Singular genuine rigidity

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Abstract

We extend the concept of genuine rigidity of submanifolds by allowing mild singularities, mainly to obtain new global rigidity results and unify the known ones. As one of the consequences, we simultaneously extend and unify Sacksteder and Dajczer-Gromoll theorems by showing that any compact n-dimensional submanifold of $\mathbb{R}^{n+p}$ is singularly genuinely rigid in $\mathbb{R}^{n+q}$, for any $q < \min\{5, n\} - p$. Unexpectedly, the singular theory becomes much simpler and natural than the regular one, even though all technical codimension assumptions, needed in the regular case, are removed.

1. Introduction

One of the most fundamental problems in submanifold theory is the (isometric) rigidity in space forms, i.e., whether an isometric immersion of a given Riemannian manifold is unique up to rigid motions. Satisfactory solutions to the local version of the problem in low codimension were obtained under certain nondegeneracy assumptions on the second fundamental form, like the ones in [12], [1], [2], [9] and [4]. Recently, the concept of rigidity was extended to the one of genuine rigidity in order to deal with deformations that arise as deformations of submanifolds of larger dimension; see [6] and [11]. This reduction is important since the difficulties in understanding rigidity aspects of submanifolds grow together with the codimensions, not with the dimensions. This concept also allowed to generalize and unify the papers mentioned above, among others, by treating them under a common framework.

Global rigidity results are considerably more difficult to obtain. The most important is the beautiful classical Sacksteder's theorem [14], which states that a compact Euclidean hypersurface is rigid provided its set of totally geodesic points does not disconnect the manifold. Outside the hypersurfaces realm there is only the 23 year old paper [8], where Dajczer and Gromoll showed that, along each connected component of an open dense subset, any compact Euclidean submanifold in codimension 2 is either genuinely rigid or a submanifold of a special kind of deformable hypersurface. Although

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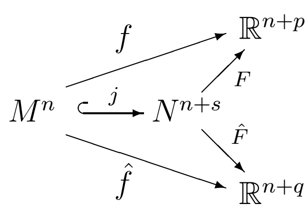
the authors did not have the tools to justify it at the time, they had to allow certain simple singularities in these hypersurfaces. The necessity to introduce singularities was justified only recently in [10], and this is precisely what motivated this work: to allow singularities in the genuine rigidity theory, mainly with the double purpose of obtaining new global results and unifying the known ones. In the process, we found out that introducing these mild singularities is quite natural and straightforward, even for local purposes, enabling us to substantially simplify the theory. In fact, after completing this work, we regard the presence of mild singularities in rigidity problems of submanifolds not only as a necessary assumption to obtain global results but, more importantly, as the natural setting for a deeper and better understanding of the phenomena in an area where singularities rarely appear.

In order to state our main results, let us introduce the main concepts. We say that a pair of isometric immersions $f : M^n \to \mathbb{R}^{n+p}$ and $\hat{f} : M^n \to \mathbb{R}^{n+q}$ *singularly extends isometrically* when there are an embedding $j : M^n \hookrightarrow N^{n+s}$ into a manifold N^{n+s} with $s > 0$, and isometric maps $F : N^{n+s} \to \mathbb{R}^{n+p}$ and $\hat{F} : N^{n+s} \to \mathbb{R}^{n+q}$ such that $f = F \circ j$ and $\hat{f} = \hat{F} \circ j$, with the set of points where F and $\hat{F}$ fail to be immersions (that may be empty) contained in $j(M)$. In other words, the isometric extensions F and $\hat{F}$ in the following commutative diagram are allowed to be singular, but only along $j(M)$:



(1)

An isometric immersion $\hat{f} : M^n \to \mathbb{R}^{n+q}$ is a *strongly genuine deformation* of a given isometric immersion $f : M^n \to \mathbb{R}^{n+p}$ if there is no open subset $U \subset M^n$ along which the restrictions $f|_U$ and $\hat{f}|_U$ singularly extend isometrically. Accordingly, the isometric immersion f is said to be *singularly genuinely rigid* in $\mathbb{R}^{n+q}$ for a fixed integer q if, for any given isometric immersion $\hat{f} : M^n \to \mathbb{R}^{n+q}$, there is an open dense subset $U \subset M^n$ such that $f|_U$ and $\hat{f}|_U$ singularly extend isometrically.

More geometrically, an isometric deformation of a Euclidean submanifold M^n is strongly genuine if no open subset of M^n is a submanifold of a higher dimensional (possibly singular) isometrically deformable submanifold, in such a way that the isometric deformation of the former is induced by an isometric deformation of the latter, while (possibly) including singularities along M^n. The key point here is that, since all our extensions are ruled, the singularities that eventually appear are quite mild and easy to understand, as it is classically done in the classification of flat and ruled surfaces in $\mathbb{R}^3$.

The following is our main global result. Recall that an immersion f is called D^d*-ruled* if $D^d \subset TM$ is a totally geodesic distribution whose leaves are mapped by f to (open subsets of) d-dimensional affine subspaces.

Theorem 1. *Let $f : M^n \to \mathbb{R}^{n+p}$ and $\hat{f} : M^n \to \mathbb{R}^{n+q}$ be isometric immersions of a compact Riemannian manifold with $p + q < n$. Then, along each connected component of an open dense subset of M^n, either f and $\hat{f}$ singularly extend isometrically, or f and $\hat{f}$ are mutually D^d-ruled, with $d \geq n - p - q + 3$.*

In particular, for $p+q \leq 4$, Theorem 1 easily unifies Sacksteder and Dajczer-Gromoll Theorems in [8] and [14] cited above, states that the only way to isometrically immerse a compact Euclidean hypersurface in codimension 3 is through compositions (which in turn were classified in [3] and [10]), and provides a global version of the main result in [7]:

Corollary 2. *Any compact isometrically immersed submanifold M^n of $\mathbb{R}^{n+p}$ is singularly genuinely rigid in $\mathbb{R}^{n+q}$ for $q < min\{5, n\} - p$.*

From Theorem 1 we get the following topological criteria for singular genuine rigidity, without any *a priori* assumption on the codimensions, that extends Theorem 7 in [6]:

Corollary 3. *Let M^n be a compact manifold whose k-th Pontrjagin class satisfies that $[p_k] \neq 0$ for some $k > \frac{3}{4}(p+q-3)$. Then, any analytic immersion $f : M^n \to \mathbb{R}^{n+p}$ (with the induced metric) is singularly genuinely rigid in $\mathbb{R}^{n+q}$ in the C^∞-category.*

Our global results are based on a local analysis whose main tool is the bilinear form that we construct next. Consider a pair of isometric immersions $f : M^n \to \mathbb{R}^{n+p}$ and $\hat{f} : M^n \to \mathbb{R}^{n+q}$. Let

$$\tau \colon L^\ell \subset T_f^\perp M \to \hat{L}^\ell \subset T_{\hat{f}}^\perp M$$

be a vector bundle isometry and suppose that it preserves the second fundamental forms and the normal connections restricted to the rank ℓ vector normal subbundles L^ℓ and $\hat{L}^\ell$. Equivalently, its natural extension $\bar{\tau} = Id \oplus \tau \colon TM \oplus L^\ell \to TM \oplus \hat{L}^\ell$ is a parallel bundle isometry. Let $\phi_\tau \colon TM \times (TM \oplus L^\ell) \to L^\perp \times \hat{L}^\perp$ be the flat bilinear form given by

$$\phi_\tau(X, v) = \left((\tilde{\nabla}_X v)_{L^\perp}, (\tilde{\nabla}_X \bar{\tau} v)_{\hat{L}^\perp}\right), \quad X \in TM,\ v \in TM \oplus L^\ell,$$

where $\tilde{\nabla}$ stands for the connection in Euclidean space and $L^\perp \times \hat{L}^\perp$ is endowed with the semi-Riemannian metric $\langle\, ,\rangle = \langle\, ,\rangle|_{L^\perp} - \langle\, ,\rangle|_{\hat{L}^\perp}$. A subset $S \subset L^\perp \oplus \hat{L}^\perp$ is called *null* if $\langle \eta, \xi\rangle = 0$ for all $\eta, \xi \in S$. Given a distribution D on M^n, denote by $\mathcal{O}(D)$ the smallest totally geodesic distribution of M^n that contains D.

We can now state our main local result, which applies even to $\ell = 0$ and $\tau = 0$.

Theorem 4. *Let $\hat{f} : M^n \to \mathbb{R}^{n+q}$ be a strongly genuine deformation of $f : M^n \to \mathbb{R}^{n+p}$ and $\tau : L^\ell \subset T_f^\perp M \to \hat{L}^\ell \subset T_{\hat{f}}^\perp M$ a parallel vector bundle isometry that preserves second fundamental forms. Let $D \subset TM \oplus L^\ell$ be a subbundle such that $\phi_\tau(TM, D)$ is a null subset. Then $D \subset TM$ and, along each connected component of an open dense subset of M^n, f and $\hat{f}$ are mutually $\mathcal{O}(D)$-ruled.*

The key advantage of Theorem 4 over the local rigidity results of this kind is that it deals with easily to construct null subsets instead of nullity distributions of flat bilinear forms. A good example of an application of this is the following singular version of Theorem 1 in [6] removing the technical assumption on the codimensions. Recall that $Y \in TM$ is a *regular element* of ϕ_τ if rank $(\phi_\tau^Y) = i(\phi_\tau)$, where $\phi_\tau^Y = \phi_\tau(Y, \cdot)$ and

$$i(\phi_\tau) := \max\{\text{rank}(\phi_\tau^X) \colon X \in TM\}.$$

Denote by $RE(\phi_\tau) \subset TM$ the open dense subset of regular elements of ϕ_τ. Using a well-known property of flat bilinear forms we immediately conclude from Theorem 4:

Corollary 5. *Under the assumptions of Theorem 4, along each connected component of an open dense subset of M^n, f and $\hat{f}$ are mutually $\mathcal{O}(D^d_Y)$-ruled, where $D^d_Y := \ker(\phi^Y_\tau) \subset TM$ for any $Y \in RE(\phi_\tau)$, and $d = n + \ell - i(\phi_\tau) \geq n - p - q + 3\ell$.*

As it is clear from the statements, the rulings in the above are larger and easier to compute than the ones in the main result in [6]. The bundles obtained in this work are also better suited for certain global applications.

By allowing singular extensions we recover all the corollaries in [6], even without the technical restrictions on the codimensions required there. For example, from Corollary 5 we conclude the following extension of Corollary 5 in [6].

Corollary 6. *Any isometrically immersed submanifold M^n of $\mathbb{R}^{n+p}$ with positive Ricci curvature is singularly genuinely rigid in $\mathbb{R}^{n+q}$, for every $q < n - p$.*

As we will see, the proof of the local Theorem 4 works for any simply connected space form. Moreover, the global Theorem 1 and Corollary 2 still hold for complete submanifolds, even if the ambient space is the hyperbolic space, as long as one of the immersions is bounded. For complete submanifolds in the round sphere we show:

Theorem 7. *Let $f : M^n \to \mathbb{S}^{n+p}$ and $\hat{f} : M^n \to \mathbb{S}^{n+q}$ be isometric immersions of a complete submanifold with $p + q < n - \mu_n$. Then, along each connected component of an open dense subset of M^n, either f and $\hat{f}$ singularly extend isometrically, or f and $\hat{f}$ are mutually D^d-ruled, with $d \geq n - p - q + 3$.*

In the above statement μ_n is defined as $\mu_n = \max\{k : \rho(n-k) \geq k+1\}$, where $\rho(m) - 1$ is the maximum number of pointwise linearly-independent vector fields on $\mathbb{S}^{m-1}$ and is given by $\rho((odd)2^{4d+b}) = 8d + 2^b$, for any nonnegative integer d and $b \in \{0, 1, 2, 3\}$. Some values of μ_n are: $\mu_n = n -$ (highest power of $2 \leq n$) for $n \leq 24$, $\mu_n \leq 8d - 1$ for $n < 16^d$ and $\mu_{2^d} = 0$.

From Theorem 7 we conclude the analogous to Corollary 2 when the ambient space is the sphere:

Corollary 8. *Any complete isometrically immersed submanifold M^n of $\mathbb{S}^{n+p}$ is singularly genuinely rigid in $\mathbb{S}^{n+q}$ for $q \leq 3-p$ if $4 \leq n \leq 7$, or $q \leq 4-p$ if $n \geq 8$.*

The paper is organized as follows. In Section 2 we first provide the basic properties of the bilinear form ϕ_τ, and then we show how it can be used to obtain regular and singular isometric extensions, which is all that is needed to prove our local results. Section 3 is devoted to revisit the theory of compositions using ϕ_τ. As an application we show that, generically, $(n-1)$-ruled submanifolds are compositions. In Section 4 we prove Theorem 1, and Section 5 is dedicated to the proof of Theorem 7.

2. The flat bilinear form ϕ_τ

In this section we study in detail the properties of the bilinear form ϕ_τ, which was introduced in [6] but not used in its full strength. We will see that it is a powerful tool to deal with isometric rigidity problems.

Consider two isometric immersions $f : M^n \to \mathbb{R}^{n+p}$ and $\hat{f} : M^n \to \mathbb{R}^{n+q}$ with second fundamental forms α and $\hat{\alpha}$ and normal connections $\nabla^\perp$ and $\hat{\nabla}^\perp$ defined on their normal bundles $T_f^\perp M$ and $T_{\hat{f}}^\perp M$, respectively. Endow $T_f^\perp M \times T_{\hat{f}}^\perp M$ with its natural semi-Riemannian metric $\langle\, ,\rangle$ of type (p,q) and compatible connection $\bar{\nabla}$,

$$\langle (\xi,\hat{\xi}),(\eta,\hat{\eta})\rangle = \langle \xi,\eta\rangle_{T_f^\perp M} - \langle \hat{\xi},\hat{\eta}\rangle_{T_{\hat{f}}^\perp M}, \qquad \bar{\nabla}_X(\xi,\hat{\xi}) = \left(\nabla_X^\perp \xi, \hat{\nabla}_X^\perp \hat{\xi}\right),$$

for $\xi,\eta \in T_f^\perp M$, $\hat{\xi},\hat{\eta} \in T_{\hat{f}}^\perp M$, and $X \in TM$. By the Gauss equation, the symmetric bilinear form $\beta = \alpha \oplus \hat{\alpha} : TM \times TM \to T_f^\perp M \times T_{\hat{f}}^\perp M$ is *flat*, that is,

$$\langle \beta(X,Y),\beta(Z,T)\rangle = \langle \beta(X,T),\beta(Z,Y)\rangle,\ \forall X,Y,Z,T \in TM.$$

The concept of flat bilinear forms was introduced by Moore in [13] to study isometric immersions of the round sphere in Euclidean space in low codimension, and was used afterwards in several papers about isometric rigidity, even implicitly, following a remark also in [13]. For example, it can be used to prove the classical Beez-Killing theorem in [12], in which case the objective is to show that $\operatorname{Im}(\beta)$ is everywhere a null set. Notice that flatness makes sense even for nonsymmetric bilinear forms.

Outside the realm of hypersurfaces it is important to obtain information about the normal connections too, so a different (nonsymmetric) flat bilinear form is needed. Yet, unexpectedly and in contrast to the strongest known local rigidity results, we will not make use of *a priori* nullity estimates like the one in Theorem 3 in [4] in ours since we will not deal with nullity spaces. In particular, this will allow us to get rid of the usual technical constraints on the codimensions.

Throughout this work,

$$\tau : L^\ell \subset T_f^\perp M \to \hat{L}^\ell \subset T_{\hat{f}}^\perp M$$

will denote a vector bundle isometry that preserves the induced second fundamental forms and normal connections in the rank ℓ normal subbundles L and $\hat{L}$. That is, $\tau\circ\alpha_L=\hat{\alpha}_{\hat{L}}$, and $\tau(\nabla^\perp_X\xi)_L=(\hat{\nabla}^\perp_X\tau\xi)_{\hat{L}}$ for every $X\in TM$, $\xi\in L$, where we represent the orthogonal projections onto L and $\hat{L}$ with the corresponding subindexes. Equivalently, its natural extension

$$\overline{\tau}=Id\oplus\tau:TM\oplus L\to TM\oplus\hat{L},$$

is a parallel vector bundle isometry. Let $\phi_\tau:TM\times(TM\oplus L)\to L^\perp\times\hat{L}^\perp\subset T^\perp_fM\times T^\perp_{\hat{f}}M$ be the bilinear form defined as

$$\phi_\tau(X,v)=\left((\tilde{\nabla}_Xv)_{L^\perp},(\tilde{\nabla}_X\overline{\tau}v)_{\hat{L}^\perp}\right),$$

where $\tilde{\nabla}$ denotes the connection of the Euclidean ambient spaces. Notice that, if $\ell=0$, then $\tau=0$ and $\phi_0=\beta$. On $L^\perp\times\hat{L}^\perp$ we will always consider the semi-Riemannian metric and compatible connection induced from the ones in $T^\perp_fM\times T^\perp_{\hat{f}}M$, still denoted by $\langle\,,\rangle$ and $\bar{\nabla}$, respectively.

The main two properties of ϕ_τ are given by the following.

Proposition 9. *ϕ_τ is a flat Codazzi tensor.*

Proof: For $X,Y\in TM$ and $v,w\in TM\oplus L$, using that $\bar{\tau}$ is parallel we get

$$\begin{aligned}\langle\phi_\tau(X,v),\phi_\tau(Y,w)\rangle&=\langle(\tilde{\nabla}_Xv)_{L^\perp},(\tilde{\nabla}_Yw)_{L^\perp}\rangle-\langle(\tilde{\nabla}_X\overline{\tau}v)_{\hat{L}^\perp},(\tilde{\nabla}_Y\overline{\tau}w)_{\hat{L}^\perp}\rangle\\&=\langle\tilde{\nabla}_Xv,\tilde{\nabla}_Yw\rangle-\langle\tilde{\nabla}_X\overline{\tau}v,\tilde{\nabla}_Y\overline{\tau}w\rangle\\&=-\langle v,\tilde{\nabla}_Y\tilde{\nabla}_Xw\rangle+\langle\overline{\tau}v,\tilde{\nabla}_Y\tilde{\nabla}_X\overline{\tau}w\rangle\\&=-\langle v,\tilde{\nabla}_X\tilde{\nabla}_Yw\rangle+\langle\overline{\tau}v,\tilde{\nabla}_X\tilde{\nabla}_Y\overline{\tau}w\rangle\\&=\langle\phi_\tau(Y,v),\phi_\tau(X,w)\rangle.\end{aligned}$$

The very same approach shows that ϕ_τ is a Codazzi tensor, that is,

$$(\bar{\nabla}_X\phi_\tau)(Y,v):=\bar{\nabla}_X\phi_\tau(Y,v)-\phi_\tau(\nabla_XY,v)-\phi_\tau(Y,(\tilde{\nabla}_Xv)_{TM\oplus L})=(\bar{\nabla}_Y\phi_\tau)(X,v),$$

so we left the computation to the reader. ∎

Denote the left nullity space of ϕ_τ by Δ_τ and its dimension by ν_τ, i.e.,

$$\Delta_\tau:=\{X\in TM:\phi_\tau(X,\cdot\,)=0\},\qquad\nu_\tau:=\dim\Delta_\tau,\tag{2}$$

and let $U\subset M^n$ be a connected component of an open dense subset where ν_τ is locally constant. Since ϕ_τ is a Codazzi tensor, Δ_τ is a smooth integrable distribution on U.

Corollary 10. *The space $\mathrm{Im}\,(\phi_\tau)^\perp$ is parallel along the leaves of Δ_τ in U. In particular, both the nullity space and the light cone bundle of $\langle\,,\rangle|_{\mathrm{Im}\,(\phi_\tau)^\perp}$ are smooth and parallel along these leaves on any open subset $U'\subset U$ where they have constant dimension.*

Proof: The parallelism of $\mathrm{Im}\,(\phi_\tau)^\perp$ is a consequence of the fact that ϕ_τ is a Codazzi tensor, since $\bar\nabla_X(\phi_\tau(Y,v)) = \phi_\tau([X,Y],v) + \phi_\tau(Y,(\tilde\nabla_X v)_{TM\oplus L}) \in \mathrm{span}\,\mathrm{Im}\,(\phi_\tau)$ for every $X \in \Delta_\tau$, $Y \in TM$, $v \in TM \oplus L$. The last assertion follows from the compatibility of $\bar\nabla$ with respect to $\langle\, ,\, \rangle$. ∎

Observe that such a τ as above arises naturally when f and $\hat f$ singularly extend isometrically. Indeed, with the notations in Diagram 1 in the Introduction, if F and $\hat F$ are regular we just take $L^s = F_*(T_j^\perp M)$, $\hat L^s = \hat F_*(T_j^\perp M)$, and $\tau(F_*(\xi)) = \hat F_*(\xi)$. If they are not regular, at least locally we can consider a sequence of submanifolds $j_k\colon M^n_k \to N^{n+s} \setminus j(M^n)$ smoothly converging to j as $k \to \infty$, and then take L^s and $\hat L^s$ as accumulation of $F_*(T^\perp_{j_k} M_k)$ and $\hat F_*(T^\perp_{j_k} M_k)$, respectively. In particular, we have:

Lemma 11. *The metric in* $\mathrm{Im}\,(\beta)^\perp \subset T_f^\perp M \oplus T_{\hat f}^\perp M$ *is almost everywhere definite if and only if* $\tau = 0$ *is locally the only vector bundle isometry preserving second fundamental forms. In this situation,* $\hat f$ *is a strongly genuine deformation of* f.

Proof: Both conditions are clearly equivalent to the non-existence of unit vector fields $\xi \in T_f^\perp M$ and $\hat\xi \in T_{\hat f}^\perp M$ defined on some open subset $U \subset M^n$ such that $A_\xi = \hat A_{\hat\xi}$. ∎

The form ϕ_τ and genuine rigidity

In general, we show that a pair of isometric immersions $\{f, \hat f\}$ as above is genuine, i.e., each one is a (regular) genuine deformation of the other, by explicitly constructing, locally almost everywhere, isometric immersions $j : M^n \to N^{n+s}$, $F : N^{n+s} \to \mathbb{R}^{n+p}$ and $\hat F : N^{n+s} \to \mathbb{R}^{n+q}$ as in Diagram 1, that is, satisfying $f = F \circ j$ and $\hat f = \hat F \circ j$. Usually, we also require F and $\hat F$ to be ruled extensions of f and $\hat f$ since a genuine pair must be mutually ruled by the main result in [6]. Since in this paper we work with singular extensions, the ruled ones have the additional advantage that their singularities are quite easy to characterize and deal with.

In order to build ruled extensions of f and $\hat f$, choose any smooth rank s subbundle $\Lambda \subset TM \oplus L$, and define the maps $F = F_{\Lambda,f} : \Lambda \to \mathbb{R}^{n+p}$ and $\hat F = F_{\Lambda,\hat f} : \Lambda \to \mathbb{R}^{n+q}$ as

$$F(v) = f(p) + v, \quad \hat F(v) = \hat f(p) + \overline{\tau} v, \quad v \in \Lambda_p, \;\; p \in M^n. \tag{3}$$

One of the main reasons that make the form ϕ_τ useful in any flavour of genuine rigidity is that it gives the precise condition that guarantees that these two maps are isometric:

Proposition 12. *The maps* F *and* $\hat F$ *in* (3) *are isometric if and only if* $\phi_\tau(TM, \Lambda)$ *is a null set.*

Proof: For every smooth section v of Λ and $Z \in TM$ since $\overline{\tau}$ is parallel we have $\|(F\circ v)_* Z\|^2 - \|(\hat F \circ v)_* Z\|^2 = \|Z + \tilde\nabla_Z v\|^2 - \|Z + \tilde\nabla_Z \overline{\tau} v\|^2 = \|(\tilde\nabla_Z v)_{L^\perp}\|^2 - \|(\tilde\nabla_Z \overline{\tau} v)_{\hat L^\perp}\|^2 = \langle \phi_\tau(Z,v), \phi_\tau(Z,v)\rangle$. ∎

In particular, if in addition $\Lambda \cap TM = 0$, both maps are immersions in a neighborhood N^{n+s} of the 0-section of Λ, and thus induce the same Riemannian metric on N^{n+s}. Therefore F and $\hat{F}$ are (regular) isometric ruled extensions of f and $\hat{f}$. Similarly, if $\Lambda \not\subset TM$, along each open subset $U \subset M^n$ where the subspaces $\Lambda' = \Lambda \cap (\Lambda \cap TM)^\perp$ have locally constant dimension $s' > 0$, we have that the restrictions $F'|_{\Lambda'} = F_{\Lambda', f|_U}$ and $\hat{F}'|_{\Lambda'} = F_{\Lambda', \hat{f}|_U}$ also give (regular) isometric ruled extensions of $f|_U$ and $\hat{f}|_U$ defined in a neighborhood $N^{n+s'}$ of the 0-section of Λ' along U.

We proceed to characterize singular ruled extensions, that occur above when $\Lambda \subset TM$. We say that $F = F_{\Lambda,f}$ in (3) is a *singular extension of* f if it is an immersion in some open neighborhood of the 0-section of Λ, except of course at the 0-section itself. We say that F *nowhere induces a singular extension of* f if, for every open subset $U \subset M^n$ and every subbundle $\Lambda' \neq 0$ of $\Lambda|_U$, the restriction of F to Λ' is not a singular extension of $f|_U$. We show next that F nowhere induces a singular extension of f only when the latter is $\mathcal{O}(\Lambda)$-ruled, where $\mathcal{O}(\Lambda)$ denotes the smallest totally geodesic distribution of M^n that contains Λ.

Proposition 13. *Let $f : M^n \to \mathbb{R}^{n+p}$ be an isometric immersion and $\Lambda \subset TM$ a smooth distribution. Then, $F_{\Lambda,f}$ nowhere induces a singular extension of f if and only if f is $\mathcal{O}(\Lambda)$-ruled along each connected component of an open dense subset of M^n.*

Proof: Clearly, it is enough to give a proof for the direct statement and for a rank one distribution, i.e., $\Lambda = \text{span}\,\{X\}$ for some nonvanishing vector field X on M^n. Consider the map $F : \Lambda \cong M^n \times \mathbb{R} \to \mathbb{R}^{n+p}$ given by (3), that is, $F(p,t) = f(p) + tX(p)$. This map will be a singular extension in some open neighborhood of $p \in M^n$ if and only if it is an immersion in a neighborhood of $(p, 0)$, except at the points in $M^n \times \{0\}$. Therefore, for all $p \in M^n$ there exists a sequence $(p_m, t_m) \to (p, 0)$, with $t_m \neq 0$, such that $\text{rank}(F_{*(p_m,t_m)}) = n$. Define the tensors $K(Z) = \nabla_Z X$ and $H_t(Z) = Z + tK(Z)$ for $Z \in TM$. Thus, there is $Y_m \in T_{p_m}M$ such that $F_{*(p_m,t_m)} Y_m = X(p_m)$, since $H_t \to Id$ as $t \to 0$ and

$$F_*\partial_t = X, \qquad F_*Z = H_t(Z) + t\alpha(X, Z), \quad \forall\, Z \in TM.$$

Let S_X be the K-invariant subspace generated by X,

$$S_X = \text{span}\,\{X, K(X), K^2(X), K^3(X), ...\}.$$

Observe that the equality $F_{*(p_m,t_m)} Y_m = X(p_m)$ is equivalent to $H_{t_m} Y_m = X(p_m)$ and $\alpha(X(p_m), Y_m) = 0$. In particular, if t_m is sufficiently small,

$$\alpha(X(p_m), H_{t_m}^{-1}(X(p_m))) = 0 \tag{4}$$

and $\lim_{m\to\infty} H_{t_m}^{-1}(X(p_m)) = X(p)$. Consider a precompact open neighborhood $U \subset M^n$ of p, so $\|\alpha\| < c$ and $\|K\| < c$ for some constant $c > 1$. Hence for $t \in I = (-\frac{1}{c^2}, \frac{1}{c^2})$ we have that H_t is invertible on U, and

$$H_t^{-1} = \sum_{i \geq 0} (-t)^i K^i,$$

since $H_t(\sum_{i=0}^{N}(-t)^iK^i) = Id - (-t)^{N+1}K^{N+1}$.

We claim that $\alpha(X, S_X) = 0$ along M^n. Assume otherwise, define $j := \min\{k \in \mathbb{N} : \alpha(X(q), K^k(X(q))) \neq 0,\ q \in M^n\}$ and take $p \in M^n$ such that $\alpha(X(p), K^j(X(p))) \neq 0$. By (4) we obtain that

$$\sum_{i \geq j}(-t_m)^i\alpha(X(p_m), K^i(X(p_m))) = 0.$$

Dividing the above by t_m^j and taking $m \to \infty$ we conclude that $\alpha(X(p), K^j(X(p))) = 0$, which is a contradiction.

Now, since $\alpha(X, S_X) = 0$ on M^n, for any $t \in I$ and $p \in U$ we get $F_{*(p,t)}(H_t^{-1}(X)) = X$ since $H_t^{-1}(X) \in S_X$. It follows that $\operatorname{rank}(F_*) = n$ in all $U \times I$, and therefore $F(U \times I) = f(U)$. Hence a segment of the line generated by X is contained in $f(U)$. In particular, f is $\mathcal{O}(\Lambda)$-ruled along each connected component of an open dense subset of M^n where $\mathcal{O}(\Lambda)$ has locally constant dimension. ∎

We are now able to prove our main local result.

Proof of Theorem 4. Locally, if $D \not\subset TM$ along some open set U then we have regular isometric extensions of $f|_U$ and $\hat{f}|_U$ by extending them as in (3) along any subbundle $\Lambda \subset D$ such that $D = (D \cap TM) \oplus \Lambda$. Hence, $D \subset TM$ and by Proposition 13 we conclude that f and $\hat{f}$ are mutually $\mathcal{O}(D)$-ruled almost everywhere. ∎

The following lemma due to Moore [13] immediately gives Corollary 5 by applying Theorem 4 to τ and $D^d = \ker(\phi_\tau^Y)$, since it tells us that $\phi_\tau(TM, \ker(\phi_\tau^Y))$ is null.

Lemma 14. *Let $\varphi : \mathbb{V} \times \mathbb{V}' \to \mathbb{W}$ be a flat bilinear form, and set $\varphi^X = \varphi(X, \cdot)$. Then,*

$$\varphi(\mathbb{V}, \ker(\varphi^X)) \subset \operatorname{Im}(\varphi^X) \cap \operatorname{Im}(\varphi^X)^\perp, \quad \forall X \in RE(\varphi).$$

In particular, if the inner product in $\mathbb{W}$ is definite, we have that $\ker(\varphi^X) = \mathcal{N}(\varphi)$ for all $X \in RE(\varphi)$, where $\mathcal{N}(\varphi) := \{w \in \mathbb{V}' : \varphi(\cdot, w) = 0\}$ is the (right) nullity of φ.

Remark 15. While Corollary 5 with its estimate $d \geq n - p - q + 3\ell$ is immediate from Lemma 14, the corresponding regular result, Theorem 14 in [6], requires 5 pages just to give a computer assisted proof of the estimate on d, uses the very long and technical Theorem 3 in [4], and therefore it is only valid for $\min\{p, q\} \leq 5$; see [5]. The simplifications gained with the singular theory reside in the fact that, while here we use Lemma 14 to easily obtain null subsets, the main results in [6] require the computation of estimates of ranks of several bundles and nullities of trickily constructed bilinear forms.

Remark 16. In several applications we have that $D = \mathcal{N}(\alpha_{L^\perp}) \cap \mathcal{N}(\hat{\alpha}_{\hat{L}^\perp})$ even in the singular case. For example, this is the case if $d = n - p - q + 3\ell$ in Corollary 5, or if $\ell = \min\{p, q\}$, or if one of the codimensions is low enough. In this situation, $L_D \subset L$, $\tau|_{L_D}$ is also parallel and preserves second fundamental forms, and therefore we recover the structure of the normal bundles in Theorem 1 in [6]; see Lemma 20 below.

3. Compositions revisited through ϕ_τ

In this section we revisit the theory of compositions using the form ϕ_τ.

Let $f : M^n \to \mathbb{R}^{n+p}$ be an isometric immersion of a simply connected Riemannian manifold M^n with second fundamental form α, and $L \subset T_f^\perp M$ a rank ℓ normal subbundle. Define the bilinear form

$$\phi_L : TM \times (TM \oplus L) \to L^\perp, \quad \phi_L(Z, v) = (\tilde{\nabla}_Z v)_{L^\perp}.$$

We can build another isometric immersion of M^n using ϕ_L when it is flat:

Proposition 17. *There exists an isometric immersion $\hat{f} : M^n \to \mathbb{R}^{n+\ell}$ and a parallel vector bundle isometry $\sigma : L \to T_{\hat{f}}^\perp M$ such that the second fundamental form of $\hat{f}$ is $\hat{\alpha} = \sigma \circ \alpha_L$ if and only if ϕ_L is flat. In this case, $\phi_L = \phi_\sigma$.*

Proof: By projecting the fundamental equations of f onto L we easily see that flatness of ϕ_L is equivalent to the fact that the pair $(\alpha_L, (\nabla^\perp)|_L)$ satisfies the fundamental equations of Euclidean submanifolds. Indeed, flatness of ϕ_L with the four vectors in TM is equivalent to Gauss equation, with three vectors in TM and one in L we get Codazzi equation, while two vectors in TM and two in L recovers Ricci equation. ∎

The following is a reinterpretation of Proposition 8 in [4], which is the main tool to construct compositions. Recall that, for f and $\hat{f}$ as in the previous section, we say that f is a *(regular) composition* of $\hat{f}$ when they extend isometrically as in Diagram 1 with $s = q$. In this case, $\hat{F}$ is a local isometry and thus, if $\hat{f}$ is an embedding, there is an open neighborhood $U \subset N^{n+q}$ of $j(M^n)$ and an isometric immersion $h = F|_U \circ (\hat{F}|_U)^{-1} : W \subset \mathbb{R}^{n+q} \to \mathbb{R}^{n+p}$ of the open subset $W = \hat{F}(U) \supset f(M)$ satisfying $f = h \circ \hat{f}$.

Proposition 18. *Suppose that ϕ_L is flat, and let $\hat{f}$ be given by Proposition 17. If $i(\phi_L)$ is constant and $i(\phi_L) = i(\alpha_{L^\perp})$, then f is a composition of $\hat{f}$.*

Proof: Observe that, since $\hat{L} = T_{\hat{f}}^\perp M$ and $\phi_L = \phi_\sigma$, the image of both ϕ_L and $\alpha_{L^\perp} = \phi_L|_{TM\times TM}$ are Riemannian. Thus, by Lemma 14 we have that $\ker(\phi_L^X) = \mathcal{N}(\phi_L)$ and $\ker(\alpha_{L^\perp}^X) = \mathcal{N}(\alpha_{L^\perp})$, for every $X \in RE(\phi_L) \cap RE(\alpha_{L^\perp})$. The result follows from Proposition 12 taking the rank ℓ subbundle $\Lambda = \mathcal{N}(\phi_L) \cap \mathcal{N}(\alpha_{L^\perp})^\perp$, which is transversal to M^n. ∎

By allowing singular flat extensions h, Proposition 13 provides a singular version of Proposition 18: if there is a rank ℓ subbundle $\Lambda \subset \mathcal{N}(\phi_L)$ such that $F_{D,f}$ is an immersion near the 0-section of Λ, except possibly along it, then f is locally almost everywhere a *singular composition* $f = h \circ \hat{f}$, where h is an immersion except (possibly) along $\hat{f}(M)$.

Remark 19. By Theorem 4 applied to $\tau = \sigma$, if $\hat{f}$ in Proposition 17 is a strongly genuine deformation of f, then they must be at least mutually $(n-p+2\ell)-$ruled, and by Proposition 18, $i(\alpha_{L^\perp}) < i(\phi_L) \le p - \ell$.

The $(n-1)$-ruled case

As an application of the above, here we study general $(n-1)$-ruled n-dimensional Euclidean submanifolds. We show that such a submanifold is locally a composition if its codimension is bigger than the rank of its curvature operator. Although this fact has independent interest, it will be used to prove Corollary 2.

Until the end of this section X, Y will denote vectors in a totally geodesic distribution $D \subset TM$, and $Z \in TM$.

Lemma 20. *If f is D-ruled, then the normal subbundle*

$$L_D := \operatorname{span} \alpha(TM, D) \subset T_f^\perp M \tag{5}$$

is parallel along D on any open subset V where $\ell_D := \dim L_D$ is constant.

Proof: Since D is totally geodesic, the lemma follows from Codazzi equation since

$$\nabla_X^\perp \alpha(Z, Y) = -\alpha(\nabla_X Z, Y) - \alpha(Z, \nabla_X Y) - \alpha(\nabla_Z X, Y) - \alpha(X, \nabla_Z Y) \in L_D. \quad \blacksquare$$

In particular, if rank $D = n-1$, $L_D \subset L$ and L is also parallel along D, our form ϕ_L is flat since $\phi_L(D^{n-1}, TM \oplus L) = 0$. Therefore Proposition 17 gives:

Corollary 21. *Suppose f is D^{n-1}-ruled and $L \subset T_f^\perp M$ is a rank ℓ normal subbundle parallel along D^{n-1} such that $L_D \subset L$. Then, there is a D^{n-1}-ruled isometric immersion*

$$\hat{f} : M^n \to \mathbb{R}^{n+\ell}$$

and a parallel vector bundle isometry $\sigma : L \to T_{\hat{f}}^\perp M$ such that the second fundamental form of $\hat{f}$ is $\hat{\alpha} = \sigma \circ \alpha_L$. In particular, taking V in Lemma 20 simply connected, there exists a D^{n-1}-ruled isometric immersion $f_D : V \subset M^n \to \mathbb{R}^{n+\ell_D}$ with $\hat{\alpha} = \sigma \circ \alpha_{L_D}$.

Notice that, if rank $D = n-1$, ℓ_D is intrinsic since it agrees with the rank of the curvature operator of M^n. Our purpose is to show that f is locally a composition of f_D.

Proposition 22. *Under the assumptions of Corollary 21, if $\operatorname{Im} \alpha(x) \not\subset L(x)$ for some $x \in M^n$, then f is a composition of $\hat{f}$ near x, that is, there is a neighborhood U of x and an isometric immersion $h : W \subset \mathbb{R}^{n+\ell} \to \mathbb{R}^{n+p}$ of an open set W containing $f(U)$ such that $f = h \circ \hat{f}$ on U.*

Proof: The hypothesis is equivalent to the existence of an orthogonal decomposition

$$T_f^\perp V = L \oplus \Sigma \oplus N,$$

on an open neighborhood V of x, where Σ is a line bundle and $N = \{\eta \in L^\perp : A_\eta = 0\}$. We proceed by induction on the codimension $p \geq \ell + 1$ of f. For $p = \ell + 1$ we get that f is a composition of $\hat{f}$ near x by Proposition 18 since $1 \leq i(\alpha_{L^\perp}) \leq i(\phi_L) \leq \operatorname{rank} L^\perp = 1$.

Suppose the lemma holds for $p-1$, and let $L' \subset T_f^\perp M$ be any subbundle of rank $p-1$ parallel along D with $L \subset L'$. By Corollary 21 there is a D^{n-1}-ruled isometric immersion $f' : M^n \to \mathbb{R}^{n+p-1}$ whose second fundamental form is $\sigma' \circ \alpha_{L'}$, where $\sigma' : L' \to T_{f'}^\perp M$ is a parallel bundle isometry.

Now, choosing L' such that $\Sigma(x) \not\subset L'(x)^\perp$, by the inductive hypothesis f' is a composition of $\hat{f}$ near x, i.e., $f' = h' \circ \hat{f}$ near x for some local isometric immersion h' between $\mathbb{R}^{n+\ell}$ and $\mathbb{R}^{n+p-1}$. If we further choose L' in such a way that $\Sigma(x) \not\subset L'(x)$, then f is a composition of f' near x, $f = h'' \circ f'$, again by Proposition 18. We conclude that $f = h'' \circ f' = (h'' \circ h') \circ \hat{f}$ is also a composition of $\hat{f}$ near x. ∎

Corollary 23. *On each connected component U of an open dense subset of M^n, f is a composition of f_D on U.*

Proof: Consider an open dense simply connected subset where ℓ_D is locally constant, and work on a connected component V of it where f_D exists by Corollary 21. Along the open subset of V where $\operatorname{Im}\alpha \not\subset L_D$, the corollary follows from Proposition 22 applied to $L = L_D$. On the other hand, if $\operatorname{Im}\alpha \subset L_D$ along a connected open subset $U \subset V$, Codazzi equation for $\eta \in L_D^\perp$ gives $A_{(\nabla_Z^\perp \eta)_{L_D}} X = A_{(\nabla_X^\perp \eta)_{L_D}} Z = 0$. That is, $L_D^\perp$ is a parallel normal subbundle. Since $\operatorname{Im}\alpha \subset L_D$, we have that $L_D^\perp$ is actually constant in $\mathbb{R}^{n+p}$. Thus $f(U) \subset \mathbb{R}^{n+\ell_D}$ and the result also follows on U. ∎

Remark 24. Observe that in the proof of Proposition 22 we did not apply Proposition 18 directly to f and f_D, but instead inductively. This is so because all our isometric extensions are extensions by relative nullity: applying directly Proposition 18 would give an isometric immersion h with relative nullity of codimension one only, while the relative nullity of h in Proposition 23 generically has codimension $p - \ell_D$. The reader should take this into consideration when trying to apply our results to submanifolds that are already ruled with big rulings.

Corollary 25. *If $\hat{f}$ in Corollary 21 is a genuine deformation of f, then M^n is flat, $L_D = \hat{L}_D = 0$, $\operatorname{Im}\alpha \subset L$, and $D^{n-1} = \mathcal{N}(\alpha)$ almost everywhere. Moreover, f and $\hat{f}$ singularly extend isometrically along each connected component of an open dense subset of M^n and, in particular, $\hat{f}$ is nowhere a strongly genuine deformation of f.*

Proof: By Corollary 23 we only need to prove the last assertion. Since $\operatorname{Im}\alpha \subset L$, we have that $\phi_\sigma(TM, TM) = 0$. Since f and $\hat{f}$ are nowhere totally geodesic, by Theorem 4 we singularly extend them isometrically using any vector field in M^n not in D^{n-1}. ∎

We point out that all results obtained until now remain valid when the ambient space is the simply connected space form $\mathbb{Q}_c^m$ of constant sectional curvature c, just by using the exponential map of $\mathbb{Q}_c^m$ when constructing the extensions, e.g., as in (3).

4. Global applications

The purpose of this section is to give the proof of Theorem 1 and its corollaries. To do this, we use compactness to transport information along the leaves of relative nullity to the whole manifold. The use of the intersection of relative nullities makes the proof short and straightforward, even in the hypersurface case of the original Sacksteder's theorem, without the need of induction or case by case analysis.

The intersection of the relative nullities

First, we establish some well-known properties of the splitting tensor adapted to our problem. Let M^n be a Riemannian manifold and D a smooth totally geodesic distribution on M^n. The *splitting tensor* C of D is the map $C : D \times D^\perp \to D^\perp$ defined by

$$C_Y X := C(Y, X) = -(\nabla_X Y)_{D^\perp}.$$

Let $f : M^n \to \mathbb{Q}_c^{n+p}$ be an isometric immersion of M^n with second fundamental form α and suppose further that D is contained in the relative nullity $\mathcal{N}(\alpha)$ of f. Let $\gamma : [0, b] \to M^n$ be a geodesic such that $\gamma([0, b))$ is contained in a leaf of D. Using the curvature tensor of $\mathbb{Q}_c^{n+p}$ we easily see that $C_{\gamma'}$ satisfies the Riccati type ODE

$$C'_{\gamma'} = C^2_{\gamma'} + cI, \tag{6}$$

where we denote with a $'$ the covariant derivative with respect to the parameter of γ.

Recall that the shape operator of f in the direction $\xi \in T_f^\perp M$, denoted by A_ξ, is defined as $\langle A_\xi X, Y\rangle = \langle \alpha(X, Y), \xi\rangle$ for $X, Y \in TM$. For all $Y \in D$ we easily obtain from Codazzi equation that $\nabla_Y A_\xi = A_\xi C_Y + A_{\nabla_Y^\perp \xi}$, where we understand the operators restricted to $D^\perp$. If ξ is parallel along γ this reduces to

$$A'_\xi = A_\xi \circ C_{\gamma'}. \tag{7}$$

We will use the splitting tensor of the intersection Δ_0 of the relative nullities of two isometric immersions, i.e., $\Delta_0 = \mathcal{N}(\beta)$ for $\beta = \phi_0 = \alpha \oplus \hat{\alpha}$, which is (2) for $\tau = 0$. We thus need the following two results for Δ_0.

Lemma 26. *Let $f : M^n \to \mathbb{Q}_c^{n+p}$ and $\hat{f} : M^n \to \mathbb{Q}_c^{n+q}$ be isometric immersions of a Riemannian manifold M^n. Then, along each connected component U of an open dense subset of M^n where $\nu_0 = \dim \Delta_0$ is constant, Δ_0 is an integrable distribution with totally geodesic leaves in M^n, $\mathbb{Q}_c^{n+p}$ and $\mathbb{Q}_c^{n+q}$. In particular, there is a splitting tensor associated to Δ_0 on U.*

Proof: By Proposition 9, β is a Codazzi tensor. So taking $X, Z \in \Delta_0$ and $Y \in TM$ we get $\beta(Y, \nabla_X Z) = -(\bar{\nabla}_X^\perp \beta)(Y, Z) = -(\bar{\nabla}_Y^\perp \beta)(X, Z) = 0$, and the lemma follows. ∎

Lemma 27. *Let $U \subset M^n$ be an open subset where ν_0 is constant, $\gamma : [0, b] \to M^n$ a geodesic with $\gamma([0, b))$ contained in a leaf of Δ_0 in U joining $x = \gamma(0)$ and $y = \gamma(b)$, and P_γ the parallel transport along γ beginning at $t = 0$. We have that $\Delta_0(y) = P_\gamma(\Delta_0(x))(b)$, and the splitting tensor $C_{\gamma'}$ of Δ_0 smoothly extends to $t = b$. In particular, the ODE (7) holds up to time $t = b$. Moreover, $RE(\beta(y)) = P_\gamma(RE(\beta(x)))(b)$ and $i(\beta(y)) = i(\beta(x))$.*

Proof: Let $J : \Delta_0^\perp(\gamma) \to \Delta_0^\perp(\gamma)$ be the unique solution in $[0, b)$ of the ODE

$$J' + C_{\gamma'} \circ J = 0, \quad J(0) = I. \tag{8}$$

From (6) it follows that J also satisfies the linear ODE with constant coefficients $J'' + cJ = 0$, and hence it extends smoothly to $t = b$, where it is defined in $P_\gamma(\Delta_0^\perp(x))(b)$.

For any pair of vector fields $X \in JM$ and $V \in \Delta_0^\perp$ parallel along γ, since β is Codazzi we have

$$\bar{\nabla}^\perp_{\frac{d}{dt}}(\beta(X, J(V))) = \beta(X, (J' + C_{\gamma'} \circ J)(V)) = 0.$$

Thus $\beta(X, J(V))$ is parallel along γ. Since $X(0)$ is arbitrary, J is invertible in $[0, b]$. Moreover, since $P_\gamma(\Delta_0(x))(b) \subset \Delta_0(y)$ by continuity, it follows that $P_\gamma(\Delta_0^\perp(x))(b) = \Delta_0^\perp(y)$. We conclude that $C_{\gamma'}$ extends smoothly to $[0, b]$ as $C_{\gamma'} = -J' \circ J^{-1}$ by (8). The last two assertions follow from the parallelism of $\beta(X, J(V))$. ∎

Proofs of the global statements

The only ingredient we need to easily obtain Theorem 1 from our local statements is the following.

Proposition 28. *Let $f : M^n \to \mathbb{R}^{n+p}$ and $\hat{f} : M^n \to \mathbb{R}^{n+q}$ be isometric immersions of a compact Riemannian manifold M^n with $p + q < n$. Then, at each point in M^n, either $i(\beta) \leq p + q - 3$, or there are unit vectors $\xi \in T_f^\perp M$ and $\hat{\xi} \in T_{\hat{f}}^\perp M$ such that $\hat{A}_{\hat{\xi}} = A_\xi$. Moreover, the second case holds globally if $\min\{p, q\} \leq 5$.*

Proof: Let $W \subset M^n$ be the open subset where such unit vectors do not exist, i.e., where the metric in $\operatorname{Im}(\beta)^\perp \subset T_f^\perp M \oplus T_{\hat{f}}^\perp M$ is definite, and $i(\beta) \geq p + q - 2$ if $\min\{p, q\} \geq 6$. We claim first that $\nu_0 > 0$ on W.

At a fixed a point in W, β is nondegenerate since $\operatorname{Im}(\beta)^\perp$ is definite. If $\min\{p, q\} \leq 5$, the claim is just Theorem 3 in [4]. If otherwise, this is actually the easiest case in the proof of that theorem for which no hypothesis on $\min\{p, q\}$ is needed. Indeed, if $X \in RE(\beta)$ we have that $\dim \operatorname{Im}(\beta^X) \cap \operatorname{Im}(\beta^X)^\perp \leq \dim \operatorname{Im}(\beta^X)^\perp = p + q - \dim \operatorname{Im}(\beta^X) \leq 2$. In this situation, Lemma 6 in [4] easily implies that there is $Y \in RE(\beta)$ such that $\Delta_0 = \ker(\beta^X) \cap \ker(\beta^Y) = \ker(\lambda)$, for $\lambda := \beta^Y|_{\ker(\beta^X)}$. But since β is flat we have that $\operatorname{Im}(\lambda) \subset \operatorname{Im}(\beta^X)^\perp$, and therefore $\nu_0 = \dim \ker(\lambda) = n - \dim \operatorname{Im}(\beta^X) - \dim \operatorname{Im}(\lambda) \geq n - p - q > 0$, as claimed.

Let $W' \subset W$ be the open subset where $\nu_0 > 0$ is minimal in W, and $\gamma \subset W'$ a maximally defined unit geodesic contained in a maximal leaf of Δ_0 in W'. Since, by

Lemma 26, γ is mapped onto a straight line by both f and $\hat{f}$ and M^n is compact, γ must be defined in a bounded interval (a, b). By Lemma 27 the values of ν_0 and $i(\beta)$ are constant along γ up to $t = b$, so $y := \gamma(b) \notin W$. Hence, since $i(\beta(y)) = i(\beta(\gamma(0)) \geq p + q - 2$, there are unit vectors $\xi_0 \in T^\perp_{f(y)}M$ and $\hat{\xi}_0 \in T^\perp_{\hat{f}(y)}M$ such that $A_{\xi_0} = \hat{A}_{\hat{\xi}_0}$. If ξ and $\hat{\xi}$ are their parallel transports along γ, by uniqueness of the solutions of the extended ODE (7) obtained in Lemma 27 we get $A_\xi = \hat{A}_{\hat{\xi}}$ also along the whole $\gamma \subset W$, which contradicts the definition of W. We conclude that W is empty. ∎

Remark 29. Observe that in the proof above we only used the non-existence of an unbounded geodesic contained in Δ_0. In particular, Proposition 28, and thus Theorem 1 and Corollary 2, hold for complete manifolds if we require that either $f(M)$ or $\hat{f}(M)$ contains no complete straight line instead of compactness.

Proof of Theorem 1. Let $V \subset M^n$ be the open subset where $i(\beta(x)) \geq p + q - 2$. By Proposition 28 and Corollary 10 applied to $\phi_0 = \beta$, there exists a trivially parallel isometry of line bundles parallel along Δ_0, $\tau : L = \text{span}\,\{\xi\} \to \hat{L} = \text{span}\,\{\hat{\xi}\}$, defined on an open dense subset U of V, and that preserves second fundamental forms. The result now follows from Corollary 5 applied to each connected component U' of U with this τ, and to $M^n \setminus \overline{V}$ with $\tau = 0$ since, in either case, $n + \ell + i(\phi_\tau) \geq n - p - q + 3$. ∎

Although Corollary 2 can be easily proved directly from Theorem 1 and Corollaries 5 and 10, we will use the results obtained in Section 3.1.

Proof of Corollary 2. By Theorem 1 we only need to show that, for $p+q = 4$, the immersions singularly extend isometrically almost everywhere on a subset U where $d \geq n - 1$ is constant. Clearly, this is the case if $d = n$, since both immersions would be totally geodesic in U and we isometrically extend them with $N^{n+1} = U \times \mathbb{R}$, $F = f \times Id$ and $\hat{F} = \hat{f} \times Id$.

If $d = n - 1$ on U, we have as in the proof of Theorem 1 and again by Corollary 5 that $p + q = 4$, $\ell = 1$, $i(\phi_\tau) = 2$, and $D^{n-1} = \mathcal{N}(\alpha_{L^\perp}) \cap \mathcal{N}(\hat{\alpha}_{\hat{L}^\perp})$ along an open dense subset of U. If $U' \subset U$ is the open subset where L_D in (5) is nonzero, then $L = L_D$, $\hat{L} = \hat{L}_D$, $f_D = \hat{f}_D$, and thus by Corollary 23 f and $\hat{f}$ (regularly) extend isometrically almost everywhere on U'. On $U \setminus U'$, $L_D = 0$ and $D^{n-1} = \Delta_0$ almost everywhere, so Corollaries 10 and 21 tell us that there is an isometric immersion $f' : U \to \mathbb{R}^{n+1}$ with second fundamental form $\alpha_L = \hat{\alpha}_{\hat{L}}$. By Corollary 25, the pairs $\{f, f'\}$ and $\{\hat{f}, f'\}$ both singularly extend isometrically, and since the codimension of f' is one, the pair $\{f, \hat{f}\}$ also singularly extends isometrically almost everywhere on $U \setminus U'$. ∎

Remark 30. Corollary 2 for $p = q = 2$ reduces to the main result in [8], except for the fact that singular flat extensions can occur in the former. This is a consequence of a gap in [8], whose long and involved case by case proof did not cover all possibilities.

Proof of Corollary 3. By Proposition 26 in [6] and Theorem 1, if an isometric immersion $\hat{f} : M^n \to \mathbb{R}^{n+q}$ is a strongly genuine deformation of $f : M^n \to \mathbb{R}^{n+p}$, then the k-th Pontrjagin form p_k of M^n vanishes for any k such that $4k > 3(p + q - 3)$. ∎

5. The space forms case

As we pointed out, Theorem 1 and Corollary 2 hold for compact manifolds when the ambient space is the hyperbolic space following the same proofs. In this section we show that they also hold for complete submanifolds in the sphere under a mild codimension condition.

For the following, recall that $\rho(m)-1$ is the maximum number of pointwise linearly-independent vector fields on $\mathbb{S}^{m-1}$.

Lemma 31. *Let $f: M^n \to \mathbb{S}^{n+p}$ be an isometric immersion and D^d a nontrivial totally geodesic distribution contained in the relative nullity of f. If there exists a nonconstant geodesic $\sigma: [0,\infty) \to M^n$ in D^d, then the splitting tensor $C_{\sigma'}$ associated to D^d has no real eigenvalues. In particular, such a geodesic cannot exist if $\rho(n-d) < d+1$.*

Proof: By (6), $C_{\sigma'}$ is given by

$$(P_\sigma^{-1} \circ C_{\sigma'} \circ P_\sigma)(t) = (\sin(t)I + \cos(t)C_{\sigma'}(0))(\cos(t)I - \sin(t)C_{\sigma'}(0))^{-1},$$

where P_σ is the parallel transport along σ. Since $C_{\sigma'}$ is defined for all $t \geq 0$, we easily conclude that $C_{\sigma'(0)}$ has no real eigenvalues.

For the last assertion, choose a basis $\{T_1, ..., T_d\}$ of $D(x)$. By the first assertion, for any unit vector $Z \in D^\perp(x)$ and $a, a_1, ..., a_d \in \mathbb{R}$, the equation $0 = aZ + \sum_{i=1}^d a_i C_{T_i} Z = aZ + C_T Z$ implies that $a = a_i = 0$, where $T = \sum_{i=1}^d a_i T_i$. Hence $Z, C_{T_1}Z, ..., C_{T_d}Z$ are linearly independent in $D^\perp(x)$. Since this holds for any unit vector $Z \in D^\perp(x)$, considering Z as the position vector of the unit sphere $\mathbb{S}^{n-d-1} \subset D^\perp(x)$ we get d nonvanishing linearly independent vector fields in $\mathbb{S}^{n-d-1}$. Hence, $d \leq \rho(n-d)-1$. ∎

Proof of Theorem 7. By Lemma 31, geodesics in Δ_0 cannot be defined for arbitrary large time if $\mu_n < n-p-q$ when the ambient space is the sphere. Thus, as observed in Remark 29, Proposition 28 holds for complete manifolds when the ambient spaces are spheres as long as $p+q < n-\mu_n$. ∎

Proof of Corollary 8. It is analogous to the one for Corollary 2 using Theorem 7 instead of Theorem 1, just observing that for small codimensions we can simplify the assumptions on μ_n. ∎

References

[1] Allendoerfer, C. B.: *Rigidity for spaces of class greater than one*. Amer. J. Math. **61** (1939), 633-644.

[2] do Carmo, M. and Dajczer, M.: *A rigidity theorem for higher codimensions*. Math. Ann. **274** (1986), 577-583.

[3] Dajczer, M. and Florit, L. A.: *On conformally flat submanifolds*. Comm. Anal. Geom. **4** (1996), 261–284.

[4] Dajczer, M. and Florit, L. A.: *Compositions of isometric immersions in higher codimension*. Manuscripta Math. **105** (2001), 507-517. *Erratum:* Manuscripta Math. **110** (2003), 135.

[5] Dajczer, M. and Florit, L. A.: *A counterexample to a conjecture on flat bilinear forms*. Proc. AMS **132** (2004), 3703-3704.

[6] Dajczer, M. and Florit, L. A.: *Genuine deformations of submanifolds*. Comm. Anal. Geom. **12** (2004), 1105-1129.

[7] Dajczer, M. and Florit, L. A. and Tojeiro, R.: *Euclidean hypersurfaces with genuine deformations in codimension two*. Manuscripta Math. **140** (2013), 621-643.

[8] Dajczer, M. and Gromoll, D.: *Isometric deformations of compact Euclidean submanifolds in codimension* 2. Duke Math. J. **79** (1995), 605-618.

[9] Dajczer, M. and Tojeiro, R.: *On compositions of isometric immersions*. J. Differential Geom. **36** (1992), 1-18.

[10] Florit, L. A. and de Freitas, G. M.: *Classification of codimension two deformations of rank two Riemannian manifolds*. Comm. Anal. Geom. **25** (2017), 751-797.

[11] Florit, L. A. and Tojeiro, R.: *Genuine deformations of submanifolds II: the conformal case*. Comm. Anal. Geom. **18** (2010), 397-419.

[12] Killing, W.: *Die nicht-Euklidischen Raumformen in Analytische Behandlung*. Teubner, Leipzig, 1885.

[13] Moore, J. D.: *Submanifolds of constant positive curvature. I*. Duke Math. J. **44** (1977), 449-484.

[14] Sacksteder, R.: *The rigidity of hypersurfaces*. J. Math. Mech. **11** (1962), 929-939.

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