ORBIFOLD FIBRATIONS OF ESCHENBURG SPACES

LUIS A. FLORIT AND WOLFGANG ZILLER

Compact manifolds that admit a metric with positive sectional curvature are still poorly understood. In particular, there are few known obstructions for the existence of such metrics. By Bonnet-Meyers the fundamental group must be finite, by Synge it has to be 0 or $\mathbb{Z}_2$ in even dimensions, and the $\hat{A}$-genus must vanish when the manifold is spin. For non-negative curvature, besides some results on the structure of the fundamental group, we have Gromov's Betti number theorem which states that they are bounded by a constant that only depends on the dimension. In fact, there is no known obstruction that distinguishes the class of simply connected manifolds which admit positive curvature from the ones that admit non-negative curvature.

It is therefore surprising that there are very few known examples with positive curvature. They all arise as quotients of a compact Lie group, endowed with a left invariant metric, by a subgroup of isometries acting freely. They consist, apart from the rank one symmetric spaces, of certain homogeneous spaces in dimensions $6, 7, 12, 13$ and 24 due to Berger [Be], Wallach [Wa], and Aloff-Wallach [AW], and of biquotients in dimensions $6, 7$ and 13 due to Eschenburg [E1],[E2] and Bazaikin [Ba].

A different method of searching for new positively curved examples is suggested by another property that many (but not all) of the known examples share: they are the total space of a fiber bundle where the projection onto the base is a Riemannian submersion. It is therefore suggestive to look for new examples which admit fiber bundle structures. Weinstein ([We]) studied this question by considering Riemannian submersions with totally geodesic fibers, such that the sectional curvatures spanned by a horizontal and a vertical vector are positive. Even this weaker condition on a fiber bundle, called fatness, is already strong ([DR],[Zi]).

The concept of fatness and any further curvature computations can be done in a larger category of bundles, where the spaces involved are orbifolds and the bundle structure is an orbifold one. Even if one is only interested in manifolds this is an important generalization. In fact, one can now give many of the other known examples of positive curvature metrics, apart from the rank one symmetric spaces, an orbifold bundle structure as well; see Section 2 for a summary. The only case where such an orbifold bundle structure was not known, until now, is the family of (generic) Eschenburg biquotients $E_{p,q} = \mathrm{SU}(3)/\!/\,\mathrm{S}^1$, where the $\mathrm{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ action on $\mathrm{SU}(3)$ is given by

$$z \cdot g = z^p \, g \, \overline{z}^{\,q},$$

with $p, q \in \mathbb{Z}^3$, $\sum p_i = \sum q_i$, and $z^p := \mathrm{diag}(z^{p_1}, z^{p_2}, z^{p_3}) \in \mathrm{U}(3)$. This biquotient is an orbifold if and only if $p - q_\sigma \neq 0$ for all permutations $\sigma \in S_3$, where we have set

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$q_\sigma = (q_{\sigma(1)}, q_{\sigma(2)}, q_{\sigma(3)})$. If one has that $\gcd(p_i - q_j, p_{i'} - q_{j'}) = 1$, for all $i \neq i', j \neq j'$, the quotient is a manifold. Further conditions must be satisfied for an Eschenburg metric to have positive curvature; see Section 1.

The purpose of this paper is to study all isometric circle actions on Eschenburg manifolds (and more generally orbifolds) which act almost freely, i.e., their isotropy groups are finite, and thus give rise to principal orbifold bundle structures. One easily sees that they all indeed admit such actions and we will examine in detail their geometric properties. In particular, we obtain a large new family of 6-dimensional orbifolds with positive sectional curvature and with small singular locus.

By [GSZ], an isometric circle action on $E_{p,q}$ is given by a biquotient action on SU(3) that commutes with the original one or, equivalently, a $\mathrm{T}^2 = \mathrm{S}^1 \times \mathrm{S}^1$ biquotient action on SU(3) that contains the original circle as a subgroup. So, given $a, b \in \mathbb{Z}^3$ with $\sum a_i = \sum b_i$, we define a circle action $\mathrm{S}^1_{a,b}$ on $E_{p,q}$ by

$$w \cdot [g] = [w^a g\, \overline{w}^b], \quad w \in \mathrm{S}^1 .$$

The projection onto the quotient $\hat{\pi} : E_{p,q} \to O^{a,b}_{p,q}$ is then an orbifold principal bundle if this action is almost free. We will show:

Theorem A. *The circle action $\mathrm{S}^1_{a,b}$ on $E_{p,q}$ is almost free if and only if*

$$(p - q_\sigma) \ \ \text{and} \ \ (a - b_\sigma) \ \ \text{are linearly independent, for all } \sigma \in S_3.$$

The quotient $O^{a,b}_{p,q}$ is then an orbifold whose singular locus is the union of at most nine orbifold 2-spheres and six points that are arranged according to the schematic diagram in Figure 1.

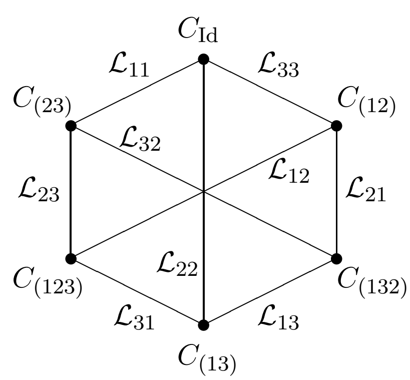

Figure 1. The structure of the singular locus.

The lift of the orbifold 2-spheres to SU(3) consists of the nine copies of U(2) inside SU(3) given by

$$\mathrm{U}(2)_{ij} = \left\{ \tau_i \begin{bmatrix} A & 0 \\ 0 & \det \overline{A} \end{bmatrix} \tau_j \ : \ A \in \mathrm{U}(2) \right\} , \quad 1 \le i, j \le 3,$$

where $\tau_r \in \mathrm{O}(3)$ is the linear map that interchanges the $r^{\underline{th}}$ vector of the canonic basis with the third one. The lift of the six singular points consists of the six copies of T^2 given by

$$\mathrm{T}^2_\sigma = \sigma \operatorname{diag}(z, w, \bar{z}\bar{w}),$$

where $\sigma \in S_3 \subset \mathrm{O}(3)$ is a permutation matrix. We define the *parity* of each one of these six singular points to be the parity of the corresponding σ. Clearly each $\mathrm{U}(2)$ contains two T^2's and each T^2 is contained in three $\mathrm{U}(2)$'s. They are also arranged according to Figure 1, where edges correspond to the $\mathrm{U}(2)$'s and vertices to the T^2's.

For the lift of the singular locus to $E_{p,q}$ under the fibration $\hat{\pi}$, the edges in Figure 1 represent totally geodesic lens spaces, and the vertices are closed geodesics. If $E_{p,q}$ is smooth, the lens spaces are smooth as well. In Section 3 we will also determine the isotropy groups corresponding to these lens spaces and closed geodesics, which can also be interpreted as the orbifold groups of the orbifold quotient. They are constant along each of these closed geodesics, and along each lens space (outside the closed geodesics).

In Section 4 we examine the question of how to minimize the singular locus in $O^{a,b}_{p,q}$ and its orbifold groups. There exist some Eschenburg spaces which admit a free circle action, but in general the most one can hope for is an isometric circle action such that the singular locus of the quotient consists of a single point. It turns out that there is a topological obstruction to the existence of such an action.

The most basic topological invariant of an Eschenburg space is the order h of the cyclic group $H^4(E_{p,q}, \mathbb{Z})$. We also associate to $E_{p,q}$ the integers (mod h) denoted by $\alpha(\sigma, \epsilon_1, \epsilon_2) \in \mathbb{Z}_h$, where $\sigma \in S_3$ and $\epsilon_1, \epsilon_2 = \pm 1$, that only depend on p, q; see (4.5) for an explicit formula. We show:

Theorem B. *Let $E_{p,q}$ be an Eschenburg manifold equipped with an Eschenburg metric. Then, there exists an isometric circle action on $E_{p,q}$ whose singular locus is composed of at most 3 points with the same parity if and only if $\alpha(\sigma, \epsilon_1, \epsilon_2) = 0$ for some choice of $\sigma \in S_3$, $\epsilon_1, \epsilon_2 = \pm 1$.*

This implies in particular that a generic Eschenburg space does not admit an isometric circle action with only one singular point. However, it is easy to find examples for which this is the case; see Section 4. Observe that all such examples have non-negative curvature, since this holds for the Eschenburg metric in general.

On the other hand, for positive curvature the situation is different. To illustrate this, we study in detail the case of general cohomogeneity one Eschenburg manifolds, that is, $E_d = E_{(1,1,d),(0,0,d+2)}$, $d \geq 0$, which have positive curvature when $d > 0$. For $d \leq 2$, it is known that E_d admits a free isometric circle action, in fact even free actions by $\mathrm{SO}(3)$ ([Sh]). For the remaining cases, we prove the following.

Theorem C. *Let E_d be any cohomogeneity one Eschenburg manifold, $d \geq 3$, equipped with a positively curved Eschenburg metric. Then:*

i) There is no isometric S^1 action on E_d with only one singular point.

In the following particular examples the singular locus of the isometric circle action $\mathrm{S}^1_{a,b}$ on E_d consists of:

ii) *Two points with equal orbifold groups* $\mathbb{Z}_{d+1}$ *if* $a = (0, -1, 1)$ *and* $b = (0, 0, 0)$;

iii) *Two points with equal orbifold groups* $\mathbb{Z}_{d-1}$ *if* $a = (0, 1, 1)$ *and* $b = (2, 0, 0)$*, and* 3 *does not divide* $d - 1$*. If it does divide, we get in addition the orbifold 2-sphere joining these two singular points, with orbifold group* $\mathbb{Z}_3$;

iv) *A smooth totally geodesic 2-sphere with orbifold group* $\mathbb{Z}_{d-1}$ *if* $a = (0, 1, 1)$ *and* $b = (0, 0, 2)$.

Hence, an interesting open question is whether part (i) in Theorem C holds more generally for all positively curved Eschenburg manifolds. Computer experiments support an affirmative answer: *There is no positively curved Eschenburg manifold with* $|H^4(E_{p,q}, \mathbb{Z})| \leq 10^6$*, altogether* 10.085.359.999 *spaces, that admits an isometric circle action whose singular locus is a single point.* If they do not exist, it would be interesting to understand the phenomena behind this difference between non-negative and positive curvature.

Finally, observe that the most regular orbifold we obtain in Theorem C is given by $\operatorname{diag}(z, zw, z^3 w) \backslash \operatorname{SU}(3) / \operatorname{diag}(1, 1, \overline{z}^5 \overline{w}^2)$, which is a compact 6-dimensional positively curved orbifold which has only two singular points with orbifold groups $\mathbb{Z}_2$.

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1. Preliminaries

Recall that an orbifold is a topological space which locally is the quotient of an open set $U \subset \mathbb{R}^n$ under the effective action of a finite group Γ that fixes $p \in U$. The group Γ is called the orbifold group at the projection of p in U/Γ, and the required natural compatibility conditions for 2 overlapping orbifold charts implies that the orbifold group is well defined. An orbifold metric is a Riemannian metric on each chart U such that Γ acts isometrically. In many ways orbifolds can be treated just like manifolds. Local geometric calculations can be done with the smooth metric since all geometric objects are invariant under isometries. The simplest examples of orbifolds are manifolds divided by a finite group (so called good orbifolds). More generally, if a compact Lie group G acts isometrically on a Riemannian manifold M such that all isotropy groups are finite (so called almost free actions), then M/G is an orbifold, as follows immediately from the slice theorem for the group action. Moreover, in this case the orbifold groups are the isotropy groups, divided by the ineffective kernel. In our case, orbifolds will be obtained as quotients of Eschenburg spaces under circle actions with finite isotropy groups.

We now introduce some notations that will be helpful and discuss general properties of Eschenburg spaces; see [E1], [E2]. We denote the diagonal matrices $\mathcal{D} = \mathbb{C}^3 \subset \mathbb{C}^{3\times 3}$ by $x = (x_1, x_2, x_3) = \operatorname{diag}(x) \in \mathbb{C}^{3\times 3}$. For an element in the symmetric group S_3 that takes $i \to j \to k \to i$, or $i \to j \to i$, we use the notation (ijk), or (ij), respectively. We have a natural action of S_3 on $\mathcal{D}$ defined by $x_\sigma = \operatorname{diag}(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$, $\sigma \in S_3$, $x \in \mathcal{D}$. If $z \in \mathrm{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $p \in \mathbb{Z}^3 \subset \mathbb{R}^3$, we denote by $z^p = \operatorname{diag}(z^{p_1}, z^{p_2}, z^{p_3}) \in \mathrm{U}(3)$.

Observe that

$$G = \{(g_1, g_2) \in \mathrm{U}(3) \times \mathrm{U}(3) : \det g_1 = \det g_2\}$$

acts on SU(3) by $(g_1, g_2)\cdot g = g_1\, g\, g_2^{-1}$. The group G has a maximal torus $\mathrm{T}^5 = (\mathcal{D}\times\mathcal{D})\cap G$, whose Lie algebra is $\mathfrak{t} = (\mathcal{D} \times \mathcal{D}) \cap \mathfrak{g}$. Let $\mathfrak{t}_{\mathbb{Z}}$ be the lattice in $\mathfrak{t}$ given by $\mathfrak{t}_{\mathbb{Z}} = \mathfrak{t} \cap (2\pi i\mathbb{Z}^3 \times 2\pi i\mathbb{Z}^3)$, that we identify with

$$\mathfrak{t}_{\mathbb{Z}} = \{(p, q) \in \mathbb{Z}^3 \times \mathbb{Z}^3 : \operatorname{tr} p = \operatorname{tr} q\}$$

via $(p, q) \to (2\pi ip, 2\pi iq)$.

For each $(p, q) \in \mathfrak{t}_{\mathbb{Z}}$ we define an S^1 action on SU(3) by

$$z \cdot g = z^p g\, \overline{z}^{\,q}, \quad z \in \mathrm{S}^1, \quad g \in \mathrm{SU}(3). \tag{1.1}$$

This action is easily seen to be almost free if and only if

$$p - q_\sigma \neq 0, \quad \forall\, \sigma \in S_3, \tag{1.2}$$

since this is clearly the case only when the action is free on the Lie algebra level. In this situation, the quotient is a 7-dimensional orbifold, which we call the *Eschenburg orbifold* $E_{p,q}$, that comes with the projection $\pi = \pi_{p,q}\colon \mathrm{SU}(3) \to E_{p,q}$. Furthermore, the action is free if and only if

$$\gcd(p_1 - q_{\sigma(1)}, p_2 - q_{\sigma(2)}) = 1, \quad \forall\, \sigma \in S_3, \tag{1.3}$$

which is equivalent to $\gcd(p_i - q_j, p_{i'} - q_{j'}) = 1$, for all $i \neq i', j \neq j'$. In this case, $E_{p,q}$ is a smooth 7–dimensional manifold, called the *Eschenburg manifold* $E_{p,q}$. Finally, the action is effective only when

$$\gcd(\{p_i - q_j : 1 \leq i, j \leq 3\}) = 1. \tag{1.4}$$

Indeed, $z \in \mathrm{S}^1$ fixes $g \in \mathrm{SU}(3)$ if and only if z^p is conjugate to z^q and if this is true for all g, we necessarily have $z^p = z^q = \lambda\,\mathrm{Id}$ for some λ. Again, one can take $1 \leq i \leq 2$ in the above since $\operatorname{tr} p = \operatorname{tr} q$.

The *Eschenburg metric* on $E_{p,q}$ is the submersion metric obtained by scaling the biinvariant metric on SU(3) in the direction of $\mathrm{U}(2)_{jj} \subset \mathrm{SU}(3)$ for some $1 \leq j \leq 3$, and it has positive sectional curvature if and only if, for all $1 \leq i \leq 3$,

$$p_i \notin [\min(q_1, q_2, q_3), \max(q_1, q_2, q_3)], \quad \text{or} \quad q_i \notin [\min(p_1, p_2, p_3), \max(p_1, p_2, p_3)]. \tag{1.5}$$

If this condition is satisfied, we will call $E_{p,q}$ a *positively curved Eschenburg space.* Since the proof of this fact is a Lie algebra computation, it still remains valid if we consider, more generally, Eschenburg orbifolds. Thus in the orbifold category there exists a much larger class of compact positively curved examples. In fact, any six integers satisfying (1.5) will determine one, since (1.5) implies (1.2).

Observe that, since (p, q), $(-p, -q)$ and $(p + k\,\mathrm{Id}, q + k\,\mathrm{Id})$, $k \in \mathbb{Z}$, induce the same action and since $\operatorname{tr} p = \operatorname{tr} q$, an Eschenburg orbifold is determined by four integers only. In fact, $\{p_{\tau(1)} - q_{\sigma(2)}, p_{\tau(1)} - q_{\sigma(3)}, p_{\tau(2)} - q_{\sigma(1)}, p_{\tau(3)} - q_{\sigma(1)}\}$ defines an Eschenburg orbifold for fixed given permutations $\tau, \sigma \in S_3$. Furthermore, notice that $E_{p,q}$ has positive sectional curvature if and only if two rows or two columns of the matrix $A_{ij} = p_i - q_j$ contain integers with the same sign, and it is a manifold if and only if any two entries not in the same row or column are relatively prime.

There are two natural subclasses of Eschenburg manifolds. One is the family of cohomogeneity two Eschenburg spaces corresponding to $(p, q) = ((c, d, e), (0, 0, c+d+e))$ with $\gcd(c, d) = \gcd(d, e) = \gcd(e, c) = 1$. They admit an isometric action of $\mathrm{T}^2 \times \mathrm{SU}(2)$ such that the quotient is two-dimensional. A further subclass is the family of cohomogeneity one Eschenburg spaces corresponding to $(p, q) = ((1, 1, d), (0, 0, d + 2))$ which admit an isometric action of $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{S}^1$ such that the quotient is one-dimensional. In general, Eschenburg spaces admit an isometric action of T^3 such that the quotient is four-dimensional. In [GSZ] it was shown that these three groups are indeed the identity component of the isometry group. In particular, the isometry has rank three in all cases.

The only homological invariant that varies for different Eschenburg manifolds is the order $h = h(E_{p,q})$ of the cohomology group $H^4(E_{p,q}, \mathbb{Z}) = \mathbb{Z}_h$. This integer is given by

$$h = |p_1p_2 + p_1p_3 + p_2p_3 - q_1q_2 - q_1q_3 - q_2q_3|$$

(see [E2]), which can be rewritten, up to sign, as

$$h = (p_{\tau(1)} - q_{\sigma(2)})(p_{\tau(1)} - q_{\sigma(3)}) - (p_{\tau(2)} - q_{\sigma(1)})(p_{\tau(3)} - q_{\sigma(1)}), \tag{1.6}$$

for any permutations $\tau, \sigma \in S_3$. Moreover, the integer h must be odd for Eschenburg manifolds (see [Kr], Remark 1.4). Notice also that, if $E_{p,q}$ is positively curved, we can assume that $p_{\tau(1)} - q_{\sigma(2)}$, $p_{\tau(1)} - q_{\sigma(3)}$, $p_{\tau(2)} - q_{\sigma(1)} > 0$ and $p_{\tau(3)} - q_{\sigma(1)} < 0$, and hence there are only finitely many positively curved Eschenburg manifolds for a given order h ([CEZ]).

Finally, we introduce notations for a few orbifolds that we will need. Given $p, q, d \in \mathbb{Z}$, $d \neq 0$, the *lens space* $L(p, q, d)$ is the quotient

$$L(p, q, d) := \mathbb{S}^3/\mathbb{Z}_d,$$

where the action of $\mathbb{Z}_d = \{\xi \in \mathrm{S}^1 : \xi^d = 1\}$ on $\mathbb{S}^3 = \{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 = 1\}$ is given by

$$\xi \cdot (x, y) = (\xi^p x, \xi^q y). \tag{1.7}$$

This orbifold is a smooth manifold when $\gcd(p, d) = \gcd(q, d) = 1$. When there is no restriction on ξ, we get the *weighted complex projective space*

$$\mathbb{CP}^1[p, q] := \mathbb{S}^3/\,\mathrm{S}^1,$$

that is, the S^1 action is still given by (1.7), for $\xi \in \mathrm{S}^1$. For convenience, we will still refer to the orbifold

$$L(p, q, 0) := \mathbb{S}^1 \times \mathbb{CP}^1[p, q]$$

as a lens space.

2. Known orbifold fibrations

In this section we collect, for the convenience of the reader, the known fibrations and orbifold fibrations where the total space is one of the known compact simply connected positively curved manifolds and the projection is a Riemannian submersion. Here we will leave out the rank one symmetric spaces with their well known Hopf fibrations.

We start with the homogeneous examples G/H, which have been classified in [AW] and [BB]. All except for one admit homogeneous fibrations of the form $K/H \to G/H \to G/K$ coming from inclusions $H \subset K \subset G$:

1. The Wallach flag manifolds each of which is the total space of the following fibrations:

- $$\mathbb{S}^2 \to \mathrm{SU}(3)/\,\mathrm{T}^2 \to \mathbb{CP}^2,$$
- $$\mathbb{S}^4 \to \mathrm{Sp}(3)/\,\mathrm{Sp}(1)^3 \to \mathbb{HP}^2,$$
- $$\mathbb{S}^8 \to \mathrm{F}_4\,/\,\mathrm{Spin}(8) \to \mathrm{Ca}\mathbb{P}^2.$$

2. The Aloff-Wallach examples $E_{0,q} = W_{q_1,q_2} = \mathrm{SU}(3)/\,\mathrm{diag}(z^{q_1}, z^{q_2}, \bar{z}^{q_1+q_2})$, which have positive curvature when $q_1q_2(q_1+q_2) \neq 0$, admit two kinds of fibrations:

- $$\mathbb{S}^1 \to W_{q_1,q_2} \to \mathrm{SU}(3)/\,\mathrm{T}^2,$$

and a lens space fibration

- $$\mathbb{S}^3/\mathbb{Z}_{q_1+q_2} \to W_{q_1,q_2} \to \mathbb{CP}^2,$$

where the fiber is $\mathrm{U}(2)/\,\mathrm{diag}(z^{q_1}, z^{q_2}) = \mathrm{SU}(2)/\mathbb{Z}_{q_1+q_2}$.

3. The Berger example $\mathrm{SU}(5)/\,\mathrm{Sp}(2)\cdot\mathrm{S}^1$ admits a fibration

- $$\mathbb{RP}^5 \to \mathrm{SU}(5)/\,\mathrm{Sp}(2)\cdot\mathrm{S}^1 \to \mathbb{CP}^4,$$

where the fiber is $\mathrm{U}(4)/\,\mathrm{Sp}(2)\cdot\mathrm{S}^1 = \mathrm{SU}(4)/\,\mathrm{Sp}(2)\cdot\mathbb{Z}_2 = \mathrm{SO}(6)/\,\mathrm{O}(5) = \mathbb{RP}^5$.

4. Finally for the homogeneous category, we have the Berger space $\mathrm{SO}(5)/\,\mathrm{SO}(3)$. This space is special since $\mathrm{SO}(3)$ is maximal in $\mathrm{SO}(5)$ and hence does not admit a homogeneous fibration. In [GKS] it was shown that $\mathrm{SO}(5)/\,\mathrm{SO}(3)$ is diffeomorphic to the total space of an $\mathbb{S}^3$ bundle over $\mathbb{S}^4$, but the fibration is not a Riemannian submersion of the positively curved metric. It was observed though by K. Grove and the last author that the subgroup $\mathrm{SU}(2) \subset \mathrm{SO}(4) \subset \mathrm{SO}(5)$ acts with only finite isotropy groups and hence gives rise to an orbifold fibration

- $$\mathbb{S}^3 \to \mathrm{SO}(5)/\,\mathrm{SO}(3) \to \mathbb{S}^4.$$

To see this, one observes that the action by $\mathrm{SO}(4) \subset \mathrm{SO}(5)$ has cohomogeneity one and from the group diagram of this action (see [GWZ]) it follows that there is a codimension two submanifold, one of the singular orbits $\mathrm{SO}(4)/\,\mathrm{O}(2)$, along which the isotropy group is a cyclic group $\mathbb{Z}_3$ and away from this orbit, the action is free. The quotient is homeomorphic to $\mathbb{S}^4$ and the metric is smooth, except along a Veronese embedding $\mathbb{RP}^2 \subset \mathbb{S}^4$ where the metric has an angle $2\pi/3$ normal to $\mathbb{RP}^2$.

The remaining known examples of positively curved compact manifolds are given by biquotients:

5. A free action of T^2 on $\mathrm{SU}(3)$ gives the inhomogeneous positively curved flag manifold discovered by Eschenburg, $\mathrm{SU}(3)/\!/\,\mathrm{T}^2 := \mathrm{diag}(z, w, zw)\backslash\,\mathrm{SU}(3)/\,\mathrm{diag}(1, 1, z^2w^2)^{-1}$. It admits the fibration

- $$\mathbb{S}^2 \to \mathrm{SU}(3)/\!/\,\mathrm{T}^2 \to \mathbb{CP}^2$$

which one obtains by extending the action of T^2 to the $\mathrm{U}(2)$ action on $\mathrm{SU}(3)$ given by $A \cdot B = \mathrm{diag}(A, \det A)\, B\ \mathrm{diag}(1, 1, \det A^2)^{-1}$. To see that the base $\mathrm{SU}(3))/\!/\,\mathrm{U}(2)$ is $\mathbb{CP}^2$, one first uses the identification $\mathrm{SU}(3)/\,\mathrm{SU}(2) \cong \mathbb{S}^5$ given by $[g] \mapsto e_3 g$. The remaining circle action of the center of $\mathrm{U}(2)$ then becomes $(v_1, v_2, v_3) \to (\bar{z}^2 v_1, \bar{z}^2 v_2, z^2 v_3)$ which effectively is conjugate to the Hopf action with quotient $\mathbb{CP}^2$.

6. For the 7-dimensional Eschenburg spaces, there are 3 subfamilies which are known to admit fibrations. One is the family of Aloff-Wallach spaces discussed above. A second one arises from the inhomogeneous flag manifold which gives rise to a fibration

- $$\mathbb{S}^1 \to E_{p,q} \to \mathrm{SU}(3)/\!/\,\mathrm{T}^2,$$

for every (p, q) of the form $(p, q) = ((p_1, p_2, p_1 + p_2), (0, 0, 2p_1 + 2p_2))$.

The third subfamily consists of the cohomogeneity two Eschenburg manifolds defined by $(p, q) = ((p_1, p_2, p_3), (0, 0, \bar{p})$ where $\bar{p} = p_1 + p_2 + p_3$ and the p_i's are pairwise relatively prime. They admit an action by $\mathrm{SU}(2)$ acting on the right since it commutes with the circle action. As was observed in [BGM], it gives rise to an orbifold fibration

- $$F \to E_{p,q} \to \mathbb{CP}^2[p_2 + p_3, p_1 + p_3, p_1 + p_2]\ ,$$

where the fiber F is $\mathbb{RP}^3$ if all p_i's are odd, and $F = \mathbb{S}^3$ otherwise. Here the base is a 2-dimensional weighted complex projective space. Indeed, if one first uses the identification $\mathrm{SU}(3)/\,\mathrm{SU}(2) \cong \mathbb{S}^5$ as above, the remaining circle action becomes $(v_1, v_2, v_3) \to (z^{\bar{p}-p_1} v_1, z^{\bar{p}-p_2} v_2, z^{\bar{p}-p_3} v_3)$. This also shows that the coordinate points in the weighted projective space correspond to $\mathrm{U}(2)$'s inside $\mathrm{SU}(3)$, and the isotropy groups are cyclic of order $\bar{p} - p_i$. On the other hand, we also have that $\gcd(\bar{p} - p_i, \bar{p} - p_j) =: a > 1$ for at least one pair i, j and hence the orbifold set also contains at least one $\mathbb{CP}^1 \subset \mathbb{CP}^2$ with orbifold group $\mathbb{Z}_a$.

7. Finally, we have the Bazaikin biquotients $B_q = \mathrm{SU}(5)/\!/\,\mathrm{Sp}(2) \cdot \mathrm{S}^1$ given by

$$B_q = \mathrm{diag}(z^{q_1}, \ldots, z^{q_5})\backslash\,\mathrm{SU}(5)/\,\mathrm{diag}(A, z^{\,q_0})^{-1},$$

where $A \in \mathrm{Sp}(2) \subset \mathrm{SU}(4) \subset \mathrm{SU}(5)$, $q = (q_1, \cdots, q_5)$ is an ordered set of odd integers and $q_0 = \sum q_i$. In this case we obtain the fibration

- $$\mathbb{RP}^5 \to B_q \to \mathbb{CP}^4[q_0 - q_1, \cdots, q_0 - q_5].$$

by enlarging the biquotient action of $\mathrm{Sp}(2)\cdot \mathrm{S}^1$ to one of $\mathrm{SU}(4)\cdot \mathrm{S}^1$. The isotropy groups and the weights are obtained as in the case 5 above.

It is an interesting fact that there are no other two tori that act freely on SU(3) besides the ones in fibrations 2 and 6; see [E2]. Besides some of the rank one symmetric spaces, it thus only remains the general family of Eschenburg spaces, which was not known, until now, to admit an orbifold fibration.

3. Circle orbifold fibrations

We now search for an almost free S^1 action on $E_{p,q}$. We also require that the circle action acts isometrically in the positively curved Eschenburg metrics. As mentioned in Section 1, the isometry group of a positively curved Eschenburg space has rank 3 and hence any circle action is conjugate to one lying in a maximal torus. This maximal torus can be chosen to be the 3-torus induced by the biquotient action of the maximal torus $\mathrm{T}^5 \subset G \subset \mathrm{U}(3)\times \mathrm{U}(3)$. We are thus forced to consider circle actions induced by a circle inside this 3-torus. This amounts to finding an S^1 action on SU(3) that commutes with the one that defines $E_{p,q}$, in such a way that they give together a $\mathrm{T}^2 = \mathrm{S}^1\times \mathrm{S}^1 \subset \mathrm{T}^5$ almost free action on SU(3).

To describe this $\mathrm{T}^2 \subset \mathrm{T}^5$, let $(a, b) \in \mathfrak{t}_{\mathbb{Z}}$, and write the T^2 action as

$$(z, w)\cdot g = w^a z^p g\, \overline{z}^q \overline{w}^b, \qquad z, w \in \mathrm{S}^1,\ g \in \mathrm{SU}(3). \tag{3.1}$$

The action is almost free if and only if it is free at the Lie algebra level. Since the Lie algebra of T^2 is spanned by $i(p, q)$ and $i(a, b) \in \mathfrak{u}(3)\times \mathfrak{u}(3)$, this holds if and only if there are no $x, y \in \mathbb{R}$ such that $xp + ya$ is conjugate to $xq + yb$. Since both are diagonal matrices, the almost free property is then equivalent to

$$(a - b_\sigma) \ \text{ and } \ (p - q_\sigma) \ \text{ are linearly independent}, \ \forall\, \sigma \in S_3. \tag{3.2}$$

This T^2 action on SU(3) defines a circle action on $E_{p,q}$, which we denote by $\mathrm{S}^1_{a,b}$ and its quotient by $O^{a,b}_{p,q}$. This circle action is clearly almost free if and only if the T^2 action is almost free, and, in this case, $O^{a,b}_{p,q}$ is an orbifold. Recall that we have the projection $\pi_{p,q}\colon \mathrm{SU}(3) \to E_{p,q}$ and we define a further projection $\hat{\pi}^{a,b}_{p,q}\colon E_{p,q} \to O^{a,b}_{p,q}$. If clear from context, we also denote these projections simply by π and $\hat{\pi}$, respectively. Our purpose is to study the geometry of the orbifold fibration

$$\mathrm{S}^1 \to E_{p,q} \to O^{a,b}_{p,q}.$$

From now on we assume that (p, q) and (a, b) satisfy (3.2). Notice that this implies that (1.2) holds for $n(p, q) + m(a, b)$, for all $(n, m) \in \mathbb{Z}^2 \setminus \{0\}$. Thus $E_{a,b}$ is also an Eschenburg orbifold and we obtain a symmetry in the process and the commutative diagram of orbifold fibrations given in Figure 2.

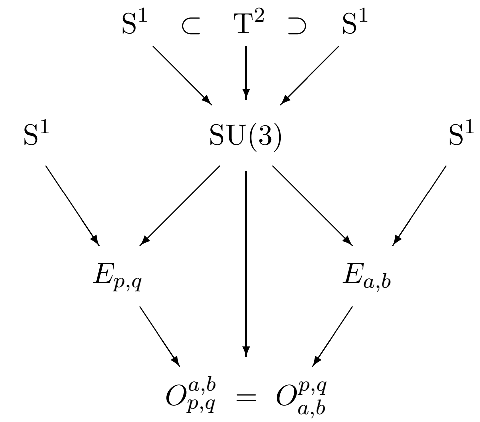

Figure 2. Orbifold fibrations of Eschenburg spaces.

Since $\mathrm{S}^1_{a,b}$ is almost free, there are only regular orbits and exceptional orbits. We define the *exceptional set* $\mathcal{S}^{a,b}_{p,q} \subset E_{p,q}$ to be the union of all exceptional orbits, which thus coincides with the set of points in $E_{p,q}$ where the action is not free.

Recall that we have the nine embeddings $\mathrm{U}(2)_{ij} \subset \mathrm{SU}(3)$, $1 \leq i, j \leq 3$, and the six two-dimensional tori T^2_σ, $\sigma \in S_3$. They give rise to their respective images in $E_{p,q}$,

$$\mathcal{L}_{ij} := \pi(\mathrm{U}(2)_{ij}), \qquad C_\sigma := \pi(T^2_\sigma).$$

While the C_σ's are clearly circles, we claim that the $\mathcal{L}_{ij}$'s are lens spaces. Indeed, if $g \in \mathrm{U}(2)_{ij}$ and $z \in S^1$, we get

$$\tau_i z^p g \bar{z}^q \tau_j = \begin{bmatrix} z^{p_{i_1}-q_{j_1}} x & z^{p_{i_1}-q_{j_2}} y & 0 \\ -z^{p_{i_2}-q_{j_1}} \lambda \bar{y} & z^{p_{i_2}-q_{j_2}} \lambda \bar{x} & 0 \\ 0 & 0 & z^{p_i-q_j} \bar{\lambda} \end{bmatrix}, \tag{3.3}$$

where $\lambda \in S^1$ and $(x, y) \in \mathbb{S}^3$ and we have used the index convention $\{i_1, i_2, i\} = \{j_1, j_2, j\} = \{1, 2, 3\}$. Taking a representative $g \in \mathrm{SU}(2) = \mathbb{S}^3$ in the orbit (i.e. $\lambda = 1$) and identifying the upper 2×2 matrix in (3.3) with its first row, we conclude that $\mathcal{L}_{ij}$ is the lens space

$$\mathcal{L}_{ij} = L(p_{i_1} - q_{j_1}, p_{i_1} - q_{j_2}, p_i - q_j).$$

From their very definition, we see that each lens space $\mathcal{L}_{ij}$ contains precisely two of the circles $C_\sigma \subset \mathcal{L}_{ij}$, where σ is one of the two permutations that satisfy $\sigma(i) = j$. Moreover, each circle is then obtained as the intersection of three lens spaces. They are arranged as shown in Figure 1 in the Introduction.

⋄ *The exceptional set.* We now proceed to investigate the structure of $\mathcal{S}^{a,b}_{p,q}$. The isotropy group $\mathrm{S}^1_{[g]} \subset \mathrm{S}^1$ at $[g] \in E_{p,q}$ is the finite cyclic group given by the elements $w \in \mathrm{S}^1$ such that there is $z \in \mathrm{S}^1$ with

$$g^{-1} z^p w^a g = z^q w^b. \tag{3.4}$$

If $1 \neq w \in \mathrm{S}^1_{[g]}$, there must be $z \in \mathrm{S}^1$ and $\sigma \in S_3$ such that

$$z^p w^a = z^{q_\sigma} w^{b_\sigma}. \tag{3.5}$$

If there exists a w such that the two matrices in (3.5) have a triple eigenvalue, then it acts trivially on all of $E_{p,q}$ and hence belongs to the ineffective kernel, whose order we denote by $\kappa_0 \in \mathbb{N}$.

Otherwise, there are two possibilities. If all eigenvalues are distinct, (3.4) implies that $g(e_i) = \mu_i e_{\sigma^{-1}(i)}$ with $\mu_1\mu_2\mu_3 = 1$ and thus $[g] \in C_\sigma$. Furthermore, it follows that there exists a $\kappa_\sigma \in \mathbb{N}$ such that $S^1_{[g]} = \mathbb{Z}_{\kappa_\sigma}$ for all $[g] \in C_\sigma$, i.e. the isotropy group has constant order along the exceptional subset C_σ.

If, on the other hand, the matrices in (3.5) have a double eigenvalue, the third eigenvalue must coincide as well since (p, q) and (a, b) belong to $\mathfrak{t}_{\mathbb{Z}}$. Thus there exist i, j with $1 \leq i, j \leq 3$ such that $\bar{z}^{p_{i'}-q_{j'}} = w^{a_{i'}-b_{j'}}$, for all $i' \neq i$, $j' \neq j$. It follows that $[g] \in \mathcal{L}_{ij}$, and that w fixes all of $\mathcal{L}_{ij}$. Hence there exists a $\kappa_{ij} \in \mathbb{N}$ such that the isotropy group along $\mathcal{L}_{ij}$ is $\mathbb{Z}_{\kappa_{ij}}$, except along the two circles $C_\sigma \subset \mathcal{L}_{ij}$ where it is $\mathbb{Z}_{\kappa_\sigma}$, with σ being one of the two permutations that satisfy $\sigma(i) = j$.

We thus have shown that the exceptional set is given by

$$\mathcal{S}^{a,b}_{p,q} = \bigcup_{(i,j)\in\Gamma} \mathcal{L}_{ij} \bigcup_{\sigma\in\Sigma} C_\sigma \ , \tag{3.6}$$

where $\Gamma = \{(i, j) : \kappa_{ij} > \kappa_0,\ 1 \leq i, j \leq 3\}$ and $\Sigma = \{\sigma \in S_3 : \kappa_\sigma > \kappa_0\}$.

Notice that each lens space $\mathcal{L}_{ij}$ in this exceptional set must be totally geodesic. To see this, it is sufficient to show that $\mathrm{U}(2)_{ij}$ is totally geodesic in $\mathrm{SU}(3)$ with respect to the Eschenburg metric since $\mathrm{U}(2)_{ij}$ is invariant under the circle action $\mathrm{S}^1_{a,b}$. This in turn follows since $\mathrm{U}(2)_{ij}$ is the fixed point set of an isometry on $\mathrm{SU}(3)$ of the form $g \to z^v g \bar{z}^{v_\sigma}$ for some z^v with two equal diagonal entries and some permutation σ. The circles C_σ, being intersections of two totally geodesic submanifolds, are thus closed geodesics.

The metric on the lens spaces are in general not homogeneous, in fact they are homogeneous if and only if the circle action on $\mathrm{U}(2)_{ij}$ is one sided; see the proof of Theorem 4.1 in [GSZ]. This is only possible when the Eschenburg space has cohomogeneity two. In the case where the lens space is homogeneous, the metric is a Berger type metric, i.e. induced from a metric on $\mathbb{S}^3$ shrunk in direction of the Hopf fibers. In the cohomogeneity one case there are 6 such homogeneous lens spaces and in the remaining cohomogeneity two spaces 3 of them are homogeneous. When the lens space is not homogeneous, its isometry group still contains a 2-torus.

$\diamond$ *The isotropy groups.* We next determine the order of the isotropy groups of our circle actions. To do so, for $v \in \mathbb{Z}^n$, set $\gcd(v) = \gcd(\{v_1, \ldots, v_n\})$ and define

$$\kappa(v, w) := \gcd(v)^{-1} \gcd(\{v_i w_j - v_j w_i : 1 \leq i < j \leq n\}).$$

Proposition 3.7. *For the almost free circle action $\mathrm{S}^1_{a,b}$ on $E_{p,q}$ given by* (3.1) *the following holds:*

(a) *The ineffective kernel is* $\mathbb{Z}_{\kappa_0}$*,* $\kappa_0 = \kappa(P, A)$*, where* $P, A \in \mathbb{Z}^6$ *are the vectors whose components are* $p_i - q_j$ *and* $a_i - b_j$*, respectively, with* $i \neq j$ *(the same index ordering for both).*

(b) *The isotropy group along* C_σ *is* $\mathbb{Z}_{\kappa_\sigma}$*, where*

$$\kappa_\sigma = \frac{|(p_1 - q_{\sigma(1)})(a_2 - b_{\sigma(2)}) - (p_2 - q_{\sigma(2)})(a_1 - b_{\sigma(1)})|}{\gcd(p_1 - q_{\sigma(1)}, p_2 - q_{\sigma(2)})}.$$

(c) *The isotropy group along* $\mathcal{L}_{ij}$*, outside* $C_\sigma \subset \mathcal{L}_{ij}$ *with* $\sigma(i) = j$*, is* $\mathbb{Z}_{\kappa_{ij}}$*,* $\kappa_{ij} = \kappa(V, W)$*, where* $V, W \in \mathbb{Z}^4$ *are the vectors whose components are* $p_{i'} - q_{j'}$ *and* $a_{i'} - b_{j'}$*, respectively, with* $i' \neq i, j' \neq j$ *(the same index ordering for both).*

Proof. We need the next elementary lemma concerning the lattice points $\mathbb{Z}^n$ inside the parallelogram spanned by $v, w \in \mathbb{Z}^n$, $\mathcal{P}_{v,w} = \{tv + sw : t, s \in [0, 1)\}$.

Lemma 3.8. *If* $p = (p_1, \ldots, p_n), a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ *are linearly independent, the projection of the lattice points* $\mathcal{P}_{p,a} \cap \mathbb{Z}^n$ *inside* $\mathcal{P}_{p,a}$ *to the s-coordinate is the set* $\mathbb{Z}_\kappa = \{i/\kappa : i = 0, \ldots, \kappa - 1\} \subset [0, 1)$*, with* $\kappa = \kappa(p, a)$*.*

Proof. We start with the case of $n = 2$. The number of lattice points $\mathbb{Z}^2$ inside $\mathcal{P}_{p,a}$ is equal to its area, that is, $\#(\mathcal{P}_{p,a} \cap \mathbb{Z}^2) = |p_1a_2 - p_2a_1|$. Indeed, using translations, we can assume that $p_i, a_i \geq 0$ and, if we inscribe $\mathcal{P}_{p,a}$ inside the rectangle $\mathcal{P}_{(p_1+a_1,0),(0,p_2+a_2)}$, it follows that $\pm\#(\mathcal{P}_{p,a} \cap \mathbb{Z}^2) = (p_1 + a_1)(p_2 + a_2) - a_1a_2 - p_1p_2 - 2p_2a_1 = p_1a_2 - p_2a_1$.

Setting $p' = d^{-1}p$, observe that $\mathcal{P}_{p,a}$ is the union of the $d = \gcd(p_1, p_2)$ disjoint rectangles $kp' + \mathcal{P}_{p',a}$, $k = 0, \ldots, d - 1$, since $tp + s_0a, t'p + s_0a \in \mathbb{Z}^2$ implies that $(t - t')p \in \mathbb{Z}$. Thus the number of points in the projection of $\mathcal{P}_{p,a} \cap \mathbb{Z}^2$ to the s-coordinate is $\gcd(p_1, p_2)^{-1}|p_1a_2 - p_2a_1|$. This proves our claim if $n = 2$.

For $n > 2$, by the above argument we can assume $\gcd(p) = 1$, and thus the number of points of the projection of the lattice to the s-coordinate is equal to $\kappa = \#(\mathcal{P}_{p,a} \cap \mathbb{Z}^n)$. In the plane $\Pi = \mathrm{span}\{p, a\}$, consider the lattice $L = \Pi \cap \mathbb{Z}^n$. From the case $n = 2$ it follows that $\kappa = |p \wedge a|/|v \wedge w|$, where $\{v, w\}$ is a base of L, or, equivalently, $\kappa = \max\{|p \wedge a|/|v \wedge w| : v, w \in L \text{ are linearly independent}\}$. Since $\gcd(p) = 1$, we can take $v = p$ and thus $a = rp \pm \kappa w$, for some $r \in \mathbb{Z}$. This implies that κ divides $(a_i - rp_i)p_j$ and $(a_j - rp_j)p_i$ and thus $a_ip_j - a_jp_i$ for every i, j, i.e., κ divides $\kappa(p, a)$. On the other hand, if we choose $u \in \mathbb{Z}^n$ with $\langle u, p\rangle = 1$, $\kappa(p, a)$ divides $\sum_j u_j(a_ip_j - a_jp_i) = a_i - \langle u, a\rangle p_i$ for every i, and hence $w' = \kappa(p, a)^{-1}(a - \langle u, a\rangle p) \in L$. Therefore, $\kappa \geq |p \wedge a|/|p \wedge w'| = \kappa(p, a)$, and the lemma follows. ∎

Assume the action is not effective. Then, there exists $1 \neq w \in \mathrm{S}^1$ such that, for every $g \in \mathrm{SU}(3)$, there exists $z = z_g \in \mathrm{S}^1$ with $g^{-1}z_g^p w^a g = z_g^q w^b$. By choosing different g, it is easy to see that there exist $z, \lambda \in \mathrm{S}^1$ such that $z^p w^a = z^q w^b = \lambda\,\mathrm{Id}$. We can write this as $\overline{z}^{(p_i - q_j)} = w^{(a_i - b_j)}$, for all $1 \leq i, j \leq 3$. If we set

$$z = e^{2\pi i t}, \quad w = e^{2\pi i s}, \quad (t, s) \in [0, 1) \times [0, 1), \tag{3.9}$$

this means that $tP + sA \in \mathbb{Z}^6$ and the claim follows from Lemma 3.8.

From our discussion above it follows that the determination of the isotropy groups falls into 2 cases, depending on whether the matrices in (3.5) have simple eigenvalues or a double eigenvalue.

Case 1: Simple eigenvalues. Using (3.9), there is $t \in [0,1)$ such that

$$t(p - q_\sigma) + s(a - b_\sigma) \in \mathbb{Z}^3. \tag{3.10}$$

We need only to consider the first two coordinates since $(p,q),(a,b) \in \mathfrak{t}_\mathbb{Z}$ and hence we look for the lattice points $\mathbb{Z}^2$ inside the two dimensional parallelogram determined by $v_\sigma = (p_1 - q_{\sigma(1)}, p_2 - q_{\sigma(2)}), w_\sigma = (a_1 - b_{\sigma(1)}, a_2 - b_{\sigma(2)})$ and its projections to the s-coordinate. Thus the claim follows from Lemma 3.8.

Case 2: Double eigenvalue. Since (p,q) and (a,b) belong to $\mathfrak{t}_\mathbb{Z}$, the third eigenvalue must coincide as well. Thus there are $1 \le i,j \le 3$ such that $\bar{z}^{p_{i'}-q_{j'}} = w^{a_{i'}-b_{j'}}$, for all $i' \ne i$, $j' \ne j$. This is equivalent to $tV + sW \in \mathbb{Z}^4$ and we apply Lemma 3.8. ∎

Remark 3.11. Notice that, by Proposition 3.7 (b) and (3.2), the circle action given by (3.1) is almost free if and only if $\kappa_\sigma \ne 0$ for all $\sigma \in S_3$. Observe also that the order of the isotropy groups are in terms of the possibly ineffective action. The orders need to be divided by κ_0 to obtain the isotropy groups of the action when made effective. In explicit computations it is useful to notice that since $(p,q),(a,b) \in \mathfrak{t}_\mathbb{Z}$, the number of entries in the definition of $\kappa(P,A)$ in part (a) can be reduced from 6 to 4 numbers, and for κ_{ij} from 4 to 3.

⋄ *The singular locus.* We finally discuss the singular locus of the orbifold $O^{a,b}_{p,q}$, i.e., $\hat{\pi}(\mathcal{S}^{a,b}_{p,q}) \subset O^{a,b}_{p,q}$. Each $\hat{\pi}(C_\sigma)$ will be called a *vertex* of $\hat{\pi}(\mathcal{S}^{a,b}_{p,q})$, while $\hat{\pi}(\mathcal{L}_{ij})$ will be called a *face*.

For this purpose, we assume for simplicity that the Eschenburg space $E_{p,q}$ is a smooth manifold. In this case (1.3) implies that the lens spaces $\mathcal{L}_{ij}$ are all smooth manifolds as well. This is evident if all p_i are distinct from p_j. If two of them agree, $\mathcal{L}_{ij}$ is either equal to $L(0,a,1) = \mathbb{S}^3$ or to $L(1,1,0) = \mathbb{S}^2 \times \mathbb{S}^1$ (the latter two are not possible in positive curvature).

Clearly, by Proposition 3.7 and the slice theorem, each $\hat{\pi}(C_\sigma)$ is a point with orbifold group $\mathbb{Z}_{\kappa_\sigma/\kappa_0}$, while the orbifold group of $\hat{\pi}(\mathcal{L}_{ij})$ is $\mathbb{Z}_{\kappa_{ij}/\kappa_0}$, outside $\hat{\pi}(C_\sigma)$ and $\hat{\pi}(C_{\sigma'})$ with $\sigma(i) = \sigma'(i) = j$.

On the other hand, each face $\hat{\pi}(\mathcal{L}_{ij})$ is two dimensional and is itself an orbifold which is totally geodesic in $O^{a,b}_{p,q}$. We claim that it is homeomorphic to $\mathbb{S}^2$ and that it has only two orbifold points, namely the vertices $\hat{\pi}(C_\sigma), \hat{\pi}(C_{\sigma'}) \in \hat{\pi}(\mathcal{L}_{ij})$ with orbifold angles $2\pi\kappa_{ij}/\kappa_\sigma$ and $2\pi\kappa_{ij}/\kappa_{\sigma'}$, respectively. Indeed, $\mathrm{S}^1_{a,b}$ preserves $\mathcal{L}_{ij}$, with C_σ and $C_{\sigma'}$ as two of its orbits. We now apply the slice theorem of the action restricted to $\mathcal{L}_{ij}$ at a point in C_σ where the isotropy group is $\mathbb{Z}_{\kappa_\sigma}$. This isotropy group acting on the two dimensional slice has $\mathbb{Z}_{\kappa_{ij}}$ as its ineffective kernel, while the quotient group $\mathbb{Z}_{\kappa_\sigma}/\mathbb{Z}_{\kappa_{ij}}$ acts effectively via a finite rotation group. Thus, $\mathbb{Z}_{\kappa_\sigma/\kappa_{ij}}$ is the orbifold group of $\hat{\pi}(C_\sigma)$ as a singular point of the two dimensional orbifold $\hat{\pi}(\mathcal{L}_{ij})$. Away from C_σ, $C_{\sigma'} \subset \mathcal{L}_{ij}$, the circle action is free, modulo its ineffective kernel $\mathbb{Z}_{\kappa_{ij}}$, and hence $\hat{\pi}(\mathcal{L}_{ij})$ has only two orbifold points. As a consequence, the singular two sphere $\hat{\pi}(\mathcal{L}_{ij})$ is smooth if and only if $\kappa_{ij} = \kappa_\sigma = \kappa_{\sigma'}$. Since the metric on the lens space has T^2 as its isometry group, the orbifold still admits a circle of isometries, i.e., it is rotationally symmetric.

4. Minimizing the singular set

Given an Eschenburg space $E_{p,q}$, it is natural to try to find quotients $O = O^{a,b}_{p,q}$ as regular as possible by minimizing the exceptional set $\mathcal{S} = \mathcal{S}^{a,b}_{p,q}$. This can be achieved in several ways, e.g., by making the orders κ_σ and κ_{ij} as small as possible, or $\kappa_{ij} = 1$ for most pairs $1 \le i, j \le 3$ to eliminate the corresponding lens spaces from $\mathcal{S}$.

Recall that there exist precisely two 2-tori $\mathrm{T}^2 \subset \mathrm{U}(3) \times \mathrm{U}(3)$ that act freely on $\mathrm{SU}(3)$ as a biquotient action. In particular, there are two infinite families of Eschenburg spaces which admit a free S^1 action; see Section 2. For the general case, we can try to minimize the singular set as follows.

Proposition 4.1. *For each Eschenburg manifold $E_{p,q}$ endowed with an Eschenburg metric, there exists an isometric circle action whose exceptional set is composed of at most 3 totally geodesic lens spaces, intersecting along one closed geodesic, and the order of the cyclic isotropy groups of these lens spaces is bounded by $h = |H^4(E_{p,q}, \mathbb{Z})|$.*

Proof. Fix $\sigma \in S_3$, and let $\sigma' = \sigma \circ (123)$, $\sigma'' = \sigma \circ (132) \in S_3$ be the two permutations with the same parity of σ. Observe that if $\kappa_{\sigma'} = 1$, by (3.6) and Proposition 3.7 the three lens spaces that contain $C_{\sigma'}$ consist of regular points. If, in addition, $\kappa_{\sigma''} = 1$, then 6 of the lens spaces are regular, and therefore $\mathcal{S}$ would be composed of at most the 3 lens spaces that meet at the circle C_σ. In particular, in this situation all the κ_{ij} must divide κ_σ. Now, we claim that this can always be done, and with $\kappa_\sigma \le h$.

Fix $\epsilon_1, \epsilon_2 = \pm 1$. Since $E_{p,q}$ is smooth, there are $x, y, z, w \in \mathbb{Z}$ such that

$$\left\{ \begin{aligned} & x(p_1 - q_{\sigma(2)}) - y(p_2 - q_{\sigma(3)}) = \epsilon_1, \\ & w(p_1 - q_{\sigma(3)}) - z(p_2 - q_{\sigma(1)}) = \epsilon_2. \end{aligned} \right. \tag{4.2}$$

In view of (1.3), the set of all solutions of (4.2) is given by $x' = x + k_1(p_2 - q_{\sigma(3)})$, $y' = y + k_1(p_1 - q_{\sigma(2)})$, $w' = w + k_2(p_2 - q_{\sigma(1)})$ and $z' = z + k_2(p_1 - q_{\sigma(3)})$, with $k_1, k_2 \in \mathbb{Z}$. Now, we define $(a, b) \in \mathfrak{t}_{\mathbb{Z}}$, which depends on $\sigma, \epsilon_1, \epsilon_2, k_1, k_2$, by $a = (-z', -x', y' + w')$, $b_\sigma = (w' - x', y' - z', 0)$. By definition this implies that $\kappa_{\sigma'} = \kappa_{\sigma''} = 1$ for $\mathrm{S}^1_{a,b}$. The orders of the other 4 circles are given by

$$\left\{ \begin{aligned} \kappa_\sigma \quad &= |sh + (x + y - z)(p_1 - q_{\sigma(1)}) - (w + z - x)(p_2 - q_{\sigma(2)})|, \\ \kappa_{\sigma\circ(12)} &= |s(p_1 - q_{\sigma(2)})(p_2 - q_{\sigma(1)}) - w(p_1 - q_{\sigma(2)}) + y(p_2 - q_{\sigma(1)})|, \\ \kappa_{\sigma\circ(23)} &= |s(p_2 - q_{\sigma(3)})(p_3 - q_{\sigma(2)}) + (z + w)(p_2 - q_{\sigma(3)}) + x(p_3 - q_{\sigma(2)})|, \\ \kappa_{\sigma\circ(13)} &= |s(p_1 - q_{\sigma(3)})(p_3 - q_{\sigma(1)}) - (x + y)(p_1 - q_{\sigma(3)}) - z(p_3 - q_{\sigma(1)})|, \end{aligned} \right. \tag{4.3}$$

where $s = k_1 - k_2$ and h has the sign in (1.6). Since (a, b) and $(a+np+m\,\mathrm{Id}, b+nq+m\,\mathrm{Id})$, $n, m \in \mathbb{Z}$, induce the same circle action on $E_{p,q}$, we can assume $k_1 = s$, $k_2 = 0$ and we get

$$(4.4)\qquad \begin{cases} a = \big(-z, -x - s(p_2 - q_{\sigma(3)}), y + w + s(p_1 - q_{\sigma(2)})\big)\,, \\ b_\sigma = \big(w - x - s(p_2 - q_{\sigma(3)}), y - z + s(p_1 - q_{\sigma(2)}), 0\big)\,. \end{cases}$$

Now, by Remark 3.11, $\mathrm{S}^1_{a,b}$ is almost free if and only if the orders in (4.3) are nonzero. On the other hand, using (1.3) it is easy to check that $\kappa_{\sigma\circ(12)}, \kappa_{\sigma\circ(23)}, \kappa_{\sigma\circ(13)} \neq 0$, unless at least two of the integers $p_i - q_j$ with $i \neq j$ are ± 1, and $\kappa_\sigma = 0$ or 4. Therefore, since h is odd, we can make $0, 4 \neq \kappa_\sigma \leq h$ by choosing s appropriately, which proves our claim. ∎

In order for an action to have only one singular point, there should be 3 vertices with the same parity that are regular, where recall that the *parity* of the vertex $\hat{\pi}(C_\sigma) \in O$ is the parity of σ. That is, there should exist 3 permutations σ, σ', σ'' with the same parity such that $\kappa_\sigma = \kappa_{\sigma'} = \kappa_{\sigma''} = 1$. Notice that, by the last observation in the proof of Proposition 4.1, in this situation the action is automatically almost free. In addition, when such permutations exist, the singular locus of O is composed of at most the 3 vertices whose parity is the opposite to that of σ.

To see when such an action exists, define the following integers mod $h = |H^4(E_{p,q}, \mathbb{Z})|$:

$$(4.5)\qquad \alpha(\sigma, \epsilon_1, \epsilon_2) := \big((x + y - z)(p_1 - q_{\sigma(1)}) - (w + z - x)(p_2 - q_{\sigma(2)}) + 1\big) \mod h$$

where x, y, z, w are defined in (4.2). One easily verifies that $\alpha(\sigma, \epsilon_1, \epsilon_2)$ does not depend on the choice of x, y, z, w, i.e., they do not depend on k_1, k_2, and are hence well defined for every Eschenburg manifold $E_{p,q}$. Now, Theorem B in the Introduction is an immediate consequence of the first equation in (4.3).

Notice that the condition in Theorem B depends only on the parity of σ, hence giving 8 tests to check. Moreover, it is trivially satisfied when $h = 1$, i.e., $H^4(E_{p,q}, \mathbb{Z}) = 0$. Using (1.6) one easily sees that there are infinitely many Eschenburg manifolds with $h = 1$, for example, $p = (2k+2, k, 0)$, $q = (2k+1, k+2, -1)$, with $k \in \mathbb{Z}$ such that $\gcd(k-1, 3) = 1$. One can further try to minimize the singular set to get only one point. It turns out that such actions exist in abundance. For example, on $E_{(3,2,1),(4,2,0)}$, the almost free circle action given by $a = (1, 1, 0)$, $b = (2, 0, 0)$ has only one singular point of order 3.

However, no Eschenburg manifold with positive curvature seems to satisfy the condition in Theorem B, aside from the ones that already admit a free circle action. By means of a computer program (whose C code can be found in www.impa.br/~luis/eschenburg), we searched among all positively curved spaces with $h \leq 10^6$, in total 10.085.359.999 Eschenburg manifolds. It turns out that none of these spaces satisfies the condition, apart from the 314.617 ones that admit free actions. Moreover, none of these 314.617 spaces admit an isometric circle action with only one singular point. On the other hand, we will see that there are many spaces that admit circle actions with only two singular points of opposite parity.

We can use the above methods to obtain a nice singular locus for a general positively curved Eschenburg space. We give two typical examples here, obtaining in particular the proof of Theorem C.

Cohomogeneity one. Consider an arbitrary cohomogeneity one Eschenburg manifold, that is, $E_d = E_{(1,1,d),(0,0,d+2)}$, $d \geq 0$. It has positive curvature if $d > 0$ and satisfies $h = 2d + 1$. For $d \leq 2$, E_d is known to admit a free S^1 action; cf. Section 2. So, assume $d \geq 3$. We want to study all isometric S^1 actions on E_d for which at least 3 of the vertices are regular. We will see that, for such an action, the regular vertices do not have the same parity. As a consequence of this analysis, we will obtain the proof of Theorem B in the introduction.

If there are 3 regular vertices, at least two of them correspond to permutations σ', σ'' with the same parity as the one of, say, σ. Thus we are in the situation of the proof of Proposition 4.1, and we have (4.2), (4.3) and (4.4). It is not difficult to compute all the possibilities in (4.3), getting two cases:

Case (a): $\sigma(3) = 3$. The integers $x = \epsilon_1$, $z = -\epsilon_2$, $y = w = 0$ solve (4.2), we can assume that $\sigma = \mathrm{Id}$, and (4.3) becomes

$$\kappa_{\mathrm{Id}} = |2\lambda - s|,\ \kappa_{(12)} = |s|,\ \kappa_{(23)} = |d\lambda - \epsilon_2|,\ \kappa_{(13)} = |d\lambda - \epsilon_1|,$$

with $\lambda = s(d+1) - \epsilon_1 - \epsilon_2$. The only other order that can be 1 is the second, $\kappa_{(12)}$, that is, we can assume $s = 1$ and then $a = (\epsilon_2, d+1-\epsilon_1, 1)$, $b = (d+1-\epsilon_1, \epsilon_2+1, 0)$. Since there is no lens space connecting $C_{(23)}$ and $C_{(13)}$, the only possibly singular lens spaces are $\mathcal{L}_{11}$ and $\mathcal{L}_{22}$ whose orders are $\gcd(d-1,3)$ and $\gcd(d-2,5)$ if $\epsilon_1 = -\epsilon_2$, or both equal to 1 if $\epsilon_1 = \epsilon_2$. Therefore, the minimal singular locus we can get in case (a) is given by 3 isolated singular points for $\epsilon_1 = \epsilon_2 = 1$, whose orbifold orders are $2d-3$, $d(d-1)-1$ and $d(d-1)-1$. Notice that for $d \leq 2$ we recover the known free S^1 actions.

Case (b): $\sigma(3) \neq 3$. The integers $y = -\epsilon_1$, $w = \epsilon_2$, $x = z = 0$ solve (4.2), we can assume that $\sigma(3) = 1$, and we get for (4.3)

$$\kappa_\sigma = |(2d+1)s - \epsilon_1(d+1) + \epsilon_2|,\ \kappa_{\sigma\circ(12)} = |(d+1)(s-\epsilon_1) + \epsilon_2|,$$
$$\kappa_{\sigma\circ(23)} = |2s - \epsilon_1|,\ \kappa_{\sigma\circ(13)} = |ds + \epsilon_2|.$$

Notice that, again, $\kappa_\sigma > 1$, which says that no cohomogeneity one Eschenburg manifold with $d \geq 3$ satisfies the condition in Theorem B. To get one of the other 3 orders equal to one, we should have either

- (b_1) $s = 0$, with $a = (0, 0, \epsilon_2 - \epsilon_1)$, $b_\sigma = (\epsilon_2, -\epsilon_1, 0)$, and $\kappa_\sigma = \kappa_{\sigma\circ(12)} = d + 1 - \epsilon_1\epsilon_2$ and the other 4 orders equal to one; or
- (b_2) $s = \epsilon_1$, with $a = (0, -\epsilon_1, \epsilon_2)$, $b_\sigma = (\epsilon_2 - \epsilon_1, 0, 0)$, $\kappa_\sigma = \kappa_{\sigma\circ(23)} = d + \epsilon_1\epsilon_2$ and the other 4 orders also equal to one.

For (b_1), since $b_1 = 0$ and $b_2, b_3 = \pm 1$, the orbifold order of the only possibly singular lens space $\mathcal{L}_{31}$ is $\kappa_{31} = \kappa((1, -d-1, 1), (-b_2, -b_3, -b_2)) = d + 1 - \epsilon_1\epsilon_2 = \kappa_\sigma$. So, it coincides with the orbifold order of the two exceptional circles it contains. Therefore, the singular locus is a smooth totally geodesic 2-sphere with constant orbifold group $\mathbb{Z}_{d+1-\epsilon_1\epsilon_2}$.

Similarly, for (b_2), the orbifold order of the only possible singular lens space $\mathcal{L}_{1\sigma(1)}$ is $\kappa_{1\sigma(1)} = \kappa((1, 1 - q_{\sigma(2)}, d), (-\epsilon_1, -\epsilon_1, \epsilon_2)) = \gcd(q_{\sigma(2)}, d + \epsilon_1\epsilon_2)$.

For $\sigma = (13)$ we get as before that $\kappa_{13} = d + \epsilon_1\epsilon_2 = \kappa_\sigma$. So, also in this case the singular locus is a smooth totally geodesic 2-sphere with constant orbifold group $\mathbb{Z}_{d+\epsilon_1\epsilon_2}$, and the exceptional set is the homogeneous lens space $\mathcal{L}_{13} = \mathbb{S}^3/\mathbb{Z}_{d+1}$. Notice also that this action gives the smallest possible value for κ_σ under the assumption that $\kappa_{\sigma'} = \kappa_{\sigma''} = 1$.

On the other hand, for $\sigma = (123)$, it holds that $\kappa_{12} = \gcd(d+2, 2-\epsilon_1\epsilon_2)$. Therefore, either $\kappa_{12} = 1$ if $\epsilon_1 = \epsilon_2$, in which case the singular locus has only two vertices with the same orbifold group $\mathbb{Z}_{d+1}$, or $\kappa_{12} = \gcd(d-1, 3)$ if $\epsilon_1 = -\epsilon_2$, for which we get a 2-sphere as singular locus if and only if 3 divides $d-1$, that is smooth if only if $d = 4$.

Cohomogeneity two. Observe that $(p, q) = ((1, c, d), (0, 0, c + d + 1))$ lies in the same plane generated in Theorem B *iv*) for E_d. So, for any element in this cohomogeneity two subfamily, the same action has as its exceptional set the smooth lens space $\mathcal{L}_{13} = \mathbb{S}^3/\mathbb{Z}_{d+c}$ with constant isotropy group $\mathbb{Z}_{d-c}$.

For a general cohomogeneity two Eschenburg manifold $E_{p,q}$, that is, $p = (c, d, e)$, $q = (0, 0, c + d + e)$, with $\gcd(c, d) = \gcd(d, e) = \gcd(e, c) = 1$, we can consider the action generated by $a = (0, 0, 0)$, $b = (1, -1, 0)$. In this case, we get $\kappa_{\mathrm{Id}} = \kappa_{(12)} = |c + d|$, $\kappa_{(123)} = \kappa_{(23)} = |c + e|$, $\kappa_{(13)} = \kappa_{(132)} = |d + e|$. The orders of the lens spaces are $\kappa_{13} = \gcd(2, d+e)$, $\kappa_{23} = \gcd(2, c+e)$, $\kappa_{33} = \gcd(2, c+d)$, and $\kappa_{11} = \kappa_{12} = \gcd(c+d, c+e)$, $\kappa_{21} = \kappa_{22} = \gcd(c+d, d+e)$, $\kappa_{31} = \kappa_{32} = \gcd(c+e, d+e)$. In particular, for the positively curved Eschenburg space given by $(c, d, e) = (1, 2, 3)$ the singular locus consists of one 2-sphere and 4 isolated points, whereas for $(c, d, e) = (1, 2, -3)$ it consists of one 2-sphere and 2 isolated points. On the other hand, $(c, d, e) = (1, 3, 5)$ has the full set in Figure 1 as its singular locus.

Remark. With a simplified version of the arguments in Section 3, it is easy to compute the singular locus of the circle action that defines a general Eschenburg orbifold $E_{p,q}$. Again, it is given by Figure 1, composed of the C_σ's and the $\mathcal{L}_{ij}$'s, but now the orders of their cyclic orbifold groups are $\gcd(p - q_\sigma)$ and $\gcd(p - q_\sigma, p - q_{\sigma'})$, respectively. In particular, unlike in the case of circle actions on Eschenburg spaces, the order of the lens spaces is the gcd of the orders of the two circles which are contained in them. There is now no difficulty to obtain positively curved Eschenburg orbifolds that are not manifolds with the smallest possible singular locus. For example, $E_{(5,3,-5),(2,1,0)}$ has only one circle as singular locus with order 3.

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IMPA: Est. Dona Castorina 110, 22460-320, Rio de Janeiro, Brazil
E-mail address: luis@impa.br

University of Pennsylvania: Philadelphia, PA 19104, USA
E-mail address: wziller@math.upenn.edu