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Compositions of isometric immersions in higher codimension[3]

Abstract. Given a submanifold M^n of Euclidean space $\mathbb{R}^{n+p}$ with codimension $p \le 6$, under generic conditions on its second fundamental form, we show that any other isometric immersion of M^n into $\mathbb{R}^{n+p+q}$, $0 \le q \le n-2p-1$ and $2q \le n+1$ if $q \ge 5$, must be locally a composition of isometric immersions. This generalizes several previous results on rigidity and compositions of submanifolds. We also provide conditions under which our result is global.

An isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ of a connected n-dimensional Riemannian manifold into Euclidean space with codimension p is said to be *rigid* if any other isometric immersion into the same ambient space is congruent to f by an Euclidean motion. But rigidity is lost once we allow new immersions to have *higher* codimension than the given one. In fact, for given $q \ge 1$, an abundance of isometric immersions $g\colon M^n \to \mathbb{R}^{n+p+q}$ can be produced by composing f with isometric immersions into $\mathbb{R}^{n+p+q}$ of open subsets $V \subset \mathbb{R}^{n+p}$ so that $f(M) \subset V$. We recall that the study of the large set of local isometric immersions between Euclidean spaces goes back to Cartan ([**Ca**]). Furthermore, complete descriptions for codimensions one and two were given in [**DG**] and [**DF**], respectively.

In this paper we answer a rigidity question already considered in [**DT**] for the special case of hypersurfaces ($p = 1$). Namely, we find sufficient (generic) conditions which, for f and g as above, imply that g must be a composition in the sense of the following definition.

Definition. Given an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ we say that an isometric immersion $g\colon M^n \to \mathbb{R}^{n+p+q}$, $q \ge 0$, is a *composition* when there is an isometric embedding $f'\colon M^n \hookrightarrow N_0^{n+p}$ into a flat manifold N_0^{n+p}, an

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isometric immersion $j\colon N_0^{n+p} \to \mathbb{R}^{n+p}$ (that is, a local isometry) satisfying $f = j \circ f'$ and an isometric immersion $h\colon N_0^{n+p} \to \mathbb{R}^{n+p+q}$ such that $g = h \circ f'$.

For any open subset $U \subset M^n$ where f as in the definition is an embedding, it follows that there exists an isometric immersion $h\colon V \subset \mathbb{R}^{n+p} \to \mathbb{R}^{n+p+q}$ of a tubular neighborhood V of $f(U)$ such that

$$g = h \circ f. \tag{1}$$

In particular, (1) holds globally if f itself is an embedding.

Observe that for $q = 0$ being a composition just means that the two immersions are congruent. Hence, the notion of composition extends the one of rigidity. In fact, we believe that considering rigidity results within the more general setting of compositions leads to a deeper understanding of the theory.

Next we deal with second fundamental forms which carry the structure corresponding to compositions. Given $f\colon M^n \to \mathbb{R}^{n+p}$, we say that the second fundamental form α_g of $g\colon M^n \to \mathbb{R}^{n+p+q}$ *decomposes* at $x \in M^n$ if there are a subspace $L^p \subset T_{g(x)}^\perp M$ and an isometry $\tau\colon T_{f(x)}^\perp M \to L^p$ so that

$$\alpha_g(x) = \tau \circ \alpha_f(x) \oplus \gamma, \tag{2}$$

where $\gamma\colon T_xM \times T_xM \to L^\perp$. If α_g decomposes at each point of M^n, we call the decomposition *regular* when the image $S(\gamma)$ and nullity $N(\gamma)$ of γ have both constant dimension.

Besides the aforementioned result in [**DT**] for hypersurfaces, our main result also generalizes the rigidity theorem in [**CD**]. There, it was shown that $f\colon M^n \to \mathbb{R}^{n+p}$ is rigid for $p \leq 5$ if the *s-nullity* of f satisfies everywhere $\nu_s^f \leq n - 2s - 1$ for all $1 \leq s \leq p$, where

$$\nu_s^f(x) = \max\{\dim N(\alpha_{\Gamma^s})(x)\colon \Gamma^s \subset T_{f(x)}^\perp M\},$$

with α_{Γ^s} denoting the orthogonal projection of α_f onto the subspace Γ^s. Here we prove the following for compositions.

Theorem 1. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ and $g\colon M^n \to \mathbb{R}^{n+p+q}$, $q \geq 0$, be isometric immersions. Suppose $p \leq 6$, and assume that f satisfies everywhere*

$$\nu_s^f \leq n - q - 2s - 1 \quad \text{for all} \quad 1 \leq s \leq p.$$

When $q \geq 5$, assume further that $\nu_1^f \leq n - 2q + 1$. Then α_g decomposes everywhere and g is a composition if α_g decomposes regularly.

A rather simple example due to Henke ([**He**]) shows that the preceding global result does not hold without the regularity assumption even if f is an embedding. Nevertheless, we always have regularity and that f is an embedding along connected components of an open dense subset of M^n. We conclude that (1) holds along each one of these components under assumptions on f only.

Application. By Theorem 1 an isometric immersion $g\colon \mathbb{S}^r \times \mathbb{S}^k \to \mathbb{R}^{n+q+2}$, for $r \leq k$ and $n = r+k$, must be a composition of $\mathbb{S}^r \times \mathbb{S}^k \hookrightarrow \mathbb{R}^{r+1} \times \mathbb{R}^{k+1} = \mathbb{R}^{n+2}$ if either $3 \leq r \leq 7$ and $q \leq r-3$ or $r \geq 8$ and $q \leq (r+1)/2$. Of course, the same conclusion holds if spheres are replaced by convex hypersurfaces.

We also have the following rigidity result for minimal immersions.

Theorem 2. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ be a minimal isometric immersion which satisfies $\nu_s^f(x_0) \leq n-q-2s-1$ at $x_0 \in M^n$ for all $1 \leq s \leq p \leq 6$. Then, any minimal isometric immersion $g\colon M^n \to \mathbb{R}^{n+p+q}$ is congruent to f in $\mathbb{R}^{n+p+q}$.*

A stronger result was obtained for hypersurfaces in [**BDJ**]. See also [**Da**] for the case $q = 1$ for both results.

The paper is organized as follows. In Section 1, we generalize in two directions by means of a simpler proof a result in [**CD**] on flat symmetric bilinear forms. In particular, the conformal rigidity theorem there remains valid for an extra unit in the codimension. Moreover, we have good reasons to believe that this result will be crucial in the understanding of isometric deformations of submanifolds in low codimension. The proofs are given in Section 2.

§1 Flat bilinear forms.

Given a symmetric bilinear form $\beta\colon V \times V \to W$ between finite dimensional real vector spaces, the nullity $N(\beta) \subset V$ of β is defined as

$$N(\beta) = \{X \in V \colon \beta(X,Y) = 0, \;\; Y \in V\},$$

and the image $S(\beta) \subset W$ of β is given by

$$S(\beta) = \operatorname{span}\{\beta(X,Y), \;\; X, Y \in V\}.$$

A vector $Y \in V$ is called a *regular element* of β if the linear map $B_Y \colon V \to W$, defined as $B_Y(X) = \beta(X,Y)$, satisfies

$$\dim B_Y(V) = \max\{\dim B_Z(V), \quad Z \in V\}.$$

It is easy to check that the subset $RE(\beta) \subset V$ of regular elements is open and dense; see [**Mo**] or [**D**] for details.

We denote by $W^{p,q}$ a $(p+q)$–dimensional vector space with a possible indefinite inner product of type (p,q), where $q \geq 0$ is the maximal dimension of a negative definite subspace. We call a subspace $U \subset W^{p,q}$ *degenerate* if the restriction of the metric of $W^{p,q}$ to U is degenerate, and denote by $\operatorname{rank} U$ the rank of the metric induced on U. Thus, $\operatorname{rank} U = \dim U - \dim U \cap U^\perp$ and U is *nondegenerate* if $\operatorname{rank} U = \dim U$. Finally, we say that U is *null* when $\operatorname{rank} U = 0$, that is, $U \subset U^\perp$.

Theorem 3. *Let $\beta \colon V^n \times V^n \to W^{p,q}$, where $p \leq q$ and $p+q < n$, be a nonzero symmetric bilinear form which is flat, that is,*

$$\langle \beta(X,Y), \beta(Z,T)\rangle - \langle \beta(X,T), \beta(Z,Y)\rangle = 0 \quad \textit{for all} \quad X,Y,Z,T \in V.$$

Assume $1 \leq p \leq 6$ and $\dim N(\beta) \leq n-p-q-1$. Then there is an orthogonal decomposition $W^{p,q} = W_1^{\ell,\ell} \oplus W_2^{p-\ell,q-\ell}$, $1 \leq \ell \leq p$, such that the W_j-components β_j of β satisfy

i) $\beta_1 \neq 0$ and $S(\beta_1)$ is null.

ii) β_2 is flat and $\dim N(\beta_2) \geq n - \dim W_2$.

For the proof of the above result we need several lemmas. The first one is a consequence of Theorem 3.8 in [**Ar**].

Lemma 4. *Given a degenerate subspace $U \subset W^{p,q}$, set $\mathcal{E} = U \cap U^\perp$ and let $\mathcal{S} \subset U$ be a subspace such that $\mathcal{E} \oplus \mathcal{S} = U$. Then, there is a (not necessarily unique) subspace $\widehat{\mathcal{E}} \subset W$ with $\dim \widehat{\mathcal{E}} = \dim \mathcal{E}$ so that $\mathcal{E} \oplus \widehat{\mathcal{E}}$ is nondegenerate and $\mathcal{E} \oplus \widehat{\mathcal{E}} \subset \mathcal{S}^\perp$.*

The following result is due to Moore (see [**Mo**] **or** [**Da**]).

Lemma 5. *Let $\beta \colon V \times V \to W^{p,q}$ be a flat bilinear form. Then,*

$$\beta(\ker B_X, V) \subset B_X(V) \cap B_X(V)^\perp$$

for any $X \in RE(\beta)$.

Finally, we recall an elementary fact from [**CD**] which is proved using that the subset of *not asymptotic* regular elements

$$RE^*(\beta) = \{X \in RE(\beta)\colon \beta(X,X) \neq 0\},$$

is open and dense in V.

Lemma 6. *Let $\beta\colon V \times V \to W^k$ be a symmetric bilinear form such that $S(\beta) = W^k$. Given $X \in RE^*(\beta)$, take $X = X_1, \dots, X_r \in V$ such that $B_X(V) = span\{B_X(X_j),\ 1 \leq j \leq r\}$, where $r = \dim B_X(V)$. Then,*

$$S(\beta) = span\{\beta(X_i, X_j),\ \ 1 \leq i \leq j \leq r\}.$$

In particular, $r(r+1) \geq 2k$.

Proof of Theorem 3: First suppose that $S(\beta)$ is degenerate. By Lemma 4, there is a decomposition $W^{p,q} = \mathcal{E} \oplus \widehat{\mathcal{E}} \oplus \mathcal{V}$ such that $S(\beta) \subset \mathcal{E} \oplus \mathcal{V}$, where $\mathcal{E} = S(\beta) \cap S(\beta)^\perp \neq 0$ and $\mathcal{V}^\perp = \mathcal{E} \oplus \widehat{\mathcal{E}}$. Accordingly, there is a decomposition $\beta = \beta_1 + \beta_2$, where $S(\beta_1) = \mathcal{E}$ and $S(\beta_2) \subset \mathcal{V}$. Hence, $\beta_1 \neq 0$ is null and $\beta_2 = \beta - \beta_1$ is flat. Moreover, $S(\beta_2)$ is nondegenerate. Otherwise, there is $0 \neq \eta = \sum_j \beta_2(X_j, Y_j) \subset \mathcal{V}$ so that $\langle \eta, S(\beta_2)\rangle = 0$, that is,

$$0 = \langle \sum \beta_2(X_j, Y_j), \beta_2(Z,T)\rangle = \langle \sum \beta(X_j, Y_j), \beta(Z,T)\rangle \quad \text{for all} \quad Z, T \in V.$$

Hence, $\sum_j \beta(X_j, Y_j) \in \mathcal{E}$. Thus $\eta = 0$, which is a contradiction.

To complete the proof of the theorem it suffices to assume that $S(\beta)$ is nondegenerate and conclude that $\dim N(\beta) \geq n - p - q$. We claim that $\mathcal{U}(X) = B_X(V) \cap B_X(V)^\perp$ satisfies $\mathcal{U} = \mathcal{U}(X) \neq 0$ for any $X \in RE(\beta)$. Otherwise, $\mathcal{N} \subset N(\beta)$ from Lemma 5, where $\mathcal{N} = \mathcal{N}(X) = \ker B_X$. Since $N(\beta) \subset \mathcal{N}$, we conclude that $\dim N(\beta) = \dim \mathcal{N} \geq n - p - q$, which is a contradiction and proves the claim.

Set $\tau = \min\{\dim \mathcal{U}(X) : X \in RE(\beta)\}$. The subset

$$\mathcal{R}(\beta) = \{X \in RE(\beta) : \dim \mathcal{U}(X) = \tau\}$$

is open and dense in V; see [**DR**] or [**D**]. We fix an element $X \in \mathcal{R}(\beta)$ for the remaining of the proof. Lemma 4 yields a decomposition

$$W^{p,q} = \mathcal{U} \oplus \widehat{\mathcal{U}} \oplus \mathcal{V}, \tag{3}$$

where $\mathcal{V}^\perp = \mathcal{U} \oplus \widehat{\mathcal{U}}$ and $B_X(V) \subset \mathcal{U} \oplus \mathcal{V}$. Let $\widehat{\beta}: V^n \times V^n \to \widehat{\mathcal{U}}$ be the $\widehat{\mathcal{U}}$-component of β according to (3). Set $\widehat{B}_X = \widehat{\beta}(X, \cdot\,)$ and $\kappa = \dim \widehat{B}_Y(V)$ for $Y \in RE(\widehat{\beta})$. Being $S(\beta)$ nondegenerate, for any vector $\xi \in \mathcal{U}$ there are vectors $Y, Z \in V$ such that $0 \neq \langle \xi, \beta(Y,Z)\rangle = \langle \xi, \widehat{\beta}(Y,Z)\rangle$. It follows that

$$S(\widehat{\beta}) = \widehat{\mathcal{U}}. \tag{4}$$

Lemma 7. *Given $Y \in \mathcal{R}(\beta) \cap RE(\widehat{\beta})$, we define $\rho \geq 0$ by*

$$2\rho = \mathit{rank}\,(B_Y(V) \cap \mathcal{U}) \oplus \widehat{B}_Y(V).$$

Then, $\rho \leq p - \tau$ and $\dim B_Y(\mathcal{N}) \leq p - \kappa$.

Proof: Set $V^n = \mathcal{L} \oplus \widetilde{\mathcal{L}}$, where $\widetilde{\mathcal{L}} = \ker \widehat{B}_Y$. Thus, $B_Y(\widetilde{\mathcal{L}}) \subset \mathcal{U} \oplus \mathcal{V}$ and $B_Y(\mathcal{L}) \cap (\mathcal{U} \oplus \mathcal{V}) = 0$. Hence, $\dim B_Y(\mathcal{L}) = \kappa$. The matrix of inner products of the elements of a basis of $B_Y(V)$ associated to the decomposition

$$B_Y(V) = B_Y^0 \oplus B_Y(\mathcal{L}) \oplus B_Y^1,$$

where $B_Y^0 = B_Y(\widetilde{\mathcal{L}}) \cap \mathcal{U} = B_Y(V) \cap \mathcal{U}$ and $B_Y(\widetilde{\mathcal{L}}) = B_Y^0 \oplus B_Y^1$, has the form

$$\begin{bmatrix} 0 & A & 0 \\ A^t & B & C \\ 0 & C^t & D \end{bmatrix}.$$

It follows that $\operatorname{rank} B_Y(V) \geq 2\rho + \operatorname{rank} B_Y^1$ since $\rho = \operatorname{rank} A$. We obtain from $B_Y^1 \subset \mathcal{U} \oplus \mathcal{V}$ and $B_Y^1 \cap \mathcal{U} = 0$ that $\operatorname{rank} B_Y^1 \geq \dim B_Y^1 - p + \tau$. Therefore, $\dim B_Y(V) - \tau = \operatorname{rank} B_Y(V) \geq 2\rho + \dim B_Y^1 - p + \tau$. We conclude that

$$2\rho \leq \dim B_Y(V) \cap \mathcal{U} + \kappa + p - 2\tau. \tag{5}$$

Clearly, $\rho \geq \dim B_Y(V) \cap \mathcal{U} + \kappa - \tau$. We get using (5) that

$$\dim B_Y(V) \cap \mathcal{U} \leq p - \kappa. \tag{6}$$

Then, the first statement follows from (5) and (6) whereas the second one from Lemma 5 and (6). ∎

Fix $Y_1 \in \mathcal{R}(\beta) \cap RE^*(\widehat{\beta})$. Lemma 6 and (4) yield $\kappa(\kappa+1) \geq 2\tau$ and

$$\widehat{\mathcal{U}}(X) = \operatorname{span}\{\widehat{\beta}(Y_i, Y_j) :\ 1 \leq i \leq j \leq \kappa\}, \tag{7}$$

where $\widehat{B}_{Y_1}(V) = \text{span}\{\widehat{B}_{Y_1}(Y_j)\}\colon 1 \leq j \leq \kappa\}$. Given any $N \in \mathcal{N}$, we have from Lemma 5 that $\beta(N, Z) \in \mathcal{U}$ for all $Z \in V$. It follows from (7) that

$$\beta(N, Z) = 0 \quad \text{if and only if} \quad \langle \beta(N, Z), \widehat{\beta}(Y_i, Y_j)\rangle = 0,\ 1 \leq i, j \leq \kappa. \tag{8}$$

We conclude the proof arguing for the most difficult case $p = 6$, being the cases $p \leq 5$ similar. Suppose first that $4 \leq \tau \leq 6$, and assume that $\kappa = 3$, which is its lowest possible value. Thus, there are vectors $Y_1, Y_2, Y_3 \in \mathcal{R}(\beta) \cap RE^*(\widehat{\beta})$ such that

$$\widehat{\mathcal{U}}(X) = \text{span}\{\widehat{\beta}(Y_i, Y_j)\colon 1 \leq i \leq j \leq 3\}. \tag{9}$$

Using Lemma 6 again, we choose Y_2 so that in (9) we may drop the element corresponding to $(i, j) = (3, 3)$ when $\tau = 5$, and when $\tau = 4$ the ones for $(i, j) = (2, 3), (3, 3)$. Hence, $\widehat{\mathcal{U}} = \widehat{B}_{Y_1}(V) + \widehat{B}_{Y_2}(V)$ for $\tau = 4, 5$. Consider the linear map $B_1 = B_{Y_1}|_{\mathcal{N}}\colon \mathcal{N} \to B_{Y_1}(\mathcal{N})$. Then, $\dim B_{Y_i}(\mathcal{N}) \leq 3$ by Lemma 7. Hence, $N_1 = \ker\, B_1$ satisfies

$$\dim N_1 \geq \dim \mathcal{N} - 3. \tag{10}$$

Flatness gives $\langle \beta(N_1, V), \widehat{B}_{Y_1}(V)\rangle = 0$. In particular,

$$\text{rank}\, B_{Y_2}(N_1) \oplus \widehat{B}_{Y_1}(V) = 0. \tag{11}$$

We use Lemma 7 again. If $\tau = 6$, then $\rho = 0$ and

$$\text{rank}\, B_{Y_2}(N_1) \oplus (\widehat{B}_{Y_1}(V) + \widehat{B}_{Y_2}(V)) = 0. \tag{12}$$

If $\tau = 5$, then $\rho \leq 1$. Thus,

$$\text{rank}\, B_{Y_2}(N_1) \oplus \widehat{\mathcal{U}} \leq 2. \tag{13}$$

In fact, it follows from (11) that (13) holds for $\tau = 4$. We get from (12) and (13) that $\dim B_{Y_2}(N_1) \leq 1$ for $4 \leq \tau \leq 6$. Set $B_2 = B_{Y_2}|_{N_1}\colon N_1 \to B_{Y_2}(N_1)$. It follows using (10) and $\dim \mathcal{N} \geq n - 6 - q + \tau$ that $N_2 = \ker\, B_2$ satisfies

$$\dim N_2 \geq \dim N_1 - \dim B_{Y_2}(N_1) \geq \dim \mathcal{N} - 4 \geq n - q - 6.$$

By a similar argument as above, $B_{Y_3}(N_2) = 0$ when $\tau = 6$. It follows from (8) that $N_2 \subset N(\beta)$. In particular, $\dim N(\beta) \geq n - q - 6$ as we wished. Finally, one can easily check that the estimate for $\dim N_2$ is even larger if $\kappa > 3$, and this concludes the proof for $4 \leq \tau \leq 6$. The argument for the remaining cases is similar and easier. ∎

§2 The proofs.

It is easy to see that the following result is equivalent to Theorem 5 in [**DT**] when f is an embedding.

Proposition 8. *Given an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$, suppose that the second fundamental form of an isometric immersion $g\colon M^n \to \mathbb{R}^{n+p+q}$, $q \geq 1$, decomposes regularly. Assume that τ is parallel for the induced connection on L. If*

$$W = span\{(\widetilde{\nabla}_X \xi)_{TM\oplus L}\colon X \in TM \text{ and } \xi \in L^\perp\}$$

satisfies $W \cap L = 0$, then g is a composition.

Proof: Observe that $W \subset N(\gamma)^\perp \oplus L$, where $TM = N(\gamma) \oplus N(\gamma)^\perp$. From $W \cap L = 0$, we easily see that $\dim W = \dim N(\gamma)^\perp$. Being $N(\gamma)$ smooth, it follows that the subspaces $\Gamma = (N(\gamma)^\perp \oplus L) \cap W^\perp$ form a rank-p vector bundle. Moreover, we have that $\Gamma \cap TM = 0$. Let $F\colon \Gamma \to \mathbb{R}^{n+p}$ be defined as

$$F(\xi_x) = f(x) + \xi_x, \quad \xi_x \in \Gamma(x).$$

Thus, there is an open neighborhood N_0^{n+p} of the 0-section $f'\colon M^n \to N_0^{n+p}$ of Γ such that $F|_{N_0^{n+p}}$ is a local diffeomorphism onto the open subset $F(N_0^{n+p}) \subset \mathbb{R}^{n+p}$. We take in N_0^{n+p} the metric induced by F and identify $(\mathrm{Id} \oplus \tau)^{-1}\Gamma$ with Γ. Let $h\colon N_0^{n+p} \to \mathbb{R}^{n+p+q}$ be the immersion $h(\xi_x) = g(x) + \xi_x$. A straightforward computation shows that F and h are isometric; see the proof of Theorem 5 in [**DT**]. Hence, h is an isometric immersion and $g = h \circ f'$. ∎

Proof of Theorem 1: Let $\beta\colon TM \times TM \to T_f^\perp M \oplus T_g^\perp M$ be defined as $\beta = \alpha_f \oplus \alpha_g$. Endow $T_f^\perp M \oplus T_g^\perp M$ with the indefinite inner product of type $(p, p+q)$ given by

$$\langle\langle\ ,\ \rangle\rangle_{T_f^\perp M \oplus T_g^\perp M} = \langle\ ,\ \rangle_{T_f^\perp M} - \langle\ ,\ \rangle_{T_g^\perp M}.$$

Then β is a symmetric bilinear form which is flat from the Gauss equation for f and g. Theorem 3 applies to $\beta(x)$ at each $x \in M^n$. With the notations there, suppose that $\ell < p$. It follows that

$$\nu_{p-\ell}^f(x) \geq \dim N(\pi_{T_f^\perp M} \circ \beta_2(x)) \geq \dim N(\beta_2(x)) \geq n - q - 2(p-\ell).$$

This is a contradiction to our hypothesis on ν_s^f for $s = p - \ell$, and implies that $\ell = p$. We conclude that at each point α_g decomposes as in (2) and $\Omega = N(\gamma)$ satisfies

$$\dim \Omega \geq n - q. \tag{14}$$

Assume now that the subspaces $S(\gamma)$ have constant dimension. Clearly, the same holds for the subspaces $S(\alpha_g)$ since $\nu_1^f < n$ everywhere. We claim that the decomposition is smooth in the sense that the subspaces L form a vector subbundle and that $\tau\colon T_f^\perp M \to L \subset T_g^\perp M$ is a bundle isometry. To prove the claim, observe that $S(\beta) \cap S(\beta)^\perp \subset T_f^\perp M \oplus T_g^\perp M$ is a smooth subbundle of rank p and that $L = S(\alpha_f) \subset T_g^\perp M$ is its orthogonal projection onto $T_g^\perp M$.

We identify $T_f^\perp M$ and L by means of τ. Let $K\colon TM \to \mathrm{End}(L)$ be the linear map into the skew-symmetric endomorphisms of L defined by

$$K(X)\eta = \nabla_X^\perp \eta - (\widehat{\nabla}_X^\perp \eta)_L,$$

where $\nabla^\perp$ and $\widehat{\nabla}^\perp$ denote the normal connection for f and g, respectively, and writing a linear space as subscript indicates taking the orthogonal projection of the vector onto that subspace. We need the following result.

Lemma 9. *The tensor K satisfies $K(Z) = 0$ and*

$$K(X)\alpha_f(Y, Z) = K(Y)\alpha_f(X, Z)$$

for all $Z \in \Omega$ and $X, Y \in TM$.

Proof: Since $\nu_1^f < n - q$ by assumption, we get using (14) that

$$L = \mathrm{span}\{\alpha_f(\Omega, X)\colon X \in TM\}. \tag{15}$$

It follows easily from the Codazzi equation for f and g that

$$K(Z_1)\alpha_f(Z_2, Z_3) = K(Z_2)\alpha_f(Z_1, Z_3)$$

if either $Z_1, Z_2 \in \Omega$ or $Z_3 \in \Omega$. Denote

$$(X_1, X_2, X_3, X_4, X_5) = \langle K(X_1)\alpha_f(X_2, X_3), \alpha_f(X_4, X_5)\rangle,$$

and take $Z_1, Z_2, Z_3 \in \Omega$. We have,

$$
\begin{aligned}
(Y, Z_1, Z_2, Z_3, X) &= -(Y, Z_3, X, Z_1, Z_2) = -(X, Z_3, Y, Z_1, Z_2) \\
&= (X, Z_1, Z_2, Z_3, Y) = (Z_2, Z_1, X, Z_3, Y) = -(Z_2, Z_3, Y, Z_1, X) \\
&= -(Z_3, Z_2, Y, Z_1, X) = (Z_3, Z_1, X, Z_2, Y) = (Z_1, Z_3, X, Z_2, Y) \\
&= -(Z_1, Z_2, Y, Z_3, X) = -(Y, Z_1, Z_2, Z_3, X) = 0.
\end{aligned}
$$

Hence, $\langle K(Z_1)\alpha_f(Y, Z_2), \alpha_f(Z_3, X)\rangle = 0$, and this concludes the proof. ∎

Proceeding with the proof of the theorem, we claim that $K = 0$, that is, τ is parallel. Let us assume otherwise. Define $\phi \colon \Omega \times TM \to S$ by $\phi = \alpha_S|_{\Omega \times TM}$, where

$$S = \operatorname{span}\{K(X)L \colon X \in TM\}$$

and $\alpha_S = \pi_S \circ \alpha_f$. It follows from (15) that

$$S = \operatorname{span}\{\phi(\Omega, X) \colon X \in TM\}. \tag{16}$$

Take $Y \in TM$ so that $K(Y)$ has maximal rank. Set $\Omega_Y = \ker \phi(\,\cdot\,, Y)$. Then,

$$\dim \Omega_Y \geq n - q - k \tag{17}$$

from (14), where $k = \dim \phi(\Omega, Y)$. By Lemma 9, we have

$$K(Y)\phi(\Omega_Y, X) = K(X)\phi(\Omega_Y, Y) = 0 \quad \text{for all} \quad X \in TM.$$

Being $K(Y)$ skew-symmetric, we easily obtain that

$$\alpha_{K(Y)L}(\Omega_Y, X) = 0 \quad \text{for all} \quad X \in TM.$$

Set $\dim K(Y)L = r \geq 2$. From our assumption on ν_r^f and (17), we get

$$n - q - 2r - 1 \geq \nu_r^f \geq \dim \Omega_Y \geq n - q - k.$$

Hence, $5 \leq 2r + 1 \leq k \leq p \leq 6$. Thus, $r = 2$, $k = 5, 6$ and $\dim S = 5, 6$. From (16), there is $Y_1 = Y, Y_2 \in TM$ such that $S = \phi(\Omega, Y_1) + \phi(\Omega, Y_2)$. By Lemma 9, we have

$$
\begin{aligned}
S &= \operatorname{span}\{K(X)S \colon X \in TM\} \\
&= \operatorname{span}\{K(X)(\phi(\Omega, Y_1) + \phi(\Omega, Y_2)) \colon X \in TM\} \\
&\subseteq K(Y_1)L + K(Y_2)L,
\end{aligned}
$$

which implies that $\dim S \leq 4$, a contradiction. This proves the claim.

To reduce the proof of the theorem to Proposition 8, all we have to show is that the condition $W \cap L = 0$ is satisfied. This is divided in two cases.

Case 1. Assume $\dim \Omega > n - q$. We claim that L is parallel along Ω. The subspaces $D = S(\gamma)$ have constant dimension by assumption, hence, there is a smooth orthogonal splitting $L^\perp = D \oplus D^\perp$. Taking the difference between the Codazzi equations for f and g and using that τ is parallel gives for the second fundamental form of g that

$$A_{(\widehat{\nabla}^\perp_X \xi)_D} Y = A_{(\widehat{\nabla}^\perp_Y \xi)_D} X \quad \text{for all} \quad X, Y \in TM \;\text{ and }\; \xi \in L. \tag{18}$$

Taking $X \in \Omega$ in (18), we get

$$\widehat{\nabla}^\perp_X \xi \subset D^\perp \quad \text{for all} \quad X \in \Omega \;\text{ and }\; \xi \in L. \tag{19}$$

The Codazzi equation also yields

$$A_{(\widehat{\nabla}^\perp_X \eta)_L} Y = A_{(\widehat{\nabla}^\perp_Y \eta)_L} X \quad \text{for all} \quad X, Y \in \Omega \;\text{ and }\; \eta \in D^\perp. \tag{20}$$

For $\eta \in D^\perp$, consider the linear map $\phi_\eta \colon \Omega \to L$ defined as $\phi_\eta(X) = (\widehat{\nabla}^\perp_X \eta)_L$, and set $r = \dim \mathrm{Im}\ \phi_\eta$. It follows from (20) that

$$\langle \alpha(\ker \phi_\eta, TM), \mathrm{Im}\ \phi_\eta \rangle = 0.$$

Thus, $\nu_r^f \geq \dim \Omega - r$. This is not possible by (14) and the hypothesis on ν_r^f unless $r = 0$, that is,

$$\widehat{\nabla}^\perp_X \xi \subset D \quad \text{for all} \quad X \in \Omega \;\text{ and }\; \xi \in L, \tag{21}$$

and the claim follows from (19) and (21).

From the Codazzi equation and the claim, we have that

$$\langle \nabla_Z Y, A_\xi X \rangle = \langle \alpha(Z, Y), \widehat{\nabla}^\perp_X \xi \rangle$$

for all $Z, Y \in \Omega$ and $\xi \in L^\perp$, or equivalently,

$$\widetilde{\nabla}_Z Y \perp W \quad \text{for all} \quad Z, Y \in \Omega. \tag{22}$$

Assume that there is a normal vector $0 \neq \xi \in W \cap L$. Then (22) implies that

$$\langle A_\xi Y, Z \rangle = 0 \quad \text{for all} \quad Y, Z \in \Omega.$$

It follows easily that $\nu_1^f \geq 2\dim\Omega - n \geq n-2q+2$, which is in contradiction with our assumption on ν_1^f.

Case 2. Assume $\dim\Omega = n-q$. We follow closely the argument in [**DT**]. It is shown there that γ smoothly decomposes as the orthogonal sum of one-dimensional orthogonal forms of rank one, namely,

$$\gamma = \gamma_1 \oplus \cdots \oplus \gamma_q,$$

with correspondent linearly independent unit eigenvectors $Z_1, \ldots, Z_q$ and nonzero eigenvalues $\lambda_1, \ldots \lambda_q$. Therefore, there exists an orthonormal basis $\{\xi_1, \ldots, \xi_q\}$ of $L^\perp = D$ such that

$$\gamma_j(X,Y) = \lambda_j \langle X, Z_j\rangle\langle Y, Z_j\rangle \xi_j, \quad 1 \leq j \leq q.$$

In fact, the existence of such a decomposition goes back to Cartan ([**Ca**]). It follows from (18) that

$$\sum_{k=1}^{q} \lambda_k \left(\langle \widehat{\nabla}^\perp_{Z_i}\eta, \xi_k\rangle\langle Z_j, Z_k\rangle - \langle \widehat{\nabla}^\perp_{Z_j}\eta, \xi_k\rangle\langle Z_i, Z_k\rangle \right) Z_k = 0$$

for any $\eta \in L$. We easily get that $(\widehat{\nabla}^\perp_Y \xi_k)_L = 0$, for any $Y \perp Z_k$. We conclude that $W = \text{span}\{\lambda_j Z_j - (\widehat{\nabla}^\perp_{Z_j}\xi_j)_L,\ 1 \leq j \leq q\}$, and thus $W \cap L = 0$. ∎

When $p = 1$ and $q \geq 5$, the additional assumption in [**DT**] is that M^n is not $(n-q+1)$–ruled. Our additional assumption on ν_1^f is used in Case 1 to make sure that Γ has dimension p. Thus, when $p = 1$ it just assures that $\Gamma \neq 0$. But if $\Gamma = 0$, then Ω is clearly totally geodesic in the ambient space, that is, the submanifold is ruled by the leaves of Ω. Hence, both assumptions are equivalent.

Proof of Theorem 2: Let $U \subset M^n$ be an open connected neighborhood of x_0 where $\nu_s^f \leq n-q-2s-1$ for $1 \leq s \leq p$. By Theorem 3, we have that α_g decomposes as in (2) along U. Moreover. since γ is flat and traceless, an elementary argument shows that $\gamma = 0$. From Theorem A in [**DR**] (or Theorem 4.5 in [**D**]), it follows that g reduces codimension to p. Thus g is congruent to f along U, and the result follows using that minimal Euclidean submanifolds are real-analytic. ∎

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