# Submanifolds and Isometric Immersions: class guide

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**Prerequisites:** Basics about manifolds, tensors, at least up to page 12 here. A bit of Riemannian geometry, fundamental group and covering maps.

**Bibliography:** [**DT**], [**dC**], [**ON**], [**Pe**], [**Sp**], [**KN**]....

## DO ALL THE EXERCISES IN [**DT**] !!!!

## Clickable index

## §1. Notations

Top. manifolds: Hausdorff + countable basis. Partitions of unity.
$n$-dimensional differentiable manifolds: $M^n$. Everything is $C^\infty$.
$\mathcal{F}(M) := C^\infty(M, \mathbb{R})$; $\mathcal{F}(M, N) := C^\infty(M, N)$.
$(x, U)$ chart $\Rightarrow$ coordinate vector fields $= \partial_i := \partial/\partial x_i \in \mathfrak{X}(U)$.
Tangent bundle $TM$, vector fields $\mathfrak{X}(M) := \Gamma(TM) \cong \mathcal{D}(M)$.
Submersions, immersions, embeddings, local diffeomorphisms.
Vector bundles, trivializing charts, transition functions, sections.
Tensor fields $\mathfrak{X}^{r,s}(M)$, $k$-forms $\Omega^k(M)$, orientation, integration.
Pull-back of a vector bundle $\pi : E \to N$ over $N$: $f^*(E)$.
Vector fields along a map $f : M \to N \Rightarrow \mathfrak{X}_f \cong \Gamma(f^*(TN))$.
$f$-related vector fields.

Distributions: Definition. Integrable and involutive distributions.

**Theorem 1** (Frobenius)**.** A distribution $D \subset TM$ is integrable if and only if it is involutive, i.e., $[X, Y] \in \Gamma(D), \forall X, Y \in \Gamma(D)$.

## §2. Riemannian metrics

Gauss, 1827: $M^2 \subset \mathbb{R}^3 \Rightarrow \langle\, , \rangle|_{M^2}$, $K_M = K_M(\langle\, , \rangle)$, distances, areas, volumes... Non-Euclidean geometries.
Riemann, 1854: $\langle\, , \rangle \Rightarrow K_M$ (relations proved decades later).
Slow development. General Relativity pushed up!
Riemannian metric, Riemannian manifold: $(M^n, \langle\, , \rangle) = M^n$.
$g_{ij} := \langle \partial_i, \partial_j \rangle \in \mathcal{F}(U) \Rightarrow (g_{ij}) \in C^\infty(U, S(n, \mathbb{R}) \cap Gl(n, \mathbb{R}))$.
Isometries, local isometries, isometric immersions.
Product metric. $T_p\mathbb{V} \cong \mathbb{V}$, $T\mathbb{V} \cong \mathbb{V} \times \mathbb{V}$.

*Examples:* $(\mathbb{R}^n, \langle\, , \rangle_{can})$, Euclidean submanifolds. Nash.

*Example:* (bi-)invariant metrics on Lie groups.

**Proposition 2.** *Every differentiable manifold admits a Riemannian metric.*

Angles between vectors at a point. Norm.
Riemannian vector bundles: $(E, \langle\,,\rangle)$.
It always exists <u>local</u> orthonormal frames: $\{e_1, \ldots, e_n\}$.
Length of a piecewise differentiable curve $\Rightarrow$ Riem. distance $d$.
The topology of $d$ coincides with the original one on $M$.

## §3. Linear connections

If $M^n = \mathbb{R}^n$, or even if $M^n \subset \mathbb{R}^N$, there is a natural way to differentiate vector fields. And this depends only on $\langle\,,\rangle$.

**Def.:** An *affine connection* or a *linear connection* or a *covariant derivative* on $M$ is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

with $\nabla_X Y$ being $\mathbb{R}$-bilinear, tensorial in $X$ and a derivation in $Y$.

Tensoriality in $X \Rightarrow (\nabla_X Y)(p) = \nabla_{X(p)} Y$ makes sense.
Local oper.: $Y|_U = 0 \Rightarrow (\nabla_X Y)|_U = 0 \Rightarrow (\nabla_X Z)|_U = \nabla^U_{X|_U}(Z|_U)$
$\Rightarrow$ The *Christoffel symbols* $\Gamma^k_{ij}$ of $\nabla$ in a coordinate system $\Rightarrow$
Christoffel symbols completely determine the connection: all that is needed is to have <u>local basis of sections</u> $\Rightarrow$
Connections on vector bundles: formally exactly the same.
The above property on $U$ is a particular case of the following:

**Proposition 3.** *(or "Everything I know about connections!") Let* $\nabla$ *be a linear connection on a vector bundle* $\pi : E \to M$*. Then, for every smooth map* $f : N \to M$*, there exists a unique linear connection* $\nabla^f$ *on* $f^*(E)$ *such that*

$$\nabla^f_Y(\xi \circ f) = \nabla_{f_*Y}\xi, \quad \forall\ Y \in \mathfrak{X}(N), \xi \in \Gamma(E).$$

We will omit the superindex $f$ in $\nabla^f$.
In particular, Proposition 3 holds for any smooth curve $\alpha(t) = \alpha : I \subset \mathbb{R} \to M$, and if $V \in \mathfrak{X}_\alpha$ we denote $V' := \nabla_{\partial_t} V \in \mathfrak{X}_\alpha$. So, if $\alpha'(0) = v$, $\nabla_v Y = (Y \circ \alpha)'(0)$. But beware of "$\nabla_{\alpha'}\alpha'$"!!

**Def.:** $V \in \mathfrak{X}_\alpha$ is *parallel* if $V' = 0$. We denote by $\mathfrak{X}''_\alpha$ the set of parallel vector fields along $\alpha$.

**Proposition 4.** *Let* $\alpha : I \subset \mathbb{R} \to M$ *be a piecewise smooth curve, and* $t_0 \in I$*. Then, for each* $v \in T_{\alpha(t_0)}M$*, there exists a unique parallel vector field* $V_v \in \mathfrak{X}_\alpha$ *such that* $V_v(t_0) = v$*.*

The map $v \mapsto V_v$ is an isomorphism between $T_{\alpha(t_0)}M$ and $\mathfrak{X}''_\alpha$, and the map $(v, t) \mapsto V_v(t)$ is smooth when $\alpha$ is smooth $\Rightarrow$

**Def.:** The *parallel transport of* $v \in T_{\alpha(t)}M$ *along* $\alpha$ *between* $t$ *and* $s$ is the map $P^\alpha_{ts} : T_{\alpha(t)}M \to T_{\alpha(s)}M$ given by $P^\alpha_{ts}(v) = V_v(s)$.

Notice that $\mathcal{F}(M) = \mathfrak{X}^0(M) = \mathfrak{X}^{0,0}(M)$ and $\mathfrak{X}(M) = \mathfrak{X}^{0,1}(M)$.
Covariant differentiation of 1-forms and tensors: $\forall r, s \geq 0$,

$$\nabla \Rightarrow \begin{cases} \nabla : \mathfrak{X}^r(M) \to \mathfrak{X}^{r+1}(M); \\ \nabla : \mathfrak{X}^{r,s}(M) \to \mathfrak{X}^{r+1,s}(M); \\ \nabla : \mathfrak{X}^{r,s}(E, \hat{\nabla}) \to \mathfrak{X}^{r+1,s}(E, \hat{\nabla}); \end{cases}$$

for any affine vector bundle $(E, \hat{\nabla})$ (in partic., for $E = (TM, \nabla)$).

### 3.1 The Levi-Civita connection

**Def.:** A linear connection $\nabla$ on a Riemannian manifold $(M, \langle\, ,\rangle)$ is said to be *compatible* with $\langle\, ,\rangle$ if, for all $X, Y, Z \in \mathfrak{X}(M)$,

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

<u>*Exercise*</u>. $\nabla$ is compatible with $\langle\, ,\rangle \iff \forall\, V, W \in \mathfrak{X}_\alpha,\ \langle V, W\rangle' = \langle V', W\rangle + \langle V, W'\rangle \iff \forall\, V, W \in \mathfrak{X}''_\alpha, \langle V, W\rangle$ is constant $\iff P^\alpha_{ts}$ is an isometry, $\forall\, \alpha, t, s \iff \nabla\langle\, ,\rangle = 0$.

**Def.:** The tensor $T_\nabla(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ is called the *torsion* of $\nabla$. We say that $\nabla$ is *symmetric* if $T_\nabla = 0$.

**Miracle:** *Every Riemannian manifold $(M, \langle\, ,\rangle)$ <u>has</u> a <u>unique</u> linear connection that is symmetric and compatible with $\langle\, ,\rangle$, called the <u>Levi-Civita connection</u> of $(M, \langle\, ,\rangle)$.*

This is a consequence of the *Koszul formula*: $\forall\, X, Y, Z \in \mathfrak{X}(M)$,

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle - \langle Y, [X, Z]\rangle + \langle Z, [X, Y]\rangle.$$

<u>*Exercise*</u>. Verify that this formula defines a linear connection with the desired properties.

This is the only connection that we will work with. In coordinates, if $(g^{ij}) := (g_{ij})^{-1}$,

$$\Gamma^k_{ij} = \frac{1}{2}\sum_r \left(\frac{\partial g_{ir}}{\partial x_j} + \frac{\partial g_{jr}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_r}\right) g^{rk}\ .$$

<u>*Exercise*</u>. Show that, for $(\mathbb{R}^n, \langle\, ,\rangle_{can})$, $\Gamma^k_{ij} = 0$ and $\nabla$ is the usual vector field derivative.

<u>*Exercise*</u>. Use Koszul formula to show that the Levi-Civita connection of a bi-invariant metric of a Lie Group satisfies, and is characterized, by the property that $\nabla_X X = 0\ \forall\, X \in \mathfrak{g}$.

**Lemma 5.** *(Symmetry and Compatibility Lemma) Let $N$ be any manifold, and $f : N \to M$ a smooth map into a Riemannian manifold $M$. Then:*

- $\nabla^f$ *is symmetric, that is,* $\nabla^f_X f_* Y - \nabla^f_Y f_* X = f_*[X,Y]$, $\forall\ X, Y \in \mathfrak{X}(N)$;
- $\nabla^f$ *is compatible with the natural metric on* $f^*(TM)$.

*Example:* $f\colon N \to M$ an isometric immersion $\Rightarrow f^*(TM) = f_*(TN) \oplus^{\perp} T^{\perp}_f N \ \Rightarrow \forall\ Z \in \mathfrak{X}_f, Z = Z^{\top} + Z^{\perp} \ \Rightarrow$ the relation between the Levi-Civita connections is $f_* \nabla^N_X Y = (\nabla^f_X f_* Y)^{\top}$.

## §4. Geodesics

When do we have minimizing curves? What are those curves? Critical points of the arc-length funct. $L : \Omega_{p,q} \to \mathbb{R}$: *geodesics*:

$$\gamma'' := \nabla_{\frac{d}{dt}} \gamma' = 0.$$

Geodesics = second order nonlinear nice ODE $\Rightarrow$

**Proposition 6.** $\forall\ v \in TM$, $\exists\ \epsilon > 0$ *and a unique geodesic* $\gamma_v : (-\epsilon, \epsilon) \to M$ *such that* $\gamma_v'(0) = v$ $(\Rightarrow \gamma_v(0) = \pi(v))$.

$\gamma$ a geodesic $\Rightarrow \|\gamma'\| =$ constant.
$\gamma$ and $\gamma \circ r$ nonconstant geodesics $\Rightarrow r(t) = at + b$, $a, b \in \mathbb{R} \Rightarrow$ $\gamma_v(at) = \gamma_{av}(t)$; $\gamma_v(t+s) = \gamma_{\gamma_v'(s)}(t) \Rightarrow$ *geodesic field* $G$ of $M$:

**Proposition 7.** *There is a unique vector field* $G \in \mathfrak{X}(TM)$ *such that its trajectories are* $\gamma'$, *where* $\gamma$ *are geodesics of* $M$.

The local flux of $G$ is called the *geodesic flow* of $M$. In particular:

**Corollary 8.** *For each* $p \in M$, *there is a neighborhood* $U_p \subset M$ *of* $p$ *and positive real numbers* $\delta, \epsilon > 0$ *such that the map*

$$\gamma : T_\epsilon U_p \times (-\delta, \delta) \to M, \quad \gamma(v,t) = \gamma_v(t),$$

*is differentiable, where* $T_\epsilon U_p := \{v \in TU_p \colon \|v\| < \epsilon\}$.

Since $\gamma_v(at) = \gamma_{av}(t)$, changing $\epsilon$ by $\epsilon\delta/2$ we can assume $\delta = 2 \Rightarrow$ We have the ***exponential map*** of $M$ (terminology from $O(n)$):

$$\exp : T_\epsilon U_p \to M, \quad \exp(v) = \gamma_v(1).$$

$\Rightarrow \exp(tv) = \gamma_v(t) \Rightarrow \exp_p = \exp|_{T_pM} \colon B_\epsilon(0_p) \subset T_pM \to M \Rightarrow$

**Proposition 9.** *For every* $p \in M$ *there is* $\epsilon > 0$ *such that* $B_\epsilon(p) := \exp_p(B_\epsilon(0_p)) \subset M$ *is open and* $\exp_p \colon B_\epsilon(0_p) \to B_\epsilon(p)$ *is a diffeomorphism.*

An open set $p \in V \subset M$ onto which $\exp_p$ is a diffeomorphism as above is called a *normal neighborhood* of $p$, and when $V = B_\epsilon(p)$ it is called a *normal* or ***geodesic ball*** centered at $p$.

Proposition 9 $\Rightarrow \left(\exp_p|_{B_\epsilon(0_p)}\right)^{-1}$ is a chart of $M$ in $B_\epsilon(p) \Rightarrow$ We always have (local!) ***polar coordinates*** for any $(M, \langle\,,\rangle)$:

$$\varphi : (0, \epsilon) \times \mathbb{S}^{n-1} \to B_\epsilon(p) \backslash \{p\}, \qquad \varphi(s, v) = \gamma_v(s), \qquad (1)$$

where $\mathbb{S}^{n-1} = \{v \in T_pM : \|v\| = 1\}$ is the unit sphere in $T_pM$.

*Examples:* $(\mathbb{R}^n, can)$; $(\mathbb{S}^n, can)$.

*Exercise*. Show that for a bi-invariant metric on a Lie Group, it holds that $exp_e = exp^G$.

### 4.1 Geodesics are (local) arc-length minimizers

**Lemma 10.** *(Gauss' Lemma) Let* $p \in M$ *and* $v \in T_pM$ *such that* $\gamma_v(s)$ *is defined up to time* $s = 1$. *Then,*

$$\langle (\exp_p)_{*v}(v), (\exp_p)_{*v}(w) \rangle = \langle v, w \rangle, \quad \forall\ w \in T_pM.$$

*Proof.* If $f(s,t) := \gamma_{v+tw}(s) = \exp_p(s(v+tw))$ then, for $t=0$, $f_s = (\exp_p)_{*sv}(v)$, $f_t = (\exp_p)_{*sv}(sw)$ and $\langle f_s, f_t\rangle_s = \langle v, w\rangle$. ∎

Gauss' Lemma $\Rightarrow \mathbb{S}_\epsilon(p) := \partial B_\epsilon(p) \subset M$ is a regular hypersurface of $M$ orthogonal to the geodesics emanating from $p$, called the *geodesic sphere* of radius $\epsilon$ centered at $p$.
Now, $B_\epsilon(p) := \exp_p(B_\epsilon(0_p)) \subset M$ as in Proposition 9 agrees with the metric ball of $(M,d)$ !!!!! More precisely:

**Proposition 11.** *Let $B_\epsilon(p) \subset U$ a normal ball centered at $p \in M$. Let $\gamma : [0,a] \to B_\epsilon(p)$ be the geodesic segment with $\gamma(0) = p$, $\gamma(a) = q$. If $c : [0,b] \to M$ is another piecewise differentiable curve joining $p$ and $q$, then $l(\gamma) \leq l(c)$. Moreover, if equality holds, then $c$ is a monotone reparametrization of $\gamma$.*

*Proof.* In polar coordinates, $c(t) = \exp_p(s(t)v(t))$ in $B_\epsilon(p)\backslash\{p\}$, and if $f(s,t) := \exp_p(sv(t)) = \gamma_{v(t)}(s)$, we have that $c' = s'f_s + f_t$. Now, use that $f_s \perp f_t$, by Gauss' Lemma. ∎

**Corollary 12.** *$d$ is a distance on $M$, $d_p := d(p,\cdot)$ is differentiable in $B_\epsilon(p)\backslash\{p\}$, and $d_p^2$ is differentiable in $B_\epsilon(p)$.*

*Exercise*. Compute $\|\nabla d_p\|$ and the integral curves of $\nabla d_p$ inside $B_\epsilon(p)\backslash\{p\}$.

**Remark 13.** Proposition 11 is LOCAL ONLY, and $\epsilon = \epsilon(p)$: $\mathbb{R}^n$; $\mathbb{S}^n$; $\mathbb{R}^n \setminus \{0\}$.

## §5. Curvature

Gauss: $K(M^2 \subset \mathbb{R}^3) = K(\langle\,,\rangle)$. Riemann: $K(\sigma) = K_p(\exp_p(\sigma))$.

**Def.:** The *curvature tensor* or *Riemann tensor* of $M$ is (sign!)

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

We also call $R$ the (4,0) tensor given by

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W \rangle.$$

Curvature tensor $R_{\hat{\nabla}}$ of a vector bundle $E$ with a connection $\hat{\nabla}$: exactly the same.

**Proposition 14.** *For all $X, Y, Z, W \in \mathfrak{X}(M)$, it holds that:*

- *$R$ is a tensor;*
- *$R(X,Y,Z,W)$ is skew-symmetric in $X, Y$ and in $Z, W$;*
- *$R(X,Y,Z,W) = R(Z,W,X,Y)$;*
- *$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ (first Bianchi id.);*
- $R^s_{ijk} = \sum_l \Gamma^l_{ik}\Gamma^s_{jl} - \sum_l \Gamma^l_{jk}\Gamma^s_{il} + \partial_j \Gamma^s_{ik} - \partial_i \Gamma^s_{jk}$ ($\Rightarrow R \cong \partial^2 \langle\,,\rangle$).

*Proof.* Exercise. ∎

$\langle\,,\rangle \Rightarrow \mathfrak{X}(M) \cong \Omega^1(M)$ and $\langle\,,\rangle$ extends to the tensor algebra $\Rightarrow$ the *curvature operator* $R : \Omega^2(M) \to \Omega^2(M)$ is self-adjoint.

**Def.:** If $\sigma \subset T_pM$ is a plane, then the *sectional curvature* of $M$ in $\sigma$ is given by

$$K(\sigma) := \frac{R(u,v,v,u)}{\|u\|^2\|v\|^2 - \langle u,v\rangle^2}, \qquad \sigma = \operatorname{span}\{u,v\}.$$

**Proposition 15.** *If $R$ and $R'$ are tensors with the symmetries of the curvature tensor and Bianchi such that $R(u,v,v,u) = R'(u,v,v,u)$ for all $u, v$, then $R = R'$ (i.e., $K$ determines $R$).*

**Corollary 16.** *If $M$ has constant sectional curvature $c \in \mathbb{R}$, then $R(X,Y,Z,W) = c(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle)$.*

**Def.:** The <u>*Ricci tensor*</u> is the symmetric (2,0) tensor given by

$$Ric(X,Y) := \frac{1}{n-1}\mathrm{trace}\, R(X,\cdot,\cdot,Y),$$

and the <u>*Ricci curvature*</u> is $Ric(X) = Ric(X,X)$ for $\|X\| = 1$.

*Example:* $\mathbb{CP}^n$ as $\mathbb{S}^{2n+1}/\mathbb{S}^1$ has $K(X,Y) = 1 + 3\langle JX,Y\rangle^2$ and $Ric \equiv (n+2)/(n-1)$.

**Def.:** The <u>*scalar curvature*</u> of $M$ is $\frac{1}{n}\mathrm{trace}\, Ric$.

**Lemma 17.** *(Compare with Lemma 5) Let $f : U \subset \mathbb{R}^2 \to M$ be a map into a Riemannian manifold and $V \in \mathfrak{X}_f$. Then,*

$$\nabla_{\partial_t}\nabla_{\partial_s}V - \nabla_{\partial_s}\nabla_{\partial_t}V = R(f_*\partial_t, f_*\partial_s)V.$$

*Equivalently, $R_{\nabla f}(\cdot,\cdot)V = R_{\nabla}(f_*\cdot, f_*\cdot)V$, $\forall\, f \colon N \to M$.*

*Proof.* Since $R_{\nabla f}$ is a tensor, it is enough to check the lemma for coordinate vector fields on $N$ and for $V = \overline{V} \circ f$, $\overline{V} \in \mathfrak{X}(M)$. ∎

<u>*Exercise*</u>. Let $\pi : E \to M$ be a vector bundle of rank $k$ with a linear connection $\nabla$. Then $\nabla$ is flat if and only if each $\xi \in E$ has a (unique!) local parallel extension. If $M$ is simply connected, such an extension exists globally and therefore $E \cong M \times \mathbb{R}^k$ is trivial.

## §6. Isometric immersions (finally!)

As we have seen in the Example in page 5, if $f : M \to N$ is an isometric immersion $\Rightarrow f^*(TN) = f_*(TM) \oplus^\perp T_f^\perp M$, and $\nabla_X^M Y = (\nabla_X^f f_* Y)^\top$, $\forall X, Y \in TM$. Moreover, we have that

$$\alpha(X, Y) := \left(\nabla_X^f f_* Y\right)^\perp$$

is a symmetric tensor, called the *second fundamental form of $f$*. In addition, $\nabla^\perp : TM \times \Gamma(T_f^\perp M) \to \Gamma(T_f^\perp M)$ given by

$$\nabla_X^\perp \eta = \left(\nabla_X^f \eta\right)^\perp$$

is a connection in $T_f^\perp M$, called the *normal connection of $f$*.
Identifications.

*Exercise*. Show that $\nabla^\perp$ is a compatible connection with the induced metric on $T_f^\perp M$.

$\alpha(p)$ is the quadratic approximation of $f(M) \subset N$ at $p \in M$.
$\alpha(v, v) = \gamma_v'(0)$: Picture!
$\eta \in T_{f(p)}^\perp M \Rightarrow$ (self-adjoint!) *shape operator* $A_\eta : T_pM \to T_pM$.
The Fundamental Equations. Particular case: $K =$ constant $\Rightarrow$ the *Fundamental Theorem of Submanifolds*.
Gauss equation $\Leftrightarrow K(\sigma) = \overline{K}(\sigma) + \langle \alpha(u, u), \alpha(v, v) \rangle - \|\alpha(u, v)\|^2$
$\Rightarrow$ Riemann notion of sectional curvature agrees with ours.

*Example:* $\mathbb{S}^{n-1}(r) \subset \mathbb{R}^n \Rightarrow K \equiv 1/r^2$ (it *had* to be constant!).

*Example:* Height functions and the graph of a real function.

Model of the hyperbolic space $\mathbb{H}^n$ as a submanifold of $\mathbb{L}^{n+1}$.

## §7. Hypersurfaces

Principal curvatures and directions; mean curvature; Gauss-Kronecker curvature; Gauss map.
Fundamental equations in this simpler case.
Locally convex and strictly convex hypersurfaces.

**Proposition 18.** *Given a compact Euclidean hypersurface $M^n \subset \mathbb{R}^{n+1}$, for every $0 \neq v \in \mathbb{R}^{n+1}$ there exists $x \in M^n$ such that $v$ is normal to $M^n$ at $x$ and $A_v > 0$.*

**Theorem 19.** *For a compact Euclidean hypersurface $M^n$:*
*The Gauss-Kronecker curvature never vanishes* $\iff$
*$M$ is orientable and the Gauss map is a diffeomorphism* $\iff$
*The second fundamental form is definite everywhere* $\iff$
*$M$ is a convex hypersurface ($M = \partial B$ for a convex body $B$).*

## §8. Totally geodesic and umbilic submanifolds

$\mathbb{Q}_{\tilde{c}}^m \subset \mathbb{Q}_c^{m+p}$ for $\tilde{c} \geq c$.
Axioms of $r$-planes and $r$-spheres.

## §9. Nullity distributions

The (relative) nullity distribution ($\Delta$) $\Gamma_c$ and the index of (relative) nullity ($\nu = \dim \Delta$) $\mu_c = \dim \Gamma_c$. $\Gamma = \{X : R(X, ...) = \tilde{R}(X...)\}$ and $\mu = \dim \Gamma$ are extrinsic.

**Proposition 20.** *For an isometric immersion $f : M \to \tilde{M}$, the following assertions hold:*

*i)* $\nu$, $\mu$ *and* $\mu_c$ *are upper semicontinuous. Hence, the subsets where* $\nu$, $\mu$ *and* $\mu_c$ *attain their minimum values are open, and there is an open and dense subset of* $M^n$ *where* $\nu$ *(also* $\mu$ *and* $\mu_c$*) is locally constant;*

*ii)* $\Delta$ $(\Gamma_c)$ *is smooth on any open subset of* $M^n$ *where* $\nu$ *(also* $\mu$ *and* $\mu_c$*) is constant;*

*iii) If* $\tilde{M}$ *has constant sectional curvature, then* $\Delta$ *is a totally geodesic (hence integrable) distribution on any open subset where* $\nu$ *is constant, and the restriction of* $f$ *to each leaf of* $\Delta$ *is totally geodesic.*

**Remark 21.** $\Gamma_c$ is always an intrinsic totally geodesic foliation where $\mu_c$ is constant (why?). Moreover, $\Delta \subset \Gamma$.

*Exercise*. Every umbilical distribution of a Riemannian manifold is integrable and its leaves are umbilical submanifolds.

## §10. Principal Normals and flat normal bundle

Principal and Dupin principal normals. Eigendistributions. Submanifolds with flat normal bundle.

## §11. Reduction of codimension

First normal spaces $N_1(x) := \text{span}\{\alpha(X,Y) : X, Y \in T_xM\}$.

**Proposition 22.** *Let* $f : M^n \to \mathbb{Q}_c^{n+p}$ *be an isometric immersion. Suppose that there exists a parallel normal subbundle* $L^q \subset T^\perp M$ *of rank* $q < p$ *such that* $N_1(x) \subset L^q(x)$ *for all* $x \in M^n$*. Then the codimension of* $f$ *reduces to* $q$*.*

$s$-nullities $\nu_s$ and $\nu_s^*$. 1-regular isometric immersions.

**Proposition 23.** *Let $f : M^n \to \mathbb{Q}_c^m$ be a 1-regular isometric immersion such that rank $N_1 = q \leq n-1$. If $\nu_s^*(x) < n-s$ for all $1 \leq s \leq q$ at any point $x \in M^n$, then $N_1$ is parallel and thus $f$ reduces codimension to $q$.*

## §12. Minimal submanifolds

Let $f_t : M^n \to \bar{M}$ be an isotopy of $f = f_0$. Write $T = f_0' = f_* Z + \eta \in \mathfrak{X}_f$, $Z \in TM$ and $\eta \in T_f^\perp M$. We will denote by $H = \text{trace } \alpha/n \in \Gamma(T_f^\perp M)$ the *mean curvature vector* of $f$. Then,

$$(dvol_t)'(0) = (-n\langle H, \eta\rangle + div\ Z)\, dvol.$$

**Proposition 24.** *$M^n$ compact with boundary and $Z|_{\partial M} = 0$, then $V(t) := Vol(f_t(M))$ satisfies $V'(0) = -n\int_M \langle H, \eta\rangle dvol$. In particular, minimal submanifolds are the critical points of the volume functional for compactly supported variations.*

$f : M^n \to \mathbb{R}^m \Rightarrow \Delta f = nH$. Hence, minimal $\Rightarrow$ harmonic $\Rightarrow$ *There are no compact minimal Euclidean submanifolds.* But:

**Proposition 25.** *A compact minimal Euclidean submanifold with boundary is contained in the interior of the convex hull of its boundary.*

If $f : M^n \to \mathbb{Q}_c^{n+p}$ is minimal, then $Ric_M \leq c$ since

$$Ric_M(X) = c + \frac{n}{n-1}\langle A_H X, X\rangle - \frac{1}{n-1}\sum_{i=1}^{p}\langle A_{\xi_i}^2 X, X\rangle.$$

In particular, $scal_M = c + \frac{n}{n-1}\|H\|^2 - \frac{1}{n(n-1)}\|\alpha\|^2$.

**Lemma 26.** *Given $F : M^n \to \mathbb{R}^{m+1}$, there exists a minimal $f : M^n \to \mathbb{S}^m_c$ such that $F = inc \circ f \iff \Delta F = -ncF$.*

*Construction*. Let $\mathcal{H}(m,d)$ be the vector space of homogeneous harmonic polynomials of degree $d$ in $(m+1)$ real variables. Then, $\dim \mathcal{H}(m,d) = n+1$, where $n = n(m,d) = \frac{(2d+m-1)(d+m-2)!}{d!(m-1)!} - 1$. Then, $W = W(m,d) = \{f|_{\mathbb{S}^m} : f \in \mathcal{H}(m,d)\}$ is contained in (actually, it is equally to) the eigenspace of $\Delta_{\mathbb{S}^m}$ with eigenvalue $\lambda(m,d) = -d(m+d-1)$. Fix $\langle\, ,\rangle$ the $L^2$-inner product on $W$, and $\{f_0, \ldots, f_n\}$ an orthonormal basis of $W$. Set $G := O(m+1)$,

$$F := (f_0, \ldots, f_n) : \mathbb{S}^m \to \mathbb{R}^{n+1}.$$

Since $\langle\, ,\rangle$ is invariant under the $G$-action $A \cdot f = f \circ A$, the basis $\{A \cdot f_0, \ldots, A \cdot f_n\}$ is also orthonormal. So there is $\tilde{A} \in O(n+1)$ such that $F \circ A = \tilde{A} \circ F$, and the map $A \mapsto \tilde{A}$ is a group homomorphism (such an $F$ is said to be *$G$-equivariant*). In particular, $G$ acts isometrically and transitively with the metric induced by $F$, and the isotropy groups $O(m)$ act transitively on the grassmanians of each tangent space. Thus, there exists $\tilde{c} > 0$ such that $F^*\langle, \rangle = \tilde{c}\langle, \rangle$, and hence $F$ induces an isometric immersion of $\mathbb{S}^m_{1/\tilde{c}}$ into $\mathbb{R}^{n+1}$ with $\Delta F = -(1/\tilde{c})\lambda(m,d)F$. We conclude by Lemma 26 that there is a minimal equivariant isometric immersion $f : \mathbb{S}^m_{1/\tilde{c}} \to \mathbb{S}^n_c$, $c = \lambda(m,d)/m\tilde{c}$, $F = inc \circ f$. We have constructed the minimal and equivariant *Veronese embeddings*,

$$f : \mathbb{S}^m_r \to \mathbb{S}^n, \quad r = r(m,d) := \frac{m}{d(m+d-1)}$$

(these are embeddings if $d$ is odd, and embeddings of the projective space if $d$ is even). And they are (essentially) *unique!!*

## §13. Minimal rigidity of hypersurfaces

Deformability and rigidity. The associated family.

**Theorem 27.** *Let $f : M^n \to \mathbb{Q}_c^{n+1}$ be a minimal immersion of a Riemannian manifold with $\mu_c \not\geq n-2$. Then, $f$ is rigid among minimal immersions $g : M^n \to \mathbb{Q}_c^{n+p}$, i.e., $g = inc \circ f$.*

*Proof.* Diagonalize the shape operator of $f$, $Ae_i = \lambda_i e_i$, and set $\alpha_{ij} := \alpha_g(e_i, e_j)$. By Gauss equation, $\lambda_i^2 = \sum_k \|\alpha_{ik}\|^2$ and

$$(\langle \alpha_{ii}, \alpha_{jj}\rangle - \|\alpha_{ij}\|^2)^2 = \lambda_i^2 \lambda_j^2 = \sum_k \|\alpha_{ik}\|^2 \sum_k \|\alpha_{jk}\|^2$$
$$\geq (\|\alpha_{ii}\|^2 + \|\alpha_{ij}\|^2)(\|\alpha_{jj}\|^2 + \|\alpha_{ij}\|^2) \geq (\langle \alpha_{ii}, \alpha_{jj}\rangle + \|\alpha_{ij}\|^2)^2.$$

So, $\alpha_{ij} \neq 0 \ \Rightarrow \langle \alpha_{ii}, \alpha_{jj}\rangle \leq 0 \Rightarrow \lambda_i \lambda_j \leq -\|\alpha_{ij}\|^2 < 0$. Thus, at a point with $\nu = \mu_c \leq n-3$, there should be a pair with $\alpha_{ij} = 0$. The above equation implies that $\alpha_{ii}$ and $\alpha_{jj}$ are linearly dependent, and $\alpha_{is} = 0$ for $i \neq s \neq j$. Changing the roles of $s$ and $j$ we get $\alpha_{ij} = 0$. We conclude that $(\alpha_g)_{N_g^1} = \pm \alpha_f$. Done, since $N_g^1$ is parallel (e.g. by Proposition 23) and $g$ is analytic. ∎

## §14. Local rigidity and flat bilinear forms

In local coordinates, an isometric immersion is a solution of a nonlinear PDE, so if the codimension is small it should be overdetermined, hence rigidity should be true under generic conditions. Analyze the proof of Theorem 27: It's just Gauss equation!

But: $f$ rigid $\Rightarrow$ Find $\tau : T_f^\perp M \to T_g^\perp M$ satisfying

$$\tau \circ \alpha_f = \alpha_g.$$

Such $\tau$ is unique if $f$ is full (or unique in $N_1^f$), and its parallelism is not hard to see. Now, a necessary condition for the existence such a bundle isometry $\tau$ is that

$$\|\alpha_f(X,Y)\| = \|\alpha_g(X,Y)\|, \quad \forall X, Y \in TM. \tag{2}$$

which is equivalent by polarization to that, $\forall\, X, Y, X', Y' \in TM$,

$$\langle \alpha_f(X,Y), \alpha_f(X',Y') \rangle = \langle \alpha_g(X,Y), \alpha_g(X',Y') \rangle.$$

But this is also sufficient(!!): just define $\tau$ as $\tau \circ \alpha_f = \alpha_g$ and extend linearly. In other words, we need to understand when the flat bilinear form (FBF) $\beta = (\alpha_f, \alpha_g)$ is null, where

$$\beta = (\alpha_f, \alpha_g) : TM \times TM \to (T_f^\perp M \times T_g^\perp M, \langle\,,\rangle_f - \langle\,,\rangle_g).$$

### 14.1 Flat bilinear forms

Let $\beta : \mathbb{V} \times \mathbb{V}' \to \mathbb{W}^{p,q}$ a FBF.

**Def.:** $RE(\beta)$. $S(\beta)$. $\beta_X$ for $X \in \mathbb{V}$. Isotropic (null) subspaces. $\nu_\beta := \dim N(\beta)$.

**Proposition 28.** *The subset $RE(\beta)$ is open and dense in $V$.*

Observe that, by flatness: *if $\beta_X(\mathbb{V}') \subset \mathbb{W}$ is isotropic for all $X$ in a dense subset, then $\beta$ is null.*

**Proposition 29.** *For any bilinar form $\beta$ and $X \in RE(\beta)$, $\beta(\mathbb{V}, \operatorname{Ker}\beta_X) \subset \operatorname{Im}\beta_X$. If $\beta$ is also flat, then $\beta|_{\mathbb{V}\times\operatorname{Ker}\beta_X}$ is null since $\beta(\mathbb{V}, \operatorname{Ker}\beta_X) \subset \mathcal{U}(X) := \operatorname{Im}\beta_X \cap \operatorname{Im}\beta_X^\perp$.*

*Proof.* For any $Y \in \mathbb{V}$ and $t$ small, $L_t = \operatorname{Im} \beta_{X+tY} \subset \mathbb{W}$ is a *continuous* family of subspaces that contain $\beta_{X+tY}(\operatorname{Ker} \beta_X) = \beta_Y(\operatorname{Ker} \beta_X)$, which does not depend on $t$. ∎

**Corollary 30.** $\beta \colon \mathbb{V}\times\mathbb{V}' \to \mathbb{W}^{p,0}$ *FBF* $\Rightarrow \nu_\beta \geq \dim \mathbb{V}' - \dim \mathbb{W}$.

**Theorem 31** (Chern-Kuiper)**.** $M^n \subset \tilde{M}^{n+p} \Rightarrow \nu \leq \mu \leq \nu + p$.

**Corollary 32.** $M^n \subset \mathbb{R}^{n+p}$ *compact* $\Rightarrow \mu (= \mu_0) \not> p$.

**Corollary 33.** $M^n \subset \mathbb{R}^{n+p}$ *compact and flat* $\Rightarrow p \geq n$.

### 14.2 Uniqueness of the normal connection

**Proposition 34.** *Let $f, f' : M^n \to \mathbb{Q}_c^{n+p}$ be isometric immersions and let $\tau : T_f^\perp M \to T_{f'}^\perp M$ be a vector bundle isometry that preserves the second fundamental forms. Then it also preserves the normal connections on the first normal bundles. In particular, it is parallel if either immersion is full.*

## §15. Local algebraic rigidity

**Lemma 35** (Lorenzian version of Corollary 30)**.** *If $\beta$ is a FBF with $S(\beta) = \mathbb{W}^{p,1}$ Lorenzian $\Rightarrow \nu_\beta \geq \dim \mathbb{V}' - \dim \mathbb{W}$.*

**Def.:** Type number $\tau$ of $f$. Obs: $\tau \geq 1 \Rightarrow f$ is full.

**Theorem 36** (Beez-Killing)**.** $M \subset \mathbb{Q}_c^{n+1}$ *with $\tau \geq 3$ is rigid.*

**Corollary 37.** *Let $f, f' : M^n \to \mathbb{Q}_c^{n+1}$ be nowhere congruent isometric immersions of a Riemannian manifold with no points of constant sectional curvature $c$. Then $f$ and $f'$ carry a common relative nullity distribution of rank $n-2$.*

**Theorem 38** (Allendoerfer)**.** *Any* $f : M^n \to \mathbb{Q}_c^{n+p}$ *with* $\tau \geq 3$ *everywhere is rigid.*

*Proof.* By Proposition 34, we only need to show that $\beta = \alpha \oplus \alpha'$ is null, since $\tau \geq 3$ implies that $f$ is null.
Let $k := \min\{\dim \mathcal{U}(X) \colon X \in RE(\beta)\}$. Similarly to $RE(\beta)$, $RE^o(\beta) := \{X \in \mathbb{V} \colon \dim \mathcal{U}(X) = k\}$ is also open and dense in $\mathbb{V}$. So, we only need to show that $k = p$.
$\tau \geq 3 \;\Rightarrow \exists\, L^{n-3p} := (\mathrm{span}\{A_{\xi_j} X_i \colon 1 \leq j \leq p, 1 \leq i \leq 3\})^{\perp} = \cap_{i=1}^3 \mathrm{Ker}\, \alpha_{X_i}$. But $\dim \mathrm{Ker}\, \beta_{X_1} = n - \mathrm{rank}\, \beta_{X_1} \geq n - 2p + k$. Proposition 29 $\Rightarrow \dim \mathrm{Ker}\, \beta_{X_1} \cap \mathrm{Ker}\, \beta_{X_2} \geq \dim \mathrm{Ker}\, \beta_{X_1} - \dim \mathcal{U}(X_1) \geq n - 2p$ and similarly $\dim \cap_{i=1}^3 \mathrm{Ker}\, \beta_{X_i} \geq n - 2p - k$. Done, since $\cap_{i=1}^3 \mathrm{Ker}\, \beta_{X_i} \subset L^{n-3p}$. ∎

## §16. The Main Lemma

## §17.

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