

MANIFOLDS WITH CONULLITY AT MOST TWO AS GRAPH MANIFOLDS

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ABSTRACT. We find necessary and sufficient conditions for a complete Riemannian manifold M^n of finite volume, whose curvature tensor has nullity at least $n - 2$, to be a geometric graph manifold. In the process, we show that Nomizu's conjecture, well known to be false in general, is true for manifolds with finite volume.

The nullity space Γ of the curvature tensor R of a Riemannian manifold M^n is defined for each $p \in M$ as $\Gamma(p) = \{X \in T_p M : R(X, Y) = 0 \ \forall Y \in T_p M\}$, and its dimension $\mu(p)$ is called the *nullity* of M^n at p . It is well known that the existence of points with positive nullity has strong geometric implications. For example, on an open subset of M^n where μ is constant, Γ is an integrable distribution with totally geodesic leaves. In addition, if M^n is complete, its leaves are also complete on the open subset where μ is minimal; see e.g. [Ma]. Riemannian n -manifolds with conullity at most 2, i.e., $\mu \geq n - 2$, which we call *CN2 manifolds* for short, appear naturally and frequently in several different contexts in Riemannian geometry, e.g.:

- Gromov's 3-dimensional graph manifolds admit a complete CN2 metric with nonpositive sectional curvature and finite volume whose set of flat points consists of a disjoint union of flat totally geodesic tori ([Gr]). These were the first examples of Riemannian manifolds with geometric rank one. Interestingly, any complete metric of nonpositive curvature on such a graph manifold is necessarily CN2 and quite rigid, as was shown in [Sch];
- A Riemannian manifold is called semi-symmetric if at each point the curvature tensor is orthogonally equivalent to the curvature tensor of some symmetric space, which is allowed to depend on the point. CN2 manifolds are semi-symmetric since they have pointwise the curvature tensor of an isometric product of a Euclidean space and a surface with constant curvature. Conversely, Szabó showed in [Sz] that a complete simply connected semi-symmetric space is isometric to a Riemannian product $S \times N$, where S is a symmetric space and N is, on an open and dense subset, locally a product of CN2 manifolds;
- Isometrically deformable submanifolds tend to have large nullity. In particular, by the classic Beez-Killing theorem, any locally deformable hypersurface in a space form has to be CN2. Yet, generically, CN2 hypersurfaces are locally rigid, and the classification of

The first author was supported by CNPq-Brazil, and the second author by a grant from the National Science Foundation, by IMPA, and CAPES-Brazil.

the deformable ones has been carried out a century ago in [Sb, Ca]; see [DFT] for a modern version and further results. The corresponding classification of locally deformable CN2 Euclidean submanifolds in codimension two is considerably more involved, and was obtained only recently in [DF] and [FF];

- A compact immersed submanifold $M^3 \subset \mathbb{R}^5$ with nonnegative sectional curvature not diffeomorphic to the 3-sphere \mathbb{S}^3 is necessarily CN2, and either isometric to $(\mathbb{S}^2 \times \mathbb{R})/\mathbb{Z}$ for some metric of nonnegative curvature on \mathbb{S}^2 , or diffeomorphic to a lens space $\mathbb{S}^3/\mathbb{Z}_p$; see [FZ1]. In the case of lens spaces, the set of points with vanishing curvature has to be nonempty with Hausdorff dimension at least two. However, it is not known yet if they can be isometrically immersed into \mathbb{R}^5 ;
- I. M. Singer asked in [Si] whether a Riemannian manifold is homogeneous if the curvature tensor at any two points is orthogonally equivalent. The first counterexamples to this question were CN2 manifolds with constant scalar curvature, which clearly have this property, and are typically not homogeneous; see [Se1, BKV].

The most trivial class of CN2 manifolds is given by cylinders $L^2 \times \mathbb{R}^{n-2}$ with their natural product metrics, where L^2 is any (not necessarily complete) connected surface. More generally, we call a *twisted cylinder* any quotient

$$C^n = (L^2 \times \mathbb{R}^{n-2})/G,$$

where $G \subset \text{Iso}(L^2 \times \mathbb{R}^{n-2})$ acts properly discontinuously and freely. The natural quotient metric is clearly CN2, and we call L^2 the *generating surface* of C^n , and the images of the Euclidean factor its *nullity leaves*. Observe that C^n fails to be complete only because L^2 does not need to be. Yet, what is important for us is that C^n is foliated by complete, flat, totally geodesic, and locally parallel leaves of codimension 2.

Our first goal is to show that these are the basic building blocks of complete CN2 manifolds with finite volume:

THEOREM A. *Let M^n be a complete CN2 manifold. Then each finite volume connected component of the set of nonflat points of M^n is globally isometric to a twisted cylinder.*

The hypothesis on the volume of M^n is essential, since complete locally irreducible Riemannian manifolds with constant conullity two abound in any dimension; see [Se1], [BKV] and references therein. These examples serve also as counterexamples to the Nomizu conjecture in [No], which states that a complete locally irreducible semi-symmetric space of dimension at least three must be locally symmetric. However, Theorem A together with Theorem 4.4 in [Sz] yield:

COROLLARY 1. *Nomizu's conjecture is true for manifolds with finite volume.*

For the 3-dimensional case, the fact that the set of nonflat points of a finite volume CN2 manifold is locally reducible was proved in [Se2] and [SW] with a longer and more delicate proof; see also [Se3] for the 4-dimensional case. Notice also that in dimension 3 the

CN2 condition is equivalent to the assumption, called $cvc(0)$ in [SW], that every tangent vector is contained in a flat plane, or to the condition that the Ricci endomorphism has eigenvalues $(\lambda, \lambda, 0)$. Furthermore, in [BS] it was shown that a complete 3-manifold with (geometric) rank one is a twisted cylinder.

Observe that we are free to change the metric in the interior of the generating surfaces of the twisted cylinders in Theorem A, still obtaining a complete CN2 manifold. Moreover, they are nowhere flat with Gaussian curvature vanishing at their boundaries. Of course, these boundaries can be quite complicated and irregular.

In general it is very difficult to fully understand how the twisted cylinders in Theorem A can be glued together through the set of flat points in order to build a complete Riemannian manifold. An obvious way of gluing them is through compact totally geodesic flat hypersurfaces. Indeed, when the boundary of each generating surface L^2 in the twisted cylinder $C = (L^2 \times \mathbb{R}^{n-2})/G$ is a disjoint union of complete geodesics γ_j along which the Gaussian curvature of L^2 vanishes to infinity order, the boundary of C is a disjoint union of complete totally geodesic flat hypersurfaces $H_j = (\gamma_j \times \mathbb{R}^{n-2})/G_j \subset M^n$, where G_j is the subgroup of G preserving γ_j . We can now use each H_j to attach another finite volume twisted cylinder C' to C along H_j , as long as C' has a boundary component isometric to H_j . Repeating and iterating this procedure with each boundary component we construct a complete CN2 manifold M^n . As we will see, the hypersurfaces H_j have to be compact if M^n has finite volume. This motivates the following concept of geometric graph manifold of dimension $n \geq 3$, which by definition is endowed with a CN2 Riemannian metric:

Definition. A connected Riemannian manifold M^n is called a *geometric graph manifold* if M^n is a locally finite disjoint union of twisted cylinders C_i glued together through disjoint compact totally geodesic flat hypersurfaces H_λ of M^n . That is,

$$M^n \setminus W = \bigsqcup_{\lambda} H_{\lambda}, \quad \text{where } W := \bigsqcup_i C_i.$$

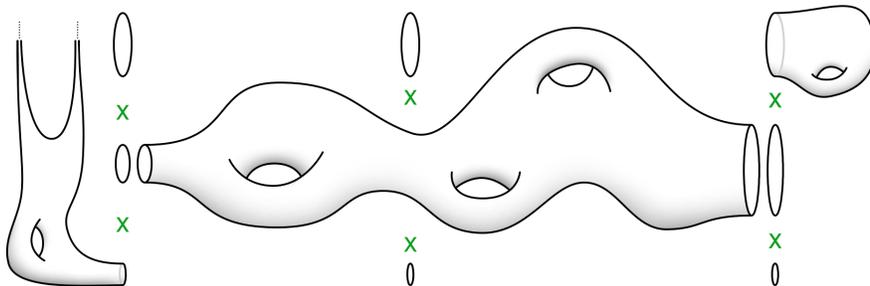


FIGURE 1. An irreducible 4-dimensional CN2 geometric graph manifold with three cylinders and two (finite volume) ends

Here we allow the possibility that a hypersurface H_λ is one-sided, even when M^n is orientable. We also assume, without loss of generality, that the nullity leaves of two cylinders C and C' , glued along H_λ , have distinct limits in H_λ . This implies in particular that for each cylinder C , the Gauss curvature vanishes along ∂C to infinite order. Notice that the locally finiteness condition is equivalent to the assumption that each H_λ is a common boundary component of two twisted cylinders C_i and C_j , that may even be globally the same, each lying at a local side of H_λ .

Observe that the complement of W is contained in the set of flat points of M^n , but we do not require that the generating surfaces of C_i have nonvanishing Gaussian curvature. In particular the sectional curvature of C_i , or equivalently its scalar curvature, can change sign. More importantly, W carries a well defined complete flat totally geodesic parallel distribution of constant rank $n-2$ contained in the nullity of M^n . Furthermore, W is dense and locally finite in the sense that it has a locally finite number of connected components (see Section 4 for a precise definition). These two topological properties will be crucial in what follows, so for convenience we say that a dense locally finite set is *full*.

A natural way to try to see if a Riemannian manifold M^n as in Theorem A is indeed a geometric graph manifold is the following. Theorem A implies that on the open set of nonflat points V we have the well defined parallel nullity distribution Γ of rank $n-2$, as in W above. Now, consider any open set $\hat{V} \supset V$ carrying a complete flat totally geodesic distribution $\hat{\Gamma}$ with $\hat{\Gamma}|_V = \Gamma$, which we call an *extension* of V . We will show that each connected component of \hat{V} is still a twisted cylinder, and call \hat{V} *maximal* if it has no larger extension. Clearly, by definition V always has a maximal extension, but it may not be unique. More importantly, all extensions of V may fail to be either dense, or locally finite, or both; see Examples 2 and 3 in Section 1.

Our second main goal is to prove that all we need to ask in order for M^n as in Theorem A to be a geometric graph manifold is that some extension of V is full:

THEOREM B. *Let M^n be a complete CN2 manifold with finite volume. Then M^n is a geometric graph manifold if and only if its set of nonflat points V admits a full extension. In particular, if V itself is full, then M^n is a geometric graph manifold.*

We point out that here we do not require a full extension \hat{V} of V to be maximal, but clearly any maximal extension of \hat{V} is also full. We can for example introduce complicated sets of flat points in the twisted cylinders, even as Cantor sets in the generating surfaces, but these flat sets will be absorbed by a maximal full extension. As we will show, any maximal full extension will satisfy the properties of W in the definition of geometric graph manifold, see Theorem 4.12. We expect that the methods developed to prove this can be extended for distributions of arbitrary rank.

The assumption of local finiteness in Theorem B can be regarded as a mild regularity condition. But we believe that even without regularity conditions it should be possible to understand the gluing between the twisted cylinders. We state:

CONJECTURE 1. *If the set of nonflat points of a complete CN2 manifold with finite volume admits a dense (not necessarily locally finite) extension, then the complement of any maximal one is a disjoint union of compact totally geodesic flat hypersurfaces, possibly accumulating (see Example 3 in Section 1).*

Certainly more difficult, we can ask what happens if we remove all hypothesis on V . In particular, we do not know if the following is true:

QUESTION. *Does the set V of nonflat points of a complete CN2 manifold with finite volume admit a maximal (not necessarily dense or locally finite) extension \hat{V} such that $\partial\hat{V}$ is a union of flat totally geodesic hypersurfaces, each of which has complete totally geodesic boundary (if nonempty)? (See Example 2 in Section 1).*

On the other hand, in the case of nonnegative or nonpositive curvature we believe that no extra assumptions are needed:

CONJECTURE 2. *Every compact CN2 manifold with nonnegative or nonpositive scalar curvature and finite volume is a geometric graph manifold.*

In [FZ2] we classify all geometric graph manifolds with nonnegative scalar curvature and show that they are three dimensional up to a Euclidean factor. Moreover, they are built as the union of at most two cylinders and, in particular, are diffeomorphic to a lens space or a prism manifold.

Another interesting question is to what extent complete CN2 manifolds with finite volume differ from geometric graph manifolds from a differentiable point of view:

QUESTION. *If M^n admits a complete CN2 metric with finite volume, does it also admit a geometric graph manifold metric?*

We caution that our definition of a graph manifold in dimension 3 is more special than the usual topological one, where the pieces are allowed to be nontrivial Seifert fibered circle bundles ([Wa]). Ours is similar, although more general, to the kind of graph manifolds one studies in nonpositive curvature.

The paper is organized as follows. In Section 1 we provide some examples in order to show that the two hypothesis in Theorem B are necessary. A general semi-global version of the de Rham theorem is provided in Section 2 and will be used in Section 3 to prove Theorem A. The proof of Theorem B is carried out in Section 4.

1. EXAMPLES.

We now build some examples to help understand how geometric graph manifolds are linked with the CN2 property, and to what extent they differ. In particular, we exhibit CN2 metrics on the 3-torus T^3 which are C^∞ perturbations of the flat metric but that are not geometric graph manifold metrics.

1. *The 3-torus as a nontrivial geometric graph manifold.* Let $L^2 = [-1, 1]^2$ with metric a C^∞ perturbation of the flat metric in a small open set $U \subset L^2$ whose closure is contained in the interior of L^2 . The cube $C = L^2 \times [-1, 1]$ with its product metric serves as a building block in all further examples, where the second factor will give rise to the nullity foliations. Depending on the example, we also adjust their sizes appropriately. Notice that the metric necessarily has scalar curvature of both signs. We now glue two such cubes along a common face in such a way that the nullity distributions are orthogonal, see Figure 3. Identifying opposite faces of the resulting larger cube defines a metric on T^3 with complete nullity foliations, making it into a nontrivial graph manifold. Figure 3 shows the nonflat points on the left, together with a full maximal extension and its two (un)twisted cylinders on the right.

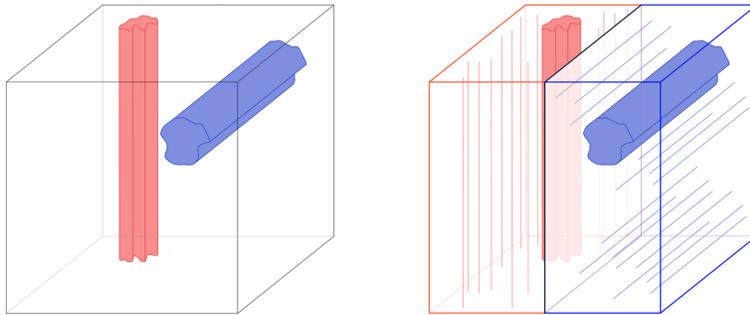


FIGURE 2. A CN2 3-torus with its set of nonflat points, and a full extension

2. *A CN2 3-torus failing to be a geometric graph manifold: no maximal dense extensions.* Here we take three basic building blocks and glue them together as in Figure 4. Adding two small flat cubes, we obtain a larger cube and identifying opposite faces defines a CN2 metric on T^3 . But this is not a geometric graph manifold since the nullity distribution cannot be extended to a dense set of T^3 . Figure 4 shows the set of nonflat points on the left, and a maximal extension of it on the right missing two octants of flat points. We point out that this example also shows that a CN2 manifold does not necessarily admit a T structure (see [CG]), as graph manifolds do.

3. *A CN2 3-torus failing to be a geometric graph manifold: no locally finite extension.* Take a sequence of building blocks $C_n = L_n \times [-1, 1]$ with $L_n = [-1/2^n, 1/2^n] \times [-1, 1]$. Glue one to the next along the squares $\{\pm 1/2^n\} \times [-1, 1] \times [-1, 1]$ as in Example 1, with nullity lines meeting orthogonally from one to the next, and accumulating at a two torus $T^2 = [-1, 1] \times [-1, 1]$. Now glue to this a copy of itself along T^2 . Identifying opposite sides in the resulting cube defines a CN2 metric on T^3 . The metric has two sequences of parallel totally geodesic flat 2-tori, approaching T^2 from both sides. It is not a geometric graph manifold since the number of connected components near T^2 of any extension of V is

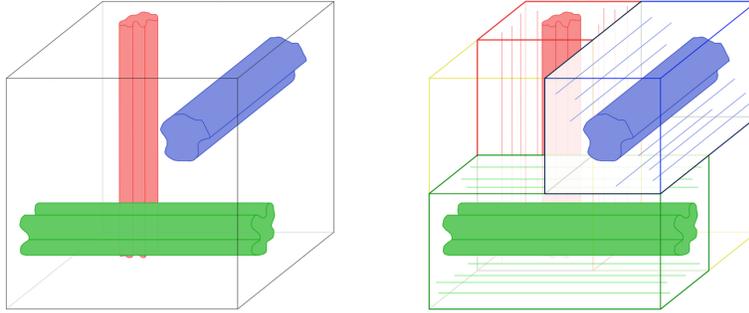


FIGURE 3. A CN2 3-torus with its nonflat points and a nondense maximal extension

infinite, even though there exist dense maximal extensions of V . Notice that T^2 is disjoint from the union of the closures of the connected components of any extension of V .

4. *Drunken cylinders.* We can modify the previous examples to obtain a more complicated behavior of the twisted cylinders, illustrating another crucial difficulty in trying to prove Conjecture 1 in the introduction. For this, we start with a flat building block $C_n = L_n \times [-1, 1]$ and identify two opposite faces to obtain a flat metric on $[-1/2^n, 1/2^n] \times T^2$. Let γ_n be a closed geodesic in the two torus $\{0\} \times T^2$ and modify the metric in a small tubular neighborhood of γ_n in $(-1/4^n, 1/4^n) \times T^2$. The boundary of the resulting manifold consists of two flat square tori and we can hence glue one to the next as in the previous example. Gluing a mirror copy of the result and identifying the remaining two faces, we obtain a CN2 metric on T^3 . We can now choose the infinite sequence of cylinders comprising the example such that the slopes of γ_n converge. Notice that in the previous examples the totally geodesic 2-tori separating components of a full extension of the set of nonflat points have the property that they are foliated by two different limits of nullity lines, whereas here the torus T^2 in the middle only has one family of limit nullity lines. The existence of two families is crucial in proving convexity properties of the boundary of a maximal full extension of the set of nonflat points; see Section 4.

We finish this section with some examples that illustrate some of the complexities of twisted cylinders.

a) Let Σ be a compact surface with boundary with universal cover L^2 , on which $G = \pi_1(\Sigma)$ acts freely. Now choose a homomorphism $\alpha: G \rightarrow \mathbb{R}$ and an action of G on $L^2 \times \mathbb{R}$ given by $(x, t) \rightarrow (gx, t + \alpha(g))$. In the cylinder $C = (L^2 \times \mathbb{R})/G$ the integral leaves of Γ^\perp are dense in C as long as the values of $\text{Im } \alpha \subset \mathbb{R}$ are not all rationally dependent. Notice though that C is diffeomorphic to $\Sigma \times S^1$ by changing the action continuously using the homomorphism $\epsilon\alpha$ and letting $\epsilon \rightarrow 0$.

b) The boundary geodesics of the leaves of Γ^\perp may not be closed. As an example, start with a complete flat strip $[-1, 1] \times \mathbb{R}$, remove infinitely many ϵ -discs centered at $(0, n)$, $n \in \mathbb{Z}$, and change the metric around them to make their boundaries totally geodesic in

such a way that the metric remains invariant under $\mathbb{Z} = \langle h \rangle$ for $h(x, t) = (x, t + 1)$. If L^2 is this surface and R is a rotation of S^1 of angle $r\pi$ for some irrational number r , then $G = \mathbb{Z} = \langle (h, R) \rangle$ acts freely and properly discontinuously on $L^2 \times S^1$. Although L^2/G_1 has three closed geodesics as boundary, the boundary of the leaves of Γ^\perp in $C = (L^2 \times S^1)/G$ has one closed geodesic and also two complete open geodesics, each of which is dense in the corresponding boundary component of C .

c) In general, a cylinder of finite volume does not necessarily have compact boundary. Indeed, consider the metric $ds^2 = dr^2 + e^{-\frac{1}{1-r^2}-t^2} dt^2$ on $L^2 = [-1, 1] \times \mathbb{R}$. Then $L^2 \times T^{n-2}$ has finite volume with two complete noncompact flat totally geodesic boundary components. Nevertheless, we will see that when we glue two twisted cylinders in a nontrivial way, their common boundary is compact.

2. A SEMI-GLOBAL DE RHAM THEOREM

The existence of a parallel smooth distribution Γ on a complete manifold M^n implies that the universal cover is an isometric product of a complete leaf of Γ with a leaf of Γ^\perp by the global deRham theorem. In this section we prove a semi-global version of this fact to be used later on, and will concentrate on flat foliations for simplicity. Although this is a special case of Theorem 1 in [PR], our proof is simpler and more direct, so we add it here for completeness.

PROPOSITION 2.1. *Let W be an open connected set of a complete Riemannian manifold M^n , and assume that Γ is a rank k parallel distribution on W whose leaves are flat and complete. If L is a maximal leaf of Γ^\perp , then the normal exponential map $\exp^\perp : T^\perp L \rightarrow W$ is an isometric covering, where $T^\perp L$ is equipped with the induced connection metric. In particular, W is isometric to the twisted cylinder $(\tilde{L} \times \mathbb{R}^k)/G$, where \tilde{L} is the universal cover of L , and G acts isometrically in the product metric.*

Proof. Since Γ is parallel, its orthogonal complement Γ^\perp is also parallel and hence integrable with totally geodesic leaves. Due to the local isometric product structure of W given by the local deRham Theorem, the normal bundle $T^\perp L$ of L is flat with respect to the normal connection of L , which in our case is simply the restriction of the connection on M^n since L is totally geodesic. Notice though that $T^\perp L$ does not have to be trivial. The normal connection defines, in the usual fashion, a connection metric on the total space. By completeness of the leaves of Γ , the normal exponential map $\exp^\perp : T^\perp L \rightarrow W$ is well defined on the whole normal bundle.

We first show that \exp^\perp is a local isometry. Indeed, if $\alpha(s) = (c(s), \xi(s))$ is a curve in $T^\perp L$ with ξ parallel along c , then $\alpha'(0)$ is a horizontal vector in the connection metric, identified with $c'(0)$ under the usual identification $H_{\xi(0)} \simeq T_{c(0)}L$. Then $d(\exp^\perp)_{\alpha(0)}(\alpha'(0)) = J(1)$ where $J(t)$ is the Jacobi field along the geodesic $\gamma(t) = \exp^\perp(t\xi(0))$ with initial conditions $J(0) = c'(0)$ and $J'(0) = 0$ since ξ is parallel along c . Since $\gamma' \in \ker R$, the Jacobi operator $R(\cdot, \gamma')\gamma'$ vanishes and thus $J(1)$ is the parallel translate of $J(0) \in T_{c(0)}L$, which in turn implies that \exp^\perp is an isometry on the horizontal space. On the vertical space it

is an isometry since it agrees with the exponential map of the fiber which is a complete flat totally geodesic submanifold of W . The image of horizontal and vertical space under \exp^\perp are also orthogonal, since the first is the parallel translate of TL and the second the parallel translate of $T^\perp L$ along γ . Hence \exp^\perp is a local isometry. Notice that this also implies that if ξ is a (possibly only locally defined) parallel section of $T^\perp L$, then $\{\exp_p(t\xi(p)) \mid p \in L\} \subset W$ are integral manifolds of Γ^\perp for all $t \in \mathbb{R}$.

Let $q \in W$ and denote by L_q the maximal leaf of Γ^\perp containing q . Since the normal exponential map of L_q is also a local isometry, there exists an $\epsilon = \epsilon(q) > 0$ such that $B_\epsilon(q) \subset L_q$ is a normal ball and the set $\hat{V}_q := \{\xi \in T^\perp B_\epsilon(q) : \|\xi\| < \epsilon\} \subset T^\perp L_q$ is isometric to the Riemannian product $B_\epsilon(q) \times B_\epsilon \subset L_q \times \mathbb{R}^k \cong L_q \times T_q^\perp L_q$, and $\exp^\perp : \hat{V}_q \rightarrow V_q := \exp^\perp(\hat{V}_q)$ is an isometry. In particular, $q \in V_q \subset W$. We identify $B_\epsilon(q) \times B_\epsilon$, \hat{V}_q and V_q via the normal exponential map of L_q . Accordingly, we denote the local leaf of Γ^\perp through $x = (p, v) \in V_q$ by $L_{x,q} := B_\epsilon(q) \times \{v\} \subset L_x \cap V_q$. Moreover, for each $y \in V_q$ and $v \in T_y^\perp L_{y,q}$ there is a parallel vector field ξ in V_q with $\xi(y) = v$, and an isometric flow $\phi_t^\xi(x) = \exp_x(t\xi(x))$ for $x \in V_q$. Notice that this flow is defined for all $t \in \mathbb{R}$, and that the images of the leaves $L_{x,q}$ are again leaves of Γ^\perp for all $t \in \mathbb{R}$, $x \in V_q$.

We now claim that \exp^\perp is surjective onto W . Take a point $q \in W$ in the closure of the open set $U = \exp^\perp(T^\perp L)$ in W , and choose $y \in V_q \cap U$. Since the leaf of Γ containing y is also contained in the image of \exp^\perp , we can assume that $y \in L_{q,q}$. Then $y = \gamma(1)$, where $\gamma(t) = \exp^\perp(t\xi)$, for $x \in L$ and $\xi \in T_x^\perp L$. If η is the parallel vector field in V_q with $\eta(y) = \gamma'(1) \in T_y^\perp L_{q,q}$, then $\phi_{-1}^\eta(L_{q,q}) \subset L$ by maximality of L and hence $q \in U$. Hence, U is closed in W , so $U = W$.

In order to finish the proof that \exp^\perp is a covering map, we show that it has the curve lifting property. Let $\alpha : [a, b] \rightarrow W$ be a smooth curve, and assume there exists a lift of $\alpha|_{[a,r]}$ to a curve $\tilde{\alpha} : [a, r] \rightarrow T^\perp L$. As usual, to extend $\tilde{\alpha}$ past r we only need to show that $\lim_{t \rightarrow r} \tilde{\alpha}(t)$ exists. Write $\tilde{\alpha}(t) = (c(t), \xi(t)) \in T^\perp L$. Since \exp^\perp is a local isometry, $|\tilde{\alpha}'| = |\alpha'|$ and hence by the local product structure c and ξ have bounded length. Thus, $\lim_{t \rightarrow r} c(t) = x_\infty \in M$ and $\lim_{t \rightarrow r} \xi(t) = \xi_\infty$ exist by completeness of M^n and \mathbb{R}^k . We only need to show that $x_\infty \in L$ since then $\exp^\perp(x_\infty, \xi_\infty) = \alpha(r)$.

Consider $\delta > 0$ such that $\alpha((r-\delta, r]) \subset V_{\alpha(r)}$. For $t \in (r-\delta, r)$ let η_t be the parallel vector field in $V_{\alpha(r)}$ with $\eta_t(\alpha(t)) = \gamma'_t(1) \in T^\perp L_{\alpha(t)}$ where $\gamma_t(s) = \exp^\perp(s\xi(t))$. We then have $\lim_{t \rightarrow r} \eta_t = \eta_\infty \in T^\perp L_{\alpha(r)}$ with $\exp_{\alpha(r)}(-\eta_\infty) = x_\infty \in W$ and hence $\phi_{-1}^{\eta_\infty}(L_{\alpha(r), \alpha(r)}) \subset L_{x_\infty}$. Since we also have that $\phi_{-1}^{\eta_t}(L_{\alpha(t), \alpha(r)}) \subset L$ for $t < r$, it follows that $\lim_{t \rightarrow r} T_{c(t)} L = T_{x_\infty} L_{x_\infty}$ and thus $L \cup L_{x_\infty}$ is an integral leaf of Γ^\perp . By the maximality of L we conclude that $L_{x_\infty} \subset L$ and therefore $x_\infty \in L$, as we wished.

Finally, if $\pi : \tilde{L} \rightarrow L$ is the universal cover, then $\pi^*(T^\perp L) \rightarrow T^\perp L$ is also a cover and since $\pi^*(T^\perp L)$ is again a flat vector bundle over a simply connected base, it is isometric to $\tilde{L} \times \mathbb{R}^k$. This proves the last claim. \square

3. THE STRUCTURE OF THE SET OF NONFLAT POINTS

This section is devoted to the proof of the following stronger version of Theorem A.

THEOREM 3.1. *Let M^n be a complete CN2 manifold, and V a connected component of the set of nonflat points of M^n . If V has finite volume, then its universal cover is isometric to $L^2 \times \mathbb{R}^{n-2}$, where L^2 is a simply connected surface whose Gauss curvature is nowhere zero and vanishes at its boundary.*

Proof. First, recall that, since $\Gamma = \ker R$ is a totally geodesic distribution, we have its *splitting tensor* $C : \Gamma \rightarrow \text{End}(\Gamma^\perp)$ defined as

$$C_T X = -(\nabla_X T)_{\Gamma^\perp},$$

where ∇ is the Levi-Civita connection of M^n , and a distribution as a subscript means to take the corresponding orthogonal projection. Clearly, Γ^\perp is totally geodesic if and only if $C \equiv 0$, that is equivalent to the parallelism of Γ . Since the set of nonflat points in a CN2 manifold agrees with the points of minimal nullity, V is saturated by the flat complete leaves of Γ . Therefore, by Proposition 2.1, all we need to show is that C vanishes.

Let $U, S \in \Gamma$ and $X \in \Gamma^\perp$. Since Γ is totally geodesic,

$$\begin{aligned} C_{\nabla_U S} X &= -(\nabla_X \nabla_U S)_{\Gamma^\perp} = -(\nabla_U \nabla_X S)_{\Gamma^\perp} - (\nabla_{[X, U]} S)_{\Gamma^\perp} \\ &= (\nabla_U (C_S X))_{\Gamma^\perp} + C_S([X, U]_{\Gamma^\perp}) = (\nabla_U C_S)X + C_S(\nabla_U X) - C_S([U, X]_{\Gamma^\perp}) \\ &= (\nabla_U C_S)X + C_S(\nabla_X U) = (\nabla_U C_S)X - C_S C_U X, \end{aligned}$$

or

$$(3.2) \quad \nabla_U C_S = C_{\nabla_U S} + C_S C_U, \quad \forall U, S \in \Gamma.$$

We now consider the so called *nullity geodesics*, i.e. complete geodesics γ with $\gamma'(0) \in \Gamma$, which are hence contained in a leaf of Γ . Along such a geodesic γ , by (3.2) the splitting tensor $C_{\gamma'}$ satisfies the Riccati type differential equation $\nabla_{\gamma'} C_{\gamma'} = C_{\gamma'}^2$ over the entire real line. That is, with respect to a parallel basis,

$$(3.3) \quad C'_{\gamma'} = C_{\gamma'}^2, \text{ whose solutions are } C(t) = C_0(I - tC_0)^{-1}, \text{ for } C_0 := C_{\gamma'(0)}.$$

Therefore, along each nullity geodesic γ in V , all real eigenvalues of $C_{\gamma'}$ vanish. Since Γ^\perp is 2-dimensional, for every $S \in \Gamma$ either all eigenvalues of C_S are complex and nonzero, or all eigenvalues are 0, i.e. C_S is nilpotent. In particular, C_S vanishes if it is self adjoint.

Let $W = \{p \in V : C \neq 0 \text{ at } p\}$, i.e. on $V \setminus W$ all splitting tensors vanish. Since the space of self-adjoint endomorphisms of Γ^\perp is pointwise 3-dimensional and intersects $\text{Im } C \subset \text{End}(\Gamma^\perp)$ only at 0, it follows that $\dim \text{Im } C = 1$ in W , and hence $\ker C$ is a smooth codimension 1 distribution of Γ along W . Accordingly, write

$$\Gamma = \ker C \oplus^\perp \text{span}\{T\},$$

for a unit vector field $T \in \Gamma$, which is well defined, up to sign, on W . By going to a two-fold cover of W if necessary, we can assume that T can be chosen globally on W .

Observe that if U, S are two sections of $\ker C$, then (3.2) implies that $\nabla_U S \in \ker C$, i.e. $\ker C$ is totally geodesic, and $\nabla_T U = 0$ as well. Since Γ is totally geodesic it follows that $\nabla_T T = 0$, that is, the integral curves γ of T are nullity geodesics. Therefore, from now on let for convenience $C = C_T$, $C(t) = C_{\gamma'(t)}$, and denote by $'$ the derivative in direction of T . In particular,

$$(3.4) \quad \operatorname{div} T = \operatorname{tr} \nabla T = -\operatorname{tr} C.$$

By (3.3) we have

$$(3.5) \quad \operatorname{tr} C(t) = \frac{\operatorname{tr} C_0 - 2t \det C_0}{1 - t \operatorname{tr} C_0 + t^2 \det C_0}, \quad \text{and} \quad \det C(t) = \frac{\det C_0}{1 - t \operatorname{tr} C_0 + t^2 \det C_0}.$$

Take $B \subset W$ a small compact neighborhood. Since either $\det C > 0$ or $\det C = \operatorname{tr} C = 0$ on W , by (3.5) there is $t_0 \in \mathbb{R}$ such that $\operatorname{tr} C(t)(q) \leq 0$ for every $q \in B$ and every $t \geq t_0$. In addition, defining $B_t := \phi_{t+t_0}(B)$ and $v(t) := \operatorname{vol} B_t$ we have that

$$v'(t) = \int_B \frac{d}{dt} \phi_t^*(d\operatorname{vol}) = \int_B \operatorname{div} T = - \int_B \operatorname{tr} C \geq 0, \quad \forall t \geq 0.$$

So, the sequence of compact neighborhoods $\{B_n, n \in \mathbb{N}\}$ has nondecreasing volume in the set V of finite volume, and thus there is a strictly increasing sequence $\{n_k : k \geq 0\}$ such that $B_{n_k} \cap B_{n_0} \neq \emptyset$ for all $k \geq 1$. We will refer to this property as *weak recurrence*. In particular, there exists a sequence $p_k := \phi_{t_0+n_k}(q_k) \in B_{n_k} \cap B_{n_0}$, with $q_k \in B$, which has an accumulation point $p \in B_{n_0} \subset W$.

Consider the open subset $W' \subset W$ on which C has nonzero complex eigenvalues and notice that, by (3.3), W' is invariant under the flow ϕ_t of T . Using the above recurrence and sequence of points $p_k \rightarrow p$, (3.5) implies that

$$\det C_{T(p)} = \lim_{k \rightarrow +\infty} \det C_{T(p_k)} = \lim_{k \rightarrow +\infty} \frac{\det C_{T(q_k)}}{1 - (t_0 - n_k) \operatorname{tr} C_{T(q_k)} + (t_0 - n_k)^2 \det C_{T(q_k)}} = 0,$$

since $n_k \rightarrow +\infty$ and q_k lies in the compact set B . But this contradicts the fact that $p \in B_{n_0} \subset W'$, where $\det C > 0$. Thus C vanishes on W' , which is a contradiction and shows that C is nilpotent on W .

We thus have a well defined 1-dimensional distribution on W spanned by the kernel of $C = C_T$, which is parallel along nullity lines by (3.3). Replacing W , if necessary, by the two-fold cover where this distribution has a section, and by a further cover to make W orientable, we can assume that there exists an orthonormal basis e_1, e_2 of Γ^\perp , defined on all of W , and parallel along nullity lines with

$$C(e_1) = 0, \quad C(e_2) = ae_1.$$

Hence

$$\begin{aligned} \nabla_T e_1 &= \nabla_T e_2 = \nabla_T T = 0, & \nabla_{e_1} T &= 0, & \nabla_{e_2} T &= -ae_1, \\ \nabla_{e_1} e_1 &= \alpha e_2, & \nabla_{e_2} e_2 &= \beta e_1, & \nabla_{e_1} e_2 &= -\alpha e_1, & \nabla_{e_2} e_1 &= aT - \beta e_2, \end{aligned}$$

for some smooth functions α, β on W . A calculation shows that

$$\begin{aligned} R(e_2, e_1)e_1 &= (e_1(\beta) + e_2(\alpha) - \alpha^2 - \beta^2)e_2 + (a\beta - e_1(a))T, \\ R(e_1, e_2)e_2 &= (e_1(\beta) + e_2(\alpha) - \alpha^2 - \beta^2)e_1 + \alpha aT, \end{aligned}$$

and hence

$$(3.6) \quad \alpha = 0, \quad \text{Scal}_M = e_1(\beta) - \beta^2, \quad \text{and} \quad e_1(a) = a\beta,$$

where Scal_M stands for the scalar curvature of M^n . The differential equation (3.3) implies that $a' = 0$, i.e. a is constant along nullity lines. Thus

$$e_2(a)' = e_2(a') + [T, e_2](a) = (\nabla_T e_2 - \nabla_{e_2} T)(a) = ae_1(a),$$

and $e_1(a)' = e_1(a') + [T, e_1](a) = 0$. So, $e_2(a) = ae_1(a)t + d$ for some smooth function d independent of t . By the weak recurrence property, $e_1(a)(p) = 0$ for $p \in B_{n_0}$ as above, and hence $e_1(a)(p') = 0$ as well, for $p' = \phi_{-n_0-t_0}(p) \in B$. Since B is arbitrary small, we have that $e_1(a) \equiv 0$ on W . But then (3.6) implies that $\beta = 0$ and thus $\text{Scal}_M = 0$, contradicting that along V we have no flat points. Altogether, C vanishes everywhere on V and hence Γ is parallel. \square

The proof becomes particularly simple for $n = 3$ since then Γ is one dimensional and only the differential equation $C' = C^2$ along the unique nullity geodesics is needed. The local product structure in this case was proved earlier in [SW] with more delicate techniques.

Let us finish this section with some observations about the geometric structure when the manifold is complete but the finite volume hypothesis is removed.

Remark. a) If C_T is nilpotent, then Scal is constant along nullity leaves. If this constant is positive, and M^n is complete, then the universal cover is isometric to $L^2 \times \mathbb{R}^{n-2}$. This follows since by (3.6) a^{-1} satisfies the Jacobi equation $(a^{-1})'' + a^{-1} \text{Scal} = 0$ along integral curves of e_1 , which cannot be satisfied for all $t \in \mathbb{R}$. For $n = 3$ this splitting was also proved in [AM].

b) If C_T does not vanish and has complex eigenvalues, then $\text{Scal}(t) = \text{Scal}(0)/(1 - t \text{tr} C_0 + t^2 \det C_0)$ along nullity lines since one easily sees that $\text{Scal}' = (\text{tr} C_T) \text{Scal}$. As was shown in [Sz], if M^n is complete and C_T has no real eigenvalues, then the universal cover of M^n splits off a Euclidean space of dimension $(n - 3)$, i.e. all locally irreducible examples are 3-dimensional.

4. CN2 MANIFOLDS AS GEOMETRIC GRAPH MANIFOLDS

The purpose of this section is to prove Theorem B in the introduction. As we will see, the proof is quite delicate and technical due to the lack of any *a priori* regularity of the boundary of a maximal full extension. The strategy is to consider a maximal extension of the set of nonflat points which, by Theorem A, is a union of twisted cylinders, and then to analyze their geometric properties at contact points.

We begin with some definitions.

Definition 4.1. We say that an open subset W of a topological space M is *locally finite* if, for every $p \in \partial W$, there exists an integer m such that, for every neighborhood $U \subset M$ of p , there exist a neighborhood $U' \subset U$ of p such that $U' \cap W$ has at most m connected components. We denote by $m(p)$ the minimum of such integers m .

In this situation, for every $p \in \partial W$, there exists a neighborhood $U_p \subset M$ of p such that $U_p \cap W$ has precisely $m(p)$ connected components. Notice that each component contains p in its boundary since by finiteness all other components have distance to p bounded away from 0. We call these components W_i the *local connected components of W at p* . Notice also that U_p can be chosen arbitrarily small. In fact, given any neighborhood U of p with $U \subset U_p$ we can construct a new neighborhood $U_p(U)$ of p as follows. Let X be the union of the connected components of $U \cap W$ that contain p in their boundary. Then $U_p(U) = (\overline{X})^\circ$ is a neighborhood of p (by local finiteness) and $U_p(U) \cap W$ has $m(p)$ connected components. Observe also that $U_p(U) \cap W_i$ are the connected components of $U_p(U) \cap W$ and all contain p in their boundary. Throughout this section U_p will always denote such a neighborhood of p and W_i the connected components of $U_p \cap W$.

In particular, by taking any $\delta > 0$ such that $B_\delta(p) \subset U_p(B_\epsilon(p))$ we get:

LEMMA 4.2. *If M^n is a Riemannian manifold, $W \subset M^n$ is locally finite and $p \in \partial W$, then for every ball $B_\epsilon(p) \subset U_p$ there is $0 < \delta = \delta(\epsilon, p) < \epsilon$ such that $W_i \cap B_\delta(p)$ is arc-connected in $W_i \cap B_\epsilon(p)$, for all $1 \leq i \leq m$.*

Let M^n be a complete CN2 manifold whose nullity distribution is parallel along the set of nonflat points V of M^n , as is the case when M^n has finite volume by Theorem A. Suppose V has a full extension $W \subset M^n$, that is, $W \supset V$ is open, dense, locally finite, and W possesses a smooth parallel distribution Γ of rank $n - 2$ whose leaves are flat and complete. Since any extension of a full extension is also full, we will assume in addition that W is maximal. Clearly, along $V \subset W$, Γ coincides with the nullity of M^n . Observe that maximal extensions always exist by definition, but they are not necessarily unique, and may fail to be dense or locally finite as shown in Examples 3 and 4 in Section 1. We call the leaves of Γ nullity leaves and, for simplicity, use $\Gamma(p)$ both for the distribution at p and for the leaf of Γ through p .

Observe that, for each sequence $\{p_n\}$ in W approaching a point $p \in \partial W$, $\Gamma(p_n)$ accumulates at an $(n - 2)$ -dimensional subspace of $T_p M$ whose image under the exponential map gives a complete totally geodesic submanifold of M^n , by completeness of the leaves of Γ . We still denote the set of all these limit submanifolds by $\Gamma(p)$, and call each of them a *boundary nullity leaf* at p , or BNL for short. In addition, given $U \subset W$ with $p \in \partial U$, we denote by $\Gamma_U(p) \subset \Gamma(p)$ the BNL's at p that arise as limits of nullity leaves in U . In particular, if W_1, \dots, W_m are the local connected components of W at p , we have

$$\Gamma(p) = \Gamma_{W_1}(p) \cup \dots \cup \Gamma_{W_m}(p).$$

We start with the following observation.

LEMMA 4.3. *Let $p \in \partial W$ such that $\Gamma(p)$ has only one BNL μ . Then, $\Gamma(q) = \{\mu\}$ for all $q \in \mu$.*

Proof. By definition, there is a unique BNL $\mu \in \Gamma(p)$. The hypersurface $B'_\epsilon(p) := \exp(\mu^\perp \cap B_\epsilon(0_p))$ is then transversal to Γ in $W'_\epsilon(p) = W \cap B'_\epsilon(p)$, which is thus dense in $B'_\epsilon(p)$.

Take $q \in \mu$, and write it as $q = \gamma_v(T)$ for some $v \in T_p\mu$, $T \in \mathbb{R}$, $\|v\| = 1$. By continuity of the geodesic flow at v , given $\delta > 0$, there is $\epsilon > 0$ such that all the nullity leaves of W through $W'_\epsilon(p)$ are C^0 δ -close to μ inside a compact ball of radius, say, $2T$, centered at p . In particular, these nullity lines of $W'_\epsilon(p)$ stay close to μ at q and form an open dense subset. This implies that there cannot be two different BNL's at q . Indeed, a second $\mu' \in \Gamma(p)$ is a limit of leaves of Γ of a local connected component W' at q . Thus all leaves in W' are close to μ' which implies that leaves of Γ on W , where it is an actual foliation, would intersect near q . \square

For the next two lemmas we need a relationship between curvature bounds and local parallel transport for Riemannian vector bundles over surfaces.

LEMMA 4.4. *Let E^k be a Riemannian vector bundle with a compatible connection ∇ over a surface S , and let $D \subset S$ be a region diffeomorphic to a closed 2-disk with piecewise smooth boundary α . If the curvature tensor of E^k is bounded by $\delta > 0$ along D , then the angle between any vector $\xi \in E_{\alpha(0)}$ and its parallel transport along α is bounded by $(k-1)\delta \text{Area}(D)$.*

Proof. Let us consider polar coordinates on D through a diffeomorphism with a 2-disk. We can assume that ξ is a unit vector and we complete $\xi = \xi_1, \dots, \xi_k$ to an orthonormal basis of $E_{\alpha(0)}$. By radially parallel transporting them first to p , and then radially to all of D , we get an orthonormal basis, which we again denote by ξ_1, \dots, ξ_k , defined on D . If we consider the connection 1-forms $w_{ij}(X) = \langle \nabla_X \xi_i, \xi_j \rangle$ on D , then one easily sees that $dw_{ij} = \langle R_\nabla(\cdot, \cdot) \xi_i, \xi_j \rangle$ since $\dim D = 2$ and $w_{ij}(Y) = 0$ for the radial direction Y .

Let $\xi(t) = \sum_i a_i(t) \xi_i(\alpha(t))$ be the parallel transport of ξ along α between 0 and t . Then, since $\nabla_{\alpha'} \xi = 0$, we have $a'_1 = \langle \xi_1, \xi \rangle' = \langle \nabla_{\alpha'} \xi_1, \xi \rangle = \langle \nabla_{\alpha'} \xi_1, \sum_{i=1}^k a_i \xi_i \rangle = \sum_{i=2}^k a_i w_{1i}(\alpha')$. Therefore, since $|a_i| \leq 1$ we obtain

$$0 \leq 1 - \langle \xi(0), \xi(1) \rangle = - \int_0^1 a'_1 = - \sum_{i=2}^k \int_0^1 a_i w_{1i}(\alpha') \leq \sum_{i=2}^k \int_\alpha |w_{1i}|.$$

For each i , choose a partition of α into countable many segments $\alpha_1, \alpha_2, \dots$ with $w_{1i}|_{\alpha_{2j}} \geq 0$ and $w_{1i}|_{\alpha_{2j+1}} \leq 0$. Then, α_j together with the radial curves (along which $w_{1i} = 0$) encloses a triangular region T_j where we apply Stokes Theorem to get

$$\int_\alpha |w_{1i}| = \sum_j \int_{T_{2j}} dw_{1i} - \sum_j \int_{T_{2j+1}} dw_{1i} \leq \int_D |\langle R_\nabla(\cdot, \cdot) \xi_1, \xi_i \rangle| \leq \delta \text{Area}(D).$$

□

In the following lemmas we study the behavior of the nullity leaves and BNL's in $U_p \subset B_\epsilon(p)$. To do this, we will be able to restrict the discussion to a single surface S transversal to the nullity leaves and BNL's near p due to the following result; see Figure 5.

LEMMA 4.5. *For each $p \in \partial W$ there exist a 2-plane $\tau \subset T_p M$, a sufficiently small convex ball $B_\epsilon(p)$, and a neighborhood $U_p \subset B_\epsilon(p)$ of p such that the surface $S := \exp(\tau \cap B_\epsilon(0_p)) \subset B_\epsilon(p)$ satisfies:*

- a) S intersects all nullity leaves and BNL's in U_p , and does so transversely;
- b) $W_i \cap S$ is connected and its closure is not contained in $U_p \cap S$;
- c) $U_p \cap S$ is diffeomorphic to an open disc.

Proof. For a fixed $1 \leq i \leq m(p)$, choose some BNL $\mu \in \Gamma_{W_i}(p)$ and for some convex ball $B_\epsilon(p)$ consider the surface $L = \exp(\mu^\perp \cap B_\epsilon(0_p))$. Given $\delta > 0$, we will fix $\epsilon > 0$ such that $(n-1) \max\{|\text{Scal}_M(x)| : x \in B_\epsilon(p)\} \text{Area}(L) < \delta$ and choose $U_p \subset B_\epsilon(p)$. Notice also that a bound on Scal_M gives a bound on the full curvature tensor since M^n is CN2. Take a sequence $p_i \in W_i$ converging to p such that $\Gamma(p_i) \rightarrow \mu$. For i large enough $\Gamma(p_i)$ is transversal to L and we can thus assume that $p_i \in W_i \cap L$. Furthermore, fix i large enough such that for $q := p_i$ the parallel translate of $\Gamma(q)$ along the minimal geodesic \overline{qp} from q to p has angle less than δ with μ . Let W'_i be the arc-connected component of $W_i \cap L$ that contains q . Thus for any $q' \in W'_i$, we can choose a curve $\alpha \subset W'_i$ connecting q to q' . If $\beta = \overline{pq}$ and $\beta' = \overline{q'p}$, we form the closed curve $\varphi = \beta * \alpha * \beta' \subset L$. We can also choose α such that φ is simple and hence bounds a disc $D \subset L$. According to Lemma 4.4 the parallel transport of μ along φ forms an angle less than δ with μ . In other words, the parallel transport of $\Gamma(q')$ along $\overline{q'p}$ has angle less than 2δ with μ for any $q' \in W'_i$. We can thus choose $\epsilon' < \epsilon$ sufficiently small, and $U_p \subset B_{\epsilon'}(p)$ such that all nullity leaves in W'_i intersect L transversely at an angle bounded away from 0. We now claim that this implies that $W'_i = W_i$, i.e. $W_i \cap L$ is connected. Otherwise, there exists an $x \in W_i$ with $x \in \partial W'_i$. Since $\Gamma(x)$ still intersects L transversely, we can choose a small product neighborhood $U \subset W_i$ as in the proof of Theorem 2.1 such that $x \in U$ and all nullity leaves in U intersect L in a unique point and transversely. But then any two points in $U \cap L$ can be connected in U and then projected along nullity leaves to lie in L . Thus $U \cap L$ is also contained in W'_i .

Since there are only finitely many connected components, and all components of $U_p \cap W$ contain p in their boundary, there exists a common ϵ' and BNL's $\mu_i \in \Gamma_{W_i}(p)$ satisfying the above properties. We can now choose a 2-plane $\tau \subset T_p M$ transversal to all μ_i and set $S := \exp(\tau \cap B_{\epsilon'}(0_p))$. Repeating the above argument for this surface S , we see that ϵ' can be chosen sufficiently small such that all nullity leaves in $U_p = U_p(B_{\epsilon'}(p))$ intersect S transversely and that for all its connected components $S \cap W_i$ is connected as well. Since in addition we can assume that the angle between the nullity lines and S is bounded away

from 0, all BNL's are transversal to S as well. Notice also that now the components of $W \cap U_p \cap S$ are precisely $W_i \cap S$.

So far S and U_p satisfy the properties in part (a) and the first part of (b). For the second part of (b), since W is locally finite, we simply choose $\epsilon' < \epsilon$ small enough such that the closure of the connected components $W_i \cap S$ are not strictly contained in $B_{\epsilon'}(p)$. But then $U_p(B_{\epsilon'}(p))$ is the desired neighborhood (see Figure 5).

We now claim that such a neighborhood is also simply connected. Let α be a closed curve in U_p which bounds a disc $D \subset S$. If D contains a point in another component W' of $W \cap S$, then W' is fully contained inside D since it does not touch α . Hence the closure of W' is contained in U_p , which contradicts (b). Thus D is contained in U_p , and we can find a null homotopy of α in $D \subset U_p$. \square

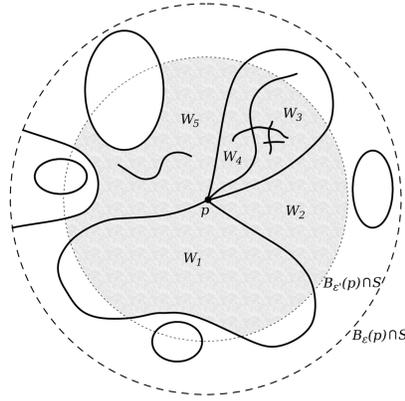


FIGURE 4. A point $p \in \partial W$ with $m(p) = 5$, the dark lines represent $\partial W \cap S$, while the shaded area corresponds to $U_p \cap S$

We will use $\epsilon > 0$, S , and U_p as in Lemma 4.5 for the remainder of this section. We point out though that the open sets $W_i \cap S$ can have quite complicated boundary. In fact, ∂W_i may not even be a Jordan curve and hence may not consist of a union of continuous arcs. Furthermore, for $p \in \partial W_i$ there may not even be a continuous curve in W_i with endpoint p . We thus carefully avoid using any such assumptions on properties of these boundaries.

It already follows from the proof of Lemma 4.5 that all BNL's in $\Gamma_{W_i}(p)$ form a small angle. We will now show that it is in fact unique.

LEMMA 4.6. *If W_i are the local connected components at p , then $\Gamma_{W_i}(p)$ is a single BNL, for each $1 \leq i \leq m$.*

Proof. Let $\mu_1, \mu_2 \in \Gamma_{W_i}(p)$ be two BNL's at p and two sequences $p_{r,k} \in W_i \cap S$, $r = 1, 2$, converging to p with $\Gamma(p_{r,k}) \rightarrow \mu_r$. For any $\epsilon' > 0$ choose $0 < \delta(\epsilon', p) < \epsilon'$ as in Lemma 4.2. For k large enough $p_{r,k} \in B_{\delta'}(p)$ and we can choose a curve $\alpha_k \subset W_i \cap B_{\epsilon'}(p)$ connecting $p_{1,k}$ to $p_{2,k}$ and by Lemma 4.5 we can also assume that α_k lies in S . Now define the loop

$\varphi_k = \beta_{2,k} * \alpha_k * \beta_{1,k}^{-1} \subset S \cap B_{\epsilon'}(p)$, where $\beta_{r,k} = \overline{p_{r,k}p}$. We can assume it is a simple closed curve and hence encloses a 2-disk $D \subset S \cap B_{\epsilon'}(p)$. Therefore, Lemma 4.4 implies that the angle between μ_1 and its parallel transport along φ_k is bounded by $(n-1)Area(S)s(\epsilon')$, where $s(\epsilon') := \max\{|\text{Scal}_M(x)| : x \in B_{\epsilon'}(p)\}$. On the other hand, the parallel transport of $\Gamma(p_{r,k})$ along $\beta_{r,k}$ converges to μ_r as $k \rightarrow \infty$ and the parallel transport of $\Gamma(p_{1,k})$ along α_k is equal to $\Gamma(p_{2,k})$. Hence the angle between μ_2 and the parallel transport of μ_1 along φ_k goes to 0 as $k \rightarrow \infty$. Finally, $s(\epsilon') \rightarrow 0$ as $\epsilon' \rightarrow 0$ since $\text{Scal}_M(p) = 0$ and we conclude that $\mu_1 = \mu_2$ as $\epsilon' \rightarrow 0$. \square

LEMMA 4.7. *For $q \in \partial W_i \cap \partial W_j \cap S$, both $\Gamma_{W_i}(q)$ and $\Gamma_{W_j}(q)$ also contain a unique BNL, and the angle between them coincides with the angle between $\Gamma_{W_i}(p)$ and $\Gamma_{W_j}(p)$.*

Proof. Fix $\delta > 0$ and let $M_\delta = \{x \in M^n : |\text{Scal}_M(x)| < \delta\}$. Then $p \in \partial W \subset V \subset M_\delta$. Choosing U_p and S as in Lemma 4.5 we can study Γ in U_p in terms of its intersection with S , and in the following drop S for clarity. In addition, assume that $\pm\delta$ are regular values of Scal_M restricted to S , and observe that $M_\delta \cap U_p$ is an open neighborhood of $\partial W_i \cap \partial W_j$. We denote by r either i or j .

Since $\pm\delta$ are regular values, the set $\{|\text{Scal}_M| \geq \delta\} \cap U_p$ is contained in the union of finitely many closed disjoint 2-discs (or half discs) D_ℓ . If we remove from W_r those discs D_ℓ which are contained in it, we obtain the open set $W'_r \subset W_r$, which is connected since W_r is. Let $\mu_r \in \Gamma_{W_r}(q)$ be a BNL and set $\nu_r = \Gamma_{W_r}(p)$, which contains only one element by Lemma 4.6. Choose two sequences $p_{r,k}, q_{r,k} \in W'_r$ such that $p_{r,k} \rightarrow p$, $q_{r,k} \rightarrow q$, with $\Gamma(q_{r,k}) \rightarrow \mu_r$ and $\Gamma(p_{r,k}) \rightarrow \nu_r$. Choose smooth simple curves $\alpha_{r,k} \subset W'_r$ joining $p_{r,k}$ to $q_{r,k}$, and let $\beta_{r,k} = \overline{p_{r,k}p}$ and $\gamma_{r,k} = \overline{q_{r,k}q}$, which, since ∂M_δ has positive distance to p and q , we can assume to lie in M_δ for k sufficiently large. Thus we get two curves $\varphi_{r,k} = \beta_{r,k}^{-1} * \alpha_{r,k} * \gamma_{r,k}$ from p to q , and hence a closed curve $\varphi_k := \varphi_{j,k}^{-1} * \varphi_{i,k} \subset M_\delta \cap U_p$, which we can also assume to be simple. By part (c) in Lemma 4.5, φ_k bounds a 2-disk $D \subset U_p$.

We claim that $\alpha_{r,k} \subset M_\delta \cap W_r$ can be modified in such a way that $D \subset M_\delta \cap U_p$. First observe that any closed disc $D_\ell \subset D$ as above must be contained in either W_i or W_j since $\partial D_\ell \cap \varphi_k = \emptyset$ and no component has its closure contained in U_p . For each $D_\ell \subset W_r$, by means of a smooth curve $\phi_\ell \subset W'_r \cap D$ connecting the boundary of D_ℓ with a point y_ℓ in $\alpha_{r,k}$ we can contour D_ℓ from the interior of D by following $\alpha_{r,k}$ up to y_ℓ , $\phi_\ell * \partial D_\ell * \phi_\ell^{-1}$, and the remaining part of $\alpha_{r,k}$. We can repeat this procedure for each D_ℓ and can also arrange this in such a way that all curves ϕ_ℓ are disjoint. Observe that this new curve, that we still call $\alpha_{r,k}$, is contained in $W_r \cap M_\delta$, and the claim is proved (see Figure 6).

As $k \rightarrow \infty$, the parallel transport of ν_r along $\varphi_{r,k}$ approaches μ_r since Γ is parallel in W_r . By Lemma 4.4, the angle between the parallel transport of ν_i along $\varphi_{i,k}$ and along $\varphi_{j,k}$ can be bounded by $(n-1)\delta Area(S)$. Since the angle between ν_i and ν_j and their parallel transport along ν_j is the same, the claim follows by taking $\delta \rightarrow 0$.

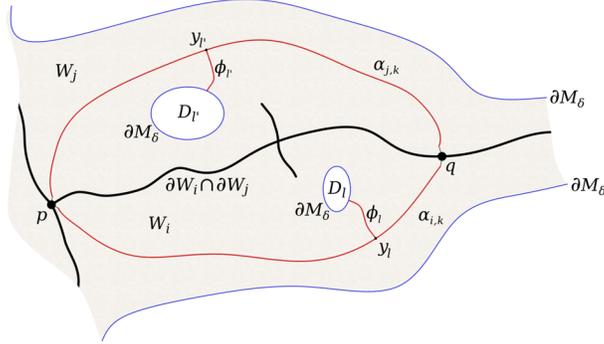


FIGURE 5. A neighborhood of $\partial W_i \cap \partial W_j$ in S with the shaded area representing M_δ

Finally, assume that there are two BNL's in $\Gamma_{W_r}(q)$. We can repeat the above argument with curves lying only in W_r since we did not assume that $i \neq j$, and it follows from Lemma 4.6 that the angle between them is 0. \square

LEMMA 4.8. *The distribution Γ does not extend continuously to any neighborhood of any $p \in \partial W$.*

Proof. Suppose that Γ extends continuously to $B_\delta(p)$. Let $\hat{\Gamma}$ be the smooth distribution in $B_\delta(p)$ obtained by parallel transporting $\Gamma(p)$ along geodesics emanating from p . Choosing a surface S centered at p as in Lemma 4.5, and contained in $B_\delta(p)$, we first claim that on S the distribution Γ agrees with $\hat{\Gamma}$. To see this, let $\alpha \subset S$ be a geodesic starting at p , and consider the angle function $\alpha(t)$ between $\Gamma(\gamma(t))$ and $\hat{\Gamma}(\gamma(t))$. An argument similar to the one in the proof of Lemma 4.7 shows that α is locally constant along the finitely many (not necessarily connected) sets $\gamma \cap \overline{W}_i$. Since by hypothesis α is continuous with $\alpha(0) = 0$, we conclude that $\alpha = 0$, as desired. Since the leaves of both Γ and $\hat{\Gamma}$ intersect S transversely, they must agree in a neighborhood of S on which Γ is thus smooth. By Lemma 4.3, this property also holds on the union of all complete leaves going through S , which contradicts the maximality of W . \square

LEMMA 4.9. *For every $p \in \partial W$ we have that $2 \leq \#\Gamma(p) \leq m(p)$.*

Proof. First, observe that Lemma 4.7 for $i = j$ shows that $\Gamma_{W_i}(q)$ contains a unique BNL for all $q \in \partial W_i \cap S$. We claim that this implies that $\partial W \cap S = \bigcup_{i \neq j} (\partial W_i \cap \partial W_j \cap S)$. If not, there exists a local component W_i , $q \in \partial W_i \cap S$ and a neighborhood U of q such that $U \subset \overline{W}_i$ and $U \cap \overline{W}_j = \emptyset$ for $j \neq i$ (see Figure 5). But then for every $r \in U \cap \partial W \cap S \subset U \cap \partial W_i \cap S$ we have that $\Gamma_{W_i}(r)$ contains a unique BNL. By Lemma 4.3 this is then also the case for any $r \in U \cap \partial W$, i.e. Γ is continuous in U , which contradicts Lemma 4.8.

We now show that $\#\Gamma(p) \geq 2$. So assume that $\Gamma(p)$ has a unique element, i.e. $\Gamma_{W_i}(p) = \Gamma_{W_j}(p)$ for all $i \neq j$. Then Lemma 4.7 implies that the same is true for any $q \in \partial W_i \cap \partial W_j \cap S$

and hence by the above for any $q \in \partial W \cap S$. Thus Γ is continuous in U_p , which again contradicts Lemma 4.8. The second inequality follows from Lemma 4.6. \square

LEMMA 4.10. *There exists $\delta = \delta(p) > 0$ such that $W_i \cap B_\delta(p)$ is convex for all i .*

Proof. Let $\epsilon' > 0$ such that $B_{\epsilon'}(p) \subset U_p \subset B_\epsilon(p)$, and let $\delta = \delta(\epsilon', p)$ as in Lemma 4.2. Take points $q, q' \in W_i \cap B_\delta(p)$ for which the minimizing geodesic segment $\overline{qq'}$ is not contained in $W_i \cap B_\delta(p)$. Take a curve $\alpha \subset W_i \cap B_{\epsilon'}(p)$ joining q with q' , and set $s := \sup\{r : \sigma_t \subset W_i, \forall 0 \leq t < r\}$, where $\sigma_t = \overline{q\alpha(t)}$. Since $\overline{q\alpha(s)} \subset \overline{W_i} \cap B_{\epsilon'}(p)$, we have that $\overline{q\alpha(s)} \cap \partial W_i \neq \emptyset$. We claim that $m(x) = 1$ for all $x \in \overline{q\alpha(s)} \cap \partial W_i$, which contradicts the first inequality in Lemma 4.9.

To prove the claim, take $\mu \in \Gamma(x)$ and consider for each $0 < t < s$ the flat totally geodesic completely ruled hypersurface $H_t := \cup_{0 < r < 1} \Gamma(\sigma_t(r)) \subset W$ with limit $H := \lim_{t \rightarrow s} H_t$. If H intersects μ transversally, H_t would also for t close to s , which is a contradiction since $\mu \subset \partial W$. Therefore, $T_x \mu$ is a hyperplane contained in $T_x H$. Since H is foliated by complete flat hypersurfaces parallel to $\Gamma(q)$ along σ_s and $\mu \subset H$ is also a complete flat hypersurface, it follows that μ is parallel to $\Gamma(q)$ along σ_s as well. Thus μ is unique and hence $m(x) = 1$. \square

We now come to the main result about the local structure of ∂W .

LEMMA 4.11. *The set $F_{ij} := \partial W_i \cap \partial W_j \subset \partial W$ is convex for all i, j , and along every geodesic in F_{ij} the two families of BNL's induced by W_i and W_j are parallel.*

Proof. Take two points $q, r \in F_{ij}$ and, for $k = i, j$, sequences $q_{k,n}, r_{k,n} \in W_k$ such that $q_{k,n} \rightarrow q, r_{k,n} \rightarrow r$. By convexity, $\overline{q_{k,n} r_{k,n}} \subset W_k$ and since both converge to \overline{qr} , it follows that $\overline{qr} \subset F_{ij}$. For the second assertion, simply observe that the parallel transport along \overline{qr} of the BNL's agrees with the limits of the parallel transport along $\overline{q_{k,n} r_{k,n}}$. \square

We are finally in a position to prove Theorem B, which follows from Theorem A and the following.

THEOREM 4.12. *Let M^n be a complete Riemannian manifold with a parallel rank $n - 2$ distribution defined in a dense, locally finite and maximal open set W , whose leaves are complete and flat. Then $M^n \setminus W$ is a disjoint union of complete flat totally geodesic embedded hypersurfaces. If, in addition, M^n has finite volume, then these hypersurfaces are compact and M^n is a geometric graph manifold.*

Proof. Consider F_{ij} as in Lemma 4.11 with $\Gamma_{W_i}(p) \neq \Gamma_{W_j}(p)$ which exists by Lemma 4.9. Then for $r = i, j$, each point in the interior F_{ij}° of F_{ij} is contained in a unique complete BNL of W_r , and we denote by $S_r \subset \partial W_r$ the union of such BNL's with $S_i \cap S_j \supset F_{ij}^\circ$. Observe in addition that S_r is a smooth flat totally geodesic hypersurface, and completely ruled since, as seen in the proof of Lemma 4.11, it arises as a limit of $H_n := \cup_{0 < t < \epsilon} \Gamma(\sigma_n(t)) \subset W_r$ for a sequence of geodesic segments $\sigma_n \subset W_r$.

We now study the local connected components based at a point $q \in S_i \cap \partial S_j \subset \partial F_{ij}^\circ$. Clearly, W_i is one of those, with S_i smooth at q and W_i lying (locally) on one side of S_i . Let W'_1, W'_2 be any two other local components at q with BNL's $\mu_s \in \Gamma_{W'_s}(q)$ for $s = 1, 2$. Observe first that μ_s cannot be transversal to S_i since otherwise leaves of Γ in W_i and W'_s would intersect. Hence μ_1 and μ_2 are tangent to S_i , which implies that $\mu_1 = \mu_2$. Indeed, otherwise μ_1 and μ_2 would intersect transversally in S_i by dimension reasons, and then near q the leaves of W'_1 and W'_2 would again intersect since they are both locally on the same side of the hypersurface S_i . Therefore, all local connected components at q , apart from W_i , share the same BNL and thus $\#\Gamma(q) = 2$. Using a surface S at q and $U_q \subset S$ as in Lemma 4.5, we see that $W_i \cap S$ is a half disc with boundary a smooth geodesic containing q in its interior. For all remaining local connected components W'_s at q , Lemma 4.11 implies that the intersections $\partial W'_1 \cap \partial W'_2 \cap S$ are geodesics with endpoints at q . Since $\Gamma_{W'_1}(q) = \Gamma_{W'_2}(q)$, Lemma 4.7 implies that $\Gamma_{W'_1}(r) = \Gamma_{W'_2}(r)$ for all $r \in \partial W'_1 \cap \partial W'_2 \cap S$ and hence by Lemma 4.3 also in a neighborhood of S . Thus by Lemma 4.8 there can be only one such component, i.e. $m(q) = 2$, and hence W_j is the second component at q . But then S_j extends past q and hence F_{ij} is a complete flat hypersurface containing p in its interior. We conclude that the number of local connected components at ∂W is 2 everywhere and ∂W is a disjoint union of complete flat totally geodesic embedded hypersurfaces.

In order to prove the last assertion of the theorem, let $C^n = (L^2 \times \mathbb{R}^{n-2})/G$ be one of the twisted cylinders with finite volume. A component of its boundary has the form $H = (\gamma \times \mathbb{R}^{n-2})/G'$, where $\gamma \subset L^2$ is a complete boundary geodesic and $G' \subset G$ the normal subgroup that preserves H . We first assume that G' acts nontrivially on γ . By taking a two-fold cover of C^n if necessary, we can assume that there are no elements of G' which act as a reflection on γ , and hence G' contains an element which acts by translation. This implies that there exists a uniform ϵ tubular neighborhood $B_\epsilon(\gamma) \subset L^2$ of the infinite geodesic γ . On $B_\epsilon(\gamma)$ the metric is C^∞ -close to the product metric on $[0, \epsilon) \times \gamma$ since the curvature of L^2 vanishes to infinite order along γ . Hence, the ϵ -tubular neighborhood of H in C^n is given by $[0, \epsilon) \times (\gamma \times \mathbb{R}^{n-2})/G' = [0, \epsilon) \times H$, with a metric also C^∞ -close to a product metric. Since C^n has finite volume, so does H . But one easily sees that a flat manifold of finite volume is compact.

On the other hand, if G' acts trivially on γ , then $H = \gamma \times F^{n-2}$ with BNL $F^{n-2} = \mathbb{R}^{n-2}/G'$. If B is a small ball near γ such that its translates under G' are disjoint from B , then the projection of $B \times \mathbb{R}^{n-2}$ to C^n is isometric to $B \times F^{n-2}$. Thus F^{n-2} again has finite volume and is hence compact. This implies the last assertion of the theorem since H is the boundary of another finite volume twisted cylinder that induces different BNL's on H . \square

Remark. One of the difficulties in proving Conjecture 1 in the Introduction is that one needs to exclude the following situation when local finiteness fails. Let W' be a concave local connected component at p with $\partial W'$ consisting of two smooth hypersurfaces meeting along their common boundary BNL at p . Then one needs to show that the complement

of W' near p cannot be densely filled with infinitely many disjoint twisted cylinders whose diameters go to 0 as they approach p .

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