Genuine rigidity of Euclidean submanifolds in codimension two

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Abstract

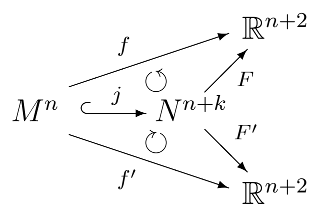
An isometric deformation of an Euclidean submanifold is called genuine if the submanifold cannot be included into a submanifold of larger dimension in such a way that the deformation of the former is given by an isometric deformation of the latter. The submanifold is said to be genuinely rigid if it has no genuine deformations. In this paper we study the deformation problem in codimension two for the classes of elliptic and parabolic submanifolds. In spite of having a second fundamental form as degenerate as possible without being flat, i.e., the Gauss map has rank two everywhere, our main result says that generically these submanifolds are genuinely rigid. An additional unexpected deformation phenomenon for elliptic submanifolds carrying a Kaehler structure is described.

It is a standard fact that an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ of an n-dimensional Riemannian manifold into flat Euclidean space with low codimension p is locally isometrically rigid provided its second fundamental form $\alpha_f\colon TM \times TM \to T_f^\perp M$ is not too degenerate; for instance see [**1**], [**2**], [**4**] and [**19**]. All these results generalize the classical Beez-Killing theorem for hypersurfaces ($p = 1$) that concludes isometric rigidity if the number of nonzero principal curvatures of f at any point is at least three. Sbrana and Cartan worked out the remaining interesting case, that is, the one of hypersurfaces with two nonzero principal curvatures. Generically these submanifolds are also locally rigid but now there are many locally deformable ones called *Sbrana-Cartan hypersurfaces*; see [**7**] and references therein.

Sbrana-Cartan hypersurfaces belonging to two of the four classes in which they can be divided, namely, *surface-like* and *ruled*, are highly deformable. For instance, the local isometric deformations of a ruled hypersurface are parametrized by the set of all smooth functions in an interval. Quite differently, the ones in the *continuous* class have only a one-parameter family of deformations while elements belonging to the remaining *discrete* class admit just a single deformation. In fact, until recently it was not clear whether hypersurfaces of the discrete class actually exist. However, it was shown in [**7**] that the transversal intersection of two flat hypersurfaces in Euclidean space is generically a Sbrana-Cartan hypersurface of discrete class obtained together with its unique isometric deformation. Moreover, a complete parametric description for those submanifolds given by this geometric procedure was obtained.

Nothing similar to the local results for hypersurfaces due to Sbrana and Cartan seems to be known for codimension higher than one. Globally, there are the ones for compact submanifolds in codimension two obtained in [**11**]. Already for codimension two the unexpected results of this paper give a clear indication of the difficulties to fully solve the local deformation problem. We need to recall from [**6**] the notion of genuine rigidity before we state our results. This concept extends the standard one of isometric rigidity and plays a fundamental role in the understanding of isometric deformations in codimension higher than one.

Observe that a hypersurface of a Sbrana-Cartan hypersurface is also deformable, and thus belongs to the class of deformable submanifolds in codimension two. This observation motivates the following definition that for $k = 2$ amounts for isometric congruence. We say that a pair of isometric immersions $f, f'\colon M^n \to \mathbb{R}^{n+2}$ *extend isometrically* if there are an isometric embedding $j\colon M^n \hookrightarrow N^{n+k}$ into a Riemannian manifold N^{n+k}, $1 \leq k \leq 2$, and a pair of isometric immersions $F, F'\colon N^{n+k} \to \mathbb{R}^{n+2}$ such that $f = F \circ j$ and $f' = F' \circ j$, that is, if the following diagram commutes:



Since our interest is the study of "real" deformations in codimension two, i.e., we want to ignore the ones given by Sbrana-Cartan hypersurfaces, the following definition of rigidity from [**6**] gives the appropriate framework.

Definition. An isometric immersion $f\colon M^n \to \mathbb{R}^{n+2}$ is *genuinely rigid* if for any given isometric immersion $f'\colon M^n \to \mathbb{R}^{n+2}$ there is an open dense subset $U \subset M^n$ such that the pair of isometric immersions $f|_U, f'|_U\colon U \to \mathbb{R}^{n+2}$ extend isometrically.

In this paper we study the local deformation problem in codimension two for the classes of elliptic and parabolic submanifolds as defined in [**5**]. Notice that the geometric structure of these submanifolds is quite different from those for which the aforementioned rigidity results apply since their second fundamental form is as degenerate as possible without being flat. We first recall the definition of the classes.

Let $f\colon M^n \to \mathbb{R}^{n+2}$ be an isometric immersion of rank two of a manifold without flat points whose second fundamental form $\alpha_f\colon TM \times TM \to T_f^\perp M$ spans the two dimensional normal space at each point. That f has *rank two* means that its Gauss map with values in the Grassmannian $G_{n,2}$ of all nonoriented n-planes in $\mathbb{R}^{n+2}$ has rank two everywhere. Equivalently, the *index of relative nullity* $\nu(x)$ of f at $x \in M^n$, that is, the dimension of the *relative nullity* vector subspace

$$\Delta(x) = \{Y \in T_xM : \alpha_f(Y, X) = 0,\ X \in T_xM\}$$

is constant $n-2$. Therefore, given a basis $X, Y \in \Delta^\perp(x)$ there are $a, b, c \in \mathbb{R}$ such that

$$a\,\alpha_f(X,X) + 2c\,\alpha_f(X,Y) + b\,\alpha_f(Y,Y) = 0. \tag{1}$$

In [**5**] we called f an *elliptic* (resp., *parabolic* or *hyperbolic*) submanifold if $ab - c^2 > 0$ (resp., $= 0$ or < 0) everywhere, a condition that is independent of the given basis.

Two (dual) alternative parametric descriptions of the large class of elliptic isometric immersions were given in [**5**]; see Theorems 10 and 14. The important family of those that are minimal, and then generically deformable, was described by Theorem 22. Below we recall from [**5**] the parametric description of the minimal ones that carry a Kaehler structure. These are either holomorphic or are always locally deformable. Examples of parabolic isometric immersions are the ruled ones, i.e., the submanifolds that carry a codimension one foliation whose leaves are (open subsets of) affine Euclidean subspaces.

In view of the Sbrana-Cartan classification it seems natural to expect the existence of a large family of elliptic or parabolic submanifolds admitting local isometric deformations that cannot be given as isometric extensions. Our results deny this expectation. In fact, we show that a locally irreducible elliptic or parabolic submanifold that cannot be realized as an hypersurface is (along connected components of an open dense subset) either rigid or any of its isometric deformations is given through isometric extensions by a deformation of a Sbrana-Cartan hypersurface.

For the elliptic case we have the following result, where a submanifold $f\colon M^n \to \mathbb{R}^{n+2}$ is called *surface-like* if either $f(M) \subset L^2 \times \mathbb{R}^{n-2}$ where $L^2 \subset \mathbb{R}^4$, or $f(M) \subset CL^2 \times \mathbb{R}^{n-3}$ where $CL^2 \subset \mathbb{R}^5$ is a cone over a surface $L^2 \subset \mathbb{S}^4$.

Theorem 1. *Let $f\colon M^n \to \mathbb{R}^{n+2}$ be a nowhere surface-like elliptic submanifold that admits no local isometric immersion in $\mathbb{R}^{n+1}$. Then f is genuinely rigid.*

We believe that the assumption that M^n does not have an isometric immersion as a hypersurface can be removed. In fact, the following result shows that this is the case for the minimal ones.

Corollary 2. *Let $f\colon M^n \to \mathbb{R}^{n+2}$, $n \geq 4$, be a substantial connected minimal isometric immersion of rank two that is not (globally) surface-like. Then f is genuinely rigid.*

To state our result for the class of parabolic submanifolds we first discuss the ones that are ruled.

Proposition 3. *Let $g\colon M^n \to \mathbb{R}^{n+1}$ be a simply connected and nowhere surface-like ruled hypersurface without flat points. Then, the family of ruled isometric immersions $f\colon M^n \to \mathbb{R}^{n+2}$ is parametrized by the set of ternary smooth arbitrary functions in an interval. Moreover, any pair of submanifolds f, f' belonging to the family, with at least one not a hypersurface, extends isometrically to either flat or Sbrana-Cartan hypersurfaces along each connected component of an open dense subset of M^n.*

Conversely, if $f\colon M^n \to \mathbb{R}^{n+2}$ is a ruled simply connected submanifold without flat points then M^n admits an isometric immersion as a ruled hypersurface.

We recall from [**6**] the following concept on isometric deformations.

Definition. An isometric immersion $\hat{f}\colon M^n \to \mathbb{R}^{n+2}$ is a *genuine deformation* of a given isometric immersion $f\colon M^n \to \mathbb{R}^{n+2}$ if there is no open subset $U \subset M^n$ along which the pair of isometric immersions $f|_U, f'|_U\colon U \to \mathbb{R}^{n+2}$ extends isometrically.

In the parabolic case we have the following complete classifying result.

Theorem 4. *Let $\hat{f}\colon M^n \to \mathbb{R}^{n+2}$ be a genuine deformation of a nowhere surface-like parabolic submanifold $f\colon M^n \to \mathbb{R}^{n+2}$. Then f is ruled along each connected component V_λ of an open dense subset of M^n and $\hat{f}|_{V_\lambda} = h \circ g|_{V_\lambda}$ is a composition of isometric immersions that has constant index of relative nullity $\nu = n-3$, where $g\colon V_\lambda \to \mathbb{R}^{n+1}$ is ruled and $h\colon U_\lambda \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+2}$ is such that $g(V_\lambda) \subset U_\lambda$.*

Notice that the condition for a composition of isometric immersions as in the theorem to have index of relative nullity $\nu = n-3$ is generic. Moreover, local flat hypersurfaces can be easily described; cf. [**8**].

It was shown in [**9**] that any simply connected minimal submanifold of rank two admits a one-parameter family of minimal isometric deformations. Moreover, by results in [**5**] the family is trivial (i.e., all immersions are congruent) only if a strong additional condition is satisfied. In view of Corollary 2, one would guess that in codimension two such one-parameter family of deformations is given by the deformations of a hypersurface that contains it and belongs to the continuous class in the Sbrana-Cartan classification. Surprisingly enough we show below that this is not true for minimal submanifolds that, in addition, carry a Kaehler structure but are not holomorphic. Moreover, we believe that our proof gives a strong indication that a result of this kind should hold for a larger class of elliptic submanifolds.

We first recall some basic properties of minimal immersions of Kaehler manifolds. We know from [**13**] that any minimal isometric immersion $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ of a Kaehler manifold is pseudoholomorphic. In terms of its second fundamental form and complex structure J this is equivalent to

$$\alpha_f(JX, Y) = \alpha_f(X, JY). \tag{2}$$

If M^{2n} is simply connected and f is not holomorphic, it was shown in [**9**] that there exists a one-parameter *associated family* of nowhere congruent isometric immersions $f_\theta\colon M^{2n} \to \mathbb{R}^{2n+p}$, where $\theta \in [0, \pi)$. They are explicitly given by the line integral

$$f_\theta = \int_{x_0}^{x} f_* \circ J_\theta,$$

where $x_0 \in M^{2n}$ is a fixed point and $J_\theta = \cos\theta I + \sin\theta J$. Moreover, the second fundamental form of f_θ is

$$\alpha_{f_\theta}(X, Y) = \alpha_f(J_\theta X, Y). \tag{3}$$

In addition, we have that $f = f_0$ can always be made the real part of its *holomorphic representative* $\mathcal{F}\colon M^{2n} \to \mathbb{C}^{n+p}$ defined as $\mathcal{F} = \frac{1}{\sqrt{2}}(f_0,\, f_{\pi/2})$.

In the following result we use that Sbrana-Cartan hypersurfaces can be of real or complex type. The definitions were given in [**7**] and will be recalled in the next section.

Theorem 5. *Let $f\colon M^{2n} \to \mathbb{R}^{2n+2}$, $n \geq 2$, be a nonholomorphic isometric immersion of rank two of a simply connected nowhere flat Kaehler manifold. The following holds:*

- (i) *The isometric immersion f is minimal. Moreover, if its holomorphic representative is substantial then any other isometric immersion $f'\colon M^{2n} \to \mathbb{R}^{2n+2}$ belongs to its (nontrivial) one-parameter associated family.*
- (ii) *The pair f, f' extend isometrically along an open dense subset of M^n to nowhere congruent Sbrana-Cartan hypersurfaces that are necessarily of discrete class and complex type.*

In particular, the question raised in [**7**] of whether there exist hypersurfaces of complex type in the discrete class now has a positive answer. In fact, an explicit parametric procedure to construct such examples follows from Theorem 5 by using the Weierstrass type representation of elliptic Kaehler submanifolds discussed below. Notice also that the Kaehler submanifold is the intersection of a one-parameter family of *nonisometric* Sbrana-Cartan hypersurfaces of the discrete class. Finally, we point out that the elements in the subclass of deformable hypersurfaces of the discrete class described earlier as being given by intersections of flat hypersurfaces are of real type.

To conclude this section, we show that the holomorphic representative of a rank two Kaehler submanifold in codimension two is (as required in part (i) of Theorem 5) generically substantial. First, we recall from [**5**] that any irreducible Kaehler submanifold $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ of rank two and substantial codimension $p \geq 2$ admits a Weierstrass type representation. In the special case of codimension $p = 2$ this representation is as follows. Let $a_0 = a_0(z)$ be a nonzero holomorphic map $a_0\colon U \to \mathbb{C}^2$ defined on a simply connected domain $U \subset \mathbb{C}$ such that $\alpha_0 = (\varphi_1, \varphi_2)$ satisfies $\varphi_2 \neq \pm i\varphi_1$. Assuming that $\alpha_r\colon U \to \mathbb{C}^{2r+2}$, $0 \leq r \leq n-1$, has been already defined, set

$$\alpha_{r+1} = \beta_{r+1}\left(1 - \phi_r^2, i(1 + \phi_r^2), 2\phi_r\right),$$

where $\beta_{r+1} = \beta_{r+1}(z) \neq 0$ is any holomorphic function defined on U, $\phi_r = \int^z \alpha_r dz$ and $\phi_r^2 = \phi_r.\phi_r$ in the standard symmetric inner product. Now let $\Psi_\theta\colon U \times \mathbb{C}^{n-1} \to \mathbb{R}^{2n+2}$ be defined by $\Psi_\theta(z,w) = \mathrm{Re}\,\{e^{i\theta}\mathcal{F}(z,w)\}$ for $\theta \in [0,\pi)$, where

$$\mathcal{F}(z,w) = \int^z \mu(z)\alpha_n(z)dz + \sum_{j=0}^{n-2} w_{j+1}\,\frac{d^j\alpha_n}{dz^j}(z) \tag{4}$$

and $\mu = \mu(z)$ is any holomorphic function defined on U. Then the maps $\Psi_\theta = \Psi_\theta(z,w)$ parametrize, at regular points, the one-parameter associated family of noncongruent substantial minimal isometric immersions of rank two of an irreducible Kaehler manifold with holomorphic representative $\mathcal{F}$. Conversely, any minimal isometric immersion of rank two and codimension two of a simply connected Kaehler manifold has such a Weierstrass type representation. The holomorphic representative $\mathcal{F}$ of f parametrized as in (4) is substantial if and only if the complex curve α_n is substantial, that is, if

$$\operatorname{span}\{d^j\alpha_n/dz^j : 0 \leq j \leq 2n+1\} = \mathbb{C}^{2n+2}. \tag{5}$$

A straightforward computation shows that generically for the functions $\alpha_0, \beta_1, \ldots, \beta_n$, the subspace on the left hand side of (5) is spanned by the linearly independent vectors: α_n, $(0, 0, \alpha_{n-1})$, $\ldots, (0, \cdots, 0, \alpha_0)$, $(-1, i, 0, \ldots, 0), \ldots, (0, \ldots, 0, -1, i, 0, 0)$ and $(0, \ldots, 0, d\alpha_0/dz)$, and hence condition (5) holds.

The proofs.

Let $f\colon M^n \to \mathbb{R}^{n+2}$ be a nowhere flat simply connected submanifold of rank two whose second fundamental form α_f spans the full normal space at each point. As seen in [**5**], at each point there are vectors $X_1, X_2 \in \Delta^\perp$ such that (1) takes the form

$$-\epsilon\, \alpha_f(X_1, X_1) + \alpha_f(X_2, X_2) = 0, \tag{6}$$

where $\epsilon = -1, 0, 1$. Moreover, the pairs $aX_1 + bX_2$, $aX_2 + \epsilon bX_1$ also satisfy (6) and, up to signs, there are no others. Then let $J\colon \Delta^\perp \to \Delta^\perp$ be the (unique up to sign) linear map defined by $JX_1 = X_2$ and $JX_2 = \epsilon X_1$. In particular, $J^2 = \epsilon I$. We conclude that f is elliptic, parabolic or hyperbolic if and only if there is a linear map $J\colon \Delta^\perp \to \Delta^\perp$ such that $J^2 = \epsilon I$ with $\epsilon = -1, 0, 1$, respectively, and

$$\alpha_f(X, JY) = \alpha_f(JX, Y) \tag{7}$$

for all $X, Y \in \Delta^\perp$ or, equivalently, for any $\xi \in T_f^\perp M$ the corresponding shape operator satisfies

$$A_\xi \circ J = {}^tJ \circ A_\xi. \tag{8}$$

We recall that the intrinsic *splitting tensor* $C\colon \Delta \times \Delta^\perp \to \Delta^\perp$ is defined as

$$C(T, X) = C_T X = -(\nabla_X T)_{\Delta^\perp}.$$

By the Codazzi equation the tensor C satisfies $A_\xi C_S = {}^tC_S A_\xi$ for any $\xi \in T_f^\perp M$ and $S \in \Delta$; see [**7**] or [**10**] for details. We obtain from (8) that

$$\operatorname{span}\{C_S : S \in \Delta\} \subset \operatorname{span}\{I, J\} \tag{9}$$

on any point of M^n. The following fact is well-known; cf. Lemma 6 in [**7**].

Lemma 6. *Assume that $C_T = \mu(T)I$ at any point and for all $T \in \Delta$. Then there is an open dense subset of M^n such that each point has a neighborhood where f is surface-like.*

It is not difficult to see that Sbrana-Cartan hypersurfaces of continuous or discrete class also satisfy (9) with $J^2 \neq 0$. In [**7**] such a hypersurface was called of complex (resp., real) type if $J^2 = -I$ (resp., $J^2 = I$).

Recall that the *index of nullity* $\mu(x)$ of the curvature tensor R at $x \in M^n$ is defined as $\mu(x) = \dim \Gamma(x)$, where

$$\Gamma(x) = \{Y \in T_xM : R(Y,X) = 0 \text{ for all } X \in T_xM\}.$$

The following result is of independent interest.

Proposition 7. *Let M^n be a Riemannian manifold with constant index of nullity $\mu(x) = n-2$, and let $f\colon M^n \to \mathbb{R}^{n+2}$ be an isometric immersion with index of relative nullity $\nu(x) \neq \mu(x)$ everywhere. Then $f = h \circ g$ is a composition of isometric immersions, where $h\colon N_0^{n+1} \to \mathbb{R}^{n+2}$ is a flat hypersurface and $g\colon M^n \to N_0^{n+1}$ an isometric embedding.*

Proof: We argue that $\nu(x) = n-3$ everywhere. Let $B\colon T_xM \to T_xM$ be any linear map such that $\Gamma(x) = \ker B$ and $\det B|_{\Gamma^\perp \times \Gamma^\perp} = -K(x)$, where $K(x) \neq 0$ is the sectional curvature of the plane $\Gamma^\perp(x)$. Let $\beta\colon TM \times TM \to W^3 := \mathbb{R} \times T^\perp_{f(x)}M$ be the symmetric bilinear map defined by

$$\beta(X,Y) = (\langle BX, Y\rangle, \alpha_f(X,Y)).$$

Then β is *flat* by the Gauss equation once W^3 is endowed with the product metric, i.e.,

$$\langle \beta(X,Y), \beta(Z,T)\rangle - \langle \beta(X,T), \beta(Z,Y)\rangle = 0,$$

and the assertion follows from Corollary 1 in [**17**].

We claim that at each point there is a vector $\eta \in T_f^\perp M$ of unit length such that $\operatorname{rank} A_\eta = 1$. Assume otherwise at $x \in M^n$, and choose B such that $B \neq A_\xi$ for any $\xi \in T^\perp_{f(x)}M$. It follows that $W^3 = \operatorname{span}\{\beta(X,Y) : X, Y \in TM\}$. By (a) of Theorem 2 in [**17**] there is an orthogonal basis $(1, \mu_j)$ of W^3 such that $\operatorname{rank}(B + A_{\mu_j}) = 1$. Hence, the three vectors $\delta_i = \mu_j - \mu_k$, where $i \neq j < k \neq i$, are pairwise linearly independent and $\operatorname{rank} A_{\delta_j} = 2$ by assumption. Take nonzero vectors $X_1, X_2, X_3 \in \Delta^\perp(x)$ such that $X_j \in \ker A_{\delta_j}$. It is clear that they are pairwise linearly independent since the δ_j are and $\nu(x) = n-3$. Moreover, we have that

$$\alpha_f(X_i, X_j) = 0 \quad \text{for all} \;\; i \neq j. \tag{10}$$

The vectors X_1, X_2, X_3 are linearly independent. If otherwise, say that $X_3 = X_1 + X_2$. It follows using (10) that $\alpha_f(X_3, X_3) = 0$, and similarly $\alpha_f(X_1, X_1) = 0 = \alpha_f(X_2, X_2)$.

We conclude that $\alpha_f|_{\pi\times\pi} = 0$, where $\pi = \text{span}\{X_1, X_2\}$. It is now easy to see that $\nu(x) = n-2$, which is not possible.

Take $0 \neq Z \in \Delta^\perp \cap \Gamma(x)$. Set $Z = \sum_{j=1}^3 a_j X_j$ and $X = \sum_{j=1}^3 y_j X_j \in \Delta^\perp(x)$. By the Gauss equation,

$$0 = K(Z,X) = \sum_{1\leq i<j\leq 3} (a_i y_j - a_j y_i)^2 \langle \alpha_f(X_i,X_i), \alpha_f(X_j,X_j)\rangle \tag{11}$$

for all $y_1, y_2, y_3 \in \mathbb{R}$. We have from $\langle \alpha_f(X_j,X_j), \delta_j\rangle = 0$ that the vectors $\alpha_f(X_j,X_j)$ must be pairwise linearly independent. Hence, at most one of the inner products in (11) may vanish. Moreover, if $\langle \alpha_f(X_i,X_i), \alpha_f(X_j,X_j)\rangle \neq 0$ then its sign coincides with the one of $K(x)$ by the Gauss equation. It follows from (11) that $Z = 0$, and this is a contradiction that proves our claim.

Clearly, $\text{span}\{\eta\} = \text{span}\{\alpha_f(Z,Z) : Z \in \Gamma\}$ and, in particular, η can be taken to be smooth. The Codazzi equation for A_η applied to pairs of vectors in $\ker A_\eta$ easily gives that η is parallel in the normal connection along $\ker A_\eta$, and the proof follows from Proposition 4 in [**11**]. ∎

Under the assumptions of Proposition 7 it also follows that there is an open dense subset $V \subset M^n$ such that along each connected component V_λ of V there are a nowhere flat hypersurface $g\colon V_\lambda \to \mathbb{R}^{n+1}$ and a flat hypersurface $h\colon U_\lambda \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+2}$ with $g(V_\lambda) \subset U_\lambda$ and $f|_{V_\lambda} = h \circ g$. This fact is used to prove our next result. Here and in the sequel we denote objects related to f' with the same symbols as for f plus a prime.

Corollary 8. *Let $f\colon M^n \to \mathbb{R}^{n+2}$ be a rank two nowhere flat isometric immersion and let $f'\colon M^n \to \mathbb{R}^{n+2}$ be another isometric immersion. Then, along each connected component V_λ of an open dense subset of M^n either $f'|_{V_\lambda}$ has rank two and $\Delta = \Delta'$, or $f'|_{V_\lambda}$ has constant index $\nu(x) = n-3$ and it is a composition of isometric immersions*

$$f'|_{V_\lambda} = h \circ g,$$

where $g\colon V_\lambda \to U_\lambda \subset \mathbb{R}^{n+1}$ and $h\colon U_\lambda \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+2}$.

Proof: We have from $\Delta = \Gamma$ that $\Delta \subseteq \Delta'$, and the proof follows from Proposition 7. ∎

Notice that for nowhere flat submanifolds of rank two $\Gamma(x) = \Delta(x)$. Riemannian manifolds with constant conullity $n - \mu(x) = 2$ were studied by Szabó in [**20**] and [**21**]. In view of Proposition 7, with the notion of hyperbolic Riemannian manifold given in ([**3**], p. 76) we have the following intrinsic version of Theorems 1 and 4. Here, that a Riemannian manifold M^n is surface-like means that it is isometric to an open subset of $\mathbb{R}^{n-3} \times \mathbb{R} \times_\rho L^2$.

Corollary 9. *Let M^n, $n \geq 3$, be a nowhere hyperbolic or surface-like Riemannian manifold of conullity two that admits no local isometric immersion in $\mathbb{R}^{n+1}$. Then any isometric immersion of M^n in $\mathbb{R}^{n+2}$ is genuinely rigid.*

For the proof of Theorem 1 we use the following result given in [**11**] as Lemma 6.

Lemma 10. *Given isometric immersions $f, f'\colon M^n \to \mathbb{R}^{n+2}$, $n \geq 3$, suppose that there are smooth orthonormal normal frames $\{\xi, \eta\}$ and $\{\xi', \eta'\}$ such that the corresponding shape operators satisfy everywhere:*

(i) $A_\xi = A'_{\xi'}$ *;*

(ii) *rank* $A_\eta = 2$ *and* $A'_{\eta'} \neq \pm A_\eta$.

Then the pair of isometric immersions f, f' extend isometrically to nowhere congruent Sbrana-Cartan hypersurfaces $F, F'\colon N^{n+1} \to \mathbb{R}^{n+2}$ with Gauss maps η, $\eta'\colon N^{n+1} \to \mathbb{S}^{n+1}$ up to parallel identification.

The isometric extensions in the preceding result were defined parametrically by Proposition 4 in [**11**], and go as follows. At each point $x \in M^n$ consider the vector subspace $\Omega(x) = \mathrm{span}\{\tilde{\nabla}_X \eta : X \in T_xM\} \subset T_x\mathbb{R}^{n+2}$, where $\tilde{\nabla}$ stands for the connection in the Euclidean ambient space. It turns out that the normal connection form $\psi(X) = \langle \nabla^\perp_X \xi, \eta\rangle = 0$ for any $X \in \ker A_\eta$. It is now easy to see that the subspaces $\Omega(x) \subset (\ker A_{\eta(x)})^\perp \oplus \mathrm{span}\{\xi(x)\}$ are two-dimensional, and hence the orthogonal complement subspaces form a line bundle $\pi\colon \Lambda \to M^n$. The extension F of f is defined as the restriction of $h\colon \Lambda \to \mathbb{R}^{n+2}$ given by

$$h(v) = f(x) + v \;\text{ if }\; x = \pi(v), \tag{12}$$

to an open neighborhood N^{n+1} of the zero section of Λ where h is an immersion. The Gauss map of F is η up to parallel identification in the ambient space along the fibers of Λ, i.e., $\eta(\lambda) = \eta(\pi(\lambda))$, and hence F is a submanifold of rank two. The hypersurface F was called in [**11**] an *extension with relative nullity* of f because the fibers of Λ are contained in its relative nullity. Let $X_0 \in TM$ be such that $\Lambda(x) = \mathrm{span}\{\xi(x) + f_{*x}(X_0)\}$. It turns out that $\Lambda'(x) = \mathrm{span}\{\xi'(x) + f'_{*x}(X_0)\}$, and hence Λ and Λ' are isometrically identified. Then the immersion F' is defined similarly along the same manifold N^{n+1} and induces the same metric as F.

Proof of Theorem 1: Assume that f admits a nowhere congruent isometric immersion $f'\colon M^n \to \mathbb{R}^{n+2}$. From the proof of standard rigidity results and the assumption that α_f spans the full normal space everywhere, there is an open dense subset of M^n where the second fundamental forms of f and f' satisfy $\alpha_f \neq \alpha_{f'}$, that is, they are not congruent by an isometry between their normal spaces; cf. [**18**]. For the purpose of this proof we may assume that this property holds at any point of M^n.

It follows from Corollary 8 that $\Delta' = \Delta$ everywhere. Moreover, f' is elliptic with $J' = J$ on an open dense subset of M^n. In fact, since f is nowhere surface-like, by Lemma 6 there is $T_0 \in \Delta$ such that $C_{T_0} = aI + J$. Hence, for any $\xi \in T^\perp_{f'}M$ we have by (8) that

$$A'_\xi J = A'_\xi C_{T_0} - aA'_\xi = {}^tC_{T_0} A'_\xi - aA'_\xi = {}^tJ A'_\xi.$$

Moreover, there is no open subset $U \subset M^n$ such that the space spanned by the second fundamental form of f' has dimension one. Otherwise, we have by Theorem 1 in [**14**] that $f'|_U$ has substantial codimension one, and that contradicts our assumptions.

For an elliptic submanifold the rank of the shape operator in any normal direction must be two. In view of Lemma 10 and our assumption that $\alpha_f \neq \alpha_{f'}$ everywhere, it remains to show that there exist smooth unit vector fields $\xi \in T_f^\perp M$ and $\xi' \in T_{f'}^\perp M$ such that $A_\xi = A'_{\xi'}$. It suffices to prove the same fact pointwise since smoothness then follows from Lemma 7 in [**11**]. Let $\beta\colon \Delta^\perp \times \Delta^\perp \to T_f^\perp M \times T_{f'}^\perp M$ be the bilinear form defined by

$$\beta(X,Y) = (\alpha_f(X,Y), \alpha_{f'}(X,Y)).$$

Since f and f' are elliptic with the same complex structure J then also β is elliptic with respect to J. Thus $S(\beta) = \text{span}\{\beta(X,Y) : X, Y \in \Delta^\perp\}$ satisfies $\dim S(\beta) = 2$. Moreover, β is flat with respect to the indefinite metric of type $(2,2)$ given by

$$\langle\langle\ ,\ \rangle\rangle_{T_f^\perp M \times T_{f'}^\perp M} = \langle\ ,\ \rangle_{T_f^\perp M} - \langle\ ,\ \rangle_{T_{f'}^\perp M}.$$

In particular, the metric induced on $S(\beta)$ cannot be definite. We conclude that there is a nonzero null vector $(\xi, \xi') \in (S(\beta))^\perp$, i.e., $0 \neq \|\xi\| = \|\xi'\|$ and $A_\xi = A_{\xi'}$, as desired. ∎

Proof of Corollary 2: Since f is real analytic there is no open subset of M^n where f is surface-like. It remains to show that M^n does not have a local isometric immersion as a hypersurface. To see this, take a local smooth orthonormal frame $\{\xi_1, \xi_2\}$ of $T_f^\perp M$ such that $\det A_{\xi_1} = \det A_{\xi_2}$ and a function $\theta \in C^\infty(M)$ satisfying $A_{\xi_2} = AJ_\theta$, where $A = A_{\xi_1}$ and $J_\theta = \cos\theta I + \sin\theta J$. In the sequel all computations are done on $\Delta^\perp$. We have that $2\det A = s < 0$ is the scalar curvature of M^n. Consider an orthonormal tangent frame $\{e_1, e_2\}$ such that $Ae_j = (-1)^{1+j}\sqrt{-s/2}\, e_j$ with $Je_1 = -e_2$. A long but straightforward computation gives that the Codazzi and Ricci equations for f reduce to

$$2s\psi \circ JJ_{-\theta} = ds \circ J - 2\gamma, \tag{13}$$

$$d\theta = -2\cos\theta\, \psi \circ J, \tag{14}$$

$$d\psi = -s\sin\theta e_1 \wedge e_2, \tag{15}$$

where γ and ψ are the connection one-forms $\gamma(X) = \langle \nabla_X e_1, e_2\rangle$ and $\psi(X) = \langle \nabla_X^\perp \xi_1, \xi_2\rangle$.

Suppose that M^n has a local isometric immersion as a hypersurface. It is easy to see that such a submanifold must have rank two and be minimal. Thus there is a function $\tau \in C^\infty(M)$ such that $2AJ_\tau$ is its second fundamental form. That AJ_τ is a Codazzi tensor implies that the one-form $(2s)^{-1}ds \circ J - 2\gamma = -d\tau$ is exact. This together with (13) and (14) give that the one-form $\sin\theta\,\psi$ must also be exact, and then (15) yields $\cos\theta\, d\theta \wedge \psi = s\, \sin^2\theta e_1 \wedge e_2$. Now using (14) again, we obtain that

$$\|d\theta\|^2\, e_1 \wedge e_2 = d\theta \wedge d\theta \circ J = 2\cos\theta\, d\theta \wedge \psi = 2s\, \sin^2\theta\, e_1 \wedge e_2.$$

But this is a contradiction since $s < 0$ and $\theta \neq 0, \pi$ because f is substantial. ∎

Proof of Proposition 3: First observe that it follows from our assumptions and the Sbrana-Cartan classification that the rulings of f are unique. Then, there are smooth orthonormal frames $\{X, Y\}$ of $\Delta^\perp$ and $\{\xi, \eta\}$ of $T_f^\perp M$ such that its second fundamental form is

$$A_\xi|_{\Delta^\perp} = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix} \quad \text{and} \quad A_\eta|_{\Delta^\perp} = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}, \tag{16}$$

the subspaces $R = \text{span}\{Y\} \oplus \Delta$ are tangent to the rulings and by the Gauss equation $b^2 = -K(X, Y) \neq 0$. Hence, we can take $b > 0$ and $\lambda \geq 0$. We obtain from the Codazzi equation for A_η that ξ and η are parallel in the normal connection along the rulings. Then any ruled isometric immersion $\tilde{f}\colon M^n \to \mathbb{R}^{n+2}$ is determined by three smooth functions a, λ and $\psi = \langle \nabla_X^\perp \xi, \eta \rangle$ such that (16) satisfies the Codazzi and Ricci equations. The Codazzi and Ricci equations involving the three functions are

$$\begin{cases} Y(a) = \langle \nabla_X X, Y \rangle a + X(b) \\ Y(\lambda) = \langle \nabla_X X, Y \rangle \lambda + b\psi \\ Y(\psi) = \langle \nabla_X X, Y \rangle \psi - b\lambda \end{cases} \quad \text{and} \quad \begin{cases} Z(a) = \langle \nabla_X X, Z \rangle a + \langle \nabla_X Y, Z \rangle b \\ Z(\lambda) = \langle \nabla_X X, Z \rangle \lambda \\ Z(\psi) = \langle \nabla_X X, Z \rangle \psi, \end{cases} \tag{17}$$

where $Z \in \Delta$. It follows that $\{a, \lambda, \psi\}$ can be arbitrarily prescribed along an integrable curve of X, and then they are completely determined by (17). By choosing $\lambda = 0$ and $\psi = 0$ we obtain all possible isometric immersion of M^n as a ruled hypersurface.

Given two ruled isometric immersions f_1, f_2 in the same family such that at least one of them, say f_1, is not a hypersurface, it is easy to see that there are local smooth orthonormal frames $\{\delta_j, \gamma_j\}$ normal to f_j, $1 \leq j \leq 2$, such that

$$A^1_{\delta_1} = A^2_{\delta_2}. \tag{18}$$

If rank $A^1_{\gamma_1} = 2$, then the pair f_1, f_2 extends isometrically by Lemma 10. Now assume that rank $A^1_{\gamma_1} = 1$. In this case each f_j corresponds to a solution $\{a_j, \lambda_j, \psi_j\}$ of (17) with $a_1 = a_2$. By Proposition 4 in [**11**], there are flat hypersurfaces $F_j\colon N_j^{n+1} \to \mathbb{R}^{n+2}$ and local isometric inclusions $h_j\colon U \subset M^n \to N_j^{n+1}$ with second fundamental form $A^j_{\delta_j}$ such that $f_j = F_j \circ h_j$, $1 \leq j \leq 2$. We conclude from (18) that there is a (local) isometry $\tau\colon N_1^{n+1} \to N_2^{n+1}$ such that $h_2 = \tau \circ h_1$. Hence $f_2 = F_2 \circ \tau \circ h_1$, and conclude that $F_1, F_2 \circ \tau\colon N_1^{n+1} \to \mathbb{R}^{n+2}$ are isometric extensions of f_1 and f_2. The proof of the converse in the statement is straightforward. ∎

Proof of Theorem 4: There exist smooth orthonormal frames $\{X, Y\}$ of $\Delta^\perp$ and $\{\xi, \eta\}$ of $T_f^\perp M$ such that the second fundamental form of f has the form (16). By Corollary 8 we have to consider two cases. Suppose that $\hat{f}$ has constant index $\nu(x) = n - 3$ on an open subset U of M^n. Then along each connected component of an open dense subset of U we have that $\hat{f} = h \circ g$ is a composition of isometric immersions. Moreover, in terms of the tangent frame $\{X, Y\}$ the second fundamental form A of g has the form

$$A|_{\Delta^\perp} = \begin{bmatrix} a' & b \\ b & 0 \end{bmatrix}. \tag{19}$$

If $a' = a$ in an open subset, we obtain from Lemma 10 that f is a composition, and that is a contradiction. After assuming $a' \neq a$ a straightforward computation using the Codazzi equation for f and $\hat{f}$ yields that f and g are ruled. Hence, f is a submanifold as described by Proposition 3. Since $\nu(x) = n - 3$, there are no unit vectors $\xi \in T_f^\perp M$ and $\xi' \in T_{\hat{f}}^\perp M$ such that $A_\xi = A_{\xi'}$, and hence $\hat{f}$ is a genuine deformation of f.

Assume now that $\Delta = \Delta'$ everywhere. In this situation the argument for the elliptic case still works if we manage to rule out the case in which there are smooth orthonormal normal frames ξ, η and ξ', η' such that $A_\xi = A'_{\xi'}$ and $\operatorname{rank} A_\eta = \operatorname{rank} A'_{\eta'} = 1$. By (8) we have $\ker A_\eta = \ker A'_{\eta'}$. Take an orthonormal tangent frame $\{X, Y\}$ spanning $\Delta^\perp$ such that $A_\eta X = \lambda X$ and $A'_{\eta'} X = \lambda' X$ with $0 \neq \lambda \neq \lambda'$. Notice that $\langle A_\xi Y, Y\rangle = 0$. Comparing the Codazzi equations for $A_\xi = A'_{\xi'}$ applied to X, Y we obtain $\lambda\psi(Y) = \lambda'\psi'(Y)$, where $\psi(Z) = \langle \nabla_Z^\perp \xi, \eta\rangle$. On the other hand, the Codazzi equations for A_η and $A'_{\eta'}$ yield $\lambda'\psi(Y) = \lambda\psi'(Y)$. It follows that $\psi(Y) = 0$. We easily conclude from the Codazzi equation that the submanifolds are ruled, and by Proposition 3 this is a contradiction. ∎

Examples. It was shown in [**10**] that the singular set Σ of a generalized Sbrana-Cartan hypersurface f of continuous or discrete class forms a codimension two deformable submanifold of rank two. Moreover, any deformation of f induces an isometric deformation of Σ that preserves the shape operator of rank two and the kernel of the shape operator of rank one. In particular, Σ cannot be elliptic. The Sbrana-Cartan hypersurface f can be recovered from Σ by taking an extension by relative nullity as in (12) but now Λ is a tangent line bundle. Moreover, given an isometric deformation of Σ' of Σ induced by an isometric deformation f' of f, it is not difficult to see that there are no other extensions by relative nullity of Σ and Σ' to isometric but not congruent hypersurfaces. Finally, we see that Σ must be hyperbolic. To rule out the parabolic case one can use the argument at the end of the preceding proof to conclude that the submanifold must be ruled. But then the same would be true for the original Sbrana-Cartan hypersurface, and that is not the case.

Of course, the examples just described are rather special. In general, we see no reason why a deformation of a hyperbolic submanifold must preserve one shape operator. On the other hand, if it does then it follows using Lemma 10 and Lemma 9 in [**11**] that there are always isometric extensions as either flat or (maybe singular as just seen) Sbrana-Cartan hypersurfaces.

Although submanifolds in codimension two that admit genuine deformations should be abundant not many examples are known. One family of examples are the complex ruled real Kaehler submanifolds in codimension two and rank four discussed in [**12**].

These submanifolds have generically and locally an associated family of deformations that are genuine.

Proof of Theorem 5: From Theorem 25 in [**5**] we have that f is minimal. Since the holomorphic representative F is substantial, by Proposition 3.2 in [**16**] any minimal isometric immersions of M^{2n} into Euclidean space for any codimension is congruent to one element in the two-parameter family of pseudoholomorphic submanifolds $\mathcal{F}_{\varphi,\theta}\colon M^{2n} \to \mathbb{R}^{2n+4}$ defined by

$$\mathcal{F}_{\varphi,\theta} = (\sin\varphi f_\theta, \, \cos\varphi f_{\theta+\pi/2}), \quad \text{where} \;\; \varphi \in [0,\pi].$$

As in the case of minimal surfaces, it is easy to see that the only elements of $\mathcal{F}_{\varphi,\theta}$ that lie in substantial codimension two are the ones that belong to the associated family of f, and that proves (i).

We have by Theorem 1 that f and f' are not genuine deformations one of the other. Thus, (along an open dense subset still called M^{2n}) there are local and nowhere congruent isometric Sbrana-Cartan hypersurfaces F, $F'\colon N^{2n+1} \to \mathbb{R}^{2n+2}$ and an isometric embedding $j\colon M^{2n} \to N^{2n+1}$ such that $f = F \circ j$ and $f' = F' \circ j$. It is left to see that F is of complex type in the discrete class.

By part (i) and the proof of Corollary 2, we know that M^{2n} admits no isometric immersion as an Euclidean hypersurface. The extensions F and F' are with relative nullity on an open dense subset, that we still assume to be all of M^{2n}. This is so because all shape operators in any normal direction of f and f' have rank two, as well as the second fundamental forms of F and F'.

We argue at a fixed point $y = j(x) \in j(M) \subset N^{2n+1}$ throughout the full argument. We have that $\Delta_F^\perp(y) = j_{*x}\Delta_f^\perp(x)$. Hence, $T' = j_{*x}T \in \Delta_F(y)$ if $T \in \Delta_f(x)$, and we obtain for the splitting tensors of f and F and $X, Y \in \Delta_f^\perp(x)$ that

$$\langle C^F_{T'} j_* X, j_* Y\rangle = -\langle \tilde\nabla_X F_* T', F_* j_* Y\rangle = -\langle \tilde\nabla_X f_* T, f_* Y\rangle = \langle C^f_T X, Y\rangle.$$

We have that f cannot be surface-like on any open subset because, otherwise, its holomorphic representative would not be substantial. It follows using $J\Delta = \Delta$ (see (2)) that

$$\operatorname{span}\{C^F_T : T \in \Delta_F(y)\} = \operatorname{span}\{C^f_T : T \in \Delta_f(x)\} = \operatorname{span}\{I, J'\},$$

where $J' = J|_{\Delta_f^\perp(x)}$ and J stands for the complex structure of M^{2n}. In particular, F is neither surface-like nor ruled, and it is of complex type since J' has no real eigenvalues. We show next that F does not belong to the continuous class.

By contradiction, assume that F admits a one-parameter family F_t of isometric deformations. Then, there is a nontrivial one-parameter family of isometric deformations $f^t = F_t \circ j$ of f that preserves a shape operator in one normal direction, that is, there are orthonormal bases $\{\xi, \eta\}$ of $T^\perp_{f(x)}M$ and $\{\xi_t, \eta_t\}$ of $T^\perp_{f^t(x)}M$ such that $A^f_\xi = A^{f^t}_{\xi_t}$. Notice that these bases are the ones that satisfy $\xi \in T_{F(y)}N$ and $\xi_t \in T_{F_t(y)}N$. On the other hand, the only isometric deformations of f are those of its associated family f_θ. This follows from Proposition 3.2 in [**16**] and the fact that the immersion (4) is substantial by assumption. Therefore, there exists a nonconstant function $\theta = \theta(t)$ such that $f^t = f_{\theta(t)}$. Since $A^f_\xi \circ J$ is symmetric by (8), there are $a, b \in \mathbb{R}$ such that $A^f_\eta = A^f_\xi(aI + bJ)$. We have that (3) is equivalent to $A^{f_\theta}_\delta = A^f_\delta \circ J_\theta$ for any $\delta \in T^\perp_{f(x)}M$. Thus there is a function $u = u(t)$ such that

$$\begin{aligned} A^f_\xi = A^{f^t}_{\xi_t} &= \cos u(t)\ A^f_\xi J_{\theta(t)} + \sin u(t)\ A^f_\eta J_{\theta(t)} \\ &= A^f_\xi J_{\theta(t)} \left(\cos u(t) I + \sin u(t)\ (aI + bJ)\right). \end{aligned}$$

Since $A^f_\xi|_{\Delta^\perp_f(x)}$ is nonsingular, we get that

$$\begin{aligned} -b \sin u(t) &= \sin \theta(t) \\ \cos u(t) + a \sin u(t) &= \cos \theta(t). \end{aligned}$$

We obtain $\sin u(t)\left((a^2 + b^2 - 1)\sin u(t) + 2a\cos u(t)\right) = 0$. Since $(a, b) \neq (0, \pm 1)$ because f is not holomorphic, we conclude that $u(t)$ is constant. Hence $\theta(t)$ is constant and that is a contradiction. ∎

To conclude, we observe that Proposition 7 can be used to prove the following unproved observation given after Theorem 1 in [**15**].

Proposition 11. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ be an isometric immersion of a Riemannian manifold with nonpositive sectional curvature. We assume $\nu(x) = n - 2p + 1$ and $\mu(x) = n - 2p + 2$ everywhere. Then, there is an open dense subset $V \subset M^n$ such that each connected component V_λ of V splits isometrically as $V_\lambda = M_1^{n_1} \times \cdots \times M_{p-1}^{n_{p-1}}$, and there are nowhere flat hypersurfaces $f_i\colon M_i^{n_i} \to \mathbb{R}^{n_i+1}$, $1 \leq i \leq p-1$ and a flat hypersurface $h\colon U_\lambda \subset \mathbb{R}^{n+p-1} \to \mathbb{R}^{n+p}$ such that $f|_{V_\lambda} = h \circ (f_1 \times \cdots \times f_{p-1})$.*

Proof: By Theorem 1 in [**15**] we may assume that $p = 2$, and hence $\nu(x) = n - 3$. The result follows from Proposition 7. ∎

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