

Submanifolds and Isometric Immersions: class guide

Luis A. Florit (luis@impa.br, office 404)

Version: 20190918.1708

Periodically [download](http://luis.impa.br/aulas/imis/aulas.pdf) the last version: <http://luis.impa.br/aulas/imis/aulas.pdf>

[Subscribe](#) to aulas-luis Google group

Prerequisites: Basics about manifolds, tensors, at least up to page 12 [here](#). A bit of Riemannian geometry, fundamental group and covering maps.

Bibliography: [DT], [dC], [ON], [Pe], [Sp], [KN]....

DO ALL THE EXERCISES IN [DT] !!!!

Clickable index

1. Notations	18. References
2. Riemannian metrics	
3. Linear connections	
4. Geodesics	
5. Curvature	
6. Isometric Immersions	
7. Hypersurfaces	
8. Tot. geod. submanifolds	
9. Nullity distributions	
10. Flat normal bundle	
11. Codimension reduction	
12. Minimal submanifolds	
13. Minimal rigidity	
14. Local rigidity and FBF	
15. Local algebraic rigidity	
16. The Main Lemma	
17. Constant curvature	

§1. Notations

Top. manifolds: Hausdorff + countable basis. Partitions of unity.

n -dimensional differentiable manifolds: M^n . Everything is C^∞ .

$\mathcal{F}(M) := C^\infty(M, \mathbb{R})$; $\mathcal{F}(M, N) := C^\infty(M, N)$.

(x, U) chart \Rightarrow coordinate vector fields $= \partial_i := \partial/\partial x_i \in \mathfrak{X}(U)$.

Tangent bundle TM , vector fields $\mathfrak{X}(M) := \Gamma(TM) \cong \mathcal{D}(M)$.

Submersions, immersions, embeddings, local diffeomorphisms.

Vector bundles, trivializing charts, transition functions, sections.

Tensor fields $\mathfrak{X}^{r,s}(M)$, k -forms $\Omega^k(M)$, orientation, integration.

Pull-back of a vector bundle $\pi : E \rightarrow N$ over N : $f^*(E)$.

Vector fields along a map $f : M \rightarrow N \Rightarrow \mathfrak{X}_f \cong \Gamma(f^*(TN))$.

f -related vector fields.

Distributions: Definition. Integrable and involutive distributions.

Theorem 1 (Frobenius). A distribution $D \subset TM$ is integrable if and only if it is involutive, i.e., $[X, Y] \in \Gamma(D), \forall X, Y \in \Gamma(D)$.

§2. Riemannian metrics

Gauss, 1827: $M^2 \subset \mathbb{R}^3 \Rightarrow \langle \cdot, \cdot \rangle|_{M^2}, K_M = K_M(\langle \cdot, \cdot \rangle)$, distances, areas, volumes... Non-Euclidean geometries.

Riemann, 1854: $\langle \cdot, \cdot \rangle \Rightarrow K_M$ (relations proved decades later).

Slow development. General Relativity pushed up!

Riemannian metric, Riemannian manifold: $(M^n, \langle \cdot, \cdot \rangle) = M^n$.

$g_{ij} := \langle \partial_i, \partial_j \rangle \in \mathcal{F}(U) \Rightarrow (g_{ij}) \in C^\infty(U, S(n, \mathbb{R}) \cap Gl(n, \mathbb{R}))$.

Isometries, local isometries, isometric immersions.

Product metric. $T_p\mathbb{V} \cong \mathbb{V}, T\mathbb{V} \cong \mathbb{V} \times \mathbb{V}$.

Examples: $(\mathbb{R}^n, \langle \cdot, \cdot \rangle_{can})$, Euclidean submanifolds. Nash.

Example: (bi-)invariant metrics on Lie groups.

Proposition 2. *Every differentiable manifold admits a Riemannian metric.*

Angles between vectors at a point. Norm.

Riemannian vector bundles: $(E, \langle \cdot, \cdot \rangle)$.

It always exists local orthonormal frames: $\{e_1, \dots, e_n\}$.

Length of a piecewise differentiable curve \Rightarrow Riem. distance d .

The topology of d coincides with the original one on M .

§3. Linear connections

If $M^n = \mathbb{R}^n$, or even if $M^n \subset \mathbb{R}^N$, there is a natural way to differentiate vector fields. And this depends only on $\langle \cdot, \cdot \rangle$.

Def.: An *affine connection* or a *linear connection* or a *covariant derivative* on M is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$$

with $\nabla_X Y$ being \mathbb{R} -bilinear, tensorial in X and a derivation in Y .

Tensoriality in $X \Rightarrow (\nabla_X Y)(p) = \nabla_{X(p)} Y$ makes sense.

Local oper.: $Y|_U=0 \Rightarrow (\nabla_X Y)|_U=0 \Rightarrow (\nabla_X Z)|_U = \nabla_{X|_U}^U (Z|_U)$

\Rightarrow The *Christoffel symbols* Γ_{ij}^k of ∇ in a coordinate system \Rightarrow Christoffel symbols completely determine the connection: all that is needed is to have local basis of sections \Rightarrow

Connections on vector bundles: formally exactly the same.

The above property on U is a particular case of the following:

Proposition 3. (or “Everything I know about connections!”)

Let ∇ be a linear connection on a vector bundle $\pi : E \rightarrow M$. Then, for every smooth map $f : N \rightarrow M$, there exists a unique linear connection ∇^f on $f^*(E)$ such that

$$\nabla_Y^f(\xi \circ f) = \nabla_{f_*Y}\xi, \quad \forall Y \in \mathfrak{X}(N), \xi \in \Gamma(E).$$

We will omit the superindex f in ∇^f .

In particular, Proposition 3 holds for any smooth curve $\alpha(t) = \alpha : I \subset \mathbb{R} \rightarrow M$, and if $V \in \mathfrak{X}_\alpha$ we denote $V' := \nabla_{\partial_t}V \in \mathfrak{X}_\alpha$.

So, if $\alpha'(0) = v$, $\nabla_v Y = (Y \circ \alpha)'(0)$. But beware of “ $\nabla_{\alpha'}\alpha'$ ”!!

Def.: $V \in \mathfrak{X}_\alpha$ is *parallel* if $V' = 0$. We denote by \mathfrak{X}''_α the set of parallel vector fields along α .

Proposition 4. Let $\alpha : I \subset \mathbb{R} \rightarrow M$ be a piecewise smooth curve, and $t_0 \in I$. Then, for each $v \in T_{\alpha(t_0)}M$, there exists a unique parallel vector field $V_v \in \mathfrak{X}_\alpha$ such that $V_v(t_0) = v$.

The map $v \mapsto V_v$ is an isomorphism between $T_{\alpha(t_0)}M$ and \mathfrak{X}''_α , and the map $(v, t) \mapsto V_v(t)$ is smooth when α is smooth \Rightarrow

Def.: The *parallel transport* of $v \in T_{\alpha(t)}M$ along α between t and s is the map $P_{ts}^\alpha : T_{\alpha(t)}M \rightarrow T_{\alpha(s)}M$ given by $P_{ts}^\alpha(v) = V_v(s)$.

Notice that $\mathcal{F}(M) = \mathfrak{X}^0(M) = \mathfrak{X}^{0,0}(M)$ and $\mathfrak{X}(M) = \mathfrak{X}^{0,1}(M)$.

Covariant differentiation of 1-forms and tensors: $\forall r, s \geq 0$,

$$\nabla \Rightarrow \begin{cases} \nabla : \mathfrak{X}^r(M) \rightarrow \mathfrak{X}^{r+1}(M); \\ \nabla : \mathfrak{X}^{r,s}(M) \rightarrow \mathfrak{X}^{r+1,s}(M); \\ \nabla : \mathfrak{X}^{r,s}(E, \hat{\nabla}) \rightarrow \mathfrak{X}^{r+1,s}(E, \hat{\nabla}); \end{cases}$$

for any affine vector bundle $(E, \hat{\nabla})$ (in partic., for $E = (TM, \nabla)$).

3.1 The Levi-Civita connection

Def.: A linear connection ∇ on a Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ is said to be *compatible* with $\langle \cdot, \cdot \rangle$ if, for all $X, Y, Z \in \mathfrak{X}(M)$,

$$X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle.$$

Exercise. ∇ is compatible with $\langle \cdot, \cdot \rangle \iff \forall V, W \in \mathfrak{X}_\alpha, \langle V, W \rangle' = \langle V', W \rangle + \langle V, W' \rangle \iff \forall V, W \in \mathfrak{X}_\alpha, \langle V, W \rangle$ is constant $\iff P_{ts}^\alpha$ is an isometry, $\forall \alpha, t, s \iff \nabla \langle \cdot, \cdot \rangle = 0$.

Def.: The tensor $T_\nabla(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ is called the *torsion* of ∇ . We say that ∇ is *symmetric* if $T_\nabla = 0$.

Miracle: *Every Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ has a unique linear connection that is symmetric and compatible with $\langle \cdot, \cdot \rangle$, called the Levi-Civita connection of $(M, \langle \cdot, \cdot \rangle)$.*

This is a consequence of the *Koszul formula*: $\forall X, Y, Z \in \mathfrak{X}(M)$,

$$2\langle \nabla_X Y, Z \rangle = X\langle Y, Z \rangle + Y\langle X, Z \rangle - Z\langle X, Y \rangle - \langle X, [Y, Z] \rangle - \langle Y, [X, Z] \rangle + \langle Z, [X, Y] \rangle.$$

Exercise. Verify that this formula defines a linear connection with the desired properties.

This is the only connection that we will work with. In coordinates, if $(g^{ij}) := (g_{ij})^{-1}$,

$$\Gamma_{ij}^k = \frac{1}{2} \sum_r \left(\frac{\partial g_{ir}}{\partial x_j} + \frac{\partial g_{jr}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_r} \right) g^{rk}.$$

Exercise. Show that, for $(\mathbb{R}^n, \langle \cdot, \cdot \rangle_{can})$, $\Gamma_{ij}^k = 0$ and ∇ is the usual vector field derivative.

Exercise. Use Koszul formula to show that the Levi-Civita connection of a bi-invariant metric of a Lie Group satisfies, and is characterized, by the property that $\nabla_X X = 0 \ \forall X \in \mathfrak{g}$.

Lemma 5. (*Symmetry and Compatibility Lemma*) *Let N be any manifold, and $f : N \rightarrow M$ a smooth map into a Riemannian manifold M . Then:*

- ∇^f is symmetric, that is, $\nabla_X^f f_* Y - \nabla_Y^f f_* X = f_*[X, Y]$, $\forall X, Y \in \mathfrak{X}(N)$;
- ∇^f is compatible with the natural metric on $f^*(TM)$.

Example: $f: N \rightarrow M$ an isometric immersion $\Rightarrow f^*(TM) = f_*(TN) \oplus^\perp T_f^\perp N \Rightarrow \forall Z \in \mathfrak{X}_f, Z = Z^\top + Z^\perp \Rightarrow$ the relation between the Levi-Civita connections is $f_* \nabla_X^N Y = (\nabla_X^f f_* Y)^\top$.

§4. Geodesics

When do we have minimizing curves? What are those curves?
Critical points of the arc-length funct. $L: \Omega_{p,q} \rightarrow \mathbb{R}$: geodesics:

$$\gamma'' := \nabla_{\frac{d}{dt}} \gamma' = 0.$$

Geodesics = second order nonlinear nice ODE \Rightarrow

Proposition 6. $\forall v \in TM, \exists \epsilon > 0$ and a unique geodesic $\gamma_v: (-\epsilon, \epsilon) \rightarrow M$ such that $\gamma'_v(0) = v$ ($\Rightarrow \gamma_v(0) = \pi(v)$).

γ a geodesic $\Rightarrow \|\gamma'\| = \text{constant}$.

γ and $\gamma \circ r$ nonconstant geodesics $\Rightarrow r(t) = at + b, a, b \in \mathbb{R} \Rightarrow \gamma_v(at) = \gamma_{av}(t); \gamma_v(t+s) = \gamma_{\gamma'_v(s)}(t) \Rightarrow$ geodesic field G of M :

Proposition 7. There is a unique vector field $G \in \mathfrak{X}(TM)$ such that its trajectories are γ' , where γ are geodesics of M .

The local flux of G is called the *geodesic flow* of M . In particular:

Corollary 8. For each $p \in M$, there is a neighborhood $U_p \subset M$ of p and positive real numbers $\delta, \epsilon > 0$ such that the map

$$\gamma: T_\epsilon U_p \times (-\delta, \delta) \rightarrow M, \quad \gamma(v, t) = \gamma_v(t),$$

is differentiable, where $T_\epsilon U_p := \{v \in TU_p : \|v\| < \epsilon\}$.

Since $\gamma_v(at) = \gamma_{av}(t)$, changing ϵ by $\epsilon\delta/2$ we can assume $\delta = 2 \Rightarrow$
 We have the exponential map of M (terminology from $O(n)$):

$$\exp : T_\epsilon U_p \rightarrow M, \quad \exp(v) = \gamma_v(1).$$

$\Rightarrow \exp(tv) = \gamma_v(t) \Rightarrow \exp_p = \exp|_{T_p M} : B_\epsilon(0_p) \subset T_p M \rightarrow M \Rightarrow$

Proposition 9. *For every $p \in M$ there is $\epsilon > 0$ such that $B_\epsilon(p) := \exp_p(B_\epsilon(0_p)) \subset M$ is open and $\exp_p : B_\epsilon(0_p) \rightarrow B_\epsilon(p)$ is a diffeomorphism.*

An open set $p \in V \subset M$ onto which \exp_p is a diffeomorphism as above is called a *normal neighborhood* of p , and when $V = B_\epsilon(p)$ it is called a *normal* or geodesic ball centered at p .

Proposition 9 $\Rightarrow (\exp_p|_{B_\epsilon(0_p)})^{-1}$ is a chart of M in $B_\epsilon(p) \Rightarrow$
 We always have (local!) polar coordinates for any (M, \langle, \rangle) :

$$\varphi : (0, \epsilon) \times \mathbb{S}^{n-1} \rightarrow B_\epsilon(p) \setminus \{p\}, \quad \varphi(s, v) = \gamma_v(s), \quad (1)$$

where $\mathbb{S}^{n-1} = \{v \in T_p M : \|v\| = 1\}$ is the unit sphere in $T_p M$.

Examples: $(\mathbb{R}^n, \text{can})$; $(\mathbb{S}^n, \text{can})$.

Exercise. Show that for a bi-invariant metric on a Lie Group, it holds that $\exp_e = \exp^G$.

4.1 Geodesics are (local) arc-length minimizers

Lemma 10. (*Gauss' Lemma*) *Let $p \in M$ and $v \in T_p M$ such that $\gamma_v(s)$ is defined up to time $s = 1$. Then,*

$$\langle (\exp_p)_* v, (\exp_p)_* w \rangle = \langle v, w \rangle, \quad \forall w \in T_p M.$$

Proof. If $f(s, t) := \gamma_{v+tw}(s) = \exp_p(s(v + tw))$ then, for $t = 0$, $f_s = (\exp_p)_{*sv}(v)$, $f_t = (\exp_p)_{*sv}(sw)$ and $\langle f_s, f_t \rangle_s = \langle v, w \rangle$. ■

Gauss' Lemma $\Rightarrow \mathbb{S}_\epsilon(p) := \partial B_\epsilon(p) \subset M$ is a regular hypersurface of M orthogonal to the geodesics emanating from p , called the geodesic sphere of radius ϵ centered at p .

Now, $B_\epsilon(p) := \exp_p(B_\epsilon(0_p)) \subset M$ as in Proposition 9 agrees with the metric ball of (M, d) !!!!! More precisely:

Proposition 11. *Let $B_\epsilon(p) \subset U$ a normal ball centered at $p \in M$. Let $\gamma : [0, a] \rightarrow B_\epsilon(p)$ be the geodesic segment with $\gamma(0) = p$, $\gamma(a) = q$. If $c : [0, b] \rightarrow M$ is another piecewise differentiable curve joining p and q , then $l(\gamma) \leq l(c)$. Moreover, if equality holds, then c is a monotone reparametrization of γ .*

Proof. In polar coordinates, $c(t) = \exp_p(s(t)v(t))$ in $B_\epsilon(p) \setminus \{p\}$, and if $f(s, t) := \exp_p(sv(t)) = \gamma_{v(t)}(s)$, we have that $c' = s'f_s + f_t$. Now, use that $f_s \perp f_t$, by Gauss' Lemma. ■

Corollary 12. *d is a distance on M , $d_p := d(p, \cdot)$ is differentiable in $B_\epsilon(p) \setminus \{p\}$, and d_p^2 is differentiable in $B_\epsilon(p)$.*

Exercise. Compute $\|\nabla d_p\|$ and the integral curves of ∇d_p inside $B_\epsilon(p) \setminus \{p\}$.

Remark 13. Proposition 11 is LOCAL ONLY, and $\epsilon = \epsilon(p)$: \mathbb{R}^n ; \mathbb{S}^n ; $\mathbb{R}^n \setminus \{0\}$.

§5. Curvature

Gauss: $K(M^2 \subset \mathbb{R}^3) = K(\langle \cdot, \cdot \rangle)$. Riemann: $K(\sigma) = K_p(\exp_p(\sigma))$.

Def.: The curvature tensor or Riemann tensor of M is (sign!)

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

We also call R the (4,0) tensor given by

$$R(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle.$$

Curvature tensor $R_{\hat{\nabla}}$ of a vector bundle E with a connection $\hat{\nabla}$: exactly the same.

Proposition 14. For all $X, Y, Z, W \in \mathfrak{X}(M)$, it holds that:

- R is a tensor;
- $R(X, Y, Z, W)$ is skew-symmetric in X, Y and in Z, W ;
- $R(X, Y, Z, W) = R(Z, W, X, Y)$;
- $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$ (first Bianchi id.);
- $R_{ijk}^s = \sum_l \Gamma_{ik}^l \Gamma_{jl}^s - \sum_l \Gamma_{jk}^l \Gamma_{il}^s + \partial_j \Gamma_{ik}^s - \partial_i \Gamma_{jk}^s$ ($\Rightarrow R \cong \partial^2 \langle \cdot, \cdot \rangle$).

Proof. Exercise. ■

$\langle \cdot, \cdot \rangle \Rightarrow \mathfrak{X}(M) \cong \Omega^1(M)$ and $\langle \cdot, \cdot \rangle$ extends to the tensor algebra \Rightarrow the curvature operator $R : \Omega^2(M) \rightarrow \Omega^2(M)$ is self-adjoint.

Def.: If $\sigma \subset T_p M$ is a plane, then the sectional curvature of M in σ is given by

$$K(\sigma) := \frac{R(u, v, v, u)}{\|u\|^2 \|v\|^2 - \langle u, v \rangle^2}, \quad \sigma = \text{span}\{u, v\}.$$

Proposition 15. *If R and R' are tensors with the symmetries of the curvature tensor and Bianchi such that $R(u, v, v, u) = R'(u, v, v, u)$ for all u, v , then $R = R'$ (i.e., K determines R).*

Corollary 16. *If M has constant sectional curvature $c \in \mathbb{R}$, then $R(X, Y, Z, W) = c(\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle)$.*

Def.: The Ricci tensor is the symmetric $(2,0)$ tensor given by

$$\text{Ric}(X, Y) := \frac{1}{n-1} \text{trace } R(X, \cdot, \cdot, Y),$$

and the Ricci curvature is $\text{Ric}(X) = \text{Ric}(X, X)$ for $\|X\| = 1$.

Example: $\mathbb{C}\mathbb{P}^n$ as $\mathbb{S}^{2n+1}/\mathbb{S}^1$ has $K(X, Y) = 1 + 3\langle JX, Y \rangle^2$ and $\text{Ric} \equiv (n+2)/(n-1)$.

Def.: The scalar curvature of M is $\frac{1}{n} \text{trace Ric}$.

Lemma 17. *(Compare with Lemma 5) Let $f : U \subset \mathbb{R}^2 \rightarrow M$ be a map into a Riemannian manifold and $V \in \mathfrak{X}_f$. Then,*

$$\nabla_{\partial_t} \nabla_{\partial_s} V - \nabla_{\partial_s} \nabla_{\partial_t} V = R(f_* \partial_t, f_* \partial_s) V.$$

Equivalently, $R_{\nabla f}(\cdot, \cdot) V = R_{\nabla}(f_ \cdot, f_* \cdot) V, \forall f : N \rightarrow M$.*

Proof. Since $R_{\nabla f}$ is a tensor, it is enough to check the lemma for coordinate vector fields on N and for $V = \bar{V} \circ f, \bar{V} \in \mathfrak{X}(M)$. ■

Exercise. Let $\pi : E \rightarrow M$ be a vector bundle of rank k with a linear connection ∇ . Then ∇ is flat if and only if each $\xi \in E$ has a (unique!) local parallel extension. If M is simply connected, such an extension exists globally and therefore $E \cong M \times \mathbb{R}^k$ is trivial.

§6. Isometric immersions (finally!)

As we have seen in the Example in page 5, if $f : M \rightarrow N$ is an isometric immersion $\Rightarrow f^*(TN) = f_*(TM) \oplus^\perp T_f^\perp M$, and $\nabla_X^M Y = (\nabla_X^f f_* Y)^\top, \forall X, Y \in TM$. Moreover, we have that

$$\alpha(X, Y) := \left(\nabla_X^f f_* Y \right)^\perp$$

is a symmetric tensor, called the *second fundamental form of f* . In addition, $\nabla^\perp : TM \times \Gamma(T_f^\perp M) \rightarrow \Gamma(T_f^\perp M)$ given by

$$\nabla_X^\perp \eta = \left(\nabla_X^f \eta \right)^\perp$$

is a connection in $T_f^\perp M$, called the *normal connection of f* .
 Identifications.

Exercise. Show that ∇^\perp is a compatible connection with the induced metric on $T_f^\perp M$.

$\alpha(p)$ is the quadratic approximation of $f(M) \subset N$ at $p \in M$.

$\alpha(v, v) = \gamma'_v(0)$: Picture!

$\eta \in T_{f(p)}^\perp M \Rightarrow$ (self-adjoint!) *shape operator* $A_\eta : T_p M \rightarrow T_p M$.

The Fundamental Equations. Particular case: $K = \text{constant} \Rightarrow$ the *Fundamental Theorem of Submanifolds*.

Gauss equation $\Leftrightarrow K(\sigma) = \bar{K}(\sigma) + \langle \alpha(u, u), \alpha(v, v) \rangle - \|\alpha(u, v)\|^2$
 \Rightarrow Riemann notion of sectional curvature agrees with ours.

Example: $\mathbb{S}^{n-1}(r) \subset \mathbb{R}^n \Rightarrow K \equiv 1/r^2$ (it *had* to be constant!).

Example: Height functions and the graph of a real function.

Model of the hyperbolic space \mathbb{H}^n as a submanifold of \mathbb{L}^{n+1} .

§7. Hypersurfaces

Principal curvatures and directions; mean curvature; Gauss-Kronecker curvature; Gauss map.

Fundamental equations in this simpler case.

Locally convex and strictly convex hypersurfaces.

Proposition 18. *Given a compact Euclidean hypersurface $M^n \subset \mathbb{R}^{n+1}$, for every $0 \neq v \in \mathbb{R}^{n+1}$ there exists $x \in M^n$ such that v is normal to M^n at x and $A_v > 0$.*

Theorem 19. *For a compact Euclidean hypersurface M^n :
The Gauss-Kronecker curvature never vanishes \iff
 M is orientable and the Gauss map is a diffeomorphism \iff
The second fundamental form is definite everywhere \iff
 M is a convex hypersurface ($M = \partial B$ for a convex body B).*

§8. Totally geodesic and umbilic submanifolds

$\mathbb{Q}_{\tilde{c}}^m \subset \mathbb{Q}_c^{m+p}$ for $\tilde{c} \geq c$.

Axioms of r -planes and r -spheres.

§9. Nullity distributions

The (relative) nullity distribution $(\Delta) \Gamma_c$ and the index of (relative) nullity $(\nu = \dim \Delta) \mu_c = \dim \Gamma_c$. $\Gamma = \{X : R(X, \dots) = \tilde{R}(X, \dots)\}$ and $\mu = \dim \Gamma$ are extrinsic.

Proposition 20. *For an isometric immersion $f : M \rightarrow \tilde{M}$, the following assertions hold:*

- i) ν , μ and μ_c are upper semicontinuous. Hence, the subsets where ν , μ and μ_c attain their minimum values are open, and there is an open and dense subset of M^n where ν (also μ and μ_c) is locally constant;*
- ii) Δ (Γ_c) is smooth on any open subset of M^n where ν (also μ and μ_c) is constant;*
- iii) If \tilde{M} has constant sectional curvature, then Δ is a totally geodesic (hence integrable) distribution on any open subset where ν is constant, and the restriction of f to each leaf of Δ is totally geodesic.*

Remark 21. Γ_c is always an intrinsic totally geodesic foliation where μ_c is constant (why?). Moreover, $\Delta \subset \Gamma$.

Exercise. Every umbilical distribution of a Riemannian manifold is integrable and its leaves are umbilical submanifolds.

§10. Principal Normals and flat normal bundle

Principal and Dupin principal normals. Eigendistributions.
Submanifolds with flat normal bundle.

§11. Reduction of codimension

First normal spaces $N_1(x) := \text{span}\{\alpha(X, Y) : X, Y \in T_x M\}$.

Proposition 22. *Let $f : M^n \rightarrow \mathbb{Q}_c^{n+p}$ be an isometric immersion. Suppose that there exists a parallel normal subbundle $L^q \subset T^\perp M$ of rank $q < p$ such that $N_1(x) \subset L^q(x)$ for all $x \in M^n$. Then the codimension of f reduces to q .*

s -nullities ν_s and ν_s^* . 1-regular isometric immersions.

Proposition 23. *Let $f : M^n \rightarrow \mathbb{Q}_c^m$ be a 1-regular isometric immersion such that $\text{rank } N_1 = q \leq n - 1$. If $\nu_s^*(x) < n - s$ for all $1 \leq s \leq q$ at any point $x \in M^n$, then N_1 is parallel and thus f reduces codimension to q .*

§12. Minimal submanifolds

Let $f_t : M^n \rightarrow \bar{M}$ be an isotopy of $f = f_0$. Write $T = f'_0 = f_*Z + \eta \in \mathfrak{X}_f$, $Z \in TM$ and $\eta \in T_f^\perp M$. We will denote by $H = \text{trace } \alpha/n \in \Gamma(T_f^\perp M)$ the *mean curvature vector* of f . Then,

$$(d\text{vol}_t)'(0) = (-n\langle H, \eta \rangle + \text{div } Z) d\text{vol}.$$

Proposition 24. *M^n compact with boundary and $Z|_{\partial M} = 0$, then $V(t) := \text{Vol}(f_t(M))$ satisfies $V'(0) = -n \int_M \langle H, \eta \rangle d\text{vol}$. In particular, minimal submanifolds are the critical points of the volume functional for compactly supported variations.*

$f : M^n \rightarrow \mathbb{R}^m \Rightarrow \Delta f = nH$. Hence, minimal \Rightarrow harmonic \Rightarrow There are no compact minimal Euclidean submanifolds. But:

Proposition 25. *A compact minimal Euclidean submanifold with boundary is contained in the interior of the convex hull of its boundary.*

If $f : M^n \rightarrow \mathbb{Q}_c^{n+p}$ is minimal, then $\text{Ric}_M \leq c$ since

$$\text{Ric}_M(X) = c + \frac{n}{n-1} \langle A_H X, X \rangle - \frac{1}{n-1} \sum_{i=1}^p \langle A_{\xi_i}^2 X, X \rangle.$$

In particular, $scal_M = c + \frac{n}{n-1}\|H\|^2 - \frac{1}{n(n-1)}\|\alpha\|^2$.

Lemma 26. *Given $F : M^n \rightarrow \mathbb{R}^{m+1}$, there exists a minimal $f : M^n \rightarrow \mathbb{S}_c^m$ such that $F = inc \circ f \iff \Delta F = -ncF$.*

Construction. Let $\mathcal{H}(m, d)$ be the vector space of homogeneous harmonic polynomials of degree d in $(m+1)$ real variables. Then, $\dim \mathcal{H}(m, d) = n+1$, where $n = n(m, d) = \frac{(2d+m-1)(d+m-2)!}{d!(m-1)!} - 1$. Then, $W = W(m, d) = \{f|_{\mathbb{S}^m} : f \in \mathcal{H}(m, d)\}$ is contained in (actually, it is equally to) the eigenspace of $\Delta_{\mathbb{S}^m}$ with eigenvalue $\lambda(m, d) = -d(m+d-1)$. Fix $\langle \cdot, \cdot \rangle$ the L^2 -inner product on W , and $\{f_0, \dots, f_n\}$ an orthonormal basis of W . Set $G := O(m+1)$,

$$F := (f_0, \dots, f_n) : \mathbb{S}^m \rightarrow \mathbb{R}^{n+1}.$$

Since $\langle \cdot, \cdot \rangle$ is invariant under the G -action $A \cdot f = f \circ A$, the basis $\{A \cdot f_0, \dots, A \cdot f_n\}$ is also orthonormal. So there is $\tilde{A} \in O(n+1)$ such that $F \circ A = \tilde{A} \circ F$, and the map $A \mapsto \tilde{A}$ is a group homomorphism (such an F is said to be G -equivariant). In particular, G acts isometrically and transitively with the metric induced by F , and the isotropy groups $O(m)$ act transitively on the grassmanians of each tangent space. Thus, there exists $\tilde{c} > 0$ such that $F^*\langle \cdot, \cdot \rangle = \tilde{c}\langle \cdot, \cdot \rangle$, and hence F induces an isometric immersion of $\mathbb{S}_{1/\tilde{c}}^m$ into \mathbb{R}^{n+1} with $\Delta F = -(1/\tilde{c})\lambda(m, d)F$. We conclude by Lemma 26 that there is a minimal equivariant isometric immersion $f : \mathbb{S}_{1/\tilde{c}}^m \rightarrow \mathbb{S}_c^n$, $c = \lambda(m, d)/m\tilde{c}$, $F = inc \circ f$. We have constructed the minimal and equivariant *Veronese embeddings*,

$$f : \mathbb{S}_r^m \rightarrow \mathbb{S}^n, \quad r = r(m, d) := \frac{m}{d(m+d-1)}$$

(these are embeddings if d is odd, and embeddings of the projective space if d is even). And they are (essentially) unique!!

§13. Minimal rigidity of hypersurfaces

Deformability and rigidity. The associated family.

Theorem 27. *Let $f : M^n \rightarrow \mathbb{Q}_c^{n+1}$ be a minimal immersion of a Riemannian manifold with $\mu_c \not\geq n - 2$. Then, f is rigid among minimal immersions $g : M^n \rightarrow \mathbb{Q}_c^{n+p}$, i.e., $g = inc \circ f$.*

Proof. Diagonalize the shape operator of f , $Ae_i = \lambda_i e_i$, and set $\alpha_{ij} := \alpha_g(e_i, e_j)$. By Gauss equation, $\lambda_i^2 = \sum_k \|\alpha_{ik}\|^2$ and

$$\begin{aligned} (\langle \alpha_{ii}, \alpha_{jj} \rangle - \|\alpha_{ij}\|^2)^2 &= \lambda_i^2 \lambda_j^2 = \sum_k \|\alpha_{ik}\|^2 \sum_k \|\alpha_{jk}\|^2 \\ &\geq (\|\alpha_{ii}\|^2 + \|\alpha_{ij}\|^2)(\|\alpha_{jj}\|^2 + \|\alpha_{ij}\|^2) \geq (\langle \alpha_{ii}, \alpha_{jj} \rangle + \|\alpha_{ij}\|^2)^2. \end{aligned}$$

So, $\alpha_{ij} \neq 0 \Rightarrow \langle \alpha_{ii}, \alpha_{jj} \rangle \leq 0 \Rightarrow \lambda_i \lambda_j \leq -\|\alpha_{ij}\|^2 < 0$. Thus, at a point with $\nu = \mu_c \leq n - 3$, there should be a pair with $\alpha_{ij} = 0$. The above equation implies that α_{ii} and α_{jj} are linearly dependent, and $\alpha_{is} = 0$ for $i \neq s \neq j$. Changing the roles of s and j we get $\alpha_{ij} = 0$. We conclude that $(\alpha_g)_{N_g^1} = \pm \alpha_f$. Done, since N_g^1 is parallel (e.g. by Proposition 23) and g is analytic. ■

§14. Local rigidity and flat bilinear forms

In local coordinates, an isometric immersion is a solution of a nonlinear PDE, so if the codimension is small it should be overdetermined, hence rigidity should be true under generic conditions. Analyze the proof of Theorem 27: It's just Gauss equation!

But: f rigid \Rightarrow Find $\tau : T_f^\perp M \rightarrow T_g^\perp M$ satisfying

$$\tau \circ \alpha_f = \alpha_g.$$

Such τ is unique if f is full (or unique in N_1^f), and its parallelism is not hard to see. Now, a necessary condition for the existence such a bundle [isometry](#) τ is that

$$\|\alpha_f(X, Y)\| = \|\alpha_g(X, Y)\|, \quad \forall X, Y \in TM. \quad (2)$$

which is equivalent by polarization to that, $\forall X, Y, X', Y' \in TM$,

$$\langle \alpha_f(X, Y), \alpha_f(X', Y') \rangle = \langle \alpha_g(X, Y), \alpha_g(X', Y') \rangle.$$

But this is also sufficient (!!): just define τ as $\tau \circ \alpha_f = \alpha_g$ and extend linearly. In other words, we need to understand when the [flat bilinear form](#) (FBF) $\beta = (\alpha_f, \alpha_g)$ is [null](#), where

$$\beta = (\alpha_f, \alpha_g) : TM \times TM \rightarrow (T_f^\perp M \times T_g^\perp M, \langle \cdot, \cdot \rangle_f - \langle \cdot, \cdot \rangle_g).$$

14.1 Flat bilinear forms

Let $\beta : \mathbb{V} \times \mathbb{V}' \rightarrow \mathbb{W}^{p,q}$ a FBF.

Def.: $RE(\beta)$. $S(\beta)$. β_X for $X \in \mathbb{V}$. Isotropic (null) subspaces.
 $\nu_\beta := \dim N(\beta)$.

Proposition 28. *The subset $RE(\beta)$ is open and dense in V .*

Observe that, by flatness: *if $\beta_X(\mathbb{V}') \subset \mathbb{W}$ is isotropic for all X in a dense subset, then β is null.*

Proposition 29. *For any bilinear form β and $X \in RE(\beta)$, $\beta(\mathbb{V}, \text{Ker } \beta_X) \subset \text{Im } \beta_X$. If β is also flat, then $\beta|_{\mathbb{V} \times \text{Ker } \beta_X}$ is null since $\beta(\mathbb{V}, \text{Ker } \beta_X) \subset \mathcal{U}(X) := \text{Im } \beta_X \cap \text{Im } \beta_X^\perp$.*

Proof. For any $Y \in \mathbb{V}$ and t small, $L_t = \text{Im } \beta_{X+tY} \subset \mathbb{W}$ is a *continuous* family of subspaces that contain $\beta_{X+tY}(\text{Ker } \beta_X) = \beta_Y(\text{Ker } \beta_X)$, which does not depend on t . ■

Corollary 30. $\beta: \mathbb{V} \times \mathbb{V}' \rightarrow \mathbb{W}^{p,0}$ FBF $\Rightarrow \nu_\beta \geq \dim \mathbb{V}' - \dim \mathbb{W}$.

Theorem 31 (Chern-Kuiper). $M^n \subset \tilde{M}^{n+p} \Rightarrow \nu \leq \mu \leq \nu + p$.

Corollary 32. $M^n \subset \mathbb{R}^{n+p}$ compact $\Rightarrow \mu (= \mu_0) \not\geq p$.

Corollary 33. $M^n \subset \mathbb{R}^{n+p}$ compact and flat $\Rightarrow p \geq n$.

14.2 Uniqueness of the normal connection

Proposition 34. Let $f, f' : M^n \rightarrow \mathbb{Q}_c^{n+p}$ be isometric immersions and let $\tau : T_f^\perp M \rightarrow T_{f'}^\perp M$ be a vector bundle isometry that preserves the second fundamental forms. Then it also preserves the normal connections on the first normal bundles. In particular, it is parallel if either immersion is full.

Def.: Type number τ of f . Obs: $\tau \geq 1 \Rightarrow f$ is full.

Remark 35. Allendoerfer: if $\tau \geq 4$, then $G \Rightarrow C+R$.

§15. Local algebraic rigidity

Lemma 36 (Lorenzian version of Corollary 30). If β is a FBF with $S(\beta) = \mathbb{W}^{p,1}$ Lorenzian $\Rightarrow \nu_\beta \geq \dim \mathbb{V}' - \dim \mathbb{W}$.

Theorem 37 (Beez-Killing). $M \subset \mathbb{Q}_c^{n+1}$ with $\tau \geq 3$ is rigid.

Corollary 38. Let $f, f' : M^n \rightarrow \mathbb{Q}_c^{n+1}$ be nowhere congruent isometric immersions of a Riemannian manifold with no points of constant sectional curvature c . Then f and f' carry a common relative nullity distribution of rank $n - 2$.

Theorem 39 (Allendoerfer). *Any $f : M^n \rightarrow \mathbb{Q}_c^{n+p}$ with $\tau \geq 3$ everywhere is rigid.*

Proof. By Proposition 34, we only need to show that $\beta = \alpha \oplus \alpha'$ is null, since $\tau \geq 3$ implies that f is null.

Let $k := \min\{\dim \mathcal{U}(X) : X \in RE(\beta)\}$. Similarly to $RE(\beta)$, $RE^o(\beta) := \{X \in \mathbb{V} : \dim \mathcal{U}(X) = k\}$ is also open and dense in \mathbb{V} . So, we only need to show that $k = p$.

$\tau \geq 3 \Rightarrow \exists L^{n-3p} := (\text{span}\{A_{\xi_j} X_i : 1 \leq j \leq p, 1 \leq i \leq 3\})^\perp = \bigcap_{i=1}^3 \text{Ker } \alpha_{X_i}$. But $\dim \text{Ker } \beta_{X_1} = n - \text{rank } \beta_{X_1} \geq n - 2p + k$. Proposition 29 $\Rightarrow \dim \text{Ker } \beta_{X_1} \cap \text{Ker } \beta_{X_2} \geq \dim \text{Ker } \beta_{X_1} - \dim \mathcal{U}(X_1) \geq n - 2p$ and similarly $\dim \bigcap_{i=1}^3 \text{Ker } \beta_{X_i} \geq n - 2p - k$. Done, since $\bigcap_{i=1}^3 \text{Ker } \beta_{X_i} \subset L^{n-3p}$. ■

§16. The Main Lemma

Def.: Nondegenerate FBFs.

Given a FBF $\beta : \mathbb{V}^n \times \mathbb{V}^n \rightarrow \mathbb{W}^{p,q}$, set $U := S(\beta) \cap S(\beta)^\perp$, $W = U \oplus \hat{U} \oplus^\perp L$ where \hat{U} is null, $S(\beta) = U \oplus L$, and $\beta = \beta_U + \beta_L$, with β_U null and β_L nondegenerate.

Lemma 40 (The Main Lemma). *β symmetric nondegenerate with $\min\{p, q\} \leq 5 \Rightarrow \nu_\beta \geq n - p - q$.*

Theorem 41. *$f : M^n \rightarrow \mathbb{Q}_c^{n+p}$ with $p \leq 5$ and $\nu_j \leq n - 2j + 1$ for all $1 \leq j \leq p \Rightarrow f$ is rigid.*

Proof. Just show that U above has dimension p . Since $\nu_1 \leq n - 3$, f is full. ■

§17. Submanifolds with constant curvature

FBF were introduced by Cartan to study $f: M_c^n \rightarrow \mathbb{Q}_c^{n+p}$. Moore.

Lemma 42. $f: M_c^n \rightarrow \mathbb{Q}_{\tilde{c}}^{n+p}$. $c < \tilde{c} \Rightarrow p \geq n - 1$. $c > \tilde{c}$ and $p \leq n - 2 \Rightarrow \alpha_f = \gamma + \sqrt{c - \tilde{c}} \langle \cdot, \cdot \rangle \xi$ with γ flat and ξ unit.

Proof. Compose f with an umbilical inclusion $\mathbb{Q}_{\tilde{c}}^{n+p} \rightarrow \mathbb{Q}_c^{n+p+1}$ (the latter Lorentzian if $c > \tilde{c}$). Apply the Main Lemma. ■

A point $x \in M$ where $\alpha_f(x) = \gamma(x) + \sqrt{c - \tilde{c}} \langle \cdot, \cdot \rangle \xi(x)$ is called a *weak umbilic f*. Weak umbilic everywhere \Rightarrow composition?

What happens if $c > \tilde{c}$, f free of weak umbilics, and $p = n - 1$?

Proposition 43 (Moore). $\beta: \mathbb{V}^n \times \mathbb{V}^n \rightarrow \mathbb{W}^{p,q}$ symmetric FBF, with $q = 0, 1$ and $\nu_\beta = n - p - q$. If $q = 1$, assume further that β is nondegenerate and that there exist a vector $e \in \mathbb{W}$ such that $\langle \beta, e \rangle > 0$. Then, β decomposes uniquely as the direct sum of $p + q$ rank one flat forms.

Proof. Let's do the case $q = 0$ only, and we may assume $p = n$, $\nu_\beta = 0$. Fix $X_0 \in RE(\beta) \Rightarrow \beta_{X_0}$ is an isomorphism $\Rightarrow C(Y) := \beta_Y \circ \beta_{X_0}^{-1} \in \text{End}(\mathbb{W})$ are all self-adjoint and commuting by flatness, $\forall Y \in \mathbb{V} \Rightarrow \exists$ and O.N.B. of \mathbb{W} such that $C(Y)\xi_i = \mu_i(Y)\xi_i$. Set $\beta_i = \langle \beta, \xi_i \rangle$, $\beta_{X_0} X_i = \xi_i \Rightarrow \beta(Y, X_i) = \mu_i(Y)\xi_i$ and $\beta_{iY} = \mu_i(Y)\beta_{iX_0} \Rightarrow \beta_{iX_0} = a_i \mu_i$ for $a_i = \beta_{iX_0} X_0 \Rightarrow a_i \mu_i(X_j) = \beta_{iX_0} X_j = \langle \xi_j, \xi_i \rangle = \delta_{ij} \Rightarrow \{\mu_i\} = \{X_i\}^*$ and $\beta_i = a_i \mu_i \otimes \mu_i$. ■

Corollary 44. Let $f: M_c^n \rightarrow \mathbb{Q}_c^{2n}$ with $\nu = 0 \Rightarrow \exists$ unique basis $\{X_i\}$ of unit vectors and o.n.b. $\{\eta_i\}$ such that $\alpha(X_i, X_j) =$

$\delta_{ij}\eta_j$. The basis $\{X_i\}$ is orthogonal if and only if $R^\perp = 0$, in which case the $\{\eta_i\}$ are the principal normals of f .

Corollary 45. Let $f: M_c^n \rightarrow \mathbb{Q}_{\tilde{c}}^{2n-1}$ with $c \neq \tilde{c}$. If $c > \tilde{c}$ assume that f has no weak umbilics. Then $R^\perp = 0$.

17.1 Principal coordinates

§18.

References

- [dC] M. do Carmo: *Riemannian Geometry*. Birkhäuser Boston, Inc., Boston, MA, 1992.
- [DT] M. Dajczer and R. Tojeiro: Submanifold theory. Book preprint (you need to be subscribed to the group!).
- [KN] S. Kobayashi, K. Nomizu: *Foundations of Differential Geometry*. Wiley Classics Library, Volume I and II.
- [ON] B. O'Neill: *Semi-Riemannian Geometry with applications to Relativity*. New York, Academic Press, 1983.
- [Pe] P. Petersen: *Riemannian geometry*. Second edition. Graduate Texts in Mathematics, 171. Springer, NY.
- [Sp] M. Spivak: *A comprehensive introduction to differential geometry*. Publish or Perish Inc., Texas, 1975.