Submanifolds and Isometric Immersions: class guide

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Prerequisites: Basics about manifolds, tensors, at least up to page 12 here. A bit of Riemannian geometry, fundamental group and covering maps.

Bibliography: [DT], [dC], [ON], [Pe], [Sp], [KN].

DO ALL THE EXERCISES IN [DT] !!!!

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§1. **Notations**

Top. manifolds: Hausdorff + countable basis. Partitions of unity. 

\( n \)-dimensional differentiable manifolds: \( M^n \). Everything is \( C^\infty \).

\( \mathcal{F}(M) := C^\infty(M, \mathbb{R}) \); \( \mathcal{F}(M, N) := C^\infty(M, N) \).

\((x, U)\) chart \( \Rightarrow \) coordinate vector fields \( \partial_i := \partial/\partial x_i \in \mathfrak{X}(U) \).

Tangent bundle \( TM \), vector fields \( \mathfrak{X}(M) := \Gamma(TM) \cong \mathcal{D}(M) \).

Submersions, immersions, embeddings, local diffeomorphisms.

Vector bundles, trivializing charts, transition functions, sections.

Tensor fields \( \mathfrak{X}^{r,s}(M) \), \( k \)-forms \( \Omega^k(M) \), orientation, integration.

Pull-back of a vector bundle \( \pi : E \to N \) over \( N \): \( f^*(E) \).

Vector fields along a map \( f : M \to N \Rightarrow \mathfrak{X}_f \cong \Gamma(f^*(TN)) \).

\( f \)-related vector fields.

**Example:** Lie Groups \( G, L_g, R_g; \mathfrak{g} := T_eG \) is an algebra;

Integral curve \( \gamma \) of \( X \in \mathfrak{g} \) through \( e \) is a homomorphism \( \Rightarrow \)

\( \exp^G : \mathfrak{g} \to G, \exp^G(X) := \gamma(1) \Rightarrow \exp^G(tX) = \gamma(t) \).

§2. **Riemannian metrics**

Gauss, 1827: \( M^2 \subset \mathbb{R}^3 \Rightarrow \langle , \rangle|_{M^2}, \ K_M = K_M(\langle , \rangle) \), distances, areas, volumes... Non-Euclidean geometries.

Riemann, 1854: \( \langle , \rangle \Rightarrow K_M \) (relations proved decades later).

Slow development. General Relativity pushed up!

Riemannian metric, Riemannian manifold: \( (M^n, \langle , \rangle) = M^n \).

\( g_{ij} := \langle \partial_i, \partial_j \rangle \in \mathcal{F}(U) \Rightarrow (g_{ij}) \in C^\infty(U, S(n, \mathbb{R}) \cap Gl(n, \mathbb{R})) \).

Isometries, local isometries, isometric immersions.

Product metric. \( T_pV \cong V, \ T\mathbb{V} \cong V \times V \).
Examples: \((\mathbb{R}^n, \langle \cdot, \cdot \rangle_{can})\), Euclidean submanifolds. Nash.

Example: (bi-)invariant metrics on Lie groups.

**Proposition 1.** Every differentiable manifold admits a Riemannian metric.

Angles between vectors at a point. Norm.
Riemannian vector bundles: \((E, \langle \cdot, \cdot \rangle)\).
It always exists local orthonormal frames: \(\{e_1, \ldots, e_n\}\).
Length of a piecewise differentiable curve \(\Rightarrow\) Riem. distance \(d\).
The topology of \(d\) coincides with the original one on \(M\).

**§3. Linear connections**

If \(M^n = \mathbb{R}^n\), or even if \(M^n \subset \mathbb{R}^N\), there is a natural way to differentiate vector fields. And this depends only on \(\langle \cdot, \cdot \rangle\).

**Def.:** An affine connection or a linear connection or a covariant derivative on \(M\) is a map

\[
\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)
\]

with \(\nabla_X Y\) being \(\mathbb{R}\)-bilinear, tensorial in \(X\) and a derivation in \(Y\).

Tensoriality in \(X\) \(\Rightarrow\) \((\nabla_X Y)(p) = \nabla_{X(p)} Y\) makes sense.
Local oper.: \(Y|_U = 0 \Rightarrow (\nabla_X Y)|_U = 0 \Rightarrow (\nabla_X Z)|_U = \nabla^U_{X|_U}(Z|_U)
\Rightarrow\) The Christoffel symbols \(\Gamma^k_{ij}\) of \(\nabla\) in a coordinate system \(\Rightarrow\) Christoffel symbols completely determine the connection: all that is needed is to have local basis of sections \(\Rightarrow\)

Connections on vector bundles: formally exactly the same.
The above property on \(U\) is a particular case of the following:
Proposition 2. (or “everything I know about connections.”) Let $\nabla$ be a linear connection on $M$ (or any vector bundle). Then, for every smooth map $f : N \to M$, there exists a unique linear connection $\nabla^f : \mathfrak{X}(N) \times \mathfrak{X}_f \to \mathfrak{X}_f$ on $f^*(TM)$ such that

$$\nabla^f_Y(X \circ f) = \nabla_{f_\ast Y} X, \quad \forall Y \in \mathfrak{X}(N), X \in \mathfrak{X}(M).$$

We will omit the superindex $f$ in $\nabla^f$.

In particular, Proposition 2 holds for any smooth curve $\alpha(t) = \alpha : I \subset \mathbb{R} \to M$, and if $V \in \mathfrak{X}_\alpha$ we denote $V' := \nabla_{\partial_t} V \in \mathfrak{X}_\alpha$.

So, if $\alpha'(0) = v$, $\nabla_v Y = (Y \circ \alpha)'(0)$. But beware of “$\nabla_{\alpha'}(\alpha'')$”!!

**Def.:** $V \in \mathfrak{X}_\alpha$ is parallel if $V' = 0$. We denote by $\mathfrak{X}_\alpha''$ the set of parallel vector fields along $\alpha$.

Proposition 3. Let $\alpha : I \subset \mathbb{R} \to M$ be a piecewise smooth curve, and $t_0 \in I$. Then, for each $v \in T_{\alpha(t_0)} M$, there exists a unique parallel vector field $V_v \in \mathfrak{X}_\alpha$ such that $V_v(t_0) = v$.

The map $v \mapsto V_v$ is an isomorphism between $T_{\alpha(t_0)} M$ and $\mathfrak{X}_\alpha''$, and the map $(v, t) \mapsto V_v(t)$ is smooth when $\alpha$ is smooth $\Rightarrow$

**Def.:** The parallel transport of $v \in T_{\alpha(t)} M$ along $\alpha$ between $t$ and $s$ is the map $P_{ts}^\alpha : T_{\alpha(t)} M \to T_{\alpha(s)} M$ given by $P_{ts}^\alpha(v) = V_v(s)$.

Notice that $\mathcal{F}(M) = \mathfrak{X}^0(M) = \mathfrak{X}^{0,0}(M)$ and $\mathfrak{X}(M) = \mathfrak{X}^{0,1}(M)$.

Covariant differentiation of 1-forms and tensors: $\forall r, s \geq 0$,

$$\nabla \Rightarrow \begin{cases} \nabla : \mathfrak{X}^r(M) \to \mathfrak{X}^{r+1}(M); \\ \nabla : \mathfrak{X}^{r,s}(M) \to \mathfrak{X}^{r+1,s}(M); \\ \nabla : \mathfrak{X}^{r,s}(E, \hat{\nabla}) \to \mathfrak{X}^{r+1,s}(E, \hat{\nabla}); \end{cases}$$

for any affine vector bundle $(E, \hat{\nabla})$ (in partic., for $E = (TM, \nabla)$).
3.1 The Levi-Civita connection

**Def.:** A linear connection $\nabla$ on a Riemannian manifold $(M, \langle , \rangle)$ is said to be *compatible* with $\langle , \rangle$ if, for all $X, Y, Z \in \mathfrak{X}(M)$,

$$X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle.$$ 

**Exercise.** $\nabla$ is compatible with $\langle , \rangle \iff \forall V, W \in \mathfrak{X}_\alpha, \langle V, W \rangle' = \langle V', W \rangle + \langle V, W' \rangle \iff \forall V, W \in \mathfrak{X}_\alpha, \langle V, W \rangle$ is constant $\iff P_{ts}^\alpha$ is an isometry, $\forall \alpha, t, s \iff \nabla \langle , \rangle = 0$.

**Def.:** The tensor $T_{\nabla}(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ is called the *torsion* of $\nabla$. We say that $\nabla$ is *symmetric* if $T_{\nabla} = 0$.

**Miracle:** Every Riemannian manifold $(M, \langle , \rangle)$ has a unique linear connection that is symmetric and compatible with $\langle , \rangle$, called the **Levi-Civita connection** of $(M, \langle , \rangle)$.

This is a consequence of the *Koszul formula*:

$$2\langle \nabla_X Y, Z \rangle = X \langle Y, Z \rangle + Y \langle X, Z \rangle - Z \langle X, Y \rangle - \langle X, [Y, Z] \rangle - \langle Y, [X, Z] \rangle + \langle Z, [X, Y] \rangle. $$ 

**Exercise.** Verify that this formula defines a linear connection with the desired properties.

This is the only connection that we will work with. In coordinates, if $(g^{ij}) := (g_{ij})^{-1}$,

$$\Gamma^k_{ij} = \frac{1}{2} \sum_r \left( \frac{\partial g_{ir}}{\partial x_j} + \frac{\partial g_{jr}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_r} \right) g^{rk}. $$

**Exercise.** Show that, for $(\mathbb{R}^n, \langle , \rangle_{\text{can}})$, $\Gamma^k_{ij} = 0$ and $\nabla$ is the usual vector field derivative.

**Exercise.** Use Koszul formula to show that the Levi-Civita connection of a bi-invariant metric of a Lie Group satisfies, and is characterized, by the property that $\nabla_X X = 0 \forall X \in \mathfrak{g}$.

**Lemma 4.** *(Symmetry and Compatibility Lemma)* Let $N$ be any manifold, and $f : N \rightarrow M$ a smooth map into a Riemannian manifold $M$. Then:
\( \nabla^f \) is symmetric, that is, \( \nabla^f_X f_* Y - \nabla^f_Y f_* X = f_* [X, Y] \), \( \forall X, Y \in \mathfrak{X}(N) \);

\( \nabla^f \) is compatible with the natural metric on \( f^*(TM) \).

Example: \( f : N \to M \) an isometric immersion \( \Rightarrow f^*(TM) = f^*(TN) \oplus \perp T^\perp f N \Rightarrow \forall Z \in \mathfrak{X}_f, Z = Z^\top + Z^\perp \Rightarrow \) the relation between the Levi-Civita connections is \( f_* \nabla^N_X Y = (\nabla^f_X f Y)^\top \).

\section*{4. Geodesics}

When do we have minimizing curves? What are those curves?

Critical points of the arc-length funct. \( L : \Omega_{p,q} \to \mathbb{R} : \text{geodesics} \):

\[ \gamma'' := \nabla_d \gamma' = 0. \]

Geodesics = second order nonlinear nice ODE \( \Rightarrow \)

**Proposition 5.** \( \forall v \in TM, \exists \epsilon > 0 \) and a unique geodesic \( \gamma_v : (-\epsilon, \epsilon) \to M \) such that \( \gamma'_v(0) = v \) \( (\Rightarrow \gamma_v(0) = \pi(v)) \).

\( \gamma \) a geodesic \( \Rightarrow \| \gamma' \| = \text{constant}. \)

\( \gamma \) and \( \gamma \circ r \) nonconstant geodesics \( \Rightarrow r(t) = at + b, a, b \in \mathbb{R} \Rightarrow \gamma_v(at) = \gamma_{av}(t); \gamma_v(t + s) = \gamma_{v(s)}(t) \Rightarrow \text{geodesic field} G \) of \( M \):

**Proposition 6.** There is a unique vector field \( G \in \mathfrak{X}(TM) \) such that its trajectories are \( \gamma' \), where \( \gamma \) are geodesics of \( M \).

The local flux of \( G \) is called the \textit{geodesic flow} of \( M \). In particular:

**Corollary 7.** For each \( p \in M \), there is a neighborhood \( U_p \subset M \) of \( p \) and positive real numbers \( \delta, \epsilon > 0 \) such that the map

\[ \gamma : T_e U_p \times (-\delta, \delta) \to M, \quad \gamma(v, t) = \gamma_v(t), \]
is differentiable, where \( T_\epsilon U_p := \{ v \in TU_p : \|v\| < \epsilon \} \).

Since \( \gamma_v(at) = \gamma_{av}(t) \), changing \( \epsilon \) by \( \epsilon\delta/2 \) we can assume \( \delta = 2 \) \( \Rightarrow \)

We have the **exponential map** of \( M \) (terminology from \( O(n) \)):

\[
\exp : T_\epsilon U_p \to M, \quad \exp(v) = \gamma_v(1).
\]

\( \Rightarrow \) \( \exp(tv) = \gamma_v(t) \Rightarrow \exp_p = \exp |_{T_p M} : B_\epsilon(0_p) \subset T_p M \to M \Rightarrow \)

**Proposition 8.** For every \( p \in M \) there is \( \epsilon > 0 \) such that \( B_\epsilon(p) := \exp_p(B_\epsilon(0_p)) \subset M \) is open and \( \exp_p : B_\epsilon(0_p) \to B_\epsilon(p) \) is a diffeomorphism.

An open set \( p \in V \subset M \) onto which \( \exp_p \) is a diffeomorphism as above is called a normal neighborhood of \( p \), and when \( V = B_\epsilon(p) \) it is called a normal or geodesic ball centered at \( p \).

Proposition 8 \( \Rightarrow \) \( (\exp_p |_{B_\epsilon(0_p)})^{-1} \) is a chart of \( M \) in \( B_\epsilon(p) \) \( \Rightarrow \)

We always have (local!) **polar coordinates** for any \( (M, \langle , \rangle) \):

\[
\varphi : (0, \epsilon) \times S^{n-1} \to B_\epsilon(p) \setminus \{p\}, \quad \varphi(s, v) = \gamma_v(s), \quad (1)
\]

where \( S^{n-1} = \{ v \in T_p M : \|v\| = 1 \} \) is the unit sphere in \( T_p M \).

**Examples:** \( (\mathbb{R}^n, \text{can}); (S^n, \text{can}) \).

**Exercise.** Show that for a bi-invariant metric on a Lie Group, it holds that \( \exp_e = \exp^G \).

### 4.1 Geodesics are (local) arc-length minimizers

**Lemma 9.** (*Gauss’ Lemma*) Let \( p \in M \) and \( v \in T_p M \) such that \( \gamma_v(s) \) is defined up to time \( s = 1 \). Then,

\[
\langle (\exp_p)_* v)(w), (\exp_p)_* w) \rangle = \langle v, w \rangle, \quad \forall \ w \in T_p M.
\]
Proof. If \( f(s, t) := γ_{v+tw}(s) = \exp_p(s(v + tw)) \) then, for \( t = 0 \), \( f_s = (\exp_p)_*sv \), \( f_t = (\exp_p)_*sw \) and \( \langle f_s, f_t \rangle_s = \langle v, w \rangle \). 

Gauss’ Lemma ⇒ \( S_ε(p) := \partial B_ε(p) \subset M \) is a regular hypersurface of \( M \) orthogonal to the geodesics emanating from \( p \), called the geodesic sphere of radius \( ε \) centered at \( p \).

Now, \( B_ε(p) := \exp_p(B_ε(0_p)) \subset M \) as in Proposition 8 agrees with the metric ball of \( (M, d) \) !!!!! More precisely:

**Proposition 10.** Let \( B_ε(p) \subset U \) a normal ball centered at \( p \in M \). Let \( γ : [0, a] → B_ε(p) \) be the geodesic segment with \( γ(0) = p \), \( γ(a) = q \). If \( c : [0, b] → M \) is another piecewise differentiable curve joining \( p \) and \( q \), then \( l(γ) ≤ l(c) \). Moreover, if equality holds, then \( c \) is a monotone reparametrization of \( γ \).

Proof. In polar coordinates, \( c(t) = \exp_p(s(t)v(t)) \) in \( B_ε(p) \{p\} \), and if \( f(s, t) := \exp_p(sv(t)) = γ_{v(t)}(s) \), we have that \( c’ = s’f_s + f_t \). Now, use that \( f_s \perp f_t \), by Gauss’ Lemma.

**Corollary 11.** \( d \) is a distance on \( M \), \( d_p := d(p, ·) \) is differentiable in \( B_ε(p) \{p\} \), and \( d_p^2 \) is differentiable in \( B_ε(p) \).

Exercise. Compute \( ∥\nabla d_p∥ \) and the integral curves of \( \nabla d_p \) inside \( B_ε(p) \{p\} \).

**Remark 12.** Proposition 10 is LOCAL ONLY, and \( ε = ε(p) \): \( \mathbb{R}^n; \mathbb{S}^n; \mathbb{R}^n \setminus \{0\} \).

§5. **Curvature**

Gauss: \( K(M^2 ⊂ \mathbb{R}^3) = K(⟨ , ⟩) \). Riemann: \( K(σ) = K_p(\exp_p(σ)) \).
**Def.:** The *curvature tensor* or *Riemann tensor* of $M$ is (sign!)

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$ 

We also call $R$ the $(4,0)$ tensor given by

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W \rangle.$$ 

Curvature tensor $R_{\hat{\nabla}}$ of a vector bundle $E$ with a connection $\hat{\nabla}$: exactly the same.

**Proposition 13.** For all $X,Y,Z,W \in \mathfrak{X}(M)$, it holds that:

- $R$ is a tensor;
- $R(X,Y,Z,W)$ is skew-symmetric in $X,Y$ and in $Z,W$;
- $R(X,Y,Z,W) = R(Z,W,X,Y)$;
- $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ (first Bianchi id.);
- $R^s_{ijk} = \sum_l \Gamma^l_{ik} \Gamma^s_{jl} - \sum_l \Gamma^l_{jk} \Gamma^s_{il} + \partial_j \Gamma^s_{ik} - \partial_i \Gamma^s_{jk}$ ($\Rightarrow R \cong \partial^2 \langle , \rangle$).

**Proof.** Exercise. $lacksquare$

$\langle , \rangle \Rightarrow \mathfrak{X}(M) \cong \Omega^1(M)$ and $\langle , \rangle$ extends to the tensor algebra $\Rightarrow$ the curvature operator $R : \Omega^2(M) \to \Omega^2(M)$ is self-adjoint.

**Def.:** If $\sigma \subset T_pM$ is a plane, then the *sectional curvature* of $M$ in $\sigma$ is given by

$$K(\sigma) := \frac{R(u,v,v,u)}{\|u\|^2\|v\|^2 - \langle u,v \rangle^2}; \quad \sigma = \text{span}\{u,v\}. $$

**Proposition 14.** If $R$ and $R'$ are tensors with the symmetries of the curvature tensor $+$ Bianchi such that $R(u,v,v,u) = R'(u,v,v,u)$ for all $u,v$, then $R = R'$ ($\Rightarrow K$ determines $R$).
Corollary 15. If $M$ has constant sectional curvature $c \in \mathbb{R}$, then $R(X, Y, Z, W) = c(\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle)$.

**Def.** The *Ricci tensor* is the symmetric $(2,0)$ tensor given by

$$Ric(X, Y) := \frac{1}{n-1} \text{trace} R(X, \cdot, \cdot, Y),$$

and the *Ricci curvature* is $Ric(X) = Ric(X, X)$ for $\|X\| = 1$.

Example: $\mathbb{C}P^n$ as $S^{2n+1}/S^1$ has $K(X, Y) = 1 + 3\langle JX, Y \rangle^2$ and $Ric \equiv (n+2)/(n-1)$.

**Def.** The *scalar curvature* of $M$ is $\frac{1}{n} \text{trace} Ric$.

**Lemma 16.** (Compare with Lemma 4) Let $f : U \subset \mathbb{R}^2 \rightarrow M$ be a map into a Riemannian manifold and $V \in \mathfrak{X}_f$. Then,

$$\nabla_{\partial_t} \nabla_{\partial_s} V - \nabla_{\partial_s} \nabla_{\partial_t} V = R(f_* \partial_t, f_* \partial_s)V.$$

Equivalently, $R_{\nabla f}(\cdot, \cdot)V = R_{\nabla}(f_* \cdot, f_* \cdot)V$, $\forall f : N \rightarrow M$.

**Proof.** Since $R_{\nabla f}$ is a tensor, it is enough to check the lemma for coordinate vector fields on $N$ and for $V = \overline{V} \circ f$, $\overline{V} \in \mathfrak{X}(M)$.
§6. **Isometric immersions**

As we have seen in the Example in page 5, if \( f : M \rightarrow N \) is an isometric immersion \( \Rightarrow f^*(TN) = f_*(TM) \oplus T^1_f M \), and

\[
\nabla^M_X Y = (\nabla^f_X f_* Y)^\top, \quad \forall X, Y \in TM.
\]

Moreover, we have that

\[
\alpha(X, Y) := \left(\nabla^f_X f_* Y\right)^\perp
\]

is a symmetric tensor, called the *second fundamental form of* \( f \).

In addition, \( \nabla^\perp : TM \times \Gamma(T^1_f M) \rightarrow \Gamma(T^1_f M) \) given by

\[
\nabla^\perp_X \eta = \left(\nabla^f_X \eta\right)^\perp
\]

is a connection in \( T^1_f M \), called the *normal connection of* \( f \).

**Identifications.**

*Exercise.* Show that \( \nabla^\perp \) is a compatible connection with the induced metric on \( T^1_f M \).

\( \alpha(p) \) is the quadratic approximation of \( f(M) \subset N \) at \( p \in M \).

Picture!

\( \eta \in T^1_{f(p)} M \Rightarrow \) (self-adjoint!) *shape operator* \( A_\eta : T_p M \rightarrow T_p M \).

The Fundamental Equations. Particular case: \( K = \text{constant} \Rightarrow \) the *Fundamental Theorem of Submanifolds.*

Gauss equation \( \iff K(\sigma) = K(\sigma) + \langle \alpha(u, u), \alpha(v, v) \rangle - \|\alpha(u, v)\|^2 \Rightarrow \) Riemann notion of sectional curvature agrees with ours.

*Example:* \( S^{n-1}(r) \subset \mathbb{R}^n \Rightarrow K \equiv 1/r^2 \) (it *had* to be constant!).

Model of the hyperbolic space \( \mathbb{H}^n \) as a submanifold of \( \mathbb{L}^{n+1} \).
§7. **Hypersurfaces**

Principal curvatures and directions; mean curvature; Gauss-Kronecker curvature; Gauss map.

**Proposition 17.**

§8.

**References**


