i=1}^m \langle \xi_i, e_j\rangle e_i$. We obtain,

$$\mathcal{F}^t\mathcal{F}(v) = \sum_{j=1}^{m} x_j\mathcal{F}^t(\xi_j - e_j) = \sum_{j=1}^{m} 2x_je_j - \sum_{i,j=1}^{m} x_j(\langle \xi_j, e_i\rangle + \langle \xi_i, e_j\rangle)e_i.$$

In matrix notation, this reads as

$$\mathcal{F}^t\mathcal{F} = 2I - (\langle \xi_j, e_i\rangle) - (\langle \xi_i, e_j\rangle).$$

Therefore $\Omega = I - (\langle \xi_i, e_j\rangle)$ satisfies (15) and (16). Moreover, since

$$\Omega e_j = e_j - \sum_{i=1}^{m}\langle \xi_j, e_i\rangle e_i = \mathcal{F}^t\xi_j,$$

we have

$$\mathcal{P}\xi_j = \xi_j - \mathcal{F}\Omega^{-1}\mathcal{F}^t\xi_j = \xi_j - \mathcal{F}e_j = \xi_j - (\xi_j - e_j) = e_j.$$

Therefore $\tilde{f} = \mathcal{R}_{\varphi,\beta,\Omega}(f)$ is such that $\tilde{f}(M^n)$ is contained in an affine subspace orthogonal to $\mathbb{R}^m$. Since $f = \mathcal{R}_{\tilde{\varphi},\tilde{\beta},\tilde{\Omega}}(\tilde{f})$ with the triple $(\tilde{\varphi}, \tilde{\beta}, \tilde{\Omega})$ given by Proposition 13, in order to complete the proof of the theorem it remains to show that $(\tilde{\beta}(e_i))_{\mathbb{R}^m} = e_i$. But this follows from

$$\langle \tilde{\beta}e_i, e_j\rangle = \langle \mathcal{P}\beta(\Omega^{-1})^te_i, e_j\rangle = \langle e_i, \Omega^{-1}\mathcal{F}^t\mathcal{P}^te_j\rangle = \langle e_i, \Omega^{-1}\mathcal{F}^t\xi_j\rangle = \langle e_i, e_j\rangle. \qquad \blacksquare$$

The case of submanifolds with flat normal bundle of the sphere now follows easily from Theorem 1.

Corollary 19. *Let $U \subset \mathbb{R}^n$, $\{\varphi_i\}_{1\leq i\leq m}$, $\mathcal{G}\colon U \to M_{n\times m}(\mathbb{R})$, and $\Omega\colon V \subset U \to Gl(\mathbb{R}^m)$ be as in Theorem 1. Then the $M_{(n+m)\times m}(\mathbb{R})$-valued map*

$$W = \begin{pmatrix} \mathcal{G}\Omega^{-1} \\ I - \Omega^{-1} \end{pmatrix}$$

satisfies $W^tW = I$ and any of its columns defines, at regular points, the position vector of an immersion with flat normal bundle into $\mathbb{S}^{n+m-1}$.

Conversely, any isometric immersion with flat normal bundle $f\colon M^n \to \mathbb{S}^{n+m-1}$ can be locally constructed in this way.

Proof: Set $V = \mathbb{R}^m$ and define $\beta \in \Gamma(V^* \oplus T^\perp M)$ by $\beta(e_i) = e_i$, where $\{e_i\}_{1 \le i \le m}$ is the canonical basis of $\mathbb{R}^m$ regarded as the orthogonal complement of $\mathbb{R}^{n+p}$ in $\mathbb{R}^{n+p+m}$. Then $\omega = d\varphi$ and β trivially satisfy (7). Moreover, if $\mathcal{F} \in \Gamma(V^* \oplus f^*T\mathbb{R}^{n+m+p})$ is given by $\mathcal{F}(v) = f_*\omega^t(v) + \beta(v)$ then

$$\mathcal{F} = \begin{pmatrix} \mathcal{G} \\ I_m \end{pmatrix},$$

where I_m denotes the $m \times m$ identity matrix. Let $\tilde{f} = \mathcal{R}_{\varphi,\beta,\Omega}(id)$, where id is the inclusion of U into $\mathbb{R}^n$, Then the isometry $\mathcal{P}$ as in (18) is given by

$$\begin{pmatrix} I - \mathcal{G}\Omega^{-1}\mathcal{G}^t & \mathcal{G}\Omega^{-1} \\ \Omega^{-1}\mathcal{G}^t & I - \Omega^{-1} \end{pmatrix}.$$

Therefore $W^tW = I$ and the $(n+p+j)^{th}$-column of W is $\mathcal{P}e_j$, $1 \le j \le m$. Therefore it is a unit parallel normal vector field to f, and hence defines, at regular points, the position vector of a submanifold with flat normal bundle of $\mathbb{S}^{n+m-1}$.

The converse follows from the converse in Theorem 1 and the fact that any isometric immersion $f\colon M^n \to \mathbb{S}^{n+m-1}$ arises as a parallel unit normal vector field of an isometric immersion $F\colon M^n \to \mathbb{R}^{n+m}$, for instance, $F = i \circ f$, where i is the canonical inclusion of $\mathbb{S}^{n+m-1}$ into $\mathbb{R}^{n+m}$. ∎

We now give a precise statement of Ferapontov's theorem referred to in the introduction for the case of holonomic submanifolds of the sphere, and show how it can be derived from Corollary 19.

Theorem 20. *On an open simply connected subset $U \subset \mathbb{R}^n$ let $\{\beta_{ij}\}_{1 \le i \ne j \le n}$ be smooth real functions satisfying the completely integrable system of PDE's*

$$\begin{cases} \dfrac{\partial \beta_{ij}}{\partial u_k} = \beta_{ik}\beta_{kj}, \quad 1 \le i \ne j \ne k \ne i \le n, \\ \dfrac{\partial \beta_{ij}}{\partial u_i} + \dfrac{\partial \beta_{ji}}{\partial u_j} + \displaystyle\sum_k \beta_{ki}\beta_{kj} = 0, \quad i \ne j, \end{cases} \tag{40}$$

let $H^\alpha = (H_1^\alpha, \ldots, H_n^\alpha)$, $1 \le \alpha \le m$, be arbitrary solutions of the linear system of PDE's

$$\frac{\partial H_j}{\partial u_i} = \beta_{ij}H_i, \quad 1 \le i \ne j \le n, \tag{41}$$

and let $X_i\colon U \to \mathbb{R}^n$, $1 \le i \le n$, satisfy

$$\frac{\partial X_i}{\partial u_j} = \beta_{ij}X_j, \;\; i \ne j, \qquad \frac{\partial X_i}{\partial u_i} = -\sum_{k \ne i} \beta_{ki}X_k \tag{42}$$

and $X^tX = I$ at some point of U, where $X = (X_1, \ldots, X_n)$, the integrability conditions of (41) and (42) being satisfied by virtue of (40). Then there exist vector functions $s^\alpha = (s_1^\alpha, \ldots, s_n^\alpha)\colon U \to \mathbb{R}^n$, $1 \le \alpha \le m$, such that $ds_i^\alpha = \sum_{k=1}^n X_{ik} H_k^\alpha du^k$, and a map $\Omega\colon U \to M_{n\times m}(\mathbb{R})$ such that

$$d\Omega = \mathcal{G}^t\, d\mathcal{G} \quad \textit{and} \quad \Omega + \Omega^t = \mathcal{G}^t\mathcal{G} + I_m,$$

where $\mathcal{G} = (s^1, ..., s^m)$. Moreover, the $(m+n)\times m$-matrix

$$W = \begin{pmatrix} \mathcal{G}\Omega^{-1} \\ \cdots \\ I_m - \Omega^{-1} \end{pmatrix}$$

satisfies $W^tW = I_m$ and any of its columns defines, at regular points, the position vector of an n-dimensional submanifold $M^n \subset \mathbb{S}^{n+m-1} \subset \mathbb{R}^{n+m}$ with flat normal bundle such that $u_1, \ldots, u_n$ are principal coordinates of M^n.

Conversely, any n-dimensional submanifold with flat normal bundle of $\mathbb{S}^{n+m-1}$ carrying a holonomic net of curvature lines can be locally constructed in this way.

Proof: It is easily checked using (42), and the fact that $X^tX = I$ at some point of U, that $X^tX = I$ everywhere on U, whence $X_1, \ldots, X_n$ determine an orthonormal frame on U. Define $\Phi^\alpha \in \Gamma(T^*U \otimes TU)$ by

$$\Phi^\alpha X_i = H_i^\alpha X_i, \ \ 1 \le i \le n, \ \ 1 \le \alpha \le m.$$

Then Φ^α is a symmetric tensor and, by (41) and (42),

$$\frac{\partial}{\partial u_j}(\Phi^\alpha X_i) = \frac{\partial H_i^\alpha}{\partial u_j} X_i + H_i^\alpha \frac{\partial X_i}{\partial u_j} = \beta_{ji} H_j^\alpha X_i + H_i^\alpha \beta_{ij} X_j = \frac{\partial}{\partial u_i}(\Phi^\alpha X_j),$$

hence Φ^α is a Codazzi tensor on U. Thus Φ^α is closed as a one-form in U with values in TU. Since U is flat, there exists $Z^\alpha \in \Gamma(TU)$ such that $\Phi^\alpha = dZ^\alpha$. Moreover, the symmetry of Φ^α implies that $Z^\alpha = \operatorname{grad}\varphi^\alpha$ for some $\varphi^\alpha \in C^\infty(U)$, and hence $\Phi^\alpha = \operatorname{Hess}\,\varphi^\alpha$ (cf. [7]). Since $\{X_i\}_{1\le i\le n}$ is a common diagonalizing basis of $\operatorname{Hess}\,\varphi^\alpha$, $1 \le \alpha \le m$, it follows that $[\operatorname{Hess}\,\varphi_\alpha, \operatorname{Hess}\,\varphi_\beta] = 0$, $1 \le \alpha, \beta \le m$. Setting $s^\alpha = \operatorname{grad}\varphi^\alpha$, the remaining of the proof follows from Corollary 19. ∎

Remark 21. Equations (40) (Lamé equations) and (41) are well-known in the theory of n-orthogonal systems (cf. [3]) where the functions H_i and β_{ij} are usually called the Lamé and rotation coefficients, respectively.

§6 Submanifolds carrying a Dupin principal normal.

A smooth normal vector field η of an isometric immersion $f\colon M^n \to \mathbb{R}^N$ is called a *principal normal* with multiplicity $m \geq 1$ if the tangent subspaces

$$\mathcal{E}_\eta = \ker(\alpha - \langle\ ,\ \rangle\,\eta)$$

have constant dimension $m \geq 1$. If η is parallel in the normal connection along the *nullity* distribution $\mathcal{E}_\eta$ then η is said to be a *Dupin* principal normal. This condition is automatic if $m \geq 2$. If η is nowhere vanishing, it is well-known that $\mathcal{E}_\eta$ is an involutive distribution whose leaves are round m-dimensional spheres in $\mathbb{R}^N$. When η vanishes identically, the distribution $\mathcal{E}_\eta = \mathcal{E}_0$ is known as the relative nullity distribution, in which case the leaves are open subsets of affine subspaces of $\mathbb{R}^N$.

Let $h\colon L^{n-m} \to \mathbb{R}^N$ be an isometric immersion carrying a parallel flat normal subbundle $\mathcal{N}$ of rank m, and let $\varphi \in C^\infty(L^{n-m})$ and $\beta \in \Gamma(T_h^\perp L)$ satisfy

$$\alpha(X, \nabla\varphi) + \nabla^\perp_X \beta = 0.$$

Assume further that the tangent subspaces

$$E(x) = \{Z \in T_xL : (\alpha(Z,X))_{\mathcal{N}^\perp} = \varphi^{-1}\beta_{\mathcal{N}^\perp}\langle Z, X\rangle \text{ for all } X \in T_xL\}$$

are everywhere trivial. Define $\mathcal{R}^{\mathcal{N}}_{\varphi,\beta}(h)\colon \mathcal{N} \to \mathbb{R}^N$ by

$$\mathcal{R}^{\mathcal{N}}_{\varphi,\beta}(h)(t) = \mathcal{R}_{\varphi,\beta+t'}(h)(x),$$

where $x = \pi(t)$ and t' is the parallel section in $\mathcal{N}$ such that $t'(x) = t$. It was shown in [6] that $\mathcal{R}^{\mathcal{N}}_{\varphi,\beta}(h)$ defines, at regular points, an immersion carrying a Dupin principal normal with integrable *conullity* distribution $\mathcal{E}_\eta^\perp$ and that, conversely, any such immersion can be locally constructed in this way.

Using the results of the previous sections we now give an explicit description of all isometric immersions carrying a Dupin principal normal of multiplicity m and integrable conullity in terms of the vectorial Ribaucour transformation, starting with an isometric immersion $g\colon L^{n-m} \to \mathbb{R}^N$ such that $g(L^{n-m})$ lies in an $(N-m)$-dimensional subspace $\mathbb{R}^{N-m} \subset \mathbb{R}^N$.

Namely, let $\mathcal{R}_{\varphi,\beta,\Omega}(g)$ be a vectorial Ribaucour transform of an isometric immersion $g\colon L^{n-m} \to \mathbb{R}^{N-m} \subset \mathbb{R}^N$ determined by (φ, β, Ω) as in Definition 9. For an orthogonal decomposition $\mathbb{R}^{m+1} = \mathbb{R} \oplus \mathbb{R}^m$, with $\mathbb{R} = \operatorname{span}\{e_0\}$, set

$$(\varphi_0, \beta_0) = (\langle\varphi, e_0\rangle, \beta(e_0)), \quad (\varphi_1, \beta_1) = (\pi_{\mathbb{R}^m} \circ \varphi, \beta|_{\mathbb{R}^m}),$$

and

$$(\bar{\varphi}_0, \hat{\beta}_0) = (\varphi_0 - \Omega_{01}\Omega_{11}^{-1}\varphi_1, \beta_0 - \beta_1(\Omega_{11}^{-1})^t\Omega_{01}^t e_0)$$

where

$$\Omega_{11} = \pi_{\mathbb{R}^m} \circ \Omega|_{\mathbb{R}^m} \quad \text{and} \quad \Omega_{01} = \pi_{\mathbb{R}} \circ \Omega|_{\mathbb{R}^m}.$$

Assume that the bilinear maps $\gamma\colon\ T_xL \times T_xL \to T_xL^\perp$ given by

$$\gamma(Z,X) = (\alpha_g(Z, D_1X) + \beta_1(\Omega_{11}^{-1})^t\Phi_1(Z)D_1(X) - \bar{\varphi}_0\langle Z, X\rangle\hat{\beta}_0)_{\mathbb{R}^{N-m}}$$

have everywhere trivial kernel. Let the subspace $\mathbb{R}^m = e_0^\perp$ be identified with the orthogonal complement of $\mathbb{R}^{N-m}$ in $\mathbb{R}^N$ and choose an orthonormal basis $\{e_0, \ldots, e_m\}$ of $V := \mathbb{R}^{m+1}$. Finally, for $t = \sum_{i=1}^m t_ie_i \in \mathbb{R}^m$ define

$$\beta_t = e_0^* \otimes (\beta_0 + t) + \sum_{i=1}^m e_i^* \otimes \beta(e_i) \quad \text{and} \quad \Omega_t = \Omega + (\langle\beta_0, t\rangle + (1/2)|t|^2)e_0^* \otimes e_0.$$

Theorem 22. *The triple $(\varphi, \beta_t, \Omega_t)$ satisfies the conditions of Definition 9 with respect to g for each $t \in \mathbb{R}^m$ and the map $G\colon\ L^{n-m} \times \mathbb{R}^m \to \mathbb{R}^N$ given by*

$$G(x,t) = \mathcal{R}_{\varphi,\beta_t,\Omega_t}(g)(x)$$

parameterizes, at regular points, an n-dimensional submanifold carrying a Dupin principal normal of multiplicity m with integrable conullity.

Conversely, any isometric immersion carrying a Dupin principal normal of multiplicity m with integrable conullity can be locally constructed in this way.

Proof: The first assertion is easily checked. By Theorem 14 we have

$$\mathcal{R}_{\varphi,\beta_t,\Omega_t}(g) = \mathcal{R}_{\bar{\varphi}_0,\bar{\beta}_t^0}(\mathcal{R}_{\varphi_1,\beta_1,\Omega_{11}}(g)),$$

where

$$\bar{\beta}_t^0 = \mathcal{P}_1(\beta_0 + t - \beta_1(\Omega_{11}^{-1})^t\Omega_{01}^te_0) = \bar{\beta}^0 + \sum_{i=1}^m t_i\eta_i,$$

with $\bar{\beta}^0 = \mathcal{P}_1\hat{\beta}_0$ and $\eta_i = \mathcal{P}_1e_i$. Then $\mathcal{N} = \mathcal{P}_1\mathbb{R}^m$ is a parallel flat normal subbundle $\mathcal{N}$ of rank m of $h = \mathcal{R}_{\varphi_1,\beta_1,\Omega_{11}}(g)$ and

$$G(x,t) = \mathcal{R}_{\bar{\varphi}_0,\bar{\beta}_0+t}(h)(x) = \mathcal{R}^{\mathcal{N}}_{\bar{\varphi}_0,\bar{\beta}_0}(h)(t),$$

where $t = \sum_{i=1}^m t_i\eta_i$. Moreover, the assumption on the bilinear map γ is easily seen to be equivalent to the subspaces

$$E(x) = \{Z \in T_xL : (\alpha_h(Z,X))_{\mathcal{N}^\perp} = -\bar{\varphi}_0^{-1}\bar{\beta}_{\mathcal{N}^\perp}\langle Z, X\rangle \text{ for all } X \in T_xL\}$$

being everywhere trivial. By the result of [6] discussed before the statement of Theorem 22, it follows that G parameterizes, at regular points, an n-dimensional submanifold carrying a Dupin principal normal of multiplicity m with integrable conullity.

Conversely, given a submanifold $f\colon\, M^m \to \mathbb{R}^N$ that carries a Dupin principal normal of multiplicity m with integrable conullity, by the aforementioned result of [6] there exist an isometric immersion $h\colon\, L^{n-m} \to \mathbb{R}^N$ carrying a parallel flat normal subbundle $\mathcal{N}$ of rank m, $\varphi \in C^\infty(L^{n-m})$ and $\beta \in \Gamma(T_h^\perp L)$ satisfying $\alpha(X, \nabla\varphi) + \nabla_X^\perp \beta = 0$, with

$$E(x) = \{Z \in T_x L : (\alpha(Z,X))_{\mathcal{N}^\perp} = -\varphi^{-1}\beta_{\mathcal{N}^\perp}\langle Z, X\rangle \text{ for all } X \in T_x L\}$$

everywhere trivial, such that f is parameterized by the map $\mathcal{R}^{\mathcal{N}}_{\varphi,\beta}(h)\colon\, L^{n-m} \times \mathbb{R}^m \to \mathbb{R}^N$ given by

$$\mathcal{R}^{\mathcal{N}}_{\varphi,\beta}(h)(x,t) = \mathcal{R}_{\varphi,\beta+t'}(h)(x),$$

where $t' = \sum_{i=1}^m t_i \eta_i$ for some orthonormal parallel frame $\eta_1, \ldots, \eta_m$ of $\mathcal{N}$.

As in the proof of Theorem 18 there is an isometric immersion $g\colon\, L^{n-m} \to \mathbb{R}^{N-m} \subset \mathbb{R}^N$ such that $h = \mathcal{R}_{\varphi_1,\beta_1,\Omega_{11}}(g)$, and hence

$$\mathcal{R}^{\mathcal{N}}_{\varphi,\beta}(h)(x,t) = \mathcal{R}_{\varphi,\beta+t'}(\mathcal{R}_{\varphi_1,\beta_1,\Omega_{11}}(g))(x).$$

In order to apply Corollary 17, we must verify that the tensor $\Phi = \operatorname{Hess}\varphi - A_\beta$ associated to (φ, β) commutes with $\tilde{\Phi}_{v_1}$ for every $v_1 \in V_1$, where $D_1\bar{\Phi}^1_{v_1} = -\Phi^1_{\Omega_{11}^{-1}v_1}$ for $\Phi^1 = \Phi(d\varphi_1, \beta_1)$. Since Φ commutes with the shape operator $\tilde{A}_{\tilde{\xi}}$ of h with respect to any normal vector field $\tilde{\xi} \in \Gamma(T_h^\perp L)$, it commutes in particular with $\tilde{A}_{\mathcal{P}_1 e_i}$. But by (21) we have

$$\tilde{A}_{\mathcal{P}_1 e_i} = D_1^{-1}\Phi^1_{\Omega_{11}^{-1}\beta_1^t e_i} = D_1^{-1}\Phi^1_{\Omega_{11}^{-1}e_i} = -\bar{\Phi}^1_{e_i},$$

and we are done. It follows from Corollary 17 that there exist (φ, β, Ω) satisfying the conditions of Definition 9 with respect to g and an orthogonal decomposition $\mathbb{R}^{m+1} = \mathbb{R} \oplus \mathbb{R}^m$ such that

$$(\varphi_1, \beta_1, \Omega_{11}) = (\pi_{\mathbb{R}^m} \circ \varphi, \beta|_{\mathbb{R}^m}, \pi_{\mathbb{R}^m} \circ \Omega|_{\mathbb{R}^m})$$

and, setting $\varphi_0 = \pi_{\mathbb{R}} \circ \varphi$, $\beta_1 = \beta|_{\mathbb{R}}$, then $(\varphi, \beta) = \mathcal{R}_{\varphi,\beta,\Omega}(\varphi_0, \beta_0)$ and

$$\mathcal{R}_{\varphi,\beta}(\mathcal{R}_{\varphi_1,\beta_1,\Omega_{11}}(g)) = \mathcal{R}_{\varphi,\beta,\Omega}(g).$$

Defining β_t and Ω_t as in the statement, we have $\mathcal{R}_{\varphi,\beta+t'}(\mathcal{R}_{\varphi_1,\beta_1,\Omega_{11}}(g)) = \mathcal{R}_{\varphi,\beta_t,\Omega_t}(g)$. ∎

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