Parametrizations in Submanifold Theory

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Preface

It is always important to obtain parametric descriptions of certain class of geometric object. First, of course, to have a classification of a complicate object by means of simpler ones, that is, to understand the class, to construct examples, etc. Secondly and perhaps more important, because of applications.

The usual way to deal with a class of submanifolds is to make use of the *fundamental equations* ((1.5), (1.6) and (1.7)) which completely determine its structure. However, these equations are very complicated and usually it requires a lot of work with them to prove simple statements. Here is where parametrizations can help, since in some sense they carry implicitly all the information of the fundamental equations in one package.

Many natural geometric restrictions on a submanifold in a space form (in reasonable codimension) imply the existence of nontrivial nullity distributions, like deformability or conformally flatness. In this setting, the hypersurface situation for relative nullity was completely understood with the Gauss parametrization ([**Sb**], [**DG**]; cf. Chapter 2), which turned out to be extremely useful as a tool. The same holds for conformal nullity and codimension less than or equal to two ([**AD**], [**DF1**]). If a certain geometric restriction implies the existence of a nullity distribution on a class of submanifolds, it thus seems to be a good idea to try first to describe parametrically the class (or even a broader one) with a Gauss-type of parametrization, and then to study the object with the strong help of this tool. The same can be said about submanifolds foliated by extrinsic subspaces or spheres.

Although there is still no satisfactory Gauss-type of parametrization for higher codimensions, we discuss in this notes some results developing and using different representations of submanifolds to solve some geometric problems, often in an unexpected way. Our purpose here is to try to convince the reader throughout several examples contained in recent research papers about the strength of this simple and useful technique.

The prerequisites for the reader are some basic knowledge about the theory of differentiable manifolds, like vector bundles, tensors, metric and curvature, connections and covariant derivatives, etc.

These notes are organized as follows. In the first chapter we included some definitions and basic facts about submanifold theory that will be used throughout this notes.

The second chapter is dedicated to develop the Gauss parametrization for hypersurfaces with relative nullity together with several applications, mostly to rank two hypersurfaces and isometric rigidity.

In Chapter 3 we construct a similar representation for hypersurfaces with conformal nullity. We call it the umbilic Gauss parametrization. Then, we use it to give a description of all conformally flat hypersurfaces, and we talk a little bit about conformal rigidity.

Generalizations of this two parametrizations for higher codimensions are done in Chapter 4. We use them to solve the problem of whether a submanifold with conformal rank two is rotational.

Chapter 5 is devoted to submanifolds with some kind of integrability of the orthogonal complement of the nullity distribution. We use it to make a quick proof of the conformal classification of all submanifolds for which the conullity is umbilic in the submanifold.

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Contents

Chapter 1

Preliminaries and notation

Although most of the topics in this notes can be done with more generality (for example in space forms of arbitrary constant sectional curvature), we have chosen to work with submanifolds in euclidean space, not only for simplicity, but also because all the ideas, problems and interesting phenomena already occur in this setting. We need first to fix some notations and to recall basic facts about the theory of submanifolds.

Let M^n be a differentiable (C^∞) manifold (superscripts will always denote dimensions). Let $\pi : TM = \cup_{x \in M} T_x M \to M^n$ be its tangent bundle whose (smooth) *sections* are $\Gamma(TM) = \{X : M^n \to TM : \pi \circ X = id\}$, the *vector fields* of M^n. A differentiable map between differentiable manifolds $f : M^n \to \tilde{M}^m$ is an *immersion* if the differential of f at x,

$$df(x) = f_{*x} : T_x M \to T_{f(x)} \tilde{M}$$

is nonsingular, for all $x \in M^n$. The number $p = m - n \geq 0$ is called the *codimension* of f. We say that f is a *hypersurface* when $p = 1$.

It is important to recall that every immersion f as above is locally an *embedding*, that is, for every $x \in M^n$ there exists a neighborhood $U \subset M^n$ of x such that $f|_U : U \to \tilde{M}^{n+p}$ is injective and the open subsets $V \subset U$ coincide with the preimage under $f|_U$ of open subsets of $f(U)$, that is, $V = (f|_U)^{-1}(f(U) \cap W)$, for some open subset $W \subset \tilde{M}^{n+p}$. This implies that $f(U)$ is an embedded submanifold of $\tilde{M}^{n+p}$, diffeomorphic to U under $f|_U$. According to this, we always identify U with $f(U)$ and the tangent space TU with $f_*TU = Tf(U)$ not to overload with notations. Therefore, the identifications $x = f(x)$, $X = f_*X$, for any $x \in M^n$ and $X \in TM$ shall not make confusion.

A *riemannian manifold* is a differentiable manifold M^n together with a symmetric tensor

$$\langle\ ,\ \rangle = \langle\ ,\ \rangle_M : TM \times TM \to \mathbb{R}$$

which is positive definite at every point, called the *(riemannian) metric* of M^n. It is a well known and remarkable fact that every riemannian manifold $(M^n, \langle\ ,\ \rangle)$ possesses a unique *Levi-Civita connection*: a $\mathbb{R}$-bilinear map

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM),$$

$\nabla(X, Y) = \nabla_X Y$, which satisfies for all $X, Y, Z \in \Gamma(TM)$, $r \in C^\infty(M)$:

- ∇ is *tensorial* in the first variable: $\nabla_{(rX)}Y = r\nabla_X Y$,
- ∇ is a *derivation* in the second variable: $\nabla_X(rY) = r\nabla_X Y + X(r)Y$,
- ∇ is *torsion-free*: $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] = 0$,
- ∇ is *compatible with the metric* of M^n:
  $(\nabla\langle\ ,\ \rangle)(X, Y, Z) := Z\langle X, Y\rangle - \langle\nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle = 0.$

We denote by $R = R_\nabla = [\nabla, \nabla] - \nabla_{[\,,\,]}$ the *curvature tensor* of ∇:

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \quad X, Y, Z \in TM.$$

An immersion $f : M^n \to \tilde{M}^{n+p}$ between riemannian manifolds is called an *isometric immersion* if $\langle\ ,\ \rangle_M = f^*\langle\ ,\ \rangle_{\tilde{M}}$, that is, if

$$\langle X, Y\rangle_M = \langle f_*X, f_*Y\rangle_{\tilde{M}}, \quad \forall\ X,\ Y \in TM. \tag{1.1}$$

Observe that if $f : M^n \to \tilde{M}^{n+p}$ is an immersion and $\tilde{M}^{n+p}$ is a riemannian manifold, we can always define a metric on M^n using (1.1), the so-called *induced metric by* f, which makes f an isometric immersion. An immersion f as above is called *conformal* if there is a positive function $r \in C^\infty(M)$ such that $\langle\ ,\ \rangle_M = rf^*\langle\ \rangle_{\tilde{M}}$.

For an isometric immersion $f : M^n \to \tilde{M}^{n+p}$, at each point $x \in M^n$ we have the ortogonal decomposition

$$T_x\tilde{M} = T_xM \oplus^\perp T_x^\perp M. \tag{1.2}$$

The subbundle $T_f^\perp M = T^\perp M = \cup_{x\in M} T_x^\perp M$ of $T\tilde M$ along M^n is called the *normal bundle* of f. Sections and vector of the tangent bundle will be denoted by upper case latin letters, $X, Y, Z... \in TM$ and lower case greek letters will stand for sections and vectors of the normal bundle: $\xi, \eta... \in T^\perp M$. We denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections of M^n and $\tilde M^{n+p}$, whose curvature tensors will be denoted by R and $\overline{R}$, respectively.

By uniqueness of the Levi-Civita connection, it is easy to check that ∇ is the tangent part of $\overline{\nabla}$ along M^n, and that the normal component of $\overline{\nabla}_X\xi$ is a well defined compatible connection over the normal bundle, the so-called *normal connection* of f:

$$\nabla_X^\perp \xi = (\overline{\nabla}_X \xi)^\perp, \quad X \in TM,\ \xi \in T^\perp M, \tag{1.3}$$

whose normal curvature tensor is denoted by $R^\perp = [\nabla^\perp, \nabla^\perp] - \nabla^\perp_{[,]}$. We say that f has *flat normal bundle* when $R^\perp \equiv 0$.

The main algebraic object in submanifold theory is the (vector valued) *second fundamental form* of the submanifold, $\alpha = \alpha_f : TM \times TM \to T^\perp M$ given by

$$\alpha(X, Y) = (\overline{\nabla}_X Y)^\perp, \quad X, Y \in TM.$$

Since Levi-Civita connections are torsion-free, we get that the second fundamental form is a symmetric tensor and then the *shape operator* $A_\xi = A_\xi^f$ of f in the direction $\xi \in T^\perp M$ given by $\langle A_\xi X, Y\rangle = \langle \alpha(X, Y), \xi\rangle$ is a self-adjoint tensor in M^n. Thus, according with (1.2) and (1.3) we have

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + \alpha(X, Y), \\ \overline{\nabla}_X \xi &= -A_\xi X + \nabla_X^\perp \xi, \end{aligned} \tag{1.4}$$

for all $X, Y \in TM$, $\xi \in T^\perp M$.

Suppose from now on that $f : M^n \to \mathbb{Q}_c^{n+p}$ is an isometric immersion into a complete simply connected space $\mathbb{Q}_c$ of constant sectional curvature $c \in \mathbb{R}$, that is,

$$\overline{R}(X, Y)Z = c(X \wedge Y)Z = c(\langle Y, Z\rangle X - \langle X, Z\rangle Y).$$

We will denote by $\mathbb{S}^n = \mathbb{Q}_1^n$ and $\mathbb{H}^n = \mathbb{Q}_{-1}^n$ the euclidean unit sphere and the hyperbolic space, respectively. Taking tangent and normal components of $\overline{R}(X, Y)Z$ and $\overline{R}(X, Y)\xi$ using (1.4), we have the *fundamental equations of the submanifold* f:

the Gauss equation

$$\begin{aligned} \langle R(X, Y)Z, W\rangle = c&\langle (X \wedge Y)Z, W\rangle \\ &+\langle \alpha(X, W), \alpha(Y, Z)\rangle - \langle \alpha(X, Z), \alpha(Y, W)\rangle, \end{aligned} \tag{1.5}$$

the Ricci equation

$$\langle R^\perp(X,Y)\xi,\eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle \tag{1.6}$$

and the Codazzi equation

$$(\nabla^\perp_X\alpha)(Y,Z) = (\nabla^\perp_Y\alpha)(X,Z), \tag{1.7}$$

where, by the usual definition of the covariant derivative of a tensor field, $(\nabla^\perp_X\alpha)(Y,Z) = (\nabla^\perp\alpha)(Y,Z,X) = \nabla^\perp_X\alpha(Y,Z) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z)$.

The *relative nullity* $\Delta(x) \subset T_xM$ of f at $x \in M^n$ is the subspace

$$\Delta(x) = \ker\alpha(x) = \{Y \in T_xM : \alpha(Y,Z) = 0 \ \ \forall\ Z \in T_xM\},$$

whose dimension $\nu(x)$ is called the *index of relative nullity of f at x*. The *rank of f at x* is the integer $n-\nu(x)$. It is easy to check that the function ν is lower semi-continuous. Thus, along connected components of an open dense subset of M^n, ν is constant and then the relative nullity is a smooth distribution. It is also a well known fact easily cheched with the Codazzi equation (1.7) that if f is an euclidean submanifold, Δ is an integrable distribution whose leaves are open subsets of affine ν-dimensional subspaces of $\mathbb{R}^{n+p}$. Moreover, along the (open!) subset of minimal index of relative nullity the leaves are complete whenever M^n is complete (see Theorem 5.3 of [**Da**]).

Similar phenomena occurs with *umbilic* distributions. For any integer $2 \le k \le n$, the submanifold f is said to be *k-umbilic* if there exists a smooth normal section $0 \neq \gamma \in \Gamma(T^\perp M)$ such that the *umbilic distribution* (associated to γ)

$$\Delta_\gamma(x) = \ker(\alpha - \langle\ ,\ \rangle\gamma) = \{Y \in T_xM : \alpha(Y,Z) = \langle Y,Z\rangle\gamma, \ \ \forall\ Z \in T_xM\}$$

has constant dimension k. Observe that $\Delta = \Delta_\gamma$ for $\gamma = 0$. In this situation, $\eta = \frac{\gamma}{\|\gamma\|}$ will be called an *umbilic direction* of f and the function $\lambda = \|\gamma\| > 0$ the *umbilic eigenvalue* (associated to η) of the submanifold. The (conformal) *conullity* of f is the distribution $\Delta_\gamma^\perp$. We denote by

$$\nu_c(x) = \max_{\gamma \in T_x^\perp M} \dim \Delta_\gamma(x)$$

the *index of conformal nullity* of f at x. The integer $n-\nu_c(x)$ is the *conformal rank* of f at x. The function ν_c is also semicontinuous and thus it is constant along connected components of an open dense subset of M^n.

The submanifold f is said to be *totally umbilic* if every point is an umbilic point, $\nu_c \equiv n$, and *totally geodesic* if every point is totally geodesic, $\nu \equiv n$, that is, $\alpha \equiv 0$. It is a classic result that totally umbilic (geodesic) euclidean submanifolds are open subsets of round spheres (affine subspaces). We resume in the following proposition the main facts about the distribution Δ_γ which will be crucial all along this notes (cf. Proposition 4 of [**DFT2**]). Similar result holds for submanifolds in spaces with constant sectional curvature.

Proposition 1. *Let $f : M^n \to \mathbb{R}^{n+p}$ be an isometric immersion, and a smooth section $\gamma \in \Gamma(T^\perp M)$ such that Δ_γ has constant dimension $k \geq 1$. Assume further that $k \geq 2$ if $\gamma \neq 0$. Then, the following hold:*

1) Δ_γ is a smooth integrable distribution.

2) The leaves of Δ_γ are (open subsets of):
- *k-dimensional affine subspaces of $\mathbb{R}^{n+p}$ if $\gamma = 0$, or,*
- *round spheres in some $(k+1)$-dimensional affine subspaces if $\gamma \neq 0$.*

3) γ is normal parallel along Δ_γ, that is, $\nabla^\perp_X \gamma = 0$ for all $X \in \Delta_\gamma$.

Chapter 2

The Gauss parametrization

This chapter is devoted to the study of the Gauss parametrization for hypersurfaces. Developped by Sbrana just for the study of local isometric deformations of euclidean hypersurfaces ([**Sb**]), it was rediscovered by Dajczer and Gromoll in [**DG**]. They already observed in that paper that the tool has a lot of applications, in particular in rigidity problems. The idea of this representation is to describe a hypersurface with relative nullity by means of its Gauss map and its 'support' function.

2.1 The parametrization

Let $f : M^n \to \mathbb{R}^{n+1}$ be an orientable euclidean hypersurface with constant index of relative nullity k. For a "saturated" open connected subset $U^n \subset M^n$ (meaning each leaf of relative nullity in U^n is maximal in M^n) we consider the quotient space U^n/Δ of leaves in U^n with projection $\pi : U^n \to U^n/\Delta$. It is an $(n-k)$-dimensional manifold (it could fail to be Hausdorff) at least in two situations:

- Locally. This is clear since we can take U^n as a saturation of some local cross section to the leaves.

- If all leaves through points in U^n are complete, in which case the projection map π is a "linear" vector bundle.

Take U^n as above orientable, and let $\xi : U^n \to \mathbb{S}^n$ be a unit normal vector field of U^n, the so-called *Gauss map* of $f|_U$. We call here $A = A_\xi^f$ its second fundamental form. The *support function* $\lambda \in C^\infty(U)$ of $f|_U$ is defined as

$$\lambda(x) = \langle x, \xi(x) \rangle.$$

Now, the key point in the parametrization: since $AY = -\overline{\nabla}_Y \xi = -\xi_* Y$, we get that ξ is constant along each leaf of U^n. Thus, there is an immersion

$$\varphi : V^{n-k} = U^n/\Delta \to \mathbb{S}^n$$

given by $\varphi \circ \pi = \xi$, that is, $\xi(U^n) = \varphi(V^{n-k}) \subset \mathbb{S}^n$ is a $(n-k)$-dimensional submanifold. For now on, we consider in V^{n-k} the metric induced by φ which makes it an isometric immersion. Moreover, for each $Y \in \Delta$ we have that $Y(\lambda) = \langle Y, \xi \rangle - \langle x, AY \rangle = 0$ and then there is a function $\mu \in C^\infty(V)$ such that $\mu \circ \pi = \lambda$.

On the other hand, for each $x \in U^n$ and $y = \pi(x) \in V^{n-k}$ we have the orthogonal decomposition

$$T_x\mathbb{R}^{n+1} = \mathbb{R}^{n+1} = \text{span}\,\{\varphi(y)\} \oplus T_{\varphi(y)}V \oplus T^\perp_{\varphi(y)}V.$$

According to this decomposition, we obtain that

$$\begin{aligned} x &= \lambda(x)\xi(x) + Z(x) + \eta(x) \\ &= \mu(y)\varphi(y) + Z(x) + \eta(x), \end{aligned}$$

for some $Z(x) \in T_{\varphi(y)}V$ and $\eta(x) \in T^\perp_{\varphi(y)}V$. But taking derivatives in the above with respect to $Y \in T_xU$ we have that

$$\begin{aligned} 0 &= \langle Y, \xi(x) \rangle \\ &= Y(\lambda) + \langle \overline{\nabla}_Y Z, \xi(x) \rangle \\ &= \pi_{*x}(Y)(\mu) - \langle Z(x), \varphi_{*y}(\pi_{*x}(Y)) \rangle. \end{aligned}$$

Hence,

$$Z = (\nabla\mu) \circ \pi, \tag{2.1}$$

where $\nabla\mu$ stands for the *gradient* vector field of μ with respect to the metric in V^{n-k}: $\langle \nabla\mu, X \rangle = X(\mu)$. In particular, Z is also parallel along the leaves of U^n and we get

$$x = (\mu\varphi + \nabla\mu)(y) + \eta(x).$$

Now, by Proposition 1 we know that when x stays in a leaf of U^n it shoud fill an open subset of an k-dimensional affine subspace. But since $\eta(x) \in T^\perp_{\varphi(y)}V$, which is itself also an k-dimensional affine subspace, we conclude that $\eta(x)$ is a local parametrization of the leaves of the normal bundle of φ. We thus have the following theorem.

Theorem 2. ([**Sb**], [**DG**]) *Let $\varphi : V^{n-k} \to \mathbb{S}^n$ be an isometric immersion and $\mu \in C^\infty(V)$. Then, on the open set of regular points, the map $\Psi : T^\perp V \to \mathbb{R}^{n+1}$ on the normal bundle $\pi : T^\perp V \to V^{n-k}$ of φ given by*

$$\Psi(\omega) = (\mu\varphi + \nabla\mu)(y) + \omega, \quad \omega \in T_y^\perp V, \tag{2.2}$$

is an immersed euclidean hypersurface with constant index of relative nullity k, Gauss map $\varphi \circ \pi$ and support function $\mu \circ \pi$. Conversely, any such hypersurface can be parametrized locally this way. Moreover, the parametrization is global provided M^n is complete and orientable.

Proof: We have already proved the converse. For the direct statement, we just invert the above process. We only have to show that Ψ maps the leaves of $T^\perp V$ into relative nullity. But this is clear since $\varphi(y)$ is the Gauss map of Ψ at any point $\omega \in T_y^\perp V$. To see this, first observe that $\Psi_{*\omega}$ is the identity on tangent space to the leaf of $T_y^\perp V$, which as usual we identify with the leaf itself. Any tangent vector of $T^\perp V$ transversal to the leaves can be written as $\gamma_* Y$, where γ is a local section of $T^\perp V$ and $Y \in TV$. We thus have that

$$\Psi_*(\gamma_* Y) = P_\gamma Y + \left(\nabla'^\perp_Y \gamma + \beta(Y, \nabla\mu)\right), \tag{2.3}$$

where β is the second fundamental form of φ, $\nabla'^\perp$ its normal connection (both in $\mathbb{S}^n$) and $P_\gamma : TV \to TV$ the tensor given by

$$P_\gamma = \mu I + \mathrm{Hess}_\mu - B_\gamma. \tag{2.4}$$

Here Hess_μ stands for the *hessian* of the function μ and B_γ for the shape operator of φ in the direction of γ. This proves that $\mathrm{Im}\ \Psi_* \perp \varphi \circ \pi$. ∎

Remarks 3. 1) According to (2.3), the regular points of Ψ are precisely the vectors $\omega \in T_y^\perp V$ such that the self adjoint operator P_ω is nonsingular, since $\mathrm{Im}\ \Psi_{*\omega} = \mathrm{Im}\ P_\omega \oplus^\perp T_y^\perp V$. Moreover, since $\varphi \circ \pi$ is the Gauss map of Ψ, we get for the second fundamental form A of Ψ along $\gamma \in \Gamma(T^\perp V)$ that

$$-Y = -\varphi_* Y = -(\varphi \circ \pi)_*(\gamma_* Y) = A(\gamma)(\Psi_* \gamma_* Y) = A(\gamma) P_\gamma Y, \quad \forall\ Y \in TV,$$

because the leaves of $T^\perp V$ are the relative nullity distribution of Ψ. Thus,

$$A(\omega) = -P_\omega^{-1} \quad \text{on} \quad \Delta^\perp(\omega). \tag{2.5}$$

2) We should point out that the Gauss parametrization naturally extends to hypersurfaces of the euclidean sphere and hyperbolic space.

2.2 Some applications

The first thing we realize about the Gauss parametrization (2.2) is that *any* immersed spherical submanifold $\varphi : V^{n-k} \to \mathbb{S}^n$ is locally the Gauss map of an n-dimensional euclidean hypersurface with constant index of nullity k. In fact, for every $y \in V^{n-k}$ and every $\omega \in T_y^\perp V$ it is not difficult to see that there is $\mu \in C^\infty(V)$ such that P_ω in (2.4) is nonsingular. We also observe that the set of euclidean hypersurfaces with the same Gauss map φ is in correspondence with support functions $\mu \in C^\infty(V)$.

2.2.1 Nonpositively curved euclidean hypersurfaces

Let $f : M^n \to \mathbb{R}^{n+1}$ be a hypersurface with principal curvatures (the eigenvalue functions of the second fundamental operator of f) $\lambda_1, \dots, \lambda_n$ and principal directions (their corresponding eigenvectors) $e_1, \dots, e_n$. The Gauss equation (1.5) implies that, for $1 \le i \ne j \le n$, the sectional curvature K_M of M^n of the plane $\sigma_{ij} = \text{span}\,\{e_i, e_j\}$ is

$$K_M(\sigma_{ij}) := \langle R(e_i, e_j)e_j, e_i\rangle = \lambda_i\lambda_j. \tag{2.6}$$

We want now to use the Gauss parametrization to characterize two important classes of euclidean submanifolds:

- M^n is *flat*, that is, $K_M \equiv 0$.
- M^n is *nonpositively curved*, that is, $K_M \le 0$.

The flat case. For a connected component U of the interior of the set of totally geodesic points of f, we know that $f(U)$ is just an open subset of an affine hyperplane of $\mathbb{R}^{n+1}$. Thus, assume that f has no totally geodesic points, i.e., $\lambda_1 \ne 0$. But from (2.6) we get that $\lambda_2 = \cdots = \lambda_n = 0$. Therefore, f has constant index of relative nullity $n-1$ and by the Gauss equation any submanifold with index of relative nullity $n-1$ should be flat. Hence, there is a smooth arc length parametrized spherical curve $c : I \subset \mathbb{R} \to \mathbb{S}^n$ and a function $r \in C^\infty(I)$ such that f can be parametrized over the normal bundle of c as

$$\Psi(t, w) = (rc + r'c')(t) + w. \tag{2.7}$$

Its regular points are the vectors $w \in T^\perp_{c(t)} I$ such that $\langle w, c''(t)\rangle \ne r(t) + r''(t)$. In particular, if M^n is complete and orientable, then $(c'')^\perp = c'' + c = 0$, that is, c parametrizes a maximal circle $\mathbb{S}^1 = \mathbb{R}^2 \cap \mathbb{S}^n$ and f is a cylinder over the plane curve $\tilde{c} = (rc + r'c') : I \to \mathbb{R}^2$: $f(M^n) = \tilde{c}(I) \times \mathbb{R}^{n-1}$.

The nonpositively curved case. Assume that M^n is nowhere flat. In view of (2.6) we can suppose that $\lambda_1 < 0 < \lambda_2$. But from (2.6) and $K_M \leq 0$ we see that $\lambda_3 = \cdots = \lambda_n = 0$, that is, f has constant index of relative nullity $n-2$. Thus, we parametrize f with a spherical surface $\varphi : V^2 \to \mathbb{S}^n$ and a smooth function μ on V^2 by (2.2). According to (2.5) and the Gauss equation, Ψ has nonpositive sectional curvature and is regular precisely at the normal vectors $\omega \in T^\perp V$ which satisfies $\det P_\omega < 0$.

In [**F1**], we showed that any isometric immersion $f : M^n \to \mathbb{R}^{n+p}$ of a nonpositively curved riemannian manifold must have index of relative nullity $\nu \geq n - 2p$ everywhere, which is a good information since by Proposition 1 to have relative nullity imposes strong restrictions on the manifold and on its isometric immersion. In this sense, the 'worst' case might be $\nu = n - 2p$. A simple example shows that this estimate is sharp:

Example. For each $1 \leq i \leq p$, let $g_i : H_i^{n_i} \to \mathbb{R}^{n_i+1}$ be a nowhere flat euclidean hypersurface of nonpositive sectional curvature which, as we saw, has constant index of relative nullity $n_i - 2$. Therefore, the product manifold $M^n = H_1^{n_1} \times \cdots \times H_p^{n_p}$ has nonpositive sectional curvature and its product immersion $g = g_1 \times \cdots \times g_p$ into $\mathbb{R}^{n+p}$ verifies $\nu \equiv n - 2p$. In fact, it is easily checked that its relative nullity is the sum of the relative nullity of the hypersurfaces because the second fundamental form of a product immersion is the direct sum of the second fundamental forms of the factors. In particular, the product immersion can be parametrized using the Gauss parametrization (2.2) for each factor.

Surprisingly, it is shown in [**FZ1**] that this is the *only* possible case: any nowhere flat euclidean submanifold with nonpositive curvature and minimal index of relative nullity should be (locally) a product of nowhere flat nonpositively curved hypersurfaces, and thus it can always be parametrized using the Gauss parametrization for each factor. The case of index of relative nullity $\nu = n - 2p + 1$ is much more complicated and was classified in [**FZ2**]. See [**F2**] for the flat normal bundle case.

2.2.2 Hypersurfaces of rank two

Let us recall the classical Beez-Killing theorem ([**B**], [**K**]). Any connected hypersurface $f : M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, with rank greater than or equal to three is locally *isometrically rigid*: for every connected open subset $U^n \subset M^n$, $f|_U : U^n \to \mathbb{R}^{n+1}$ is the unique isometric immersion (up to rigid motions of the ambient space $\mathbb{R}^{n+1}$) of U^n into $\mathbb{R}^{n+1}$. Since a hypersurface to be flat (highly deformable and classified in (2.7)) is equivalent to have $\nu \geq n-1$, the study of the local isometric deformations of hypersurfaces is restricted to the rank two case. As we saw, examples of euclidean hypersurfaces of rank two are the nowhere flat nonpositively curved ones. Sbrana ([**Sb**]) developed the Gauss parametrization precisely to classify the locally isometrically deformable hypersurfaces in euclidean space. For a modern version of his work and other applications of the Gauss parametrization to the theory of deformable hypersurfaces, see [**DFT1**].

The following result is an application of the global Gauss parametrization. Recall first that the *mean curvature vector* vector of an isometric immersion $f : M^n \to \mathbb{R}^{n+p}$ with second fundamental form α is $H = \frac{1}{n}\sum_{i=1}^n \alpha(e_i, e_i)$, where $\{e_1, \ldots, e_n\}$ is a local orthonormal tangent frame. It is independent of the frame we choose since

$$nH = \sum_{i=1}^{n} (\overline{\nabla}_{e_i} f_* e_i)^{\perp} = \sum_{i=1}^{n} (\overline{\nabla}_{e_i} df(e_i) - df(\nabla_{e_i} e_i)) = \Delta f,$$

where $\Delta = \mathrm{tr}\,\mathrm{Hess}$ is the *Laplacian* of M^n and $\Delta f = (\Delta f_1, \ldots, \Delta f_{n+p})$. Hence, every *minimal* immersion ($H = 0$) is real analytic as well as every euclidean hypersurface with constant *mean curvature* $h = (1/n)\mathrm{tr}\, A_\xi = \pm\|H\|$.

Theorem 4. ([**DG**]) *Suppose that the mean curvature h of a complete rank two hypersurface $f : M^n \to \mathbb{R}^{n+1}$ does not change sign. Then $f(M)$ splits isometrically as a product $f(M) = L^3 \times \mathbb{R}^{n-3}$, where $L^3 \subset \mathbb{R}^4$.*

Proof: We consider the global Gauss parametrization for f. First, we claim that the Gauss map φ of f is a spherical minimal surface. To see this, observe that for the second fundamental operator A of f we have using (2.5) that

$$nh(w) = \mathrm{tr}\, A(w) = -\mathrm{tr}\, P_w^{-1} = -\mathrm{tr}\, P_w \det P_w^{-1}. \tag{2.8}$$

Since $\det P_w \neq 0$ and $\mathrm{tr}\, P_w$ is <u>linear</u> in w, we conclude that h must change sign unless $\mathrm{tr}\, B_w = 0$ for every normal vector w to φ, that is, φ is minimal.

We have that the set $\mathcal{N}(y) = \{B_w : w \in T_y^{\perp} V\}$ is a linear subspace of the trace free symmetric endomorphisms of the two dimensional vector space

T_yV. In particular, $\dim \mathcal{N}(y) \le 2$. If for some $y \in V^2$ we have $\dim \mathcal{N}(y) = 2$, then the set

$$\{P_w(y) = \mu(y)I + \mathrm{Hess}_\mu(y) - B_w = B - B_w : B_w \in \mathcal{N}(y)\}$$

is the affine plane of 2×2 symmetric endomorphisms with trace $\operatorname{tr} B$. Then, there must be $B_w \in \mathcal{N}(y)$ such that $P_w(y)$ is singular, which contradicts the completeness of f. We conclude that $\dim \mathcal{N}(y) \le 1$ for all $y \in V^2$.

We claim that in this situation, the real analyticity of φ implies that $\varphi(V^2) \subset \mathbb{S}^3 \subset \mathbb{S}^n$ and then the proof follows from the Gauss parametrization. *Proof of the claim:* If φ is not totally geodesic, there is an open subset $U \subset V^2$ such that $\dim \mathcal{N}(y) = 1$ for all $y \in U$. Consider the normal subbundle $\Sigma = \{w \in T^\perp V : B_w = 0\}$ and a unit normal vector η which spans the line bundle orthogonal to Σ. The second fundamental form β of φ is then $\beta(X, Y) = \langle B_\eta X, Y\rangle \eta$. The Σ-component of the Codazzi equation (1.7) for φ is

$$\langle B_\eta Y, Z\rangle \nabla'^{\perp}_X \eta = \langle B_\eta X, Z\rangle \nabla'^{\perp}_Y \eta, \tag{2.9}$$

where $\nabla'^{\perp}$ is the normal connection $\nabla'^{\perp}$ of φ. Since $B_\eta \ne 0$ has trace 0, for all $Y \in TV$ there exist $X, Z \in TV$ such that $\langle B_\eta Y, Z\rangle = 0$ and $\langle B_\eta X, Z\rangle \ne 0$. We conclude from (2.9) that η is normal parallel, $\nabla'^{\perp}\eta = 0$, and so it is its orthogonal complement Σ: $\nabla'^{\perp}\Sigma \subset \Sigma$. But then, if w is a smooth section of Σ, as a map onto $\mathbb{R}^{n+1}$ it satisfies

$$\begin{aligned} dw(X) &= \langle dw(X), \varphi\rangle\varphi + (\overline{\nabla}_X w)^\top + \nabla^\perp_X w \\ &= -\langle w, \varphi_* X\rangle \varphi - \ B_w(X)\ + \nabla^\perp_X w \ = \ \nabla^\perp_X w \in \Sigma, \end{aligned}$$

for all $X \in TV$. We conclude that $\Sigma = \mathbb{R}^{n-3}$ is constant in $\mathbb{R}^{n+1}$ and that $\varphi(U) \subset \mathbb{S}^3 \subset \mathbb{R}^4$. The claim is proved. ∎

Remark 5. The argument that the Gauss map φ is a minimal surface at the beginning of the proof of the above theorem clearly shows the strength of the parametrization technique as a tool. We wrote in (2.8) a geometric quantity of the submanifold (the mean curvature h of f) expressed in terms of the parametrization. Then we use the hypothesis on it (that it does not change sign) to conclude that $\operatorname{tr} B_w$ could not depend on w, thus has to be zero. As another example of this idea, show using the same argument that if a rank two euclidean hypersurface f is minimal ($h = 0$) then its Gauss map φ is a minimal surface, even if M^n is not complete. This kind of simple idea is extremely useful, as we will see several times in this notes.

Corollary 6. *Let $f : M^n \to \mathbb{R}^{n+1}$ be a complete submanifold without flat points which mean curvature does not change sign. Then, $f(M) = L^3 \times \mathbb{R}^{n-3}$ splits isometrically or there exists an open subset $U \subset M^n$ such that $f|_U$ is rigid.*

Proof: It is a direct consequence of Theorem 4 and the Beez-Killing theorem. ∎

The proof of the next result also from [**DG**] is another example of what we observed in Remark 5. Look again at the simplicity of the argument.

Theorem 7. *Any isometric immersion $f : M^n \to \mathbb{R}^{n+1}$ of a connected riemannian manifold with constant mean curvature $h \neq 0$ is rigid, unless $f(M) \subset L^2 \times \mathbb{R}^{n-2}$ or $f(M) \subset \mathbb{S}^1 \times \mathbb{R}^{n-1}$ splits.*

Proof: Suppose there is a connected open subset U where $\nu = n-2$. Since $nh = -\mathrm{tr}\, P_w \det P_w^{-1}$ and $\det(A+B) = \det A + \det B + \mathrm{tr}\, A\, \mathrm{tr}\, B - \mathrm{tr}\,(AB)$ for any 2×2 matrices A, B, we have

$$nh\,(\det P_0 + \det B_w - \mathrm{tr}\, P_0\, \mathrm{tr}\, B_w + \mathrm{tr}\,(P_0 B_w)) = \mathrm{tr}\, B_w - \mathrm{tr}\, P_0. \tag{2.10}$$

Because the only term in the above quadratic in w is $\det B_w$, we obtain $\det B_w = 0$. If the Gauss map φ of f is not totally geodesic, there is w and an orthonormal basis of TV, $\{e_1, e_2\}$, such that $B_w e_2 = 0$ and $B_w e_1 = r e_1 \neq 0$. We get in (2.10) for $sw, s \in \mathbb{R}$, that

$$nh \det P_0 + \mathrm{tr}\, P_0 = sr(nh\langle P_0 e_2, e_2\rangle + 1).$$

Since the left hand side does not depend on s, both sides must vanish. Therefore, $0 = nh(nh \det P_0 + \mathrm{tr}\, P_0) = -(nh\langle P_0 e_1, e_2\rangle)^2 - 1 < 0$, which is a contradiction. Thus, φ is totally geodesic, and $f(U) \subset L^2 \times \mathbb{R}^{n-2}$. The real analyticity of f yields that $f(M) \subset L^2 \times \mathbb{R}^{n-2}$.

We left as an exercise the case that there is an open subset U with $\nu = n-1$, which is similar, easier and gives $f(M) \subset \mathbb{S}^1 \times \mathbb{R}^{n-1}$. If $\nu \leq n-3$ almost everywhere, from the Beez-Killing theorem we get that f is rigid almost everywhere. Hence, any other local immersion of M^n in $\mathbb{R}^{n+1}$ should have constant mean curvature h everywhere. We conclude from the real analyticity of f that it must be rigid. ∎

Question: What can you say about rank two euclidean hypersurfaces with constant scalar curvature $s \neq 0$? (Hint: use that, in this case, $s = \det P_w^{-1}$).

Chapter 3

Hypersurfaces with conformal nullity

We will describe here k-umbilic euclidean hypersurfaces, that is, hypersurfaces with constant index of conformal nullity k. In the process, we will see a way to obtain parametric descriptions of submanifolds caring umbilic distributions as we saw for the Gauss parametrization. Let us call this parametrization the *umbilic Gauss parametrization.*

3.1 The parametrization

Let $f : M^n \to \mathbb{R}^{n+1}$ be an connected euclidean hypersurface with Gauss map $\xi : M^n \to \mathbb{S}^n$ and second fundamental operator $A = A^f_\xi$. Assume that f is k-umbilic, $k \geq 2$, with umbilic eigenvalue $\lambda > 0$:

$$\Delta_\lambda = \ker(A - \lambda I) \quad \text{satisfies that} \quad \dim \Delta_\lambda = k \quad \text{on } M^n.$$

By Proposition 1, Δ_λ is a smooth integrable distribution whose leaves are open subsets of k-dimensional round spheres, $\mathbb{S}^k_s \subset \mathbb{R}^{k+1} \subset \mathbb{R}^{n+1}$, along which λ is constant. As in the relative nullity case, we consider the quotient space of leaves of Δ_λ in $U^n \subset M^n$ with projection π, $\pi : U^n \to V^{n-k} = U^n/\Delta_\lambda$, which is a manifold in the same conditions that of U^n/Δ. In particular, there is a function $r \in C^\infty(V)$ such that $\lambda^{-1} = r \circ \pi$. The same holds for any codimension, a fact that we are going to use in the next sections.

Define the map $h : U^n \to \mathbb{R}^{n+1}$ by

$$h = f + \lambda^{-1}\xi. \tag{3.1}$$

If $X \in \Delta_\lambda$ we obtain $dh(X) = X + X(\lambda^{-1})\xi - \lambda^{-1}AX = (I - \lambda^{-1}A)X = 0$. Therefore, h is constant along the leaves of Δ_λ and there is an immersion $g : V^{n-k} \to \mathbb{R}^{n+1}$ such that $h = g \circ \pi$. We consider on V^{n-k} the metric induced by g.

For each $x \in U^n$, $y = \pi(x)$, decompose $\xi(x) = Z(x) + \eta(x)$, where $Z(x) \in T_{g(y)}V$ and $\eta(x) \in T^{\perp}_{g(y)}V$. Since ξ is the gauss map of f, for any $Y \in TU$ we have

$$0 = \langle f_*Y, \xi\rangle = \langle h_*Y - Y(\lambda^{-1})\xi - \lambda^{-1}\xi_*Y, \xi\rangle = \langle g_*(\pi_*(Y)), Z\rangle - \pi_*(Y)(r),$$

or,

$$Z = (\nabla r) \circ \pi. \tag{3.2}$$

As in the Gauss parametrization, we know that when x stays in a leaf of Δ_λ, say, through y, it fills an open subset of a k-dimensional round sphere in some $(k+1)$-dimensional affine subspace. Since $f(x) = g(y) - r(y)(\nabla r(y) + \eta(x))$ and $\eta(x) \in T^{\perp}_{g(y)}V$ which is itself a $(k+1)$-dimensional affine subspace, we get that $\eta(x) = \eta_0(y) + s(y)\Phi(y, u)$, where $\eta_0 \in T^{\perp}V$, $0 < s \in C^\infty(V)$ and $\Phi(y, \cdot)$ is a local parametrization of the unit sphere in $T^{\perp}_{g(y)}V$. But

$$1 = \|\xi(x)\|^2 = \|\nabla r(y)\|^2 + \|\eta_0(y)\|^2 + s(y)^2 + 2s(y)\langle\eta_0(y), \Phi(y, u)\rangle.$$

Hence, $\eta_0 = 0$ and $s = \sqrt{1 - \|\nabla r\|^2}$. Taking $\mu = r^2/2$, we conclude the following result.

Theorem 8. ([**AD**]) *Let $g : V^{n-k} \to \mathbb{R}^{n+1}$ be an isometric immersion and $0 < \mu \in C^\infty(V)$ such that $\|\nabla\mu\|^2 < 2\mu$. Then, on the open set of regular points, the map on the unit normal bundle of g, $\Psi : T_1^{\perp}V \to \mathbb{R}^{n+1}$ given by*

$$\Psi(\omega) = g(y) - \nabla\mu(y) + \sqrt{2\mu(y) - \|\nabla\mu(y)\|^2}\,\omega, \qquad w \in T_1^{\perp}V(y), \tag{3.3}$$

is an immersed k-umbilic hypersurface with umbilic eigenvalue $\lambda = 1/\sqrt{2\mu}$ and Gauss map

$$\xi(w) = \frac{1}{\sqrt{2\mu}}\left(\nabla\mu - \sqrt{2\mu - \|\nabla\mu\|^2}\,\omega\right).$$

Conversely, any such hypersurface can be parametrized locally this way. The parametrization is global provided M^n is complete and orientable.

Proof: We proceed as in the proof of Theorem 2. The converse was already proved. For the direct statement, take a smooth section $\omega \in \Gamma(T_1^\perp V)$ and set $\tilde{\mu} = \sqrt{2\mu - \|\nabla\mu\|^2}$ and $\omega(y)^\perp = \{v \in T_y^\perp V : \langle v, \omega(y)\rangle = 0\}$, the tangent space to the leaf of $T_1^\perp V$ at $\omega(y)$. For any vector $v \in \omega(y)^\perp$ we have $\Psi_{*w(y)}(v) = \tilde{\mu}(y)v$ and for $Y \in TV$ we obtain

$$\Psi_*(\omega_* Y) = P_\omega Y + \frac{1}{\tilde{\mu}}\langle P_\omega Y, \nabla\mu\rangle\omega + Q_\omega(Y),$$

where $Q_\omega(Y) \in \omega^\perp$ and

$$P_\omega = I - \mathrm{Hess}_\mu - \sqrt{2\mu - \|\nabla\mu\|^2}B_\omega,$$

being B_ω the shape operator of g in the direction of ω. We conclude that ξ is the Gauss map of Ψ and $A(v) = -\overline{\nabla}_v\xi = \lambda\overline{\nabla}_v\Psi = \lambda\Psi_*(v)$, for all $v \in \omega^\perp$. ∎

As before, observe that the regular points of Ψ are precisely the vectors $\omega \in T_1^\perp V$ such that P_ω is nonsingular, and

$$Y = g_*Y = (\Psi\circ\omega + \lambda^{-1}\xi\circ\omega)_*Y = (I - \lambda^{-1}A)\Psi_*(\omega_*Y) + Y(\lambda^{-1})\xi\circ\omega.$$

Hence, $(I - \lambda^{-1}A)(U_\omega P_\omega Y) = U_\omega SY$, where S and U_ω are the nonsingular operators $SY = Y - (1/2\mu)\langle Y, \nabla\mu\rangle\nabla\mu$ and $U_\omega Y = Y + (1/\tilde{\mu})\langle Y, \nabla\mu\rangle\omega$. We conclude that

$$A = \lambda U_\omega(I - SP_\omega^{-1})U_\omega^{-1} \quad \text{in} \quad \Delta_\lambda^\perp.$$

Remark 9. We can prove the Gauss parametrization (2.2) using the umbilic Gauss parametrization. To do this, apply an inversion i_q relative to a unit sphere centered at $q \in \mathbb{R}^{n+p}\backslash M^n$, $i_q(x) = (x-q)/\|x-q\| + q$, which is a conformal map of $\mathbb{R}^{n+1}$, to an isometric immersion $f : M^n \to \mathbb{R}^{n+p}$. Then, the relative nullity of f becomes an umbilic distribution of $i_q \circ f$ all of whose umbilic leaves pass through a fixed point, and then we can apply the umbilic Gauss parametrization to it. We are going to do this for higher codimensions.

3.2 Some applications

Isometric immersions $f : M^n \to \mathbb{R}^{n+p}$ with umbilic distributions arise in several different geometric situations. As for the isometric case, there is a completely analogous theorem to that of Beez-Killing for *conformal* rigidity of hypersurfaces, due to E. Cartan ([**C1**]): any conformal immersion

$f : M^n \to \mathbb{R}^{n+1}$, $n \geq 5$, of a connected riemannian manifold with index of conformal nullity $\nu_c \leq n-3$ is locally conformally rigid, that is, $f|_U$ is the unique conformal immersion (up to conformal transformations of $\mathbb{R}^{n+1}$) of any connected open subset $U \subset M^n$ into $\mathbb{R}^{n+1}$. Moreover, Cartan locally classified all conformally deformable euclidean hypersurfaces as he did for the isometric case. More in the spirit of Sbrana's description ([**Sb**]), a parametric classification in terms of the umbilic Gauss parametrization (3.3) was done in [**DT2**], quite similar to the isometric case.

Another situation where it appears umbilic distributions is when a submanifold admits two isometric immersions into space forms with different curvature ([**DdC**]): if there are isometric immersions $f : M^n \to \mathbb{R}^{n+1}$ and $\tilde{f} : M^n \to \mathbb{S}^{n+p}$, $p \leq n-3$, then $\nu_c \geq n-p$ everywhere. With the crucial help of the umbilic Gauss parametrization (3.3), a study for the codimension two case was done in [**AD**].

But the easiest application of the umbilic Gauss parametrization is perhaps the local characterization of all conformally flat euclidean hypersurfaces, that is, each point of the submanifold has a neighborhood which is conformal to euclidean space. It is known that if $f : M^n \to \mathbb{R}^{n+p}$ is conformally flat, $n \geq 5$ and $p \leq n-3$, then there is an open dense subset of M^n so that each connected component has constant index of conformal nullity $\nu_c \geq n-p$; cf. [**Mo**]. Around 1919, nonflat conformally flat hypersurfaces for $n \geq 4$ were completely described by E. Cartan ([**C2**]) as being any envelope of a 1-parameter family of spheres. This means that the condition $\nu_c \geq n-1$ is in fact an equivalent condition for an euclidean hypersurface to be conformally flat. Since any connected open subset with $\nu_c = n$ (i.e., a totally umbilic subset) must be an open subset of a round sphere $\mathbb{S}^n_c \subset \mathbb{R}^{n+1}$, let us consider a connected open subset along which $\nu_c = n-1$. The umbilic Gauss parametrization (3.3) says in this case that any conformally flat hypersurface without umbilic and flat points can be locally described with an arc length parametrized curve $\gamma : I \to \mathbb{R}^{n+1}$ and a smooth function $k \in C^\infty(I)$ such that $(k')^2 > 2k$ by

$$\Psi(w) = \gamma(s) - k'(s)\gamma'(s) + \sqrt{2k(s) - k'(s)^2}\, w, \qquad w \in T_1^\perp I(s).$$

Chapter 4

Higher codimensions

In this section, we give generalizations to higher codimensions of the Gauss parametrization in Theorem 2 and its umbilic version, Theorem 8. We are going to see that these parametrizations are not so nice than the previous ones, since they are related to subbundles with solutions of certain partial differential equations. Nevertheless, they are still very useful. We are going to develop first the umbilic one, and then we will use it to obtain the one for relative nullity.

4.1 The parametrizations

Consider a k-umbilic isometric immersion $f : M^n \to \mathbb{R}^{n+p}$ with umbilic direction η and umbilic eigenvalue λ. Set $\gamma = \lambda\eta \neq 0$ and then

$$\Delta_\gamma = \ker(\alpha - \langle\cdot,\cdot\rangle\gamma) \subset \ker(A_\eta - \lambda I). \tag{4.1}$$

We say that f is *generic* if $\Delta_\gamma = \ker(A_\eta - \lambda\,\mathrm{I})$. Any k-umbilic hypersurface is trivially generic. As before, we have for $U^n \subset M^n$ that $V^{n-k} = U^n/\Delta_\gamma$ is a manifold and there is $0 < r \in C^\infty(V)$ such that $\lambda^{-1} = r \circ \pi$. As in (3.1), we easily see that the map

$$h = f + \lambda^{-1}\eta \tag{4.2}$$

is constant along the leaves of Δ_γ, and then there is a map $g : V^{n-k} \to \mathbb{R}^{n+p}$ such that $h = g \circ \pi$. Moreover, since f is generic g is an immersion and we endow V^{n-k} with the metric induced by g. In view of (4.2), f to be generic is equivalent to

$$T_{g\circ\pi}V \cap T_f^\perp M = 0. \tag{4.3}$$

Being η normal to f, we have that

$$0 = \langle \eta, f_* X \rangle = \langle \eta, g_*(\pi_* X) \rangle - \pi_* X(r), \quad \forall\ X \in TM.$$

Again, the TV-component of η is ∇r. Since the leaves of Δ_γ are k-dimensional umbilic spheres in $\mathbb{R}^{n+p}$ and $\|\eta\| = 1$, we obtain that there is a rank $(k+1)$ normal subbundle of g, $\Lambda^{k+1} \subset T_g^\perp V$, and a smooth section $\xi \in \Lambda^\perp$ with $\|\nabla r\|^2 + \|\xi\|^2 < 1$ so that the map ϕ: $\Lambda_1 \to \mathbb{R}^{n+p}$ given by

$$\phi(w) = g - r\eta, \qquad \eta = \nabla r + \xi + \Omega\, w, \tag{4.4}$$

is a local parametrization of f, where $\Omega = (1 - \|\nabla r\|^2 - \|\xi\|^2)^{1/2}$ and Λ_1 stands for the unit bundle of Λ^{k+1}. (From now on, we are going to omit the base point y of $w \in T_y V$ in the parametrizations, as we did in (4.4)). Although at regular points ϕ parametrizes a submanifold foliated by k-dimensional spheres (the images under ϕ of the leaves of Λ_1), it is *not* in general a k-umbilic submanifold. Our next goal is to compute the actual restrictions on (g, r, ξ, Λ) in order for f to be k-umbilic. We are not going to strongly use this in our notes, but we give it to show another way to obtain parametric descriptions using geometric data.

Let us first fix some notations. For a submanifold $G : V^{n-k} \to \mathbb{Q}_c^{n+p}$ and $\varphi \in T_G^\perp V$, let B_φ denote the second fundamental form of G in the direction φ. Define for any normal subbundle $\Lambda \subset T_G^\perp V$ the tensors $B_\Lambda : \Lambda \to \mathrm{End}(TV)$, $\alpha_G^\Lambda : TV \to \mathrm{End}(TV, \Lambda)$ and $\nabla_\Lambda^\perp : \Lambda \to \mathrm{End}(TV, \Lambda^\perp)$ by

$$\alpha_G^\Lambda(X)(Y) = \alpha_G(X, Y)^\Lambda, \quad B_\Lambda(w)(X) = B_w X, \quad \nabla_\Lambda^\perp(w)(X) = (\nabla_X^\perp w)^{\Lambda^\perp}.$$

For a tensor field $D : \Lambda \to TV$, we set $D^c(w)(X) = (Dw)^c(X) = \langle Dw, X \rangle$, and $\nabla D(w)(X) = \nabla_X Dw - D(\nabla_X^\perp w)^\Lambda$ for all $X, Y \in TV$, $w \in \Lambda^\perp$, where $\nabla^\perp$ stands for the normal connection of G, and $(\cdot)^\Lambda$ for the orthogonal projection to the Λ–component. Associated to D, we also consider the tensor field $\Psi^D : \Lambda \to \mathrm{End}(TV)$,

$$\Psi^D(w) = \Psi_w^D = D\alpha_G^\Lambda(Dw) + B_w - (\nabla D)w.$$

Theorem 10. ([**DFT3**]) *Let $g : V^{n-k} \to \mathbb{R}^{n+p}$ be an isometric immersion with a rank $(k+1)$ normal subbundle $\Lambda^{k+1} \subset T_g^\perp V$ such that there is a tensor field $T : \Lambda^\perp \to TV$ which satisfies*

$$T^* B_\Lambda = \nabla_\Lambda^\perp. \tag{4.5}$$

Assume further that there is a smooth function $\rho \in C^\infty(V)$ which satisfies $2\rho - \|\nabla\rho\|^2 - \|T^*\nabla\rho\|^2 > 0$ such that

$$T^c = d\rho \circ \Psi^T. \tag{4.6}$$

Then, at regular points, the map $\phi:\, \Lambda_1 \to \mathbb{R}^{n+p}$ given by

$$\phi(w) = g - \nabla\rho + T^*\nabla\rho - (2\rho - \|\nabla\rho\|^2 - \|T^*\nabla\rho\|^2)^{1/2} w \tag{4.7}$$

parametrizes a generic k-umbilic n-dimensional euclidean submanifold. Conversely, any such submanifold can be parametrized locally this way.

Proof: Since η is parallel in the normal connection of f along Δ_γ, the rank $(p-1)$ normal subbundle $F = (\text{span }\{\eta\})^\perp \subset T^\perp M$ is also normal parallel along Δ_γ: $\nabla^\perp_{\Delta_\gamma} F \subset F$. We see from the definition of Δ_γ that

$$A_\varphi(\Delta_\gamma) = 0, \quad \forall\ \varphi \in F. \tag{4.8}$$

Hence, $\overline{\nabla}_{\Delta_\gamma} F \subset F$, i.e., F is constant along the leaves of Δ_γ in $\mathbb{R}^{n+p}$. Thus, we can consider F as a rank $(p-1)$–subbundle of g, $F \subset T_g V \oplus T_g^\perp V$, which is normal to f and η:

$$F(\pi(x)) \oplus^\perp \text{span }\{\eta(x)\} = T^\perp_{f(x)} M, \quad \forall x \in M^n. \tag{4.9}$$

Now, for all $u, w \in \Lambda(y)$, $u \perp w$, $y = \pi(x)$, we have from (4.4) that

$$0 = \langle F(y), \phi_{*(y,w)}(0,u)\rangle = -r(y)\Omega(y)\langle F(y), u\rangle.$$

Therefore, in view of (4.3) we conclude that

$$F \subset T_g V \oplus \Lambda^\perp, \quad F \cap T_g V = 0 \tag{4.10}$$

Equation (4.8) now says that

$$0 = \langle \overline{\nabla}_X \varphi, \phi_{*(y,w)}(0,u)\rangle = -r(y)\Omega(y)\langle \overline{\nabla}_X \mu, u\rangle,$$

for all $\varphi \in F, X \in TV$, where $\overline{\nabla}$ stands for the canonical connection on $\mathbb{R}^{n+p}$. Thus,

$$\overline{\nabla} F \subset T_g V \oplus \Lambda^\perp. \tag{4.11}$$

By dimension reasons, (4.10) implies that the orthogonal projection from F onto $\Lambda^\perp$ is a bundle isomorphism. For $\mu \in \Lambda^\perp$, denote by $\overline{\mu} \in F$ the corresponding element under the inverse of this isomorphism. Thus, there exists a unique tensor $T : \Lambda^\perp \to TV$ such that

$$F = \{\overline{\mu} = T\mu + \mu : \mu \in \Lambda^\perp\}.$$

For $X \in TV$, $\mu \in \Lambda^\perp$, $w \in \Lambda$ we have from (4.11) that

$$0 = \langle \overline{\nabla}_X \overline{\mu}, w\rangle = \langle B_w T\mu, X\rangle - \langle \nabla^\perp_X w, \mu\rangle = \langle \mu, T^* B_w X - \nabla^\perp_X w\rangle,$$

which is equivalent to (4.5). From $F \perp \eta$ we easily get in (4.4) that

$$\xi = -T^* \nabla r. \tag{4.12}$$

Let $\rho := r^2/2$. Then, for $w \in \Gamma(\Lambda)$, $X \in TV$ and $\mu \in \Lambda^\perp$, we get using (4.9), (4.5) and (4.12) that

$$\begin{aligned} 0 &= \langle \overline{\mu}, \phi_*(w_* X)\rangle = X\langle \overline{\mu}, g\rangle - \langle \overline{\nabla}_X \overline{\mu}, g - r\eta\rangle \\ &= \langle T\mu, X\rangle + \langle \overline{\nabla}_X (T\mu + \mu), \nabla\rho - T^*\nabla\rho\rangle \\ &= \langle T\mu, X\rangle - \langle B_\mu \nabla\rho + B_{T^*\nabla\rho} T\mu, X\rangle + (d\rho \circ \nabla T)(\mu)(X), \end{aligned}$$

which is equivalent to (4.6). The direct statement is now straightforward. ∎

Remark 11. Equation (4.6) implies that T has kernel of codimension at most one. Thus, there exist $\varphi \in \Lambda_1^\perp$ and $Z \in TV$ such that $T(w) = \langle w, \varphi\rangle Z$. We can now write equations (4.5) and (4.6) in terms of φ, Z and ρ.

Although equations (4.5) and (4.6) are not precisely friendly, we can still use the parametrization in particular cases, even without tough work with this equations. As a simple example of this, let us answer the question of which k-umbilic submanifold have constant umbilic eigenvalue $\lambda = (2\rho)^{-1/2}$.

Corollary 12. *Let $g : V^{n-k} \to \mathbb{R}^{n+p}$ be an isometric immersion with a rank $(k+1)$ normal* parallel *subbundle $\Lambda^{k+1} \subset T_g^\perp V$, and let $r \in \mathbb{R}_+$. Then, the map $\phi : \Lambda_1 \to \mathbb{R}^{n+p}$ given by*

$$\phi(w) = g - r\,w$$

parametrizes a generic k–umbilic n–dimensional submanifold with constant umbilic eigenvalue $\lambda = 1/r$. Conversely, any such submanifold can be parametrized locally this way.

Proof: Just observe that by (4.6) we have that $T = 0$, and hence by (4.5) Λ is normal parallel. ∎

We now give the version for relative nullity of Theorem 10, that is, a generalization to higher codimensions of the Gauss parametrization. We say that a rank $n-k$ submanifold $f : M^n \to \mathbb{R}^{n+1}$ is *generic* if

$$\max\{\mathrm{rank} A^f_\varphi : \varphi \in T^\perp_f M\} = n - k.$$

Theorem 13. ([**DFT3**]) *Let $\xi : V^{n-k} \to \mathbb{S}^{n+p-1}$ be an isometric immersion with a rank k normal subbundle $\Lambda^k \subset T^\perp_\xi V$ such that there is a tensor field $T : \Lambda^\perp \to TV$ which satisfies*

$$T^* B_\Lambda = \nabla^\perp_\Lambda. \tag{4.13}$$

Assume further that there is a positive smooth function $\rho \in C^\infty(V)$ such that

$$T^c = d(-\ln \rho) \circ \Psi^T. \tag{4.14}$$

Then, at regular points, the map $\phi : \Lambda \to \mathbb{R}^{n+p}$ given by

$$\phi(w) = \rho\xi + \nabla\rho - T^*\nabla\rho + w \tag{4.15}$$

parametrizes a generic n–dimensional euclidean submanifold of rank $n-k$. Conversely, any such submanifold f can be parametrized locally this way.

Proof: For almost all inversion of a generic rank $n-k$ submanifold, we get a generic k-umbilic submanifold such that all the spheres of the umbilic foliation pass through a fixed point, say, $0 \in \mathbb{R}^{n+p}$. (In fact, these are precisely the k-umbilic immersions that come from inversion of submanifolds with index of relative nullity k). Then, we have that $\rho = \|g\|^2/2$ and there is $w_0 \in \Gamma(\Lambda_1)$ such that $g = \nabla\rho - T^*\nabla\rho + \Omega' w_0$. Now, it is not hard to check that (4.5) and (4.6) translate to (4.13) and (4.14).

Another proof: To prove the converse, split the position vector of the immersion as $f = f^\top + f^\perp \in T_f M \oplus T^\perp_f M$. Then, we have for all $X \in \Delta_\gamma$ that

$$X = f_* X = \nabla_X f^\top + \nabla^\perp_X f^\perp \in T_f M.$$

Thus, $f^\perp_* X = -A^f_{f^\perp} X + \nabla^\perp_X f^\perp = 0$ and hence $f^\perp$ is parallel along the relative nullity distribution Δ_γ. Define $\rho = \|f^\perp\|$, $\xi = -\rho^{-1} f^\perp$. Since f is generic, there is $P \in \mathbb{R}^{n+p}$ such that we can change f by $f + P$ in order to make ξ an immersion and rank $A^f_\xi = n - k$. The remaining of the proof follows as in Theorem 10. ∎

4.2 Rotational $(n-2)$-umbilic submanifolds

For an isometric immersion $f : M^n \to \mathbb{R}^{n+p}$, it was shown by B. Y. Chen ([**Ch**]) that the inequality

$$\delta_M \leq \frac{n-2}{2(n-1)}\|H\|^2 \tag{4.16}$$

holds pointwise. Here, δ_M stands for the intrinsic invariant defined as

$$\delta_M(x) = s(x) - \inf\{K_M(\sigma) : \sigma \subset T_xM\},$$

where s denotes the scalar curvature of M^n: $s = \sum_{i\neq j} K_M(e_i, e_j)$ for any local orthonormal frame $\{e_1, \ldots, e_n\}$ of TM.

It is then natural to try to understand all submanifolds for which equality in (4.16) holds everywhere. In euclidean space, Chen showed that the *trivial* examples satisfying his *basic equality* are either affine subspaces or rotational hypersurfaces obtained by rotating a straight line, that is, cones and cylinders. Nontrivial examples for $n \geq 4$ divide in two classes, namely, *any* minimal submanifold of rank two, which are completely described in [**DF3**], and the generic $(n-2)$-umbilic submanifolds satisfying

$$H = (n-1)\lambda\eta, \tag{4.17}$$

where η and λ are the umbilic direction and eigenvalue of f, respectively. At a first glance, elements in the second nontrivial class are more abundant, since the first class is just the second one for $\lambda = 0$. We are going to see here that, surprisingly, this is not true.

In fact, in [**DF2**] the authors showed that connected elements in Chen's second nontrivial class have the simplest possible geometric structure among submanifolds foliated by totally umbilic spheres, namely, they are rotational submanifolds over surfaces. This means that M^n is isometric to an open subset of a warped product $V^2 \times_\varphi \mathbb{S}^{n-2}$, $\varphi \in C^\infty(V)$ positive, and

$$f(x, y) = (h(x), \varphi(x)\, y) \tag{4.18}$$

being h: $V^2 \to \mathbb{R}^{p+1}$ an immersion. The surface $k := (h, \varphi)$: $V^2 \to \mathbb{R}^{p+2}$ is the profile of f, and we consider on V^2 the metric induced by k.

But we want to discuss first the much more general problem whether a $(n-2)$-umbilic submanifold is rotational, and present necessary and sufficient conditions for this to occur. That the nontrivial nonminimal submanifolds satisfying the basic equality (4.16) are rotational will follow immediately from the following two results. For this purpose, we are going to make use of parametrization (4.7), again in the spirit of Remark 5. Actually, we will use the weaker representation (4.4), which implies two things. First, even the simplest parametrizations are useful. Secondly, the next results hold for certain submanifolds just foliated by round spheres and not only for the k-umbilic ones.

Theorem 14. ([**DF2**]) *Let $f : M^n \to \mathbb{R}^{n+p}$, $n \geq 4$, be a generic $(n-2)$-umbilic submanifold and assume that $tr\,A_\eta \neq n\lambda$, where η and λ are the umbilic direction and umbilic eigenvalue of f, respectively. Then f is a rotational submanifold over a surface if and only if $tr\,A_\eta$ is constant along the leaves of the umbilic distribution Δ_γ, with $\gamma = \lambda\eta$.*

Proof: The direct statement is a straightforward computation. For the converse we use (4.4). It suffices to show that Λ^{n-1} is constant in ambient space. Then g reduces codimension to $p+1$, and the result easily follows.

From (4.2), we have

$$f_*X = g_*X - X(r)\,\eta - r\,\eta_*X, \quad \forall\, X \in TM, \tag{4.19}$$

where we are writing g instead of $h = g \circ \pi$ and r instead of $r \circ \pi$ not to overload with notations. Denote by P_M and P_V the orthogonal projections on TM and TV, respectively. We get

$$rP_M\eta_*X = P_M g_*X - f_*X \tag{4.20}$$

and

$$P_V f_*X = (S - rQ_w)g_*X, \tag{4.21}$$

being $S, Q_w : TV \to TV$ the tensors on V^2 given by $S = \mathrm{I} - \langle \nabla r,\ * \,\rangle \nabla r$ and

$$Q_w = \mathrm{Hess}_r - B_\xi - \Omega B_w, \quad w \in \Lambda^{n-1},$$

where B_τ denotes the second fundamental form of V^2 relative to τ.

We claim that $T = P_V P_M|_{TV}$ is a well defined tensor on V^2. From

$$g_*X = (\mathrm{I} - rA_\eta)f_*X + \nabla^\perp_X r\eta, \tag{4.22}$$

we get

$$\begin{aligned} Tg_*X &= g_*X - P_V(\mathrm{I} - P_M)g_*X = g_*X - P_V(\nabla^\perp_X r\eta) \\ &= (S - P_V P_{\langle\eta\rangle^\perp})g_*X \end{aligned}$$

where $\langle\eta\rangle = \operatorname{span}\{\eta\}$ and $\langle\eta\rangle^\perp$ stands for its orthogonal complement in $T_f^\perp M$. The claim follows from the fact that the subbundle $\langle\eta\rangle^\perp$ is constant in $\mathbb{R}^{n+p}$ along leaves of Δ_γ, as we saw in the beginning of the proof of Theorem 10.

Fix a point $x \in M^n$, and let λ_1, λ_2 be the eigenvalues of A_η different from λ corresponding to the eigenvectors X_1, X_2. We want to compute $\lambda_1 + \lambda_2$ in terms of g and r. Taking the TV-component of $-P_M\,\eta_* X_i = \lambda_i f_* X_i$ and using (4.20) and (4.21), we get

$$Tg_* X_i = (S - rQ_w)(1 - r\lambda_i) g_* X_i, \quad 1 \le i \le 2,$$

Now observe that $T > 0$. In fact, this is equivalent to $T_f^\perp M \cap T_g V = 0$, which follows from (4.22) and $\lambda_j \neq \lambda$. We conclude that $S - rQ_w$ is nonsingular.

Our assumption yields

$$0 \neq \theta := (2\lambda - \lambda_1 - \lambda_2) r = \operatorname{tr}(S - rQ_w)^{-1} T = \operatorname{tr}(P + \nu B_w)^{-1} T$$

is independent of w. Here, $P = S - r\mathrm{Hess}_r + rB_\xi$ and $\nu = r\Omega$. For a pair C, D of 2×2 matrices, we have

$$\operatorname{tr}(C^{-1}D)\,\det C = \operatorname{tr} C \operatorname{tr} D - \operatorname{tr}(CD) = \det(C + D) - \det C - \det D,$$

where we assume that C is not singular only for the first equality. Therefore,

$$\theta\,\det(P + \nu B_w) = \operatorname{tr} T \operatorname{tr}(P + \nu B_w) - \operatorname{tr}(T(P + \nu B_w)).$$

Thus,

$$\begin{aligned}\theta\nu^2 \det B_w = \nu \operatorname{tr}(T - \theta P)\operatorname{tr} B_w - \nu \operatorname{tr}((T - \theta P)B_w) + \operatorname{tr} T \operatorname{tr} P \\ -\operatorname{tr}(TP) - \theta \det P, \qquad \forall\, w \in \Lambda_1.\end{aligned}$$

Since $\dim \Lambda^{n-1} \ge 3$, we easily obtain that $\det B_w = 0$,

$$\operatorname{tr}(T - \theta P)\operatorname{tr} B_w = \operatorname{tr}((T - \theta P)B_w), \qquad \text{and} \tag{4.23}$$

$$\det(T - \theta P) = \det T > 0. \tag{4.24}$$

Suppose that $B_{w_0} \neq 0$ for some $w_0 \in \Lambda^{n-1}$. Then equation (4.23) yields $\langle (T - \theta P)v, v\rangle = 0$ for $0 \neq v \in \ker B_{w_0}$, which is in contradiction with (4.24) and proves that

$$B_w = 0, \quad \forall\, w \in \Lambda^{n-1}. \tag{4.25}$$

Since leaves of Δ_γ are the images of Λ_1 under (4.4), we have

$$\langle\eta\rangle^\perp \subset T_{g(x)}V \oplus \Lambda^\perp_{g(x)}, \quad \forall\, x \in M^n. \tag{4.26}$$

Observe that $T^\perp_{f(x)}M \cap T_{g(x)}V = 0$ implies that $\langle\eta\rangle^\perp \cap T_{g(x)}V = 0$. Hence, the orthogonal projection

$$\pi'(x)\colon \langle\eta(x)\rangle^\perp \subset T^\perp_x M \to \Lambda^\perp_{g(x)} \subset T^\perp_{g(x)}V$$

is an isomorphism. On the other hand, we have using (4.19) that

$$\langle g_*Y - r\overline{\nabla}_Y(\nabla r + \xi + \Omega\, w), \delta\rangle = 0, \quad \forall\, Y \in TV,\ w \in \Lambda^{n-1} \text{ and } \delta \in \langle\eta\rangle^\perp.$$

It follows from (4.25), (4.26) and that $\langle\eta\rangle^\perp$ is constant along the leaves that

$$\langle\overline{\nabla}_Y w, \delta\rangle = 0, \quad \forall\, Y \in TV,\ w \in \Lambda^{n-1} \text{ and } \delta \in \langle\eta\rangle^\perp.$$

Being π' an isomorphism, we conclude from (4.25) that Λ^{n-1} is constant and this proves the theorem. ∎

It is easy to prove a completely analogous theorem as the above for submanifolds in the sphere $\mathbb{S}^{n+p}$ just includding $\mathbb{S}^{n+p} \subset \mathbb{R}^{n+p+1}$. For nongeneric $(n-2)$-umbilic submanifolds we have the next result. Its proof uses standard techniques in submanifold theory, but we give it for completeness. We follow the notations of Theorem 14.

Theorem 15. ([**DF2**]) *Assume that f is a $(n-2)$-umbilic submanifold with*

$$\dim\{\ker(A_\eta - \lambda\, I)(x)\} = n-1, \quad \forall\, x \in M^n.$$

Then f is a rotational submanifold over a surface if and only if the mean curvature vector is parallel in the normal connection along the leaves of Δ_γ.

Proof: The direct statement is trivial. For the converse, let $X, Y \in \Delta^\perp_\gamma$ be orthonormal eigenvectors for A_η with eigenvalues μ and λ, respectively. By assumption, there is a smooth field of unit length $\xi \in T^\perp_f M$, $\xi \perp \eta$, parallel along Δ_γ with $A_\xi Y \neq 0$ and $\operatorname{tr} A_\xi$ constant along Δ_γ. Taking the $\Delta^\perp_\gamma$-component of the Codazzi equation (1.7) for (X, T, η), i.e.,

$$(\nabla_X A_\eta)T - A_{\nabla^\perp_X \eta}T = (\nabla_T A_\eta)X - A_{\nabla^\perp_T \eta}X, \quad T \in \Delta_\gamma,$$

we get

$$\nabla^v_X X = 0, \tag{4.27}$$

where Z^v (respectively, Z^h) denotes taking the Δ_γ (respectively, $\Delta_\gamma^\perp$) component of Z. Similarly, the X-component of the Codazzi equation for (Y, T, η) yields

$$\nabla^v_Y X = 0. \tag{4.28}$$

Now, a straightforward computation of the Codazzi equations for (X, T, ξ) and (Y, T, ξ) gives

$$\langle \nabla_Y Y, T\rangle \langle A_\xi Y, Y\rangle + \langle \nabla_X Y, T\rangle \langle A_\xi Y, X\rangle = 0$$

and

$$\langle \nabla_X Y, T\rangle \langle A_\xi Y, Y\rangle - \langle \nabla_Y Y, T\rangle \langle A_\xi Y, X\rangle = 0,$$

from which we conclude that $\nabla^v_Y Y = 0 = \nabla^v_X Y$. This, (4.27) and (4.28) say that the distribution $\Delta_\gamma^\perp$ is totally geodesic (autoparallel) in M^n. It is not difficult to see (cf. Lemma 6 of [**DF2**]) that this implies that f is a rotational submanifold and the proof is complete. ∎

Remarks 16. 1) For a connected nontrivial nonminimal euclidean submanifold $f : M^n \to \mathbb{R}^{n+p}$, $n \geq 4$, satisfying everywhere the basic equality, by Lemmas 3.2 and 3.3 of [**Ch**] there are two possibilities along each connected component of an open dense subset. Namely, f is either $(n-1)$-umbilic or is $(n-2)$-umbilic. Moreover, in both situations equation (4.17) holds. Then f is trivial in the first case and, in the second case, it follows from Theorems 14 and 15 that f is a rotational submanifold.

2) It is shown in [**DF2**] that a rotational submanifold f as in (4.18) satisfies the basic equality (4.16) if and only if φ is a solution on V^2 of the second order quasilinear elliptical differential equation

$$\varphi \,\mathrm{tr}\,(R\,\mathrm{Hess}_\varphi) + 1 = 0, \tag{4.29}$$

and the second fundamental form of h: $V^2 \to \mathbb{R}^{p+1}$ satisfies

$$\mathrm{tr}\ (R\,B^h_\xi) = 0, \quad \forall\, \xi \in T^\perp_h V, \tag{4.30}$$

where $R = I - (1 + \|\nabla\varphi\|^2)^{-1}\langle \nabla\varphi, \cdot\ \rangle \nabla\varphi$.

Chapter 5

Submanifolds with integrable conullity

Let f, γ and k as in Proposition 1. As we have seen, there is a clear difference between the Gauss parametrizations (2.2) and (3.3) and its versions for higher codimensions (4.7) and (4.15), since it appears certain PDE's associated with normal subbundles in the later ones. The main reason on the beauty of the hypersurface situation relies in the fact that the nullity distribution Δ_γ coincides with the normal space of an immersion of the quotient space $V^{n-k} = M^n/\Delta_\gamma$. We can think this phenomena as a kind of 'integrability' of the conullity of f. In this section we see how the general representations can be improved when some integrability is assumed on the conullity of f. It is interesting how parametrizations are used here to obtain better parametrizations in a sequence. We should point out that some of the results in this chapter can be generalized under even weaker concepts of integrability.

Let f as above and assume for simplicity that the quotient space of leaves of Δ_γ, $V^{n-k} = M^n/\Delta_\gamma$, is a manifold as in the Gauss parametrization. As we saw, this is not a restriction when working locally, since we can take a local cross section $\tilde{V}^{n-k} \subset M^n$ to the leaves of Δ_γ and for a saturated open subset $U^n \subset M^n$ to the leaves, the space $V^{n-k} = U^n/\Delta_\gamma$ is diffeomorphic to $\tilde{V}^{n-k}$. Therefore, we can always think that $V^{n-k} = \tilde{V}^{n-k} \subset M^n$ is a cross section to the umbilic foliation.

Although general parametrizations can be difficult to work with, they also can be used to obtain better representations when further conditions on the immersion are assumed. This is the case of the following result. We found the representation (5.2) using (4.7) for generic k-umbilic immersions, and then with the result in mind we made a nicer geometric proof (without the generic condition!), the one we give here.

We say that f has *integrable conullity at one point* if there is an immersion $h : V^{n-k} \to M^n$ such that $T_{h(y)}V = \Delta_\gamma^\perp(h(y))$, that is, if $\Delta_\gamma^\perp$ posseses just *one* leaf. For a vector subbundle $F \subset T^\perp V$ and a map $\mu : V^{n-k} \to \mathbb{R}^{n+p}$ we denote by $(\mu)^F = (\mu)^F(y)$ the orthogonal projection of $\mu(y)$ to $F(y) \subset \mathbb{R}^{n+p}$.

Theorem 17. ([**DFT3**]) *Let $h : V^{n-k} \to \mathbb{R}^{n+p}$ be an isometric immersion with a rank k parallel normal subbundle $F \subset T_h^\perp V$. Let $\mu : V^{n-k} \to \mathbb{R}^{n+p}$ be a map transversal to F^k which satisfies the first order system of PDE*

$$(d\mu)^{F^\perp} = \langle \mu, dh \rangle (\mu)^{F^\perp}. \tag{5.1}$$

Then, the map $\psi : F \to \mathbb{R}^{n+p}$ given by

$$\psi(t) = h + 2\frac{\mu + t}{\|\mu + t\|^2} \tag{5.2}$$

parametrizes, at regular points, an n-dimensional s-umbilic euclidean submanifold with conformal conullity integrable at one point and $s \geq k$. Conversely, any such n–dimensional k-umbilic euclidean submanifold f with integrable conullity at one point can be locally parametrized this way.

Proof: First observe that we may always consider $\mu \perp F$. In fact, if μ is a solution of (5.1) then also its orthogonal projection to $F^\perp$, $(\mu)^{F^\perp}$, is a solution. Moreover, by the change of parameters $\tilde{t} = t + (\mu)^F$ in (5.2) we see that the submanifold $\psi(F)$ does not change.

For the converse, consider a k-umbilic immersion f which has a conullity leaf V^{n-k}, and take $h = f|_V$. Let η and $r = \lambda^{-1}$ as in (4.2) and $\xi = \eta|_V$. We know that, for each $x \in V^{n-k}$, there is an extrinsic k-dimensional sphere $\mathbb{S}^k(x)$ passing through $h(x)$. Then $F(x) = T_{h(x)}\mathbb{S}^k(x) \subset T_h^\perp V$ since V^{n-k} is orthogonal to the umbilic distribution. Let us say that the extrinsic spheres are centered at $h + \mu/\|\mu\|^2$, where $\mu \perp F$ and $\|\mu\|^{-1}$ is its radious. This yields (5.2) but it remains to prove the assertions on F and μ.

By definition of μ, we have that $\mu = Z + \lambda\xi$ for some $Z \in T_hV$ since $\lambda\xi$ is the orthogonal projection of μ onto $T_f^\perp M$. Since, as we saw in (4.2), $\psi + r\eta$ does not depend on $t \in F$, we get from (5.2) that

$$r(\xi - \eta) = 2\|\mu + t\|^{-2}(\mu + t). \tag{5.3}$$

Now, since $(\text{span }\{\eta\})^\perp \subset T_f^\perp M$ is constant along the umbilic leaves, we have

$$(\text{span }\{\eta\})^\perp = F^\perp \cap (\text{span }\{\xi\})^\perp \subset T_h^\perp V.$$

In particular, $F^\perp \cap \text{span }\{\xi\}^\perp \perp d(\psi \circ t)$ for any smooth section $t \in \Gamma(F)$. Equation (5.2) yields

$$d(\psi \circ t) = dh + 2d(\|\mu + t\|^{-2})(\mu + t) + 2\|\mu + t\|^{-2}(d\mu + dt).$$

Hence, $(d\mu + dt) \perp F^\perp \cap (\text{span }\{\xi\})^\perp$ for all $t \in \Gamma(F)$, or equivalently,

$$(d\mu)^{F^\perp} \in \text{span }\{\xi\} = \text{span }\{(\mu)^{F^\perp}\} \tag{5.4}$$

and

$$(dt)^{F^\perp} \subset \text{span }\{\xi\}, \tag{5.5}$$

for every $t \in \Gamma(F)$, since μ does not depend on t. We also have that

$$\begin{aligned} 0 &= \langle d(\psi \circ t), r\eta\rangle = \langle d((h + r\xi) - r\eta), r\eta\rangle \\ &= \langle d((h + r\xi), r\xi - 2\|\mu + t\|^{-2}(\mu + t)\rangle - r dr \\ &= -2\|\mu + t\|^{-2}\left(\langle dh, \mu\rangle - r\langle d\mu, \xi\rangle + r\langle d\xi, t\rangle\right). \end{aligned}$$

This together with (5.4) and (5.5) imply that F is parallel in the normal connection of h and μ satisfies equation (5.1). The direct statement is easy. ∎

Remark 18. The condition on μ to be transversal to F is just for γ to be nonvanishing, that is, for the conformal nullity not to be relative nullity. If we take $\mu \in F$, or equivalently, $\mu = 0$, we get precisely the immersions with index of relative nullity k which have one leaf of the conullity distribution.

Now we use the last result to obtain a nice description of the immersions with integrable conullity. For this, we need a definition. Let $h : V \to \mathbb{R}^{n+p}$ be a isometric immersion. We say that a map $C : V \to \mathbb{R}^{n+p}$ is a *Combescure transformation of* h if $dC(X) \in T_yV$, for all $X \in T_yV$, $y \in V$, and the tensor dC is symmetric. If we write $C = Z + \beta$, $Z \in TV$, $\beta \in T^\perp V$, the symmetry of dC is equivalent to $Z = \nabla\varphi$, for some $\varphi \in C^\infty(V)$. It is easy to see that C is a Combescure transformation of h if and only if $\Phi = dC$ is a symmetric Codazzi tensor on TV that commutes with the second fundamental form of h, i.e., for all $X, Y \in TV$, $(\nabla_X\Phi)Y = (\nabla_Y\Phi)X$ and $\alpha_h(X, \Phi Y) = \alpha_h(\Phi X, Y)$. Observe that for $v \in \mathbb{R}^{n+p}$ and $a \in \mathbb{R}$ we have that $C = ah + v$ are always Combescure transformations of h, the *trivial* Combescure transformations. These are precisely the Combescure transformations with $\Phi = aI$, which are, generically, the only symmetric commuting Codazzi tensors in a submanifold.

Theorem 19. ([**DFT3**]) *With the notations of Theorem 17, we have that the submanifold has integrable conullity if and only if F is flat with respect to the induced normal connection of h and we can reparametrize the submanifold as*

$$\psi(t) = h - 2\varphi \frac{(C+t)}{\|C+t\|^2}, \tag{5.6}$$

where $C = \nabla\varphi + \beta$ is a Combescure transformation of h. Moreover, the leaves of the conullity distribution correspond to parallel sections $t \in \Gamma(F)$ in the above parametrization.

Proof: Again we consider $\mu \perp F$ in Theorem 17. We have that the conullity of ψ is integrable if and only if, for all regular point $t \in F(x_0)$, there is a section $T \in \Gamma(F)$ with $T(x_0) = t$ satisfying that $\psi \circ T$ is orthogonal to the umbilic leaves. We have,

$$d(\psi \circ T) = dh + 2\|\mu + T\|^{-2}(d\mu + dT - 2\|\mu+T\|^{-2}\langle d\mu + dT, \mu + T\rangle(\mu + T)),$$

and for any $S \in F$,

$$\psi_*(T)S = 2\|\mu + T\|^{-2}(S - 2\|\mu + T\|^{-2}\langle T, S\rangle(\mu + T)).$$

A strightforward computation yields

$$\langle d(\psi \circ T), \psi_*(T)S\rangle = 0 \iff \langle d\mu + dT, S\rangle - \langle T, S\rangle\langle dh, \mu\rangle = 0.$$

Using the parallelism of F we easily see that this condition is equivalent to

$$\nabla^\perp T \; (= (dT)^F) = -(d\mu)^F + \langle dh, \mu\rangle T. \tag{5.7}$$

Taking differences between T_1, T_2 as above we get from (5.7) that

$$\nabla^\perp(T_1 - T_2) = \langle dh, \mu\rangle(T_1 - T_2).$$

It follows easily that F is flat, i.e., $R_h^\perp(F) = 0$ and that the tangent component of μ is a gradient of the function $\tau = \|T_1 - T_2\| - \|T_1 - T_2\|(x_0)$. Hence, $\tau(x_0) = 0$. Now (5.1) and (5.7) are equivalent to

$$(d\mu)^{T_h^\perp V} = d\tau(T + (\mu)^{F^\perp}) - (dT)^F. \tag{5.8}$$

Set

$$T = e^\tau T_t + T_0,$$

where T_t is the parallel section with $T_t(x_0) = t$ and $T_0 \in \Gamma(F)$. Thus, $T_0(x_0) = 0$. The condition (5.8) is just

$$(d(\mu + T_0))^{T_h^\perp V} = d\tau(\mu + T_0)^{T_h^\perp V},$$

or equivalently,

$$d(e^{-\tau}(\mu + T_0)) \in T_h V. \tag{5.9}$$

If we call $\varphi = -e^{-\tau}$ and $\beta = e^{-\tau}(\lambda\xi + T_0)$, we conclude from (5.9) that

$$e^{-\tau}(\mu + T_0) = \nabla\varphi + \beta$$

is a Combescure transform of h. Moreover, $\mu + T = \varphi^{-1}(\nabla\varphi + \beta + T_t)$. Thus,

$$\Psi \circ T = h - 2\varphi(C + T_t)/\|C + T_t\|^2,$$

and this concludes the proof of the converse. The direct statement is straightforward. ∎

Using parametrizations it is easy in general to obtain the second fundamental form of the submanifold in terms of the original data, as we did for the hypersurface situation. We are going to give now the second fundamental form for the integrable conullity situation not only for later use, but also because its intrinsic beauty.

With the notations of the above theorem, we consider for $t \in F(x_0)$ the parallel section $T_t \in \Gamma(F)$ such that $T_t(x_0) = t$. Set $\overline{\beta} = \beta^{F^\perp}$, $\beta^t = \beta + T_t$ and $h^t = \psi \circ T_t$ the leaf of $\Delta_\gamma^\perp$ that passes through $\psi(t)$. It is easy to check that the map $\tau^t : T_h^\perp V \to T_{h^t}^\perp V$ given by

$$\tau^t(\gamma) = \gamma^t = \gamma - \langle \beta^t, \gamma \rangle \nu^t (C + T_t)$$

is a vector bundle isometry, where $\nu^t = 2\|C + T_t\|^{-2}$. Moreover, a direct computation (or see Corollary 27 *iii*) of [**DT3**]) shows that

$$A^{h^t}_{\gamma^t} = (D^t)^{-1}(A^h_\gamma + \langle \beta^t, \gamma \rangle \nu^t \Phi^t), \tag{5.10}$$

where $D^t = I - \varphi\nu^t\Phi^t$ and $\Phi^t = Hess^h_\varphi - A^h_{\beta+t}$. For $\xi \in F^\perp$ we have

$$\xi^t = \xi + \varphi^{-1}\langle \overline{\beta}, \xi \rangle (h^t - h),$$

$$\partial\xi^t/\partial t_i = \varphi^{-1}\langle \overline{\beta}, \xi \rangle \psi_*(\partial/\partial t_i) = -\langle \overline{\beta}, \xi \rangle \nu^t \xi^t_i.$$

Hence, $T^\perp_{\psi(T_t)} M = \tau^t(F^\perp)$ and

$$A^\psi_{\xi^t}\psi_*(\partial/\partial t_i) = -\varphi^{-1}\langle \overline{\beta}, \xi \rangle \psi_*(\partial/\partial t_i).$$

We conclude that $\lambda\eta^t = -\varphi^{-1}\overline{\beta}^t$.

5.1 Umbilic conullity

We give here an application of Theorem 19: we will conformally classify all k-umbilic submanifolds for which the connullity is integrable and its leaves are umbilic in the submanifold. Our proof is much shorter than the one given in [**DFT2**] with standard submanifold theory techniques. This again shows the power of parametrizations.

Theorem 20. ([**DFT2**]) *Let $f : M^n \to \mathbb{R}^{n+p}$ be a k-umbilic isometric immersion, for some $2 \leq k \leq n-2$. Suppose that the conullity distribution is umbilic in M^n. Then, up to conformal transformations of the ambient spance, the submanifold is:*

a) $M^n = L^{n-k} \times \mathbb{R}^k$, and $f = (h, id)$, for some $h : L^{n-k} \to \mathbb{R}^{n+p-k}$,

b) $M^n = L^{n-k} \times_r \mathbb{S}^k$ and $f(x,p) = (h'(x), r(x)p)$, for some immersion $h' : L^{n-k} \to \mathbb{R}^{n+p-k-1}$ and a positive function $r \in C^\infty(L)$,

c) $M^n = CL^{n-k} \times \mathbb{R}^{k-1}$, where CL is the cone over some immersion $\xi : L^{n-k} \to \mathbb{S}^{n+p-k}$.

Proof: We use parametrization (5.6) for the conullity leaves $h^t = \psi(T_t)$. The key point in the proof is the following claim.
Claim. C is a trivial Combescure transformation of h.
Since the conformal nullity distribution Δ_ξ at $h^t(x)$ is given by $\tau^t(F(x))$, we have that the conullity distribution is umbilic in the submanifold if and only if, for all $\mu \in F$, $X, Y \in \Delta_\xi^\perp(t) = T_{h^t}V$, there is $\kappa_\mu^t \in \mathbb{R}$ such that

$$\langle A_{\mu^t}^{h^t} X, Y\rangle = \langle \overline{\nabla}_X (\psi \circ T_t)_* Y, \mu^t\rangle = \kappa_\mu^t \langle X, Y\rangle.$$

At the leaf parametrized by h ($\|t\| \to \infty$), we have that $A_\mu^h = \kappa_\mu I$, for all $\mu \in F$. By (5.10) we obtain

$$\kappa_\xi I + \langle \beta^t, \xi\rangle \nu^t \Phi^t = \kappa_\xi^t (I - \varphi \nu^t \Phi^t),$$

for some $\kappa_\xi^t \in \mathbb{R}$, which is equivalent to $\Phi = dC = aI$, for some $a \in C^\infty(V)$. Since $\dim \Delta_\gamma^\perp = n - k \geq 2$, the condition for Φ to be Codazzi is precisely $a \in \mathbb{R}$. This proves our claim and we conclude that

$$C = \nabla\varphi + \beta = ah + v_0, \tag{5.11}$$

for some $v_0 \in \mathbb{R}^{n+p}$.

We only have to understand the parametrization for trivial Combescure transformations. There are two cases to consider:
Case $\kappa \equiv 0$. That is, for all $\mu \in F$, $\kappa_\mu = 0$. Since F is parallel, we conclude that h reduces codimension to $n+p-k$. Thus, we have an orthogonal splitting $\mathbb{R}^{n+p} = \mathbb{R}^{n+p-k} \oplus \mathbb{R}^k$ such that, up to translation, $h \subset \mathbb{R}^{n+p-k}$ and $F = \mathbb{R}^k$.

If $a = 0$ in (5.11), we get $\varphi = \langle h, v_0\rangle + c$, $c \in \mathbb{R}$. Hence, after translation if necessary, parametrization (5.6) becomes

$$\psi = (h, 0) - 2(\langle h, v_0'\rangle + c)(\|v_0'\|^2 + \|t\|^2)^{-1}(v_0', t), \tag{5.12}$$

where v_0' is the orthogonal projection of v_0 into $\mathbb{R}^{n+p-k} = F^\perp$. If $v_0' = 0$, we easily get $M = L \times \mathbb{R}^k$, which is of type a) in the statement. If $v_0' \neq 0$, we set $r = \varphi/\|v_0'\|$ and h' as the projection

$$h' = h + c\|v_0'\|^{-2}v_0' - \varphi\|v_0'\|^{-2}v_0' \quad \subset (\text{span } \{v_0'\})^\perp = \mathbb{R}^{n+p-k-1}.$$

In this situation, is easy to see that ψ parametrizes a submanifold of type b) in the theorem.

We analize the case $a \neq 0$, First, observe that for $c = 0$ we may write equation (5.12) as $\psi = h - 2\langle h, e\rangle(1 + \|t\|^2)^{-1}(e, t)$, for some unit vector $e \in \mathbb{R}^{n+p-k}$. If we apply to it an inversion i with respect to the unit sphere centered at e, a straightforward computation yields, up to a translation, that

$$i(\psi) = h_1 - 2(\|h_1\|^2 - 1/4)\|(2h_1, t)\|^{-2}(2h_1, t),$$

where $h_1 = \|h - e\|^{-2}(h - e) + e/2$. Applying an homothety $d/2$ to the above and setting $h_2 = (d/2)h_1$, we obtain

$$(d/2)\ i(\psi) = (h_2, 0) - 2(\|h_1\|^2 - d^2)\|(2h_2, t)\|^{-2}(2h_2, t). \tag{5.13}$$

Since $a \neq 0$, we have that $\varphi = (a/2)(\|h'\|^2 + c)$, $c \in \mathbb{R}$, where $h' = h + v_0'$, with v_0' the orthogonal projection of $a^{-1}v_0$ into $\mathbb{R}^{n+p-k} = F^\perp$. Thus (5.6) becomes (up to a translation)

$$\psi = (h', 0) - 2(\|h'\|^2 + c)\|(2h', t)\|^{-2}(2h', t). \tag{5.14}$$

If $c = 0$, the above is taken by an inversion relative to a unit sphere centered at 0 to a submanifold of type a) in the statement. If $c < 0$, then it is conformal to a submanifld of type b) by (5.13). Finally, (5.14) is conformally type c) when $c > 0$. To see this, apply an inversion i to (5.14) with respect to a unit sphere centered at $\sqrt{c}\, e_{n+p}$ to get, up to a translation,

$$i(\psi) = (s\xi, t),$$

where $s \in \mathbb{R}$, $\xi = 2\sqrt{c}\, \|h' - \sqrt{c}\, e_{n+p}\|^{-2}(h' - \sqrt{c}\, e_{n+p}) + e_{n+p} \subset \mathbb{S}^{n+p-k}$.

Case $\kappa \neq 0$. Let $\xi \in F$ be a unit generator of coker κ. Codazzi's equation and the parallelism of F easily imply that ξ is parallel in the normal connection of h, $F = \operatorname{span}\{\xi\} \oplus F'$, h reduces codimension to $n+p-k+1$, $F' = \mathbb{R}^{k-1}$ and, up to translation, $h \subset F'^{\perp} = \mathbb{R}^{n+p-k+1}$. Also from the Codazzi equation for A^h_ξ we obtain that $\kappa_\xi = -b \neq 0$ is constant, and since ξ is parallel, we get

$$h = b\xi + v_1,$$

for some $0 \neq b \in \mathbb{R}$, $v_1 \in \mathbb{R}^{n+p}$. That is, up to translation and homothety, $h \subset \mathbb{S}^{n+p-k}$. Observe that ξ is the normal space $\mathcal{N}$ of $\mathbb{S}^{n+p-k}$ in $\mathbb{R}^{n+p-k+1}$ restricted to h. Hence, an inversion i with respect to any sphere centered at any point $e \in \mathbb{S}^{n+p-k}$ sends $\mathbb{S}^{n+p-k}$ to $\mathbb{R}^{n+p-k} = \operatorname{span}\{e\}^{\perp} \subset \mathbb{R}^{n+p-k+1}$, and F to $i(\mathcal{N}) \oplus F' = \operatorname{span}\{e\} \oplus F' = \mathbb{R}^k \subset \mathbb{R}^{n+p}$. This is precisely the situation of the case $\kappa = 0$, and the proof is complete. ∎

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