GEOMETRIC GRAPH MANIFOLDS WITH NON-NEGATIVE SCALAR CURVATURE

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Abstract. We classify n-dimensional geometric graph manifolds with nonnegative scalar curvature, and first show that if $n > 3$, the universal cover splits off a codimension 3 Euclidean factor. We then proceed with the classification of the 3-dimensional case by showing that such a manifold is either a lens space or a prism manifold with a very rigid metric. This allows us to also classify the moduli space of such metrics: it has infinitely many connected components for lens spaces, while it is connected for prism manifolds.

A geometric graph manifold M^n is the union of twisted cylinders $C^n = (L^2 \times \mathbb{R}^{n-2})/G$, where $G \subset \mathrm{Iso}(L^2 \times \mathbb{R}^{n-2})$ acts properly discontinuously and freely on the Riemannian product of a surface L^2 with the Euclidean space $\mathbb{R}^{n-2}$. In addition, the boundary of each twisted cylinder is a union of compact totally geodesic flat hypersurfaces, each of which is isometric to a boundary component of another twisted cylinder. In its simplest form, as first discussed in [Gr], they are the union of building blocks of the form $L^2 \times S^1$, where L^2 is a surface, not diffeomorphic to a disk or an annulus, whose boundary is a union of closed geodesics. The building blocks are glued along common boundary totally geodesic flat tori by switching the role of the circles. Such graph manifolds have been studied frequently in the context of manifolds with nonpositive sectional curvature. In fact, they were the first examples of such metrics with geometric rank one. Furthermore, in [Sch] it was shown that if a complete 3-manifold with nonpositive sectional curvature has the fundamental group of such a graph manifold, then it is isometric to a geometric graph manifold.

One of the most basic features of the geometric graph manifolds is that their curvature tensor has nullity space of dimension at least $n-2$ everywhere. This property by itself already guarantees that each finite volume connected component of the set of non-flat points is a twisted cylinder, and under some further natural assumptions, the manifold is indeed isometric to a geometric graph manifold in the above sense; see [FZ2].

This structure also arose in a different context. In [FZ1] it was shown that a compact immersed submanifold $M^n \subset \mathbb{R}^{n+2}$ with nonnegative sectional curvature is either diffeomorphic to the sphere $\mathbb{S}^n$, isometric to a product of two convex hypersurfaces $\mathbb{S}^k \times \mathbb{S}^{n-k} \subset \mathbb{R}^{k+1} \times \mathbb{R}^{n-k+1}$, isometric to $(\mathbb{S}^{n-1} \times \mathbb{R})/\mathbb{Z}$, or diffeomorphic to a lens space $\mathbb{S}^3/\mathbb{Z}_p \subset \mathbb{R}^5$. In the latter case it was shown that each connected component of the set of nonflat points is a twisted cylinder. However, it is not known yet if such lens spaces can be isometrically immersed in $\mathbb{R}^5$. The present paper arose out of an attempt to understand the intrinsic

The first author was supported by CNPq-Brazil, and the second author by a grant from the National Science Foundation, by IMPA, and CAPES-Brazil.

geometry of such metrics. We thus want to classify all compact geometric graph manifolds with nonnegative sectional curvature, or equivalently, with nonnegative scalar curvature. Notice that under this curvature assumption compactness is equivalent to finite volume.

We first show that their study can be reduced to three dimensions.

THEOREM A. *Let M^n be a compact geometric graph manifold with nonnegative scalar curvature. Then, the universal cover $\tilde{M}^n$ of M^n splits off an $(n-3)$-dimensional Euclidean factor isometrically, i.e., $\tilde{M}^n = N^3 \times \mathbb{R}^{n-3}$. Moreover, M^n is flat or either $N^3 = \mathbb{S}^2 \times \mathbb{R}$ splits isometrically, or $N^3 = \mathbb{S}^3$ with a geometric graph manifold metric.*

In dimension 3, the simplest nontrivial example with nonnegative scalar curvature is the usual description of $\mathbb{S}^3$ as the union of two solid tori endowed with a product metric, see Figure 1. If this product metric is invariant under $\mathrm{SO}(2) \times \mathrm{SO}(2)$, we can also take a quotient by the cyclic group generated by $R_p \times R_p^q$ to obtain a geometric graph manifold metric on any lens space $L(p,q) = \mathbb{S}^3/\mathbb{Z}_p$. Here $R_p \in \mathrm{SO}(2)$ denotes the rotation of angle $2\pi/p$.

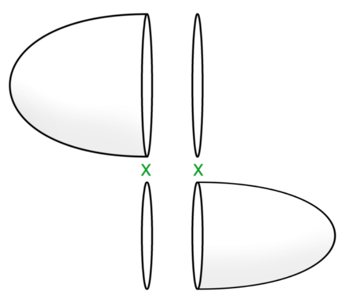

FIGURE 1. $\mathbb{S}^3 \subset \mathbb{R}^5$ with nonnegative curvature

There is a further family whose members also admit geometric graph manifold metrics with nonnegative scalar curvature: the so called *prism manifolds* $P(m,n) := \mathbb{S}^3/G_{m,n}$, which depend on two relatively prime positive integers m, n. A geometric graph manifold metric on $P(m,n)$ can be constructed as a quotient of the metric on $\mathbb{S}^3$ as above, with the two solid tori being in addition isometric, by the group generated by $R_{2n} \times R_{2n}^{-1}$ and $(R_m \times R_m) \circ J$, where J is a fixed point free isometry switching the two solid tori. Topologically $P(m,n)$ is thus a single solid torus whose boundary is identified to be a Klein bottle. Its fundamental group $G_{m,n}$ is abelian if and only if $m = 1$, and in fact $P(1,n)$ is diffeomorphic to $L(4n, 2n-1)$; see Section 1. Unlike in the case of lens spaces, the diffeomorphism type of a prism manifold is determined by its fundamental group.

Our main result is to show that these are the only compact geometric graph manifolds with nonnegative scalar curvature. We will see that the twisted cylinders in this case are of the form $C = (D \times \mathbb{R})/\mathbb{Z}$, where D is a 2-disk of nonnegative Gaussian curvature, whose boundary ∂D is a closed geodesic along which the curvature vanishes to infinite order. We fix once and for all such a metric $\langle\,,\rangle_0$ on a 2-disc D_0, whose boundary has length 1 and which is rotationally symmetric. We call a geometric graph manifold metric on a 3-manifold *standard* if the generating disk D of a twisted cylinder C as above is isometric to D_0 with metric $r^2\langle,\rangle_0$ for some $r > 0$. Observe that the projection of $\partial D \times \{s\}$ for $s \in \mathbb{R}$ is a parallel foliation by closed geodesics of the flat totally geodesic 2-torus $(\partial D \times \mathbb{R})/\mathbb{Z}$.

We provide the following classification:

THEOREM B. *Let M^3 be a compact geometric graph manifold with nonnegative scalar curvature and irreducible universal cover. Then M^3 is diffeomorphic to a lens space or a prism manifold. Moreover, we have either:*

a) M^3 is a lens space and $M^3 = C_1 \sqcup T^2 \sqcup C_2$, i.e., M^3 is isometrically the union of two nonflat twisted cylinders $C_i = (D_i \times \mathbb{R})/\mathbb{Z}$ over disks D_i glued together along their common totally geodesic flat torus boundary T^2. Conversely, any flat torus endowed with two parallel foliations by closed geodesics defines a standard geometric graph manifold metric on a lens space, which is unique up to isometry;

b) M^3 is a prism manifold and $M^3 = C \sqcup K^2$, i.e., M^3 is isometrically the closure of a single twisted cylinder $C = (D \times \mathbb{R})/\mathbb{Z}$ over a disk D, whose totally geodesic flat interior boundary is isometric to a rectangular torus T^2, and $K^2 = T^2/\mathbb{Z}_2$ is a Klein bottle. Conversely, any rectangular flat torus endowed with a parallel foliation by closed geodesics defines a standard geometric graph manifold metric on a prism manifold, which is unique up to isometry.

In addition, any geometric graph manifold metric with nonnegative scalar curvature on M^3 is isotopic, through geometric graph manifold metrics with nonnegative scalar curvature, to a standard one.

Observe that a twisted cylinder with generating surface a disc is diffeomorphic to a solid torus. In topology one constructs a lens space by gluing two such solid tori along their boundary by an element of $GL(2, \mathbb{Z})$. In order to make this gluing into an isometry, we twist the local product structure. An alternate way to view this construction is as follows. Start with an arbitrary twisted cylinder C_1 and regard the flat boundary torus as the quotient $\mathbb{R}^2$ with respect to a lattice. We can then choose a second twisted cylinder C_2 whose boundary is a different fundamental domain of the same lattice, and hence the two twisted cylinders can be glued with an isometry of the boundary tori. We note that in principle, a twisted cylinder can also be flat, but we will see that in that case it can be absorbed by one of the nonflat twisted cylinders.

The diffeomorphism type of M^3 in Theorem B is determined by the relative (algebraic) slope between the parallel foliations by closed geodesics; see Section 3 for the precise definition of algebraic slope. In case (a), M^3 is diffeomorphic to a lens space $L(p, q)$, where $q/p \in \mathbb{Q}$ is the relative slope between the foliations $[\partial D_i \times \{s\}]$, $i = 1, 2$. Analogously, the manifolds in case (b) are prism manifolds $P(m, n)$, where m/n is the relative slope between the foliation $[\partial D \times \{s\}]$ on the rectangular interior boundary torus $T^2 = S^1_{r_1} \times S^1_{r_2}$ and the foliation $S^1_{r_1} \times \{w\}$.

We can deform any geometric graph manifold metric in Theorem B to first be standard, preserving the metric on the torus T^2, and then deform T^2 to be the unit square $S^1 \times S^1$, while preserving also the sign of the scalar curvature in the process. In case (a), we can also make one of the foliations equal to $S^1 \times \{w\}$. The metric is then determined by the remaining parallel foliation of the unit square by closed geodesics whose usual slope is equal to the relative slope. Since the diffeomorphism type of a lens space $L(p,q)$ is determined by $\pm q^{\pm 1}$ mod p, we conclude:

COROLLARY. *The moduli space of geometric graph manifold metrics with nonnegative scalar curvature on a lens space $L(p,q)$ has infinitely many connected components, whereas on a prism manifold $P(m,n)$ with $m > 1$ it is connected.*

The paper is organized as follows. In Section Section 1 we recall some facts about geometric graph manifolds. In Section 2 we prove Theorem A by showing that the manifold is a union of one or two twisted cylinders over disks, while in Section 3 we classify their metrics.

1. Preliminaries

Let us begin with the definition of twisted cylinders and geometric graph manifolds.

Consider the cylinder $L^2 \times \mathbb{R}^{n-2}$ with its natural product metric, where L^2 is a connected surface. We call the quotient

$$C^n = (L^2 \times \mathbb{R}^{n-2})/G,$$

where $G \subset \mathrm{Iso}(L^2 \times \mathbb{R}^{n-2})$ acts properly discontinuously and freely, a *twisted cylinder*, and L^2 the *generating surface* of C^n. We also say that C is a twisted cylinder *over* L^2. The images of the Euclidean factor are the *nullity leaves* of C^n, that we generically denote by Γ. In fact, C^n is foliated by complete, flat, totally geodesic, and locally parallel leaves of codimension 2. These are the building blocks of the geometric graph manifolds, see Figure 2 for a typical (4-dimensional) example:

Definition. A complete connected Riemannian manifold M^n is called a *geometric graph manifold* if M^n is a locally finite disjoint union of twisted cylinders C_i glued together through disjoint compact totally geodesic flat hypersurfaces H_λ of M^n. That is,

$$M^n \setminus W = \bigsqcup_\lambda H_\lambda, \quad \text{where} \quad W := \bigsqcup_i C_i.$$

Let us first make some general remarks about this definition.

1. We allow the possibility that the hypersurfaces H_λ are one-sided, even when M^n is orientable.

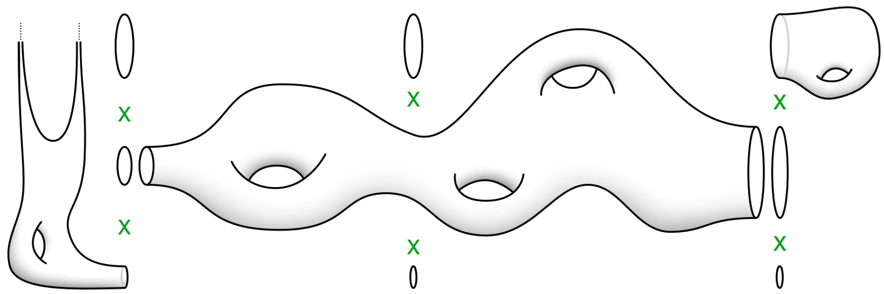

FIGURE 2. An irreducible 4-dimensional geometric graph manifold with three cylinders and two (finite volume) ends

2. The locally finiteness condition is equivalent to the assumption that each H_λ is a common boundary component of two twisted cylinders C_i and C_j, that may even be globally the same, each lying at a local side of H_λ.
3. We also assume, without loss of generality, that the nullity leaves of C_i and C_j have distinct limits in H_λ. Therefore, the generating surface L^2 of each twisted cylinder C has as boundary a union of complete geodesics along which the Gauss curvature vanishes to infinite order.
4. These boundary geodesics of L^2 do not have to be closed, even when C is compact.
5. The complement of W is contained in the set of flat points of M^n, but we do not require that the generating surfaces have nonvanishing Gaussian curvature.
6. In principle, we could ask for the hypersurfaces H_λ to be complete instead of compact. However, compactness follows when M^n has finite volume; see [FZ2].

In [FZ2] we gave a characterization of geometric graph manifolds with finite volume in terms of the nullity of the curvature tensor. But since a complete noncompact manifold with nonnegative Ricci curvature has linear volume growth by [Ya], we will assume from now on that M^n is compact.

We now recall some properties of three dimensional lens spaces and prism manifolds that will be needed later on.

One way of defining a lens space is as the quotient $L(p,q) = \mathbb{S}^3/\mathbb{Z}_p$, where $\mathbb{Z}_p \subset S^1 \subset \mathbb{C}$ acts as $g\cdot(z,w) = (gz, g^q w)$ for $(z,w) \in \mathbb{S}^3 \subset \mathbb{R}^4 = \mathbb{C}^2$. We can assume that $p,q > 0$ with $\gcd(p,q) = 1$. It is a well known fact that two lens spaces $L(p,q)$ and $L(p,q')$ are diffeomorphic if and only if $q' = q^{\pm 1} \bmod p$. An alternative description we will use is as the union of two solid tori $D_i \times S^1$, with boundary identified by a diffeomorphism f such that $\partial D_1 \times \{p_0\} \in \pi_1(\partial D_1 \times S^1)$ is taken under f into $(q,p) \in \mathbb{Z}\oplus\mathbb{Z} \simeq \pi_1(\partial D_2 \times S^1)$ with respect to its natural basis.

A prism manifold can also be described in two different ways, see e.g. [ST, HK, Ru]. One way is to define it as the quotient $\mathbb{S}^3/(H_1 \times H_2) = H_1\backslash\mathbb{S}^3/H_2$, where $H_1 \subset \mathrm{Sp}(1)$ is a cyclic group acting as left translations on $\mathbb{S}^3 \simeq \mathrm{Sp}(1)$ and $H_2 \subset \mathrm{Sp}(1)$ a binary dihedral group acting as right translations. A more useful description for our purposes is as the union of a solid torus $D \times S^1$ with the 3-manifold

$$N^3 = (S^1 \times S^1 \times I)/\langle (j, -Id) \rangle, \quad \text{where} \quad j(z, w) = (-z, \bar{w}). \tag{1.1}$$

Notice that N^3 is a bundle over the Klein bottle $K = T^2/\langle j \rangle$ with fiber an interval $I = [-\epsilon, \epsilon]$ and orientable total space. Thus ∂N^3 is the torus $S^1 \times S^1$, and we glue the two boundaries via a diffeomorphism. Here $\pi_1(N^3) \simeq \pi_1(K) = \{a, b \mid bab^{-1} = a^{-1}\}$ and $\pi_1(\partial N^3) \simeq \mathbb{Z} \oplus \mathbb{Z}$, with generators a, b^2, where a represents the first circle and b^2 the second one. Then $P(m, n)$ is defined as gluing ∂C to ∂N^3 by sending $\partial D \times \{p_0\}$ to $a^m b^{2n} \in \pi_1(\partial N^3)$. We can again assume that $m, n > 0$ with $\gcd(m, n) = 1$. Furthermore,

$$\pi_1(P(m, n)) = G_{m,n} = \{a, b \mid bab^{-1} = a^{-1},\ a^m b^{2n} = 1\}.$$

This group has order $4mn$ and its abelianization has order $4n$. Thus the fundamental group determines and is determined by the ordered pair (m, n). In addition, $G_{m,n}$ is abelian if and only if $m = 1$ in which case $P(m, n)$ is diffeomorphic to the lens space $L(4n, 2n - 1)$. Unlike in the case of lens spaces, the diffeomorphism type of $P(m, n)$ is uniquely determined by (m, n). This follows, e.g., by using the Ricci flow to deform a given metric into one of constant curvature, and the fact that the effective representations of $G_{m,n}$ in SO(4) are unique up to conjugation, and hence given by the one presented in the introduction. Prism manifolds can also be characterized as the 3-dimensional spherical space forms which contain a Klein bottle, which for $m > 1$ is also incompressible. Observe in addition that in N^3 we can shrink the length of the interval I in (1.1) down to 0, and hence $P(m, n)$ can also be viewed as a single solid torus whose rectangular flat torus boundary has been identified to a Klein bottle, as in part (b) of Theorem B.

2. A dichotomy and the proof of Theorem A

In this section we provide the general structure of geometric graph manifolds with nonnegative scalar curvature by showing a dichotomy: they are build from either one or two twisted cylinders over 2-disks. This will then be used to prove Theorem A.

Let M^n be a compact nonflat geometric graph manifold with nonnegative scalar curvature. We can furthermore assume that M^n is not itself a twisted cylinder since in this case the universal cover of M^n is isometric to $\mathbb{S}^2 \times \mathbb{R}^{n-2}$, where $\mathbb{S}^2$ is endowed with a metric of nonnegative Gaussian curvature.

By assumption, there exists a collection of compact flat totally geodesic hypersurfaces in M^n whose complement is a disjoint union of (open) twisted cylinders C_i. Let $C = (L^2 \times \mathbb{R}^{n-2})/G$ be one of these cylinders whose boundary in M^n is a disjoint union of compact flat totally geodesic hypersurfaces. There is also an *interior boundary* $\partial_i C$ of C, which we also denote for convenience as ∂C by abuse of notation. This boundary can be defined as the set of equivalence classes of Cauchy sequences $\{p_n\} \subset C$ in the interior distance function d_C of C, where $\{p_n\} \sim \{p'_n\}$ if $\lim_{n\to\infty} d_C(p_n, p'_n) = 0$. Since M^n is compact, such a Cauchy sequence $\{p_n\}$ converges in M^n, and we have a natural map $\sigma : \partial C \to M$ that sends $[\{p_n\}]$ into $\lim_{n\to\infty} p_n \in M^n$. This map is, on each component of ∂C, either an isometry or a local isometric two-fold cover since $H = \sigma(\partial C)$ consists of disjoint smooth hypersurfaces which are two-sided in the former case, and one-sided in the latter. Therefore, ∂C is smooth as well and $C \sqcup \partial C$ is a closed twisted cylinder with totally geodesic flat compact interior boundary, that by abuse of notation we still denote by C. Similarly, L^2 is a smooth surface with geodesic interior boundary components.

We first determine the generating surfaces of the twisted cylinders:

PROPOSITION 2.1. *Let $C = (L^2 \times \mathbb{R}^{n-2})/G$ be a compact twisted cylinder with nonnegative curvature. Then one of the following holds:*

i) The surface L^2 is isometric to a 2-disk D with nonnegative Gaussian curvature, whose boundary is a closed geodesic along which the curvature of D vanishes to infinite order.

ii) C is flat and there exists a compact flat hypersurface S such that C is isometric to either $[-s_0, s_0] \times S$, or to $([-s_0, s_0] \times S)/\{(s, x) \sim (-s, \tau(x))\}$ for some involution τ of S.

Proof. Since C is compact and the boundary is totally geodesic, we can apply the soul theorem to C, see [CG] Theorem 1.9. Thus there exists a compact totally geodesic submanifold $S \subset C$ and C is diffeomorphic to the disc bundle $D_\epsilon(S) = \{v \in T_pC \mid v \perp T_pS,\ |v| \leq \epsilon\}$ for some $\epsilon > 0$. Recall that S is constructed as follows. Let $C^s = \{p \in C \mid d(p, \partial C) \geq s\}$. Then C^s is convex, and the set of points C^{s_0} at maximal distance s_0 from ∂C is a totally geodesic submanifold, possibly with boundary. Repeating the process if necessary, one obtains the soul S. In our situation, let $q = [(p, v)] \in C^{s_0}$, and γ a minimal geodesic from q to ∂C. Since it meets $\partial C = ((\partial L^2) \times \mathbb{R}^{n-2})/G$ perpendicularly, we have $\gamma = [(\alpha, v)]$ where α is a geodesic in the leaf $L^2_v = [L^2 \times \{v\}]$ meeting ∂L^2_v perpendicularly. So, for every $w \in \mathbb{R}^{n-2}$, the geodesic $[(\alpha, w)]$ is also minimizing, $[(p, w)] \in C^{s_0}$ lies at maximal distance s_0 to ∂C, and hence $C^{s_0} = (T \times \mathbb{R}^{n-2})/G$ where $T \subset L^2$ is a segment, a complete geodesic or a point. Therefore $S = (T' \times \mathbb{R}^{n-2})/G$, where T' is a point or a complete geodesic (possibly closed).

We first consider the case where T' is a point and hence the soul is a single nullity leaf. Recall, that in order to show that C is diffeomorphic to the disc bundle $D_\epsilon(S)$, one constructs a gradient like vector field X by observing that the distance function to the soul has no critical points. In our case, the initial vector to all minimal geodesics from $[(p, v)] \in C$ to S lies in the leaf L^2_v and hence we can construct X such that X is tangent to L^2_v for all v. The diffeomorphism between C and $D_\epsilon(S)$ is obtained via the flow of X, which now preserves the leaves L^2_v and therefore L^2 is diffeomorphic to a disc.

If T' is a complete geodesic, the soul S is flat and has codimension 1. If X is a unit vector field in L^2 along T' and orthogonal to T', it is necessarily parallel and its image under the normal exponential map of S determines a flat surface by Perelman's solution to the soul conjecture, see [Pe]. This surface lies in L^2, and every point $q \in L^2$ is contained in such a surface since we can connect q to S by a minimal geodesic, which is contained in some L_v, and is orthogonal to T'. Thus L^2 is flat and hence either $L^2 = T' \times [-s_0, s_0]$, and hence $C = [-s_0, s_0] \times S$, or L^2 is a Moebius strip and hence $C = ([-s_0, s_0] \times S)/\{(s, x) \sim (-s, \tau(x))\}$ for some involution τ of S. $\square$

Remark 2.2. A flat twisted cylinder as in (ii) can be absorbed by any cylinder C' attached to one of its boundary components by either attaching $[-s_0, s_0]$ to the generating surface of C' in the first case, or attaching $(0, s_0]$ in the second, in which case $\{0\} \times (S/\tau) = S/\tau$ becomes a one sided boundary component of C'. Thus we can assume from now on that the generating surfaces of all twisted cylinders are 2-discs.

Remark 2.3. The properties at the boundary γ of a disk D as in Proposition 2.1 are easily seen to be equivalent to the fact that the natural gluing $D \sqcup (\gamma \times (-\epsilon, 0])$, $\gamma \cong \gamma \times \{0\}$, is smooth when we consider on $\gamma \times (-\epsilon, 0]$ the flat product metric. In fact, in Fermi coordinates $(s \geq 0, t)$ along γ, the metric is given by $ds^2 + f(t, s)dt^2$. The fact that γ is a (unparameterized) geodesic is equivalent to $\partial_s f(0, t) = 0$, while the curvature condition is equivalent to $\partial_s^k f(0, t) = 0$ for all t and $k \geq 2$. Therefore, $f(s, t)$ can be extended smoothly as $f(0, t)$ for $-\epsilon < s < 0$, which gives the smooth isometric attachment of the flat cylinder $\gamma \times (-\epsilon, 0]$ to D.

As a consequence of Proposition 2.1, $\partial C = (\gamma \times \mathbb{R}^{n-2})/G$ is connected, and so is $H = \sigma(\partial C)$. In particular, M^n contains at most two twisted cylinders with nonnegative curvature glued along H. We call such a connected compact flat totally geodesic hypersurface H a *core* of M^n. We conclude:

COROLLARY 2.4. *If M^n is not flat and not itself a twisted cylinder, then $M^n = W \sqcup H$ with core H, and either:*

a) H is two-sided, σ is an isometry, and $W = C \sqcup C'$ is the disjoint union of two open nonflat twisted cylinders as above attached via an isometry $\partial C \simeq H \simeq \partial C'$; or

b) H is one-sided, σ is a local isometric two-fold cover, $W = C$ is a single open nonflat twisted cylinder as above, and $M^n = C \sqcup H = C \sqcup (\partial C/\mathbb{Z}_2)$.

Furthermore, in case (a), if $H' \subset M^n$ is an embedded compact flat totally geodesic hypersurface then there exists an isometric product $H \times [0, a] \subset M^3$, with $H = H \times \{0\}$ and $H' = H \times \{a\}$. In particular, any such H' is a core of M^3, and hence the core is unique up to isometry. On the other hand, in case (b) the core H is unique.

Proof. We only need to argue for the uniqueness of the cores. In order to do this, given a twisted cylinder C we call boundary nullity leaf, or BNL for short, any limit of nullity leaves of C at its boundary.

For case (a), first assume that $H \cap H' \neq \emptyset$ and take $p \in H \cap H'$. Then a BNL of C at p is contained in H'. Indeed if not, the product structure of the universal cover $\pi : \tilde{C} = L^2 \times \mathbb{R}^{n-2} \to C$, together with the fact that H' is flat totally geodesic and complete, would imply that L^2, and hence C, is flat since by dimension reasons the projection of $\pi^{-1}(H' \cap C)$ onto L^2 would be a surjective submersion. Analogously, the (distinct) BNL of C' at p lies in H', and since H is the unique hypersurface containing both BNL's, we have that $H = H'$.

If, on the other hand, $H \cap H' = \emptyset$, we can assume $H' \subset C = (L^2 \times \mathbb{R}^{n-2})/G$. Again by the product structure of $\tilde{C}$ and the fact that H' is embedded we see that $H' = (\gamma' \times \mathbb{R}^{n-2})/G'$ where $\gamma' \subset L^2$ is a simple closed geodesic and $G' \subset G$ the subgroup preserving γ'. Since the boundary γ of L^2 is also a closed geodesic and L^2 is a 2-disk with nonnegative Gaussian curvature, by Gauss–Bonnet there is a closed interval $I = [0, a] \subset \mathbb{R}$ such that the flat strip $\gamma \times I$ is contained in L^2, with $\gamma = \gamma \times \{0\}$ and $\gamma' = \gamma \times \{a\}$. Thus G' acts trivially on I, which implies our claim.

In case (b) we have that $H \cap H' = \emptyset$ as in case (a) since at any point $p \in H$ we have two different BNL's at $\sigma^{-1}(p)$. Hence as before $H' = (\gamma' \times \mathbb{R}^{n-2})/G' \subset C$ and $H \times [0, a] \subset M^3$, with $H = H \times \{0\}$ and $H' = H \times \{a\}$. But then the normal bundle of H' is trivial, contradicting the fact that H is one-sided. $\square$

Remark 2.5. Any manifold in case (b) admits a two-fold cover whose covering metric is as in case (a). Indeed, we can attach to C another copy of C along its interior boundary $\partial_i C$ using the involution that generates $\mathbb{Z}_2$. Switching the two cylinders induces the two-fold cover of M^n.

We proceed by showing that our geometric graph manifolds are essentially 3-dimensional. Observe that we only use here that $M^n \setminus W$ is connected, with no curvature assumptions. In fact, the same proof shows that if $M^n \setminus W$ has k connected components, then M^n splits off an $(n-k-2)$-dimensional Euclidean factor.

Claim. If $n > 3$, the universal cover of M^n splits off an $(n-3)$-dimensional Euclidean factor.

Proof. Assume first that M^n is the union of two cylinders C and C' with common boundary H. Consider the nullity distributions Γ and Γ' on the interior of C and C', which extend uniquely to parallel codimension one distributions F and F' on H, respectively. If $F = F'$, then $\Gamma \cup \Gamma'$ is a globally defined parallel distribution, which implies that the universal cover is an isometric product $N^2 \times \mathbb{R}^{n-2}$. Otherwise $J := F \cap F'$ is a codimension two parallel distribution on H. We claim that J extends to a parallel distribution on the interior of both C and C'.

To see this, we only need to argue for C, so lift the distributions J and F to the cover $S^1 \times \mathbb{R}^{n-2}$ of H under the projection $\pi\colon L^2 \times \mathbb{R}^{n-2} \to C = (L^2 \times \mathbb{R}^{n-2})/G$, and denote these lifts by $\hat{J}$ and $\hat{F}$. They are again parallel distributions whose leaves project to those of J and F under π. At a point $(x_0, v_0) \in S^1 \times \mathbb{R}^{n-2}$ a leaf of $\hat{F}$ is given by $\{x_0\} \times \mathbb{R}^{n-2}$ and hence a leaf of $\hat{J}$ by $\{x_0\} \times W$ for some affine hyperplane $W \subset \mathbb{R}^{n-2}$. Since $\hat{J}$ is parallel, any other leaf is given by $\{x\} \times W$ for $x \in S^1$. Since G permutes the leaves of $\hat{F}$, W is invariant under the projection of G into $\mathrm{Iso}(\mathbb{R}^{n-2})$. Therefore $\pi(\{x\} \times W)$ for $x \in L^2$ are the leaves of a parallel distribution on the interior of C, restricting to J on its boundary.

Therefore, we have a global flat parallel distribution J of codimension three on M^n, which implies that the universal cover splits isometrically as $N^3 \times \mathbb{R}^{n-3}$.

Now, if M^n consists of only one open cylinder C and its one-sided boundary, by Remark 2.5 there is a two-fold cover $\hat{M}^n$ of M^n which is the union of two cylinders as above and whose universal cover splits an $(n-3)$-dimensional Euclidean factor. $\square$

We can now finish the proof of Theorem A. Since M^n is compact with nonnegative curvature, the splitting theorem implies that the universal cover splits isometrically as $\tilde{M}^n = N^k \times \mathbb{R}^{n-k}$ with N^k compact and simply connected. According to the above claim, $k = 2$ and hence $N^2 \simeq \mathbb{S}^2$, or $k = 3$ and by Theorem 1.2 in [Ha] we have $N^3 \simeq \mathbb{S}^3$. In the latter case, we claim that the metric on $\mathbb{S}^3$ is again a geometric graph manifold metric. Indeed, if $\sigma\colon \mathbb{S}^3 \times \mathbb{R}^{n-3} \to M^n$ is the covering map, and $C \subset M^n$ a twisted cylinder, then in $C' = \sigma^{-1}(C)$ the codimension 2 nullity leaves contain the $\mathbb{R}^{n-3}$ factor. Since the universal cover of C' has the form $L^2 \times \mathbb{R}^{n-2}$, the metric on $\mathbb{S}^3$ must be a geometric graph manifold metric.

3. Geometric graph 3-manifolds with nonnegative curvature

In this section we classify 3-dimensional geometric graph manifolds with nonnegative scalar curvature, giving an explicit construction of all of them. As a consequence, we show that, for each lens space, the number of connected components of the moduli space of such metrics is infinite, while for each prism manifold, the moduli space is connected.

Let M^3 be a compact nonflat geometric graph manifold with nonnegative scalar curvature. We first observe M^3 is orientable. Indeed, by Theorem A, $M^3 = \mathbb{S}^3/\Pi$ and if an element $g \in \Pi$ acts orientation reversing, the Lefschetz fixed point theorem implies that g has a fixed point. Thus every cylinder $C = (D \times \mathbb{R})/G$ is orientable as well, i.e. G acts orientation preserving.

For $g \in G$, we write $g = (g_1, g_2) \in \mathrm{Iso}(D \times \mathbb{R})$. Thus g_1 preserves the closed geodesic ∂D and fixes the soul point $x_0 \in D$. If g_1 reverses orientation, then so does g_2 and hence g would have a fixed point. Thus g_2 is a nontrivial translation, which implies that $G \simeq \mathbb{Z}$. Altogether, the twisted cylinders are of the form $C = (D \times \mathbb{R})/\mathbb{Z}$ with $\mathbb{Z}$ generated by $g = (g_1, g_2)$. If g_1 is nontrivial, then g_1 is determined by its derivative at x_0. After orienting D, $d(g_1)_{x_0}$ is a rotation R_θ of angle $2\pi\theta$, $0 \le \theta < 1$. We simply say that g_1 acts as a rotation R_θ on D. Thus g acts via

$$g(x, s) = (R_\theta(x), s + t) \in \mathrm{Iso}(D \times \mathbb{R}), \tag{3.1}$$

for a certain $t > 0$ after orienting $\Gamma \cong T^\perp D$.

In particular, we have that the interior boundary of C is a flat 2-torus. Notice also that the action of $\mathbb{Z}$ can be changed differentiably until $\theta = 0$, and hence C is diffeomorphic to a solid torus $D \times S^1$. According to Corollary 2.4, M^3 is thus either the union of two solid tori glued along their boundary, and hence diffeomorphic to a lens space, or it is a solid torus whose boundary is identified via an involution to form a Klein bottle, and therefore diffeomorphic to a prism manifold.

Remark 3.2. Let us clarify the role of orientations in our description of C in (3.1). Take a twisted cylinder C with nonnegative scalar curvature, and D a maximal leaf of $\Gamma^\perp$. Orienting Γ is then equivalent to orienting $T^\perp D$, which in turn is equivalent to choosing one of the two generators of $\mathbb{Z}$. On the other hand, orienting D is equivalent to choosing between the oriented angle θ above or $1-\theta$. In particular, these orientations are unrelated to the metric on C, i.e., changing orientations give isometric cylinders.

Next, we show that the geometric graph manifold metric on M^3 is isotopic to a standard one. In order to do this, fix once and for all a metric $\langle\ ,\ \rangle_0$ on the disc $D_0=\{x\in\mathbb{R}^2 : |x|\leq 1\}$ which is rotationally symmetric, has positive Gaussian curvature on the interior of D_0, and whose boundary is a closed geodesic of length 1 along which the Gaussian curvature vanishes to infinite order. We call the metric on M^3 *standard*, if for each twisted cylinder $C=(D\times\mathbb{R})/\mathbb{Z}$ in the complement of a core of M^3, the metric on D is isometric to $r^2\langle\ ,\ \rangle_0$ for some $r>0$. Notice that such a metric on M^3 is unique up to isometry.

Proposition 3.3. *A geometric graph manifold metric with nonnegative scalar curvature is isotopic, through geometric graph manifold metrics with nonnegative scalar curvature, to a standard one.*

Proof. We can define the isotopy separately on each cylinder $C=(D\times\mathbb{R})/\mathbb{Z}$, as long as the isometry type of the core $H=\partial C$, and the foliation of H induced by the nullity leaves of C, stays fixed. To do this, we first deform the metric $\langle\ ,\ \rangle$ on D induced from the metric on M^3.

Let $\langle\ ,\ \rangle'$ be the standard flat metric on D_0, and G_1 the projection of $\mathbb{Z}$ onto $\mathrm{Iso}(D)$. By the uniformization theorem we can write $\langle\ ,\ \rangle = f_1^*(e^{2v}\langle\ ,\ \rangle')$ for some diffeomorphism $f_1: D\to D_0$ and a smooth function v on D_0. The metric $e^{2v}\langle\ ,\ \rangle'$ is thus invariant under $C_{f_1}(G_1)=\{f_1\circ g\circ f_1^{-1} : g\in G_1\}$ which fixes $f_1(x_0)$, where $x_0\in D$ is the fixed point of the action of G_1. Equivalently, $h\in C_{f_1}(G_1)$ is a conformal transformation of $(D_0,\langle\ ,\ \rangle')$ with conformal factor $e^{2v-2v\circ h}$. Recall that the conformal transformations of $\langle\ ,\ \rangle'$ on the interior of D_0 can be viewed as the isometry group of the hyperbolic disc model. Hence there exists a conformal transformation j of D_0 with $j(f_1(x_0))=0$ and conformal factor $e^{2\tau}$. We can thus also write $\langle\ ,\ \rangle = f^*(e^{2u}\langle\ ,\ \rangle')$, where $f=j\circ f_1: D\to D_0$ and $u:=(v-\tau)\circ j$. Now the metric $e^{2u}\langle\ ,\ \rangle'$ is invariant under $C_f(G_1)$, which this time fixes the origin of D_0. So $k\in C_f(G_1)$ is a conformal transformation of $\langle\ ,\ \rangle'$ fixing the origin, with conformal factor $e^{2u-2u\circ k}$. But an isometry of the hyperbolic disc model, fixing the origin, is also an isometry of $\langle\ ,\ \rangle'$. Hence $e^{2u}=e^{2u\circ k}$, i.e. u is invariant under k. Altogether, $C_f(G_1)\subset \mathrm{SO}(2)\subset \mathrm{Iso}(D_0,\langle\ ,\ \rangle')$ and u is $C_f(G_1)$-invariant. Analogously, $r^2\langle\ ,\ \rangle_0 = f_0^*(e^{2u_0}\langle\ ,\ \rangle')$ with $f_0\in\mathrm{Diff}(D_0)$ satisfying $f_0(0)=0$ and u_0 being $\mathrm{SO}(2)$-invariant. In particular, u_0 is also $C_f(G_1)$-invariant.

We now consider the two metrics $e^{2u}\langle\ ,\ \rangle'$ and $e^{2u_0}\langle\ ,\ \rangle'$ on D_0. They both have the property that the boundary is a closed geodesic along which the curvature vanishes to infinite order. Notice that the assumption that the boundary is a closed geodesic, up to parametrization, is equivalent to the condition that the normal derivatives of u and u_0, with respect to a unit normal vector in $\langle\ ,\ \rangle'$, is equal to 1. Furthermore, since $Ke^{2u}=-\Delta u$, the curvature vanishes to infinite order if and only if Δu does. For each $0 \leq s \leq 1$, consider the $C_f(G_1)$-invariant metric on D_0 given by $\langle\ ,\ \rangle^s = e^{2(1-s)u_0+2su+a(s)}\langle\ ,\ \rangle'$, where $a(s)$ is the function that makes the boundary to have length r for all s. Clearly, for each s, the boundary is again a closed geodesic up to parametrization and K^s vanishes at the boundary to infinite order. Furthermore, since $K^s e^{2(1-s)u_0+2su+a(s)} = -(1-s)\Delta u_0 - s\Delta u$ and $\Delta u_0 < 0,\ \Delta u \leq 0$, the curvature of $\langle\ ,\ \rangle^s$ is nonnegative and positive on the interior of D_0.

Now, the metrics $f^*\langle\ ,\ \rangle^s + dt^2$ on $D \times \mathbb{R}$ are invariant under the action of $\mathbb{Z}$ and hence induce the desired one parameter family of metrics on C, since $f^*\langle\ ,\ \rangle^0 + dt^2$ is isometric to $r^2\langle\ ,\ \rangle_0 + dt^2$ via the diffeomorphism $(f_0 \circ f^{-1}) \times \mathrm{Id}$. We then glue these metrics to the core H preserving the arc length parametrization of ∂D. □

We now discuss how C induces a natural marking on $\partial_i C$. For this, let us first recall some elementary facts about lattices $\Lambda \subset \mathbb{R}^2$, where we assume that the orientation on $\mathbb{R}^2$ is fixed. A *marking* of the lattice Λ is a choice of an oriented basis $\{v, \hat v\}$ of Λ, and we call such a marking *normalized* if $\theta := \langle v, \hat v\rangle/\|v\|^2 \in [0,1)$. Notice that for any $v \in \Lambda$, there exists a unique oriented normalized marking $\{v, \hat v\}$. Indeed, if $\{v, w\}$ is some basis of Λ, then $\langle v, w+nv\rangle/\|v\|^2 = \langle v, \hat v\rangle/\|v\|^2 + n$ and hence there exists a unique $n \in \mathbb{Z}$ such that $\{v, \hat v\}$ with $\hat v = w + nv$ is normalized. If T^2 is an oriented torus with base point z_0, then $T^2 = \mathbb{R}^2/\Lambda$ for some lattice Λ with $z_0 = [0]$. Then a (normalized) marking of T^2 is a basis of $T_{z_0}T^2 \simeq \mathbb{R}^2$ which is a (normalized) marking of the lattice Λ.

Consider an oriented twisted cylinder $C = (D \times \mathbb{R})/\mathbb{Z}$ with its standard metric, where the action of $\mathbb{Z}$ is given by (3.1). The totally geodesic flat torus $T^2 = \partial_i C$, which inherits an orientation from C, has a natural marking based at $z_0 = [(p_0, t_0)]$. For this, denote by $\gamma : [0,1] \to \partial D$ the simple closed geodesic of length $r = \|\gamma'(0)\| > 0$ with $\gamma(0) = p_0$ which follows the orientation of $D = [D \times \{t_0\}] \subset C$. Let $v = \gamma'(0)$, $\hat v = \theta v + t\partial/\partial s$, and notice that the geodesic $\sigma(s) = \exp(s\hat v), 0 \leq s \leq 1$, is simple and closed with length $\|\hat v\|$. Since $\theta \in [0,1)$, the basis $\mathcal{B} := \{v, \hat v\}$ is a normalized marking of T^2 based at z_0, which we denote by $\mathcal{B}(\gamma)$. We also have a parallel oriented foliation of T^2 by the closed geodesics $[\gamma \times \{t\}] \subset T^2$, which we denote by $\mathcal{F}(C)$.

It is important for us that the above process can be reversed for standard metrics:

PROPOSITION 3.4. *Let T^2 be a flat oriented torus and $\mathcal{F}$ an oriented foliation of T^2 by parallel closed simple geodesics. Then there exists a oriented twisted cylinder $C_{\mathcal{F}} = (D \times \mathbb{R})/\mathbb{Z}$ over a standard oriented disk D, unique up to isometry, such that $\partial C_{\mathcal{F}} = T^2$ and $\mathcal{F}(C_{\mathcal{F}}) = \mathcal{F}$. Moreover, different orientations induce isometric metrics.*

Proof. Choose $\gamma \in \mathcal{F}$, and set $z_0 = \gamma(0)$, $v = \gamma'(0)$, and let $\mathcal{B}(\gamma) = \{v, \hat v\}$ be the normalized marking of T^2 based at z_0 defined as above. Set $r = \|v\|$, $\theta = \langle v, \hat v\rangle/\|v\|^2$ and $t = \|\hat v - \theta v\|$. With respect to the oriented orthonormal basis $e_1 = v/r$, $e_2 = (\hat v - \theta v)/t$ of $T_{z_0}T^2$ we have

$$T^2 = \mathbb{R}^2/\Lambda = (\mathbb{R} \oplus \mathbb{R})/(\mathbb{Z}v \oplus \mathbb{Z}\hat v) = (S^1_r \times \mathbb{R})/\mathbb{Z}\hat v,$$

where S^1_r is the oriented circle of length r. Since $v = re_1$ and $\hat{v} = \theta v + te_2$, we can also write $T^2 = (S^1_r \times \mathbb{R})/\langle g\rangle$ where $g(p, s) = (R_\theta(p), s+t)$. Now we simply attach $(D_0, r^2\langle\ ,\ \rangle_0)$ to S^1_r preserving orientations to build $C = (D_0 \times \mathbb{R})/\langle g\rangle$. Notice that any two base points of T^2 are taken to each other by an orientation preserving isometry of C, restricted to $\partial C = T^2$. Thus the construction is independent of the choice of z_0 and the choice of $\gamma \in \mathcal{F}$. By Remark 3.2, different choices of orientation induce the same metric on C, and hence $C_{\mathcal{F}}$ is unique up to isometry. □

Remark 3.5. If we do not assume that the metric on C is standard, then the construction of $C_{\mathcal{F}}$ depends on the choice of base point, and one has to assume that the metric on D is invariant under R_θ, where θ is the angle determined by the marking of T^2 induced by $\mathcal{F}$.

We can now easily classify standard geometric graph manifold metrics with two-sided core, proving case (a) of Theorem B.

THEOREM 3.6. *Let M^3 be a compact geometric graph manifold of nonnegative scalar curvature with irreducible universal cover, and assume that its core T^2 is two-sided. Then, $M^3 = C_1 \sqcup T^2 \sqcup C_2$, where $C_i = (D_i \times \mathbb{R})/\mathbb{Z}$ are twisted cylinders over 2-disks that induce two different foliations $\mathcal{F}_i = \mathcal{F}(C_i)$ of T^2 by parallel closed geodesics, $i = 1, 2$.*

Conversely, given a flat 2-torus T^2 with two different foliations $\mathcal{F}_i$ by parallel closed geodesics, there exists a standard geometric graph manifold $M^3 = C_1 \sqcup T^2 \sqcup C_2$ with irreducible universal cover whose core is T^2 and $C_i = C_{\mathcal{F}_i}$. Moreover, this data determines the standard metric up to isometries, i.e., if $h : T^2 \to \hat{T}^2$ is an isometry between flat tori, then $\hat{M}^3 = \hat{C}_1 \sqcup \hat{T}^2 \sqcup \hat{C}_2$ is isometric to M^3, where $\hat{C}_i = C_{h(\mathcal{F}_i)}$.

Proof. We only need to argue for the uniqueness. The core of a standard metric is unique since the set of nonflat points of a standard metric is dense, cf. Corollary 2.4. It is clear then that an isometry between standard geometric graph manifolds will send the core to the core, and the parallel foliations to the parallel foliations. Hence the core and the parallel foliations are determined by the isometry class of M^3.

Conversely, by uniqueness in Proposition 3.4 the standard twisted cylinders $C_{\mathcal{F}_i}$ and $C_{h(\mathcal{F}_i)}$ are isometric, which in turn induces an isometry between M^3 and $\hat{M}^3$. The only ambiguity is on which side of the torus to attach each of the twisted cylinders, but this simply gives an orientation reversing isometry fixing the core. □

Now, let us consider the one-sided core case. Here we know that $M^3 = C \sqcup K$ and that K is a nonorientable quotient of the flat torus $\partial_i C$ and hence a flat Klein bottle. It is easy to see that, if a flat torus admits an orientation reversing fixed point free isometric involution j, then T^2 has to be isometric to a rectangular torus $S^1_r \times S^1_s$ along which j is as in (1.1). The irreducibility of the universal cover of M^3 is thus equivalent to $\mathcal{F}(C)$ not to coincide with one of the two invariant parallel foliations of j, $\mathcal{F}(j) = \{S^1_r \times \{w\} : w \in S^1_s\}$ and $\{\{z\} \times S^1_s : z \in S^1_r\}$.

As in the proof of Theorem 3.6, we conclude:

Theorem 3.7. *Let M^3 be a compact geometric graph manifold of nonnegative scalar curvature with irreducible universal cover, and assume that its core K is one-sided. Then $M^3 = C \sqcup K$, where $C = (D \times \mathbb{R})/\mathbb{Z}$ is a twisted cylinder over a 2-disk with $\partial_i C = T^2$ isometric to a rectangular torus, and $\partial C = K = T^2/\mathbb{Z}_2$ a flat totally geodesic Klein bottle.*

Conversely, a rectangular flat torus $T^2 = S^1_r \times S^1_s$ and a foliation $\mathcal{F}$ of T^2 by parallel closed geodesics different from $S^1_r \times \{p\}$ or $\{p\} \times S^1_s$ define a standard geometric graph manifold with irreducible universal cover $M^3 = C_{\mathcal{F}} \sqcup K$ whose core K is one-sided. Moreover, T^2 and $\mathcal{F}$ determine M^3 up to isometry.

In order to determine the topological type of these geometric graph manifolds we introduce the concept of relative slope between two foliations of a flat torus. To define it, we first assume that the data is oriented. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two different oriented parallel foliations by closed geodesics on the oriented flat torus T^2, and choose $\gamma_i \in \mathcal{F}_i$ such that $\gamma_1(0) = \gamma_2(0) =: z_0$. If $v_i = \gamma_i'(0)$ we have the normalized markings $\mathcal{B}(\gamma_i) = \{v_i, \hat{v}_i\}$ of T^2 based at z_0 defined as above. Since $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on the set of oriented basis of a given lattice, there exist coprime integers $p, q,\ a, b \in \mathbb{Z}$ with $bq - ap = 1$ such that

$$v_2 = qv_1 + p\hat{v}_1, \quad \hat{v}_2 = av_1 + b\hat{v}_1.$$

We also have $p \neq 0$ since $v_1 \neq \pm v_2$. Notice that, since v_2 determines $\hat{v}_2$, the integers p and q determine a and b.

We call $q/p \in \mathbb{Q}$ the (algebraic) *slope* of γ_2 with respect to γ_1. Notice that the slope is independent of the choice of $\gamma_i \in \mathcal{F}_i$ since the foliations are parallel. The slope does not depend on the orientations of the geodesics either, since $\{-v, -\hat{v}\}$ is the oriented marking associated to $-\gamma$. Since $v_1 = bv_2 - p\hat{v}_2$, the slope of γ_1 with respect to γ_2 is $-b/p$. Observe though that the slopes depend on the orientation of the torus, since changing the orientation of the torus corresponds to changing $\hat{v}_i$ with $-\hat{v}_i$ which gives slopes $-q/p$ and b/p, respectively.

We denote by $[q/p] = \{\pm q/p, \mp b/p\}$ the *relative slope between the foliations*. Accordingly, if $M^3 = C_1 \sqcup T^2 \sqcup C_2$ has two-sided core, by Theorem 3.6 we have that the relative slope between $\mathcal{F}(C_1)$ and $\mathcal{F}(C_2)$ is an isometric invariant of M^3, which we call the *slope of* M^3. Analogously, if $M^3 = C \sqcup K$ has one-sided core $K = \partial_i C/\langle j \rangle$, we call the relative slope $[m/n] = \pm m/n$ of $\mathcal{F}(C)$ with respect to $\mathcal{F}(j)$ the *slope of* M^3. Notice that the slope of M^3 is well defined even when the geometric graph manifold metric is not standard.

We show next that slope of M^3 is precisely what determines its diffeomorphism type.

Theorem 3.8. *Let M^3 be a compact geometric graph manifold of nonnegative scalar curvature with irreducible universal cover and slope $[m/n]$. Then, if the core of M^3 is two-sided M^3 is diffeomorphic to the lens space $L(n, m)$, while if the core of M^3 is one-sided M^3 is diffeomorphic to the prism manifold $P(m, n)$.*

Proof. Recall that the twisted cylinders C_i are diffeomorphic to $D_i \times S^1$ by deforming α_i continuously to 0. For a two-sided core T^2, orient M^3 and T^2, choose $\gamma_i \in \mathcal{F}_i$, and let $\mathcal{B}(\gamma_i) = \{v_i, \hat{v}_i\}$ be the normalized markings of T^2 defined by C_i. Then the natural generators of $\pi_1(\partial(D_i \times S^1)) \simeq \mathbb{Z} \oplus \mathbb{Z}$ are represented by the closed geodesics γ_i and $\sigma_i(t) = \exp(t\hat{v}_i)$, $0 \leq t \leq 1$ since the marking $\{v_i, \hat{v}_i\}$ is normalized. According to the definition of slope, $v_2 = qv_1 + p\hat{v}_1$ which implies that under the diffeomorphism from $\partial D_2 \times S^1 \simeq \partial C_2$ to $\partial C_1 \simeq \partial D_1 \times S^1$, the element $(1, 0) \in \pi_1(\partial(D_2 \times S^1))$ is taken to $(q, p) \in \pi_1(\partial(D_1 \times S^1))$. By definition this is the lens space $L(p, q)$; see Section 1.

To determine the topological type in the one-sided case, we view M^3 as the union of C with the flat twisted cylinder N^3 defined in (1.1). Then $\partial N^3 = T^2$ is a rectangular torus which we glue to $\partial_i C$. Taking $\epsilon \to 0$ (or considering $T^2 \times (0, \epsilon]$ as part of C instead), we obtain M^3. We can now use our second description of prism manifolds in Section 1 and the proof finishes as in the previous case. □

Remark. Notice that if the slope of a lens space is $[q/p] = \{\pm q/p, \mp b/p\}$ and hence $bq - ap = 1$, then $b = q^{-1} \mod p$ which is consistent with the fact that $L(p, q)$ and $L(p, b)$ are diffeomorphic.

We finally classify the moduli space of metrics.

PROPOSITION 3.9. *On a lens space $L(p, q)$ the connected components of the moduli space of geometric graph manifold metrics with nonnegative scalar curvature are parametrized, up to sign, by its slope $[q/p]$, and therefore it has infinitely many components. On the other hand, on a prism manifold $P(m, n)$ with $m > 1$ the moduli space is connected.*

Proof. In Proposition 3.3 we saw that we can deform any geometric graph manifold metric into one which is standard. According to Theorem 3.6, the metric on a lens space can equivalently be defined by the triple $(T^2, \mathcal{F}_1, \mathcal{F}_2)$. We can now deform the flat metric on the torus, carrying along the foliations $\mathcal{F}_i$, which hence induces a deformation of the metric by standard metrics. In the proof of Proposition 3.4 we saw that, after choosing orientations, for $\gamma_i \in \mathcal{F}_i$ with $v_i = \gamma_i'(0)$ we have the normalized marking $\mathcal{B}(\gamma_i) = \{v_i, \hat{v}_i\}$ which represents a fundamental domain of the lattice defined by T^2. We can thus deform the flat torus to a unit square torus such that the first marking is given by $v_1 = (1, 0)$, $\hat{v}_1 = (0, 1)$. Then $v_2 = (q, p) = qv_1 + p\hat{v}_1$, which in turn determines $\hat{v}_2$, and q/p is the slope of γ_2 with respect to γ_1. After changing orientations we can furthermore assume that $p, q > 0$. If we choose the second marking to make it standard, we instead obtain the second representative of the slope. Metrics with different slope can clearly not be deformed into each other since the invariant is a rational number.

For a prism manifold, we similarly deform the metric to be standard and the rectangular torus into a unit square. But then the slope $[m/n]$ already uniquely determines its diffeomorphism type. □

Remarks. *a*) Notice that the lens space $L(4n, 2n - 1)$ has two types of geometric graph manifold metrics, one being the union of two nonflat twisted cylinders and the other being one twisted cylinder whose boundary is identified to a Klein bottle, or equivalently the union of a nonflat and a flat twisted cylinder. These clearly lie in different components of geometric graph manifold metrics.

b) One easily sees that the angle α between the nullity foliations of a lens space, i.e., the angle between v_1 and v_2, is given by $\cos(\alpha) = (q+p\theta_1)r_1/r_2 = (b-p\theta_2)r_2/r_1$, where $r_i = |v_i|$ and θ_i are the twists of the two cylinders. One can thus make the nullity leaves orthogonal if and only if $0 \le -q/p < 1$ and in that case $r_2 = pt_1,\ t_2 = t_1/p$ and $\theta_1 = -q/p,\ \theta_2 = b/p$. This determines the metric on the lens space described in the introduction as a quotient of Figure 1, and is thus the only component containing a metric with orthogonal nullity leaves.

c) We can explicitly describe the geometric graph manifold metrics on $\mathbb{S}^3 = L(1,0)$ up to deformation. We assume that the core is a unit square and that the first foliation is parallel to $(1,0)$, i.e. the first cylinder is a product cylinder. Then the second marking is given by $v_2 = (q,1),\ \hat{v}_2 = (q-1,1)$. By choosing the orientations appropriately, we can assume $q \ge 0$. According to the proof of Proposition 3.3, a marking $\{v,\hat{v}\}$ corresponds to a twisted cylinder as in (3.1) with $r = \|v\|$, $\theta = \langle v,\hat{v}\rangle/\|v\|^2$ and $t = \|\hat{v} - \theta v\|$. Thus in our case the second cylinder is given by $r = 1/t = \sqrt{1+q^2}$, $\quad \theta = (1+q^2-q)(1+q^2)$. The relative slope is $[q] = \{\pm q, \mp 1\}$, and the standard metric in Figure 1 corresponds to $q = 0$.

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