manuscripta math. 129, 127–135 (2009)

Marcos Dajczer · Luis A. Florit

The holomorphic Gauss parametrization

Received: 4 September 2008 / Revised: 13 February 2009
Published online: 17 March 2009

Abstract. We give a local parametric description of all complex hypersurfaces in $\mathbb{C}^{n+1}$ and in complex projective space $\mathbb{CP}^{n+1}$ with constant index of relative nullity, together with applications. This is a complex analogue to the parametrization for real hypersurfaces in Euclidean space known as the Gauss parametrization.

1. Introduction

Let M^n be a complete connected complex immersed hypersurface of $\mathbb{C}^{n+1}$ whose index of relative nullity, that is, the dimension of the kernel of its second fundamental form, satisfies $\nu \geq n-1$ everywhere. Equivalently, its Gauss map $\varphi\colon\ M^n \to \mathbb{CP}^n$ that assigns to each point in M^n its normal complex line in $\mathbb{C}^{n+1}$, satisfies rank $d\varphi \leq 1$. Then, it was shown by Abe [1] that the hypersurface must be an $(n-1)$-cylinder.

The situation is even more restrictive for a complete hypersurface M^n of the complex projective space $\mathbb{CP}^{n+1}$. From a general result also due to Abe [2] it follows that if $\nu > 0$ then M^n must be a totally geodesically embedded $\mathbb{CP}^n \subset \mathbb{CP}^{n+1}$.

Naturally, the situation is quite different in the local case. In fact, for any integer $\nu_0 > 0$ there are plenty of local hypersurfaces M^n in $\mathbb{C}^{n+1}$ and $\mathbb{CP}^{n+1}$ with constant index of relative nullity $\nu = \nu_0$ that are neither part of cylinders in $\mathbb{C}^{n+1}$ or totally geodesic in $\mathbb{CP}^{n+1}$.

Our main goal in this note is to give a parametric description of all complex hypersurfaces in $\mathbb{C}^{n+1}$ and $\mathbb{CP}^{n+1}$ with constant $\nu > 0$. As a consequence, the above global results will be immediate corollaries of our local construction achieved by imposing on the hypersurfaces the absence of singularities.

So far everything just said is an analogue to what happens for real hypersurfaces in Euclidean space $\mathbb{R}^{n+1}$ and the round sphere $\mathbb{S}^{n+1}$. The now called *Gauss parametrization* was introduced by Sbrana [13] as a tool to classify the locally isometrically deformable Euclidean hypersurfaces. In recent years, it has proved to be quite a useful tool, giving rise to several applications; see [3–7,9] and [11].

The parametrization of hypersurfaces in $\mathbb{C}^{n+1}$ we give here works similarly for hypersurfaces in $\mathbb{R}^{n+1}$ and provides an equivalent form of the Gauss parametrization

M. Dajczer (✉) · L. A. Florit: IMPA, Estrada Dona Castorina, 110,
22460-320 Rio de Janeiro, Brazil. e-mail: marcos@impa.br; luis@impa.br

Mathematics Subject Classification (2000): 53B25, 53C40

DOI: 10.1007/s00229-009-0260-9

given in [5]. Our parametrization for $\mathbb{CP}^{n+1}$ is a perfect analogue of the Gauss parametrization in the sphere $\mathbb{S}^{n+1}$. We point out that the holomorphicity hypothesis is redundant for submanifolds in $\mathbb{CP}^N$ with $\nu > 0$ [8].

2. The parametrization in $\mathbb{C}^N$

Let $f: M^n \to \mathbb{C}^{n+p}$ be a holomorphic isometric immersion of a Kähler Riemannian manifold $M^n = (M^n, \langle\ ,\ \rangle)$ with Levi-Civita connection ∇, normal connection $\nabla^\perp$, and second fundamental form $\alpha : TM \oplus TM \to T^\perp M$. We denote by J the complex structures of both M^n and $\mathbb{C}^{n+1}$.

Recall that the *relative nullity* subspace $\Delta(x)$ of f at $x \in M^n$ is given by

$$\Delta(x) = \{Y \in T_x M : \alpha(Y, Z) = 0, \ \ \forall\, Z \in T_x M\}\,.$$

Since f is holomorphic, $\Delta(x) \subseteq T_x M$ is a complex subspace whose (complex) dimension $\nu_f(x)$ is called the *index of relative nullity* of f at x. Along the open dense subset $M_0 \subseteq M^n$ where ν is constant, it is well known that Δ is a smooth integrable distribution whose leaves are totally geodesic in both M^n and $\mathbb{C}^{n+p}$. Locally on a saturated open subset $U \subseteq M_0$, the space of leaves of this distribution, that we denote by $\hat{M} = U/\Delta$, is naturally a complex manifold of dimension $n - \nu$ whose projection $\pi : U \to \hat{M}^{n-\nu}$ is holomorphic. This space can be naturally identified with a complex submanifold of U of dimension $n-\nu$ transversal to the leaves of relative nullity. We point out that the space of leaves is well defined globally on M_0 if M^n is complete; see [5].

Let $i: \mathbb{C}^N_* \to \mathbb{C}^N_*$ be the inversion given by $i(z) = z/\|z\|^2$, where we denote as usual $\mathbb{C}^N_* = \mathbb{C}^N \setminus \{0\}$. Its differential is

$$di_z(v) = \frac{1}{\|z\|^2} R_z v,$$

where

$$R_z v = v - 2\langle v, z\rangle i(z),$$

stands for the reflection in the z direction. Notice that R_z satisfies

$$J R_z = R_{Jz} J$$

where J is the complex structure in $\mathbb{C}^N$.

Let $f: M^n \to \mathbb{C}^{n+p}$ be a holomorphic isometric immersion of a Kähler manifold. Decompose the position vector of the immersion as

$$f = f^\perp + f^\top, \tag{1}$$

according to the orthogonal holomorphic bundle decomposition

$$\mathbb{C}^{n+p} \cong T_x\mathbb{C}^{n+p} = T_x^\perp M \oplus T_x M,$$

for each $x \in M^n$.

If there is an open subset $U \subseteq M^n$ for which the position vector f is tangent to f, then by analyticity f is everywhere tangent and, regarded as a tangent vector field, must belong to the relative nullity. Hence, f must be a (complex) cone through the origin. By means of a generic translation $f + p_0$, we assume from now on that the position vector is not tangent on an open dense subset of M^n, that we continue calling M^n.

Differentiating (1) and taking normal components, we get $\alpha_{f^\top} := \alpha(\cdot, f^\top) = -\nabla^\perp f^\perp$. Hence,

$$d f^\perp = -A_{f^\perp} - \alpha_{f^\top}, \tag{2}$$

where $A_\delta = A^f_\delta$ denotes the real shape operator of f in the direction δ. Since f is holomorphic, we have that $A_{f^\perp} J = -J A_{f^\perp}$ and $\alpha_{f^\top} J = J \alpha_{f^\top}$, and thus

$$d f^\perp J = J \left(A_{f^\perp} - \alpha_{f^\top}\right) = -J \left(d f^\perp + 2\alpha_{f^\top}\right). \tag{3}$$

Setting

$$g = i(f^\perp), \tag{4}$$

we have

$$dg = di_{f^\perp} \circ d f^\perp = \frac{1}{\|f^\perp\|^2} R_{f^\perp} d f^\perp = \|g\|^2 R_g d f^\perp, \tag{5}$$

since $R_{f^\perp} = R_g$. Therefore,

$$\begin{aligned} \|g\|^{-2} dg J &= R_g d f^\perp J = -R_g J \left(d f^\perp + 2\alpha_{f^\top}\right) \\ &= -J R_{Jg} \left(d f^\perp + 2\alpha_{f^\top}\right) \\ &= -J \left(R_g d f^\perp + 2\Pi_{g^\perp}(\alpha_{f^\top})\right) \\ &= -\|g\|^{-2} J dg - 2J\Pi_{g^\perp}\left(\alpha_{f^\top}\right) \end{aligned}$$

where $\Pi_{g^\perp} \colon T^\perp M \to T^\perp M \cap (\mathrm{span}_{\mathbb{C}}\{g\})^\perp$ is the orthogonal projection.

In particular, if the codimension is $p = 1$ we conclude that

$$dg J = -J dg,$$

that is, g is anti-holomorphic. Moreover, observe that since f is holomorphic, by (2) and (5) we have that $\ker dg = \Delta$ at each point. In other words, g is constant along the leaves of relative nullity of f and, locally, there is an anti-holomorphic immersion $\hat{f} \colon \hat{M}^{n-\nu} \to \mathbb{C}^{n+1}$ such that $g = \hat{f} \circ \pi$. We will always consider on $\hat{M}^{n-\nu}$ the Kähler metric induced by $\hat{f}$.

We have proved:

Proposition 1. *Let $f: M^n \to \mathbb{C}^{n+1}$ be a holomorphic isometric immersion of a Kähler Riemannian manifold. Then, on the open dense subset where the position vector f is not tangent, the map $g = i(f^\perp)$ is anti-holomorphic. Moreover, locally along the open dense subset which also has constant index of relative nullity ν, there is an anti-holomorphic isometric immersion $\hat{f}: \hat{M}^{n-\nu} \to \mathbb{C}^{n+1}$ such that $\hat{f} \circ \pi = g$.*

Observe that, as a consequence, the Gauss map $N: M^n \to \mathbb{CP}^n$ of f given by

$$N(x) = \text{span}_{\mathbb{C}} \left\{ \hat{f}^\perp(\pi(x)) \right\}$$

is anti-holomorphic; see [12].

Our purpose now is to describe f locally by means of the geometry of $\hat{f}$.

Theorem 2. *Let $\hat{f}: \hat{M}^{n-\nu} \to \mathbb{C}^{n+1}$ be an anti-holomorphic isometric immersion of a Kähler manifold with $\nu_{\hat{f}} = 0$ whose position vector is never tangent. Let $\mathcal{L}$ be the holomorphic vector subbundle given by*

$$\mathcal{L} = \text{span}_{\mathbb{C}} \left\{ \hat{f}^\perp \right\}^\perp \subset T_{\hat{f}}^\perp \hat{M}. \tag{6}$$

Then, the map $f: \mathcal{L} \to \mathbb{C}^{n+1}$ defined as

$$f(\xi) = i\left(\hat{f}^\perp(x)\right) + \xi, \quad \xi \in \mathcal{L}(x), \tag{7}$$

parametrizes, at regular points, a holomorphic Kähler hypersurface with constant index of relative nullity $\nu_f = \nu$. Conversely, any such hypersurface can be parametrized this way.

Proof. For the direct statement, observe first that f is holomorphic by Proposition 1. Moreover, since $\langle f, \hat{f} \circ \pi \rangle = 1$, we have that

$$0 = \langle df, \hat{f} \circ \pi \rangle + \langle f, d\hat{f} \circ \pi \rangle = \langle df, \hat{f} \circ \pi \rangle,$$

that is, $\hat{f} \circ \pi$ is normal to f. From the definition, it is clear that the fibers of $\mathcal{L}$ are contained in the relative nullity of f, and they must coincide since $\hat{f}$ is never tangent.

For the converse, we follow the arguments before Proposition 1 writing

$$f = i(g) + f^\top$$

and $\hat{f} \circ \pi = g$, where $g = i(f^\perp)$. Since g is normal to f and $\ker dg = \Delta$, by dimension reasons we conclude that the leaf of Δ through x is simply (contained in) a translation of $\mathcal{L}(\pi(x))$ defined by (6). Therefore, we set

$$f^\top = h \circ \pi + \xi$$

where $h \in \mathcal{L}^\perp$ and $\xi(x) \in \mathcal{L}(\pi(x))$. Again by dimension reasons, ξ parametrizes each leaf of $\mathcal{L}$ when x moves along a leaf of relative nullity. Now, differentiating $\langle f, g\rangle = 1$, we obtain $\langle f, dg\rangle = 0$. It follows that

$$i(\hat{f}) + h \in T^\perp \hat{M}. \tag{8}$$

By (6) and (8) we have that $i(\hat{f})+h$ and $\hat{f}^\perp$ are linearly dependent, say, $i(\hat{f})+h = \lambda \hat{f}^\perp$ Since h is tangent to f, taking the inner product with $\hat{f}$ yields $i(\hat{f}) + h = i(\hat{f}^\perp)$, as we wanted to prove. □

Remark 3. Let $\mathcal{H}_+$ and $\mathcal{H}_-$ denote the sets of hypersurfaces in $\mathbb{C}^N$ without relative nullity and whose position vectors are never tangent, that are holomorphic and anti-holomorphic, respectively. Since the roles of holomorphic and anti-holomorphic submanifolds can be reversed in the above arguments, the map defined on $\mathcal{H}_+ \cup \mathcal{H}_-$ given by

$$f \mapsto f^* = i(f^\perp)$$

is a bijection that swaps $\mathcal{H}_+$ with $\mathcal{H}_-$ such that $(f^*)^* = f$. In the case of holomorphic curves it was shown in [10] that this map is conformal.

We now compute the singular set and second fundamental form of the submanifold using the parametrization (7), the latter being completely determined by $A_{\hat{f}}$ by the holomorphicity of f. Let $P\colon\ T\hat{M} \to \Delta^\perp$ where

$$\Delta^\perp = \left(\mathcal{L} \oplus \mathrm{span}_{\mathbb{C}}\{\hat{f}\}\right)^\perp = \left(T\hat{M} \oplus \mathrm{span}_{\mathbb{C}}\{\hat{f}\}\right) \cap \left(\mathrm{span}_{\mathbb{C}}\{\hat{f}\}\right)^\perp$$

be the isomorphism given by

$$P(Z) = Z - \langle Z, \hat{f}\rangle i(\hat{f}^\perp) - \langle Z, J\hat{f}\rangle Ji(\hat{f}^\perp).$$

Proposition 4. *The singular set of f in the parametrization (7) is*

$$S = \left\{\xi \in \mathcal{L} : \hat{A}_{i(\hat{f}^\perp)+\xi} \text{ is singular}\right\},$$

where $\hat{A} = A^{\hat{f}}$. The shape operator of f in the direction $\hat{f}$ restricted to $\Delta^\perp$ is

$$A_{\hat{f}} = P\left(\hat{A}_{i(\hat{f}^\perp)+\xi}\right)^{-1} P^{-1}. \tag{9}$$

In particular, S is also the singular set of the submanifold itself.

Proof. Take $x \in \hat{M}^{n-\nu}$ and $\xi \in \mathcal{L}(x)$. From (7) we see that df_ξ is the identity on $\mathcal{L}(x)$. Notice that any vector transversal to $\mathcal{L}(x)$ at ξ can be written as $\psi_{*x} Z = d\psi_x(Z)$ for some $Z \in T_x\hat{M}$ and $\psi \in \Gamma(\mathcal{L})$ such that $\psi(x) = \xi$. Since $\hat{f}$ is always normal to f, we have

$$\begin{aligned}\left(f_{*\xi}(\psi_{*x}Z)\right)_{\Delta^\perp(x)} &= \left((f\circ\psi)_{*\xi_x}Z\right)_{\Delta^\perp(x)} = \left((i(\hat{f}^\perp)+\psi)_{*x}Z\right)_{\Delta^\perp(x)} \\ &= -P\left(\hat{A}_{i(\hat{f}^\perp)+\xi}Z\right),\end{aligned} \tag{10}$$

where a subspace as a subindex means to take its orthogonal projection. For the last equality, first observe that

$$\left((i(\hat{f}^{\perp})+\psi)_{*x}Z\right)_{\Delta^{\perp}(x)}=\left(-\hat{A}_{i(\hat{f}^{\perp})+\xi}Z+\lambda_1 i(\hat{f}^{\perp})+\lambda_2 Ji(\hat{f}^{\perp})\right)_{\Delta^{\perp}(x)}$$

and then use that $\hat{f}$ is always normal to f to compute the functions λ_j, $j=1,2$. The first claim now follows from the fact that the right-hand side of (10) depends only on Z and not on ψ.

The second part now follows since

$$\begin{aligned}f_{*\xi}\left(A_{\hat{f}}(\psi_{*x}Z)\right)&=-P\left((\hat{f}\circ\pi)_*(\psi_{*x}Z)\right)=-PZ\\&=P\left(\hat{A}_{i(\hat{f}^{\perp})+\xi}\right)^{-1}P^{-1}\left((f_{*\xi}(\psi_{*x}Z))_{\Delta^{\perp}(x)}\right),\end{aligned}$$

as we wanted. □

Recall that the *first normal space* of $\hat{f}$ at $x\in\hat{M}^{n-\nu}$ is the subspace of $N^1_{\hat{f}}(x)\subseteq T^{\perp}_x\hat{M}$ spanned by the image of the second fundamental form of $\hat{f}$ at x. Equivalently,

$$N^1_{\hat{f}}(x)=\left\{\delta\in T^{\perp}_x\hat{M}:\hat{A}_{\delta}=0\right\}^{\perp},$$

where the orthogonal complement is taken in the normal bundle.

Corollary 5. *Let $f\colon M^n\to\mathbb{C}^{n+1}$ and $\hat{f}\colon\hat{M}^{n-\nu}\to\mathbb{C}^{n+1}$ be as in Proposition 1. If M^n is complete, then $\hat{A}_{i(\hat{f}^{\perp})}$ is non-singular and $\mathcal{L}\subseteq(N^1_{\hat{f}})^{\perp}$.*

Proof. Assume that the conclusion does not hold. Hence, the polynomial

$$q(z)=\det\left(\hat{A}_{i(\hat{f}^{\perp})}+z\hat{A}_{\xi}\right)$$

has a complex root $u+iv$, associated to an eigenvector $U+iV\neq 0$ of the corresponding complexified endomorphism, that is,

$$\left(\hat{A}_{i(\hat{f}^{\perp})}+(u+iv)\hat{A}_{\xi}\right)(U+iV)=0.$$

But this is equivalent to

$$\hat{A}_{i(\hat{f}^{\perp})}U+u\hat{A}_{\xi}U-v\hat{A}_{\xi}V=0\quad\text{and}\quad\hat{A}_{i(\hat{f}^{\perp})}V+v\hat{A}_{\xi}U+u\hat{A}_{\xi}V=0.$$

In turn, using $J\hat{A}_{\delta}=-\hat{A}_{J\delta}=-\hat{A}_{\delta}J$, we easily see that this is equivalent to

$$\hat{A}_{i(\hat{f}^{\perp})+(uI-vJ)\xi}(U-JV)=0\quad\text{and}\quad\hat{A}_{i(\hat{f}^{\perp})+(uI+vJ)\xi}(U+JV)=0.$$

Since $\mathcal{L}$ is holomorphic and the leaves of relative nullity are complete, we get a contradiction with Proposition 4, because either $U-JV$ or $U+JV$ is non-zero. □

Remark 6. Observe that the previous result holds along each complete relative nullity leaf of f, even if the submanifold is not itself complete.

As an application of Theorem 2 we give a simple and direct proof of Abe's cylinder theorem [1].

Corollary 7. *Let $f\colon M^n \to \mathbb{C}^{n+1}$ be a holomorphic isometric immersion of a complete Kähler Riemannian manifold. If the index of relative nullity satisfies $\nu \geq n-1$ everywhere, then f is an $(n-1)$-cylinder, that is, $M^n = M_1^1 \times \mathbb{C}^{n-1}$, and there is $f_1\colon M_1^1 \to \mathbb{C}^2$ such that $f = f_1 \times Id$ splits.*

Proof. If f is not totally geodesic, for which the result trivially holds, by the hypothesis that $\hat{f}$ is an anti-holomorphic curve and by Corollary 5, we have

$$\mathcal{L} = \left(N_{\hat{f}}^1\right)^{\perp} \quad \text{and} \quad \operatorname{span}_{\mathbb{C}} \left\{ i(\hat{f}^{\perp}) \right\} = N_{\hat{f}}^1.$$

But since $\mathcal{L}$ is orthogonal to the position vector $\hat{f}$, we conclude that the first normal space is parallel since $0 = \langle \psi_* Z, \hat{f} \rangle = \langle \psi_* Z, i(\hat{f}^{\perp}) \rangle$ for any $\psi \in \Gamma(\mathcal{L})$. This parallelism implies that $\hat{f}$ reduces codimension, that is, it is an anti-holomorphic plane curve inside some $\mathbb{C}^2 \subset \mathbb{C}^{n+1}$, and $\mathcal{L}$ is the orthogonal complement of this plane. □

3. The parametrization in $\mathbb{CP}^N$

We show next that our parametrization in $\mathbb{C}^{n+1}$ can be used to obtain a similar parametrization for holomorphic hypersurfaces of $\mathbb{CP}^{n+1}$. The latter is cleaner than the former since it does not have the restriction about the position vectors to be nowhere tangent, and the bundle used to parametrize is the (projectivized) normal bundle itself and not a sub-bundle of it.

The condition in Theorem 2 that the position vector of a complex submanifold $f\colon M^n \to \mathbb{C}^{n+p}$ is never tangent is equivalent to the *cone f_c over f* to be an immersion, where the map $f_c\colon \mathbb{C}_* \times M^n \to \mathbb{C}^{n+p}$ is given by $f_c(z,x) = zf(x)$. Moreover, f has index of relative nullity ν if and only if f_c has index of relative nullity $\nu + 1$, and the position vector of the cone, that now is everywhere tangent, belongs to the relative nullity. Equivalently, the position vector of f is never tangent if and only if $f^{\sharp} = \hat{\pi} \circ f\colon M^n \to \mathbb{CP}^{n+p-1}$ is an immersion, where $\hat{\pi}\colon \mathbb{C}_*^N \to \mathbb{CP}^{N-1}$ denotes the projection to the quotient, and f and $f^{\sharp}$ have the same index of relative nullity. We conclude that to understand the submanifolds with constant relative nullity ν_0 in $\mathbb{CP}^N$ is equivalent to understand the cones in $\mathbb{C}^{N+1}$ with $\nu \equiv \nu_0 + 1$.

We claim that the latter are described as in (7), but without the term $i(\hat{f}^{\perp})$. Let $f_1\colon M_1^{n-1} \to \mathbb{C}^n \subset \mathbb{C}^{n+1}$ be the isometric immersion obtained as the intersection of a cone $f\colon M^n \to \mathbb{C}^{n+1}$ with constant index of relative nullity $\nu + 1$ with a hyperplane, say, $\mathbb{C}^n = \{z_{n+1} = 1\}$, so that f_1 has constant index of relative nullity ν and is never tangent. By Theorem 2, we have a parametrization of f_1 in $\mathbb{C}^n$ as

$$f_1(\xi) = i(\hat{f}_1^{\perp}) + \xi, \quad \xi \in \mathcal{L}_1.$$

Thus, $f_1 = (\eta + \xi, 1)$ in $\mathbb{C}^{n+1}$ where $\eta = i(\hat{f}_1^{\perp})$. Hence, we may parametrize f as

$$f(w, \xi) = w(\eta, 1) + \xi, \quad \xi \in \mathcal{L}_1, \; w \in \mathbb{C}_*.$$

Setting $\hat{f} = (\hat{f}_1, -1)$, we thus parametrize f as

$$f(\xi) = \xi, \quad \xi \in \mathcal{L},$$

where $\hat{f}\colon \hat{M}^{n-\nu} \to \mathbb{C}^{n+2}$ is never tangent, and

$$\mathcal{L} = \mathcal{L}_1 \oplus \operatorname{span}_{\mathbb{C}} \{(\eta, 1)\} = \operatorname{span}_{\mathbb{C}} \left\{\hat{f}^{\perp}\right\}^{\perp} \subset T_{\hat{f}}^{\perp} \hat{M}^{n-\nu}.$$

This proves our claim.

From the above description of the cones, we get for the immersion $f^{\sharp}$ a parametrization over the projectivized bundle $\mathbb{P}(\mathcal{L})$ of $\mathcal{L}$, namely, $f^{\sharp}\colon \mathbb{P}(\mathcal{L}) \to \mathbb{CP}^{n+1}$,

$$f^{\sharp}(\xi) = \xi, \quad \xi \in \mathbb{P}(\mathcal{L}). \tag{11}$$

Now, observe that $\mathcal{L}$ coincides with the normal space of $\hat{f}_c$, once we identify the fibers of the normal space of $\hat{f}_c$ when translated along the lines inside the cone that pass through 0 (we are allowed to do this because these are lines of relative nullity of $\hat{f}_c$). In other words, we have a natural identification between the normal space of $\hat{f}^{\sharp}$ and $\mathcal{L}$, and hence we can treat both as the same fiber bundle. In particular, the corresponding complex projectivized bundles are also identified: $\mathbb{P}(T_{\hat{f}^{\sharp}}^{\perp} \hat{M}) = \mathbb{P}(\mathcal{L})$. These are holomorphic fiber bundles of dimension n with $\mathbb{CP}^{\nu}$ fibers. We conclude from (11) the following.

Theorem 8. *Let $\hat{f}\colon \hat{M}^{n-\nu} \to \mathbb{CP}^{n+1}$ be an anti-holomorphic isometric immersion of a Kähler manifold with vanishing relative nullity. Then, the map $f : \mathbb{P}(T_{\hat{f}}^{\perp} \hat{M}) \to \mathbb{CP}^{n+1}$ defined as*

$$f(\xi) = \xi,$$

parametrizes, at regular points, a holomorphic Kähler hypersurface with constant index of relative nullity ν. Conversely, any such hypersurface can be parametrized this way.

We point out that the holomorphicity hypothesis in the converse is redundant when the submanifold has relative nullity. It was shown in [8] that any isometric immersion of a Kähler manifold into $\mathbb{CP}^{N}$ with positive index of relative nullity must be holomorphic.

Remark 9. Taking the Gauss map is an involution on $\bar{\mathcal{H}}_+ \cup \bar{\mathcal{H}}_-$ that swaps $\bar{\mathcal{H}}_+$ with $\bar{\mathcal{H}}_-$, where $\bar{\mathcal{H}}_+$ and $\bar{\mathcal{H}}_-$ denote the sets of holomorphic and anti-holomorphic hypersurfaces of $\mathbb{CP}^{n+1}$ with vanishing relative nullity. As opposed to the $\mathbb{C}^{N}$ case, here there is no restriction on the position vectors; see Remark 3.

References

[1] Abe, K.: A complex analogue of Hartman–Nirenberg cylinder theorem. J. Differ. Geom. **7**, 453–460 (1972)

[2] Abe, K.: Applications of a Riccati type differential equation to Riemannian manifolds with totally geodesic distributions. Tôhoku Math. J. **25**, 425–444 (1973)

[3] Dajczer, M., Florit, L.: On conformally flat submanifolds. Comm. Anal. Geom. **4**, 261–284 (1996)

[4] Dajczer, M., Florit, L., Tojeiro, R.: On deformable hypersurfaces in space forms. Ann. Mat. Pura Appl. **174**, 361–390 (1998)

[5] Dajczer, M., Gromoll, D.: Gauss parametrizations and rigidity aspects of submanifolds. J. Differ. Geom. **22**, 1–12 (1985)

[6] Dajczer, M., Gromoll, D.: Real Kaehler submanifolds and uniqueness of the Gauss map. J. Differ. Geom. **22**, 13–28 (1985)

[7] Dajczer, M., Gromoll, D.: Euclidean hypersurfaces with isometric Gauss maps. Math. Z. **191**, 201–205 (1986)

[8] Dajczer, M., Rodrguez, L.: On isometric immersions into complex space forms. Math. Ann. **299**, 223–230 (1994)

[9] Dajczer, M., Tenenblat, K.: Rigidity for complete Weingarten hypersurfaces. Trans. AMS **312**, 129–140 (1989)

[10] Dajczer, M., Tojeiro, R.: All superconformal surfaces in R^4 in terms of minimal surfaces. Math. Z. **261**, 869–890 (2009)

[11] Dajczer, M., Tojeiro, R.: Hypersurfaces with a constant support function in spaces of constant sectional curvature. Arch. Math. **60**, 296–299 (1993)

[12] Nomizu, K., Smith, B.: Differential geometry of complex hypersurfaces, II. J. Math. Soc. Japan **20**, 498–521 (1968)

[13] Sbrana, V.: Sulla varietá ad $n-1$ dimensioni deformabili nello spazio euclideo ad n dimensioni. Rend. Circ. Mat. Palermo **27**, 1–45 (1909)