# SUBMANIFOLD THEORY BEYOND 

## AN INTRODUCTION

Marcos Dajczer and Ruy Tojeiro

To our parents (In Memoriam)
Hersz and Rywka
Ruy and Jacy

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## Preface

Submanifold theory has emerged as a natural development of the classical study of curves and surfaces in Euclidean three space with the methods of differential calculus. In the last century, it has evolved into a broad subarea of differential geometry with many distinct branches, and making use of a variety of techniques. As a consequence, any book on the subject must necessarily be restricted to some particular directions in the field. Our choices in this book were certainly driven by our own research experiences and personal tastes. The initial goal was to write an updated version of the lecture notes "Submanifolds and isometric immersion", published in 1990. However, the contents of that book now cover roughly one third of the material of this book, and virtually all chapters of the original manuscript have been completely rewritten and substantially enlarged.

The focus of this book is on the general properties of isometric and conformal immersions of Riemannian manifolds into space forms, rather than on results that can be derived for some special classes of submanifolds. One main theme is the isometric and conformal deformation problem for submanifolds with arbitrary dimension and codimension. Besides providing a modern treatment of some classical theorems on this topic, a special emphasis has been given to the notion of genuine (isometric and conformal) deformations of submanifolds. A basic algebraic tool is the theory of flat bilinear forms introduced by J. D. Moore as an outgrowth of the theory of exteriorly orthogonal quadratic forms due to E. Cartan.

The first seven chapters of the book are suitable for an introductory course on submanifold theory for students with a basic background on Riemannian geometry. The language and some standard results on vector bundles are also needed and are summarized in an appendix to the book. With the omission of a few sections of a more technical nature, and of some others devoted to applications of the theory, the material of the first seven chapters could be covered in a one-semester course. More specifically, Sections 1.10 and 4.4.3, as well as the appendices to Chapters 1 and 4 , could be skipped in a first reading, and the instructor could make some choices among the materials of Sections $2.3,3.5,3.6$ and 4.3, depending on the students' interests and on his own.

The remaining chapters account for the second part of the title of the book, borrowed from Knapp's book on Lie groups. They could be used in a more advanced course by students aiming at initiating research on the subject, and are also intended to serve as a reference material for specialists in the field. A brief outline of each chapter
is given next.
Chapter 1 establishes some basic facts of the theory of submanifolds that are used throughout the rest of the book. The second fundamental form and normal connection of an isometric immersion are introduced by means of the Gauss and Weingarten formulas, and their compatibility equations are derived, namely, the Gauss, Codazzi and Ricci equations. A complete proof is provided of the so-called Fundamental theorem of submanifolds, according to which these data are sufficient to determine uniquely any submanifold of a Riemannian manifold with constant sectional curvature, up to isometries of the ambient space. Among the remaining topics covered in this chapter are totally geodesic and umbilical submanifolds of space forms, the relative nullity distribution, principal normal vector fields and submanifolds with flat normal bundle. An alternative approach to the basic equations and the Fundamental theorem of submanifolds, namely, the Burstin-Mayer-Allendoerfer theory, is summarized in an appendix to the chapter, which involve the Gauss, Codazzi and Ricci equations of higher order.

Chapter 2 discusses conditions under which the codimension of an isometric immersion of a Riemannian manifold into a space form can be reduced, that is, conditions implying that the image of the isometric immersion is contained in a totally geodesic submanifold of the ambient space. Some of them are given in terms of the $s$-nullities and the type number of the isometric immersion, which play a key role in the study of the rigidity of submanifolds in Chapter 4. The results of this chapter, as well as some of those in Chapter 1, are illustrated with the classification of constant curvature submanifolds with flat normal bundle and parallel mean curvature vector field of space forms.

In Chapter 3 some general aspects of the theory of minimal submanifolds are introduced, starting with a proof of the first variational formula. The characterization of minimal submanifolds of Euclidean space by the harmonicity of its coordinate functions is then discussed, and a few of its consequences are derived. The construction of minimal isometric immersions of spheres into spheres in terms of eigenfunctions of the Laplacian is presented. The Ricci tensor of a submanifold of a space form is computed in terms of its second fundamental form, and it is shown how this can be used to derive an obstruction to the existence of minimal isometric immersions into a space of constant sectional curvature. A strong rigidity result for minimal hypersurfaces of space forms within the class of minimal isometric immersions is given. The last section of the chapter contains a generalization for hypersurfaces with arbitrary dimension of space forms of the classical Ricci condition, which gives necessary and sufficient conditions for some neighborhood of a point with nonpositive Gaussian curvature of a two-dimensional Riemannian manifold to admit a minimal isometric immersion in $\mathbb{R}^{3}$.

Chapter 4 initiates the study of a central topic of this book, namely, the isometric rigidity/deformation problem for submanifolds of space forms. The theory of flat bilinear forms is developed, and applied in particular to prove two basic rigidity theorems due to Allendoerfer and do Carmo and Dajczer, respectively. The classical inequalities due to Chern and Kuiper are also derived. The proof of the main lemma on flat bilinear forms is provided in an appendix to the chapter.

In Chapter 5, the theory of flat bilinear forms is applied to the study of isometric
immersions of space forms into space forms. This is in fact its primary application, and goes back to Cartan's theory of exteriorly orthogonal quadratic forms. The correspondence between constant curvature submanifolds of space forms satisfying some additional conditions and solutions of certain nonlinear systems of partial differential equations is also discussed. In the surface case, this reduces to the classical correspondence between constant curvature surfaces and solutions of the sine-Gordon, sinhGordon, Laplace and wave equations. We conclude the chapter with Nikolayevsky's proof of the nonexistence of an isometric immersion of a complete non-simply connected $n$-dimensional Riemannian manifold of constant sectional curvature $c<0$ into a complete and simply connected $(2 n-1)$-dimensional Riemannian manifold of constant sectional curvature $\tilde{c}>c$.

Chapter 6deals with the geometric restrictions that are imposed on submanifolds with nonpositive extrinsic curvatures of space forms. The main tools are, on the one hand, an algebraic lemma due to Otsuki and, on the other hand, a maximum principle due to Omori and Yau and generalized by Pigola, Rigoli and Setti. The latter is a key ingredient to replace compactness. Also discussed in this chapter is Florit's estimate on the index of relative nullity of a submanifold whose extrinsic curvatures are nonpositive at some point, under the assumption that its codimension does not exceed half of its dimension.

Chapter 7 studies complete submanifolds of low codimension of space forms with positive index of relative nullity everywhere. Among the main results of the chapter are Hartman's splitting theorem for complete Euclidean submanifolds with nonnegative Ricci curvature and Dajczer-Gromoll's generalization of the rigidity of the totally geodesic inclusion of a round sphere $\mathbb{S}^{n}$ into $\mathbb{S}^{n+p}, p \leq n-1$. The proofs of both results rely on the fact that the leaves of the minimum relative nullity foliation of a complete submanifold are also complete. This is proved in detail, after developing the necessary tools, especially the splitting tensor of a totally geodesic foliation. The chapter proceeds with a discussion of the Gauss parametrization of oriented hypersurfaces with constant index of relative nullity. Besides several applications presented in the chapter, the Gauss parametrization plays a key role in the classifications in Chapters 11 and 14 , respectively, of hypersurfaces of space forms that are isometrically deformable or infinitesimally bendable. The chapter ends with a discussion of intrinsically homogeneous hypersurfaces of space forms.

Chapter 8 introduces the notion of extrinsic products of immersions, which is the simplest way of constructing an immersion of a product manifold into a space form. It is a basic fact that an isometric immersion of a Riemannian product into a space form must be an extrinsic product of isometric immersions of the factors whenever its second fundamental form is adapted to the product structure of the manifold, in the sense that the tangent spaces to each factor are preserved by all shape operators. This was proved by Moore for isometric immersions into Euclidean space, and extended by Molzan for any space form as ambient space. After giving a proof of these results, the remainder of the chapter is devoted to proving several results which assure that this condition is satisfied under assumptions of both local and global natures.

In Chapter 9 we initiate the study of conformal immersions. Our approach is
based on the fact that, to any conformal immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ of a Riemannian manifold $M^{n}$ into Euclidean space one can naturally associate an isometric immersion $F: M^{n} \rightarrow \mathbb{V}^{m+1}$ into the light-cone $\mathbb{V}^{m+1}$ of Lorentzian space $\mathbb{L}^{m+2}$, called its isometric light-cone representative. This relies on the fact that $\mathbb{R}^{m}$ can be isometrically embedded into $\mathbb{V}^{m+1}$, giving rise to a model of Euclidean space as a hypersurface of $\mathbb{V}^{m+1}$, which is a very suitable setting for dealing with Moebius geometric notions. We provide an elementary and self-contained account of the fact that Moebius transformations of Euclidean space are given by linear orthogonal transformations of $\mathbb{L}^{m+2}$ that preserve the upper half of $\mathbb{V}^{m+1}$, as well as of the natural identification of the space of oriented hyperspheres of $\mathbb{R}^{m}$ with de Sitter space $\mathbb{S}_{1,1}^{m+1} \subset \mathbb{L}^{m+2}$. This is first applied to study envelopes of congruences of spheres in Euclidean space and their relation to Dupin principal normal vector fields. A conformal version of the Gauss parametrization is presented, which allows to parametrize a Euclidean hypersurface that envelops a $k$ parameter congruence of hyperspheres in terms of the locus of their centers and their radii. As another illustration of this approach, it is shown that the classical theorem of Liouville on the classification of conformal maps between open subsets of Euclidean space of dimension $m \geq 3$ is equivalent to the isometric rigidity of the light-cone hypersurface model of $\mathbb{R}^{m}$.

The chapter proceeds by defining the notions of Moebius metric, Moebius second fundamental form, Blaschke tensor and Moebius form of a Euclidean submanifold. These concepts have been introduced by C. Wang, and have proved to be useful tools in the study of Moebius geometric properties of submanifolds. A Fundamental theorem of submanifolds within the context of Moebius geometry is derived, according to which a Euclidean submanifold is completely determined, up to Moebius transformations of the ambient space, by its Moebius metric, Moebius second fundamental form and normal connection. Our main interest in this chapter is on the study of conformal deformations of Euclidean submanifolds, which is pursued in the last chapters of the book. In particular, a proof of do Carmo-Dajczer conformal rigidity theorem is given, which provides sufficient conditions, in terms of the so-called conformal s-nullities, for a Euclidean submanifold with codimension less than five to be conformally rigid, a generalization of a well-known conformal rigidity criterion for hypersurfaces due to Cartan. The chapter also includes a conformal version, due to Tojeiro, of Moore's decomposition theorem.

In Chapter 10 two other useful ways of constructing immersions of product manifolds from immersions of the factors are discussed, namely (extrinsic) warped products of immersions and, more generally, partial tubes over extrinsic products of immersions. Both types of immersions share with extrinsic products of immersions the property that their second fundamental forms are adapted to the product structure of the manifold. Nölker's and Tojeiro's decomposition theorems are then presented, showing that, once this condition is satisfied, immersions of each kind are characterized by the special types of metrics they induce on the product manifold. These are, respectively, warped product metrics and metrics called polar. The remainder of the chapter presents Dajczer and Vlachos' sufficient conditions, in terms of the s-nullities, for the second fundamental form of an isometric immersion of a product manifold endowed with a warped
product metric to be adapted to the product structure of the manifold.
The purpose of Chapter 11 is to provide a proof of the parametric description of hypersurfaces of space forms that admit nontrivial isometric deformations. The study of such hypersurfaces in Euclidean space goes back to Sbrana in 1909 and Cartan in 1916, and their classification has been extended to the case of nonflat ambient space forms by Dajczer, Florit and Tojeiro. Apart from the hypersurfaces with the same constant curvature as the ambient space, they split into four distinct classes. The main goal of the chapter is to show how the hypersurfaces in the two most interesting classes, which admit either a one-parameter family of isometric deformations or a single one, can be parametrized in terms of the Gauss parametrization studied in Chapter 7 .

Chapter 12 deals with one of the central concepts of the book, namely, that of a genuine deformation. It comes from the observation that any submanifold of a deformable submanifold already possesses the isometric deformations induced by the latter. Therefore, when studying the isometric deformations of a submanifold with codimension greater than one, one should look for the "genuine" ones, that is, those which are not induced by isometric deformations of an "extended" submanifold of higher dimension. One of the main results of the chapter is Dajczer-Florit's theorem showing that only ruled submanifolds admit genuine deformations, among Euclidean submanifolds with low codimension. A conformal version of this result, due to Florit and Tojeiro, is also discussed. The notions of genuine isometric and conformal deformations give rise to corresponding notions of isometric and conformal genuine rigidity. Dajczer-Florit's theorem, and its conformal version by Florit andTojeiro, provide sufficient conditions for a submanifold to be genuinely rigid. These results are of local nature, and their unifying character is then explored by showing how they imply several previously known results in the literature.

The theory of genuine isometric deformations is also applied to the problem of determining the isometric immersions $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ into a complete and simply connected Riemannian manifold of dimension $(n+p)$ and constant sectional curvature $c$ for which $M^{n}$ also admits an isometric immersion $\hat{f}: M^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+q}$ with $\tilde{c} \neq c$. The chapter proceeds by discussing a weaker version of genuine isometric rigidity, in which one allows isometric deformations of a submanifold that are induced by isometric deformations of an extended submanifold whose singular set may intersect the initial submanifold. This version of genuine rigidity turns out to be essential when studying in Chapter 13 global rigidity aspects of compact submanifolds of higher codimension. The final section of this chapter is devoted to the description of submanifolds that have a nonparallel first normal bundle of low rank. Although apparently unrelated to the main subject of this chapter, the description of such submanifolds relies on similar techniques, namely, the study of conditions under which a given submanifold can be isometrically extended to a ruled submanifold.

The results of Chapter 13 are global in nature and show that complete Euclidean submanifolds with low codimension can be isometrically deformed only in very special ways. A simple proof is given of a basic theorem due to Sacksteder, according to which any compact Euclidean hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, is isometrically rigid, provided that the subset of totally geodesic points of $f$ does not disconnect
$M^{n}$. A proof of a generalization by Dajczer and Gromoll of Sacksteder's result for complete Euclidean hypersurfaces of dimension $n \geq 3$ is provided, which is preceded by a description of the geometric structure of complete Euclidean submanifolds whose rank is at most two. A far-reaching extension of Sacksteder's result, due to Florit and Guimarães, is then presented, which implies that an isometric immersion $f: M^{n} \rightarrow$ $\mathbb{R}^{n+p}$ of a compact Riemannian manifold is genuinely rigid in $\mathbb{R}^{n+q}$ in the singular sense if $p+q \leq \min \{4, n-1\}$. As a particular case, it includes a previous theorem of Dajczer and Gromoll for $p=2=q$.

Chapter 14 discusses a linearized version of the notion of an isometric bending of a submanifold, that is, of a continuous isometric deformation, namely, the notion of infinitesimal bending. Local rigidity results due to Dajczer and Rodríguez that constitute the infinitesimal counterparts of the theorems of Allendoerfer and do Carmo and Dajczer on isometric rigidity are presented. A complete local parametric description, based on the work by Dajczer and Vlachos, is then given of the nonflat infinitesimally deformable hypersurfaces, which completes and provides a modern presentation of work by Sbrana. The chapter also contains a description of the Sbrana-Cartan hypersurfaces of the continuous class as envelopes of certain two-parameter congruences of affine hyperplanes.

The purpose of Chapter 15 is to present several general results on isometric immersions of Kaehler manifolds into space forms. Most of them are about real Kaehler submanifolds, that is, isometric immersions of a Kaehler manifold $M^{2 n}$ of complex dimension $n \geq 2$ into Euclidean space. The main interest is on those real Kaehler submanifolds that are not holomorphic, specially on minimal real Kaehler submanifolds. It is shown that these submanifolds enjoy many of the basic properties of minimal surfaces in Euclidean three space, like the local existence of one-parameter families of isometric submanifolds with the same generalized Gauss map. Another property that any such submanifold shares with minimal surfaces is the fact it can be realized as the real part of a holomorphic isometric immersion, called its holomorphic representative. Real Kaehler submanifolds with type number greater than or equal to three at any point are shown to be holomorphic. In particular, this implies that real Kaehler hypersurfaces that are free of flat points have rank two. For these, a parametric description is given in terms of the Gauss parametrization.

Chapter 16 is devoted to conformally flat submanifolds in Euclidean space with low codimension. First, the characterization of conformally flat manifolds in terms of the Weyl and Schouten tensors is derived as a consequence of the fact that conformally flat manifolds are precisely those Riemannian manifolds that admit locally (globally, if simply connected) an isometric immersion with codimension one into the light cone of Lorentzian space. This basic fact is also used to prove that any simply connected compact conformally flat manifold is conformally diffeomorphic to the sphere. Then, it is shown that a generic conformally flat submanifold $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ of dimension $n \geq 4$ and codimension $p \leq n-3$ is the envelope of a $p$-parameter family of spheres. A geometric explanation of this fact is provided by means of a nonparametric description of such submanifolds. The chapter ends with a discussion of conformally flat hypersurfaces of dimension $n \geq 3$ of Euclidean space. After providing an explicit
parametrization of them for $n \geq 4$ by means of the conformal version of the Gauss parametrization, the interesting class of conformally flat Euclidean hypersurfaces of dimension three having three distinct principal curvatures is discussed in the last section of the chapter, where they are characterized as holonomic hypersurfaces satisfying some additional conditions.

The aim of Chapter 17 is to present the classification of Euclidean hypersurfaces $M^{n}$ of dimension $n \geq 5$ that admit nontrivial conformal deformations, whose study goes back to Cartan in 1917. The most interesting classes of such hypersurfaces are envelopes of some two-parameter congruences of hyperspheres that are determined by certain space-like surfaces in the de Sitter space $\mathbb{S}_{1,1}^{n+2}$.

As a general rule, with a few exceptions, results are stated along the text without giving credits to their authors. This is done at the Notes that are included at the end of each chapter, where the corresponding references are provided. The Notes also mention some further developments and open problems on the subject of the chapter. Also, with the exception of a few survey articles, in the Bibliography we have only included articles and books that are cited along the text or at the Notes to the chapters.

At the end of each chapter, we include a list of exercises. While some of them are routine exercises, others are additional results on the subject of the chapter that might as well be part of the text. Some exercises are taken from research articles on the topic, in which case a reference to the original source may be found at the Notes to that chapter. Hints, and in some cases almost complete solutions, are provided for a good part of them.

As anticipated in the first paragraph of this preface, the topics covered in the book have left aside some important (and even central) subjects of submanifold theory, even if the latter is not regarded in the broader sense it has acquired nowadays, finding applications in many areas far beyond Riemannian geometry. To mention just a few, as a means of indicating to the reader the richness of the area, neither the beautiful theory of isoparametric submanifolds of the sphere nor is that of the more general class of Dupin submanifolds is discussed. The fruitful connection between the submanifold theory and Morse theory is also barely touched, leaving important classes of submanifolds such as tight, taut and equifocal submanifolds out of the scope of this book. Extrinsically homogeneous submanifolds, that is, submanifolds that appear as orbits of linear Lie group actions, form another important class of submanifolds that is almost completely ignored in this book, except for a few examples and exercises. Among other important omissions we mention submanifolds of complex space forms, including complex, Lagrangian and CR submanifolds, and of other ambient spaces, such as symmetric spaces and, more generally, homogeneous spaces. Of course, many other topics could be added to this list. Fortunately, however, some of the preceding topics have already excellent expositions in existing books in the literature. For instance, for the theory of isoparametric and Dupin hypersurfaces of the sphere, as well as that of hypersurfaces of complex space forms, among other related topics, we refer to [74]. Applications of Morse theory to submanifold theory are well illustrated in [74], [75] and [286]. A nice exposition of the theory of extrinsically homogeneous submanifolds and, more generally, of the applications of methods involving the holonomy group
of the normal connection to the study of submanifolds, also including a discussion on isoparametric submanifolds of higher codimension, is given in [34]. The fact that such books are already available in the literature provides a strong additional support for the choices we have made.

It is a pleasure to conclude this introduction by acknowledging our indebtedness to Theodoros Vlachos, Miguel Jimenez and Sergio Chion for useful suggestions and for proofreading various portions of the book. We are also grateful to Steven Zylberman for his support with editing the English text, and to Guillermo Lobos, Felippe Guimarães, Athina Eleni Kanellopoulou, Marcos Tassi, Amalia-Sofia Tsouri and Ion Moutinho for pointing out several corrections while reading the earlier versions of the manuscript. Last but not least, the completion of this work would not have been possible without the encouragement and understanding of our spouses, Maria José and Vanessa.

## Chapter 1

## The basic equations of a submanifold

In this chapter we establish several basic facts of the theory of submanifolds that will be used throughout the book. We first introduce the second fundamental form and normal connection of an isometric immersion by means of the Gauss and Weingarten formulas. Then we derive their compatibility conditions, namely, the Gauss, Codazzi and Ricci equations. The main result of the chapter is the Fundamental theorem of submanifolds, which asserts that these data are sufficient to determine uniquely a submanifold of a Riemannian manifold with constant sectional curvature, up to isometries of the ambient space. As an application, we classify totally geodesic and umbilical submanifolds of space forms. We introduce the relative nullity distribution as well as the notion of principal normal vector fields of an isometric immersion, and derive some of their elementary properties. Submanifolds with flat normal bundle are briefly discussed.

An appendix to this chapter summarizes the Burstin-Mayer-Allendoerfer theory, which provides an alternative approach to the basic equations and the Fundamental theorem of submanifolds in the spirit of the Frenet equations for a curve.

### 1.1 Gauss and Weingarten formulas

Let $M^{n}$ and $\tilde{M}^{m}$ denote differentiable manifolds of dimensions $n$ and $m$. Here, and throughout the book, manifolds are assumed to be connected. A differentiable map $f: M^{n} \rightarrow \tilde{M}^{m}$ is called an immersion if the differential $f_{*}(x): T_{x} M \rightarrow T_{f(x)} \tilde{M}$ is injective for all points $x \in M^{n}$. The number $p=m-n$ is called the codimension of $f$. It is also usual to refer to $f$, or to $f(M)$, as an immersed submanifold, or simply a submanifold, of $\tilde{M}^{m}$.

An immersion $f: M^{n} \rightarrow \tilde{M}^{m}$ between Riemannian manifolds with metrics $\langle,\rangle_{M}$ and $\langle,\rangle_{\tilde{M}}$ is said to be an isometric immersion if

$$
\begin{equation*}
\langle X, Y\rangle_{M}=\left\langle f_{*}(x) X, f_{*}(x) Y\right\rangle_{\tilde{M}} \tag{1.1}
\end{equation*}
$$

for all $x \in M^{n}$ and $X, Y \in T_{x} M$.
If $f: M^{n} \rightarrow \tilde{M}^{m}$ is an immersion and $\langle,\rangle_{\tilde{M}}$ is a Riemannian metric on $\tilde{M}^{m}$, then (1.1) defines a Riemannian metric on $M^{n}$, called the metric induced by $f$, with respect to which $f$ becomes an isometric immersion. We will often drop the subscript and denote a Riemannian metric simply by $\langle$,$\rangle , assuming that the underlying manifold$ will be clear from the context.

Notice that $f: M^{n} \rightarrow \tilde{M}^{m}$ is an isometric immersion if and only if for all local coordinates $\varphi=\left(\varphi_{1}, \ldots, \varphi_{n}\right): U \rightarrow \mathbb{R}^{n}$ and $\psi=\left(\psi_{1}, \ldots, \psi_{m}\right): V \rightarrow \mathbb{R}^{m}$ on $M^{n}$ and $\tilde{M}^{m}$, respectively, with $f(U) \subset V$, the map

$$
\bar{f}=\left(\bar{f}_{1}, \ldots, \bar{f}_{m}\right)=\psi \circ f \circ \varphi^{-1}: \varphi(U) \rightarrow \psi(V)
$$

is a solution of the nonlinear system of $(1 / 2) n(n+1)$ partial differential equations

$$
\begin{equation*}
\sum_{i, \ell=1}^{m} \frac{\partial \bar{f}_{i}}{\partial x_{j}}(\bar{x}) \frac{\partial \bar{f}_{\ell}}{\partial x_{k}}(\bar{x}) \tilde{g}_{i \ell}(\bar{f}(\bar{x}))=g_{j k}(\bar{x}), \quad 1 \leq j, k \leq n \tag{1.2}
\end{equation*}
$$

for all $\bar{x} \in \varphi(U)$, where $g_{j k}$ and $\tilde{g}_{i \ell}$ are the coefficients of the metrics of $M^{n}$ and $\tilde{M}^{m}$ with respect to $\varphi$ and $\psi$, respectively.

Given an isometric immersion $f: M^{n} \rightarrow \tilde{M}^{m}$, we denote by $f^{*} T \tilde{M}$ the induced bundle over $M^{n}$ whose fiber at $x \in M^{n}$ is $T_{f(x)} \tilde{M}$. The orthogonal complement of $f_{*} T_{x} M$ in $T_{f(x)} \tilde{M}$ is called the normal space of $f$ at $x$ and is denoted by $N_{f} M(x)$. The normal bundle $N_{f} M$ of $f$ is the vector subbundle of $f^{*} T \tilde{M}$ whose fiber at a point $x \in M^{n}$ is $N_{f} M(x)$.

In the sequel, the set of smooth local vector fields on a manifold $M^{n}$ is denoted by $\mathfrak{X}(M)$, whereas for a general vector bundle $E$ the set of its smooth local sections is denoted by $\Gamma(E)$.

The Levi-Civita connection $\tilde{\nabla}$ of $\tilde{M}^{m}$ naturally induces a unique connection $\hat{\nabla}$ on $f^{*} T \tilde{M}$ such that

$$
\hat{\nabla}_{X}(Z \circ f)=\tilde{\nabla}_{f_{*} X} Z
$$

for all $x \in M^{n}, X \in T_{x} M$ and $Z \in \mathfrak{X}(\tilde{M})$. We always identify $\hat{\nabla}$ and $\tilde{\nabla}$, and denote the former also by $\tilde{\nabla}$. Given vector fields $X, Y \in \mathfrak{X}(M)$, decompose

$$
\tilde{\nabla}_{X} f_{*} Y=\left(\tilde{\nabla}_{X} f_{*} Y\right)^{T}+\left(\tilde{\nabla}_{X} f_{*} Y\right)^{\perp}
$$

with respect to the orthogonal decomposition

$$
f^{*} T \tilde{M}=f_{*} T M \oplus N_{f} M
$$

It is easy to verify (see Exercise 1.1) that

$$
\nabla_{X} Y=f_{*}^{-1}\left(\tilde{\nabla}_{X} f_{*} Y\right)^{T}
$$

defines a compatible torsion-free connection on $T M$, which must therefore coincide with the Levi-Civita connection of $M^{n}$.

The map $\alpha^{f}: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \Gamma\left(N_{f} M\right)$ defined by

$$
\alpha^{f}(X, Y)=\left(\tilde{\nabla}_{X} f_{*} Y\right)^{\perp}
$$

is called the second fundamental form (or shape tensor) of $f$.
We write simply $\alpha$, instead of $\alpha^{f}$, when there is no ambiguity to which immersion it refers to. Thus we can write the first basic formula of the theory of submanifolds, known as the

## Gauss formula:

$$
\tilde{\nabla}_{X} f_{*} Y=f_{*} \nabla_{X} Y+\alpha(X, Y)
$$

Since

$$
\tilde{\nabla}_{X} f_{*} Y-\tilde{\nabla}_{Y} f_{*} X=f_{*}[X, Y],
$$

where $[X, Y]$ is the Lie-bracket of $X$ and $Y$, it follows that $\alpha$ is symmetric. One can easily check that $\alpha$ is $C^{\infty}(M)$-bilinear, hence the value of $\alpha(X, Y)$ at $x \in M^{n}$ depends only on the values of $X$ and $Y$ at that point. In other words, we can regard $\alpha$ as a section of $\operatorname{Hom}^{2}\left(T M, T M ; N_{f} M\right)$.

The shape operator $A_{\xi}$ of $f$ at $x \in M^{n}$ with respect to $\xi \in N_{f} M(x)$ is defined by

$$
\left\langle A_{\xi} X, Y\right\rangle=\langle\alpha(X, Y), \xi\rangle
$$

for all $X, Y \in T_{x} M$.
Given vector fields $X, Y \in \mathfrak{X}(M)$ and $\xi \in \Gamma\left(N_{f} M\right)$, we have

$$
\begin{aligned}
\left\langle\tilde{\nabla}_{X} \xi, f_{*} Y\right\rangle & =-\left\langle\xi, \tilde{\nabla}_{X} f_{*} Y\right\rangle \\
& =-\langle\xi, \alpha(X, Y)\rangle \\
& =-\left\langle A_{\xi} X, Y\right\rangle
\end{aligned}
$$

Hence the tangent component of $\tilde{\nabla}_{X} \xi$ is $-f_{*} A_{\xi} X$. On the other hand, it is easily seen (see Exercise 1.2) that the normal component

$$
\nabla_{X}^{\perp} \xi=\left(\tilde{\nabla}_{X} \xi\right)^{\perp}
$$

defines a compatible connection on $N_{f} M$, called the normal connection of $f$. This gives us our second basic formula, namely, the

## Weingarten formula:

$$
\tilde{\nabla}_{X} \xi=-f_{*} A_{\xi} X+\nabla_{X}^{\perp} \xi .
$$

The mean curvature vector of $f$ at $x \in M^{n}$ is the normal vector defined by

$$
\begin{equation*}
\mathcal{H}(x)=\frac{1}{n} \sum_{j=1}^{n} \alpha\left(X_{j}, X_{j}\right) \tag{1.3}
\end{equation*}
$$

in terms of an orthonormal basis $X_{1}, \ldots, X_{n}$ of $T_{x} M$.
The preceding expression implies that

$$
n\langle\mathcal{H}, \xi\rangle=\operatorname{tr} A_{\xi}
$$

for any $\xi \in N_{f} M(x)$, hence the right-hand side of (1.3) does not depend on the choice of the orthonormal basis.

The immersion $f$ is said to be minimal at $x$ if $\mathcal{H}(x)=0$. We call $f$ a minimal immersion if $\mathcal{H}$ is identically zero.

For a hypersurface $f: M^{n} \rightarrow \tilde{M}^{n+1}$, a smooth unit normal vector field $\xi$ is locally unique up to sign. Once it has been fixed, we simply write $A$ for the shape operator $A_{\xi}$. Then the Gauss formula becomes

$$
\tilde{\nabla}_{X} f_{*} Y=f_{*} \nabla_{X} Y+\langle A X, Y\rangle \xi,
$$

whereas the Weingarten formula reduces to

$$
\tilde{\nabla}_{X} \xi=-f_{*} A X
$$

### 1.2 Interpretations of the second fundamental form

A smooth variation of an isometric immersion $f: M^{n} \rightarrow \tilde{M}^{m}$ is a smooth mapping $F: I \times M^{n} \rightarrow \tilde{M}^{m}$, where $0 \in I \subset \mathbb{R}$ is an open interval, such that

$$
f_{t}=F(t, \cdot): M^{n} \rightarrow \tilde{M}^{m}
$$

is an immersion for any $t \in I$ and $f_{0}=f$.
Let $\partial / \partial t$ denote the canonical vector field along the $I$ factor and set

$$
\mathcal{T}=F_{*} \partial /\left.\partial t\right|_{t=0}
$$

regarded as a section of $f^{*} T \tilde{M}$. We say that $F$ is a normal variation if the variational vector field $\mathcal{T}$ is everywhere normal to $f$. The following result gives, in particular, an interpretation of the second fundamental form.

Proposition 1.1. Let $F: I \times M^{n} \rightarrow \tilde{M}^{m}$ be a smooth variation of an isometric immersion $f: M^{n} \rightarrow \tilde{M}^{m}$. Decompose the variational vector field

$$
\mathcal{T}=f_{*} Z+\eta
$$

into its tangent and normal components. Then

$$
\begin{align*}
\left.\frac{d}{d t}\right|_{t=0}\left\langle f_{t *} X, f_{t_{*}} Y\right\rangle & =\left\langle\tilde{\nabla}_{X} \mathcal{T}, f_{*} Y\right\rangle+\left\langle f_{*} X, \tilde{\nabla}_{Y} \mathcal{T}\right\rangle \\
& =-2\langle\alpha(X, Y), \eta\rangle+\left\langle\nabla_{X} Z, Y\right\rangle+\left\langle\nabla_{Y} Z, X\right\rangle \tag{1.4}
\end{align*}
$$

for all $x \in M^{n}$ and $X, Y \in T_{x} M$. In particular, if $F$ is a normal variation, then

$$
\langle\alpha(X, Y), \eta\rangle=-\left.\frac{1}{2} \frac{d}{d t}\right|_{t=0}\left\langle f_{t_{*}} X, f_{t *} Y\right\rangle .
$$

Proof: Extend $X, Y \in T_{x} M$ to vector fields $X, Y \in \mathfrak{X}(M)$, and then to $\mathfrak{X}(I \times M)$ in the standard way, that is, at each $(t, x) \in I \times M^{n}$ consider the unique vectors in $T_{(t, x)}(I \times M)$ that project to $X(x)$ and $Y(x)$ under $\pi_{2 *}(t, x)$, respectively, and project to 0 under $\pi_{1 *}(t, x)$. Here $\pi_{1}$ and $\pi_{2}$ stand for the projections of $I \times M^{n}$ onto $I$ and $M^{n}$, respectively. Denote the extensions of $X$ and $Y$ to $I \times M^{n}$ still by $X$ and $Y$, respectively, and note that

$$
[X, \partial / \partial t]=0=[Y, \partial / \partial t],
$$

for the flows of $X$ (or $Y$ ) and $\partial / \partial t$ clearly commute. Then

$$
\begin{aligned}
\frac{d}{d t}\left\langle f_{t *} X, f_{t *} Y\right\rangle & =\left\langle\tilde{\nabla}_{\partial / \partial t} F_{*} X, F_{*} Y\right\rangle+\left\langle F_{*} X, \tilde{\nabla}_{\partial / \partial t} F_{*} Y\right\rangle \\
& =\left\langle\tilde{\nabla}_{X} F_{*} \partial / \partial t, F_{*} Y\right\rangle+\left\langle F_{*} X, \tilde{\nabla}_{Y} F_{*} \partial / \partial t\right\rangle
\end{aligned}
$$

Hence, using the Gauss and Weingarten formulas, we obtain

$$
\begin{aligned}
\left.\frac{d}{d t}\right|_{t=0}\left\langle f_{t_{*}} X, f_{t_{*}} Y\right\rangle & =\left\langle\nabla_{X} Z, Y\right\rangle-\left\langle A_{\eta} X, Y\right\rangle+\left\langle\nabla_{Y} Z, X\right\rangle-\left\langle X, A_{\eta} Y\right\rangle \\
& =-2\langle\alpha(X, Y), \eta\rangle+\left\langle\nabla_{X} Z, Y\right\rangle+\left\langle\nabla_{Y} Z, X\right\rangle,
\end{aligned}
$$

which proves (1.4).

### 1.2.1 The second fundamental form in Euclidean space

Let $M^{n}$ be a Riemannian manifold. The gradient of $g \in C^{\infty}(M)$ is the vector field $\operatorname{grad} g$ on $M^{n}$ given by

$$
\langle\operatorname{grad} g(x), X\rangle=X(g)
$$

for all $x \in M^{n}$ and $X \in T_{x} M$. The Hessian of $g$ is defined by

$$
\text { Hess } \begin{aligned}
g(X, Y) & =\left(\nabla_{X} d g\right) Y \\
& =X Y(g)-\nabla_{X} Y(g)
\end{aligned}
$$

for all $X, Y \in \mathfrak{X}(M)$.
Note that the Hessian is symmetric and that

$$
\text { Hess } g(X, Y)=\left\langle\nabla_{X} \operatorname{grad} g, Y\right\rangle
$$

which shows that it can be regarded as a section of $T^{*} M \otimes T^{*} M$.
For another useful interpretation of the second fundamental form of an isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$, we first prove the following general result that will also be needed later.

Proposition 1.2. Let $f: M^{n} \rightarrow \tilde{M}^{m}$ be an isometric immersion and let $g \in C^{\infty}(\tilde{M})$. Then the gradient and Hessian of $g$ and $h=g \circ f$ are related by

$$
\begin{equation*}
f_{*} \operatorname{gradh}=(\operatorname{gradg})^{T} \tag{1.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{Hess} h(X, Y)=\operatorname{Hess} g\left(f_{*} X, f_{*} Y\right)+\langle\operatorname{gradg}, \alpha(X, Y)\rangle \tag{1.6}
\end{equation*}
$$

for all $x \in M^{n}$ and $X, Y \in T_{x} M$.
Proof: We have

$$
\langle\operatorname{grad} h, X\rangle=X(h)=f_{*} X(g)=\left\langle\operatorname{grad} g, f_{*} X\right\rangle
$$

for all $x \in M^{n}$ and $X \in T_{x} M$, which proves (1.5). Using this and the Gauss formula we obtain

$$
\begin{aligned}
\operatorname{Hess} h(X, Y) & =\left\langle\nabla_{X} \operatorname{grad} h, Y\right\rangle \\
& =\left\langle\tilde{\nabla}_{X} f_{*} \operatorname{grad} h, f_{*} Y\right\rangle \\
& =\left\langle\tilde{\nabla}_{X} \operatorname{grad} g, f_{*} Y\right\rangle-\left\langle\tilde{\nabla}_{X}(\operatorname{grad} g)^{\perp}, f_{*} Y\right\rangle \\
& =\operatorname{Hess} g\left(f_{*} X, f_{*} Y\right)+\left\langle(\operatorname{grad} g)^{\perp}, \tilde{\nabla}_{X} f_{*} Y\right\rangle \\
& =\operatorname{Hess} g\left(f_{*} X, f_{*} Y\right)+\langle\operatorname{grad} g, \alpha(X, Y)\rangle
\end{aligned}
$$

for all $x \in M^{n}$ and $X, Y \in T_{x} M$.
Corollary 1.3. Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be an isometric immersion and let $h^{v} \in C^{\infty}(M)$ be the height function

$$
h^{v}(x)=\langle f(x), v\rangle
$$

with respect to the hyperplane normal to $v \in \mathbb{R}^{m}$. Then

$$
\begin{equation*}
\operatorname{Hessh}^{v}(x)(X, Y)=\langle\alpha(X, Y), v\rangle \tag{1.7}
\end{equation*}
$$

for all $x \in M^{n}$ and $X, Y \in T_{x} M$. Moreover, a point $x_{0} \in M^{n}$ is a critical point of $h^{v}$ if and only if $v \in N_{f} M\left(x_{0}\right)$.

Proof: Apply Proposition 1.2 to $g \in C^{\infty}\left(\mathbb{R}^{m}\right)$ given by

$$
g(x)=\langle x, v\rangle
$$

for which we have grad $g=v$ at any point and $\operatorname{Hess} g=0$. The last assertion follows easily by differentiating $h^{v}$.

A hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ is said to be locally convex at a point $x \in M^{n}$ if there exists a neighborhood $U$ of $x$ such that $f(U)$ lies on one side of the affine hyperplane that is tangent to $f$ at $x$. We say that $f$ is strictly locally convex at $x$ if, in addition, $f(x)$ is the unique point in

$$
f(U) \cap\left(f(x)+f_{*} T_{x} M\right)
$$

For instance, the sphere is strictly locally convex everywhere, but a cylinder over a plane curve is locally convex but not strictly locally convex at any point.

Corollary 1.4. If the second fundamental form of a hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ is definite at a point $x_{0} \in M^{n}$, then $f$ is strictly locally convex at $x_{0}$.

Proof: Let $\xi \in N_{f} M\left(x_{0}\right)$ and let $h^{\xi} \in C^{\infty}(M)$ be the height function

$$
h^{\xi}(x)=\left\langle f(x)-f\left(x_{0}\right), \xi\right\rangle .
$$

By the last assertion in Corollary 1.3, the point $x_{0}$ is a critical point of $h^{\xi}$. Moreover, since the second fundamental form of $f$ is definite at $x_{0}$, it follows from (1.7) that $h^{\xi}$ has a strict local maximum or minimum at $x_{0}$. Since $h^{\xi}$ vanishes at $x_{0}$, we conclude that it is either strictly positive or strictly negative in a neighborhood $U$ of $x_{0}$.

Another useful consequence of Proposition 1.2 is the following.
Corollary 1.5. Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be an isometric immersion. Given $q \in \mathbb{R}^{m}$, let $h \in C^{\infty}(M)$ be defined by

$$
h(x)=\frac{1}{2}\|f(x)-q\|^{2} .
$$

Then

$$
\operatorname{Hessh}(x)(X, Y)=\langle\alpha(X, Y), f(x)-q\rangle+\langle X, Y\rangle
$$

for all $x \in M^{n}$ and $X, Y \in T_{x} M$. Moreover, a point $x_{0} \in M^{n}$ is a critical point of $h$ if and only if $f\left(x_{0}\right)-q \in N_{f} M\left(x_{0}\right)$.

Proof: Apply Proposition 1.2 to $g \in C^{\infty}\left(\mathbb{R}^{m}\right)$ given by

$$
g(x)=\frac{1}{2}\|x-q\|^{2} .
$$

Then

$$
\operatorname{grad} g(x)=x-q
$$

and

$$
\text { Hess } g(x)(X, Y)=\langle X, Y\rangle
$$

for all $X, Y \in \mathbb{R}^{m}$. Differentiating $h$ yields the last assertion.
As an application of the preceding corollary, we prove the following extension of the fact that any compact surface in $\mathbb{R}^{3}$ has an elliptic point, that is, a point with positive Gaussian curvature.

Corollary 1.6. Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be an isometric immersion of a compact Riemannian manifold. Then there exist a point $x_{0} \in M^{n}$ and a normal vector $\xi \in N_{f} M\left(x_{0}\right)$ such that the shape operator $A_{\xi}$ is positive definite.

Proof: Let $x_{0} \in M^{n}$ be a point where

$$
h(x)=\frac{1}{2}\|f(x)\|^{2}
$$

attains its maximum. It follows from Corollary 1.5 that $f\left(x_{0}\right) \in N_{f} M\left(x_{0}\right)$. Moreover, for $\xi=-f\left(x_{0}\right)$ we have

$$
\begin{aligned}
\left\langle A_{\xi} X, X\right\rangle & =-\left\langle\alpha(X, X), f\left(x_{0}\right)\right\rangle \\
& =-\operatorname{Hess} h\left(x_{0}\right)(X, X)+\|X\|^{2} \\
& \geq\|X\|^{2}
\end{aligned}
$$

for all $X \in T_{x_{0}} M$.

### 1.2.2 The Gauss map of a Euclidean hypersurface

Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be an orientable hypersurface and let $\xi$ be a globally defined smooth unit normal vector field along $f$.

The Gauss map of $f$ is the map $\phi: M^{n} \rightarrow \mathbb{S}^{n}$ into the unit sphere whose value at $x \in M^{n}$ is $\xi_{x} \in N_{f} M(x)$.

Since the vector subspaces $f_{*} T_{x} M$ and $i_{*} T_{\phi(x)} \mathbb{S}^{n}$ both have $\phi(x)=\xi_{x}$ as a normal vector, where $i: \mathbb{S}^{n} \rightarrow \mathbb{R}^{n+1}$ is the inclusion map, we may identify them and write the Weingarten formula as

$$
\begin{equation*}
i_{*} \phi_{*}=-f_{*} A \tag{1.8}
\end{equation*}
$$

The Gauss-Kronecker curvature $\mathcal{K}$ of $f$ at $x \in M^{n}$ is defined as the determinant of its shape operator $A=A_{\xi_{x}}$.

Proposition 1.7. If $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ is a compact hypersurface, then the following assertions are equivalent:
(i) The second fundamental form is definite at every point of $M^{n}$.
(ii) $M^{n}$ is orientable and the Gauss map is a diffeomorphism.
(iii) The Gauss-Kronecker curvature $\mathcal{K}$ is nonzero at every point.

Proof: $(i) \Rightarrow(i i)$. First notice that, if $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ is any hypersurface and there exists at $x \in M^{n}$ a unit normal vector $\xi_{x}$ such that $A_{\xi_{x}}$ is negative definite, then one can extend $\xi_{x}$ to a smooth unit normal vector field $\xi$ in a neighborhood $V$ of $x$ such that $A_{\xi(y)}$ is negative definite for any $y \in V$. Simply choose an orthonormal basis $X_{1}, \ldots, X_{n}$ of $T_{x} M$ such that $\xi_{x}$ is the cross-product

$$
\xi_{x}=f_{*} X_{1} \times \cdots \times f_{*} X_{n}
$$

extend $X_{1}, \ldots, X_{n}$ to a smooth orthonormal frame in a neighborhood $U$ of $x$ and define

$$
\xi=f_{*} X_{1} \times \cdots \times f_{*} X_{n}
$$

on $U$. Then $\xi$ is a smooth unit normal vector field and $A_{\xi(y)}$ is also negative definite for any $y$ in a possibly smaller neighborhood $V \subset U$.

Now, under the assumption in part $(i)$, at every point $x \in M^{n}$ there exists a unique unit normal vector $\xi_{x}$ such that $A_{\xi_{x}}$ is negative definite. Therefore, assigning to each $x \in M^{n}$ such unique unit normal vector $\xi_{x}$ defines a global unit normal vector field on $M^{n}$. Uniqueness implies that it locally coincides with the smooth unit normal vector field constructed in the preceding paragraph. Therefore it is smooth, and hence $M^{n}$ is orientable.

Since $A=A_{\xi}$ is nonsingular, it follows from (1.8) that $\left(\phi_{*}\right)_{x}$ is injective for any $x \in M^{n}$. Hence $\phi$ is a local diffeomorphism. Actually, $\phi$ is a covering map because $M^{n}$ is compact. We conclude that $\phi$ is a diffeomorphism from the fact that $\mathbb{S}^{n}$ is simply connected for $n \geq 2$.
$(i i) \Rightarrow(i i i)$. Since $\phi$ is a diffeomorphism, its differential is everywhere nonsingular. We conclude from (1.8) that the Gauss-Kronecker curvature is nonzero at any point.
$($ iii $) \Rightarrow(i)$. Since the Gauss-Kronecker curvature is nowhere vanishing, the second fundamental form is everywhere nondegenerate. On the other hand, we know from Corollary 1.6 that there exist $x_{0} \in M^{n}$ and $\xi_{x_{0}} \in N_{f} M\left(x_{0}\right)$ such that $A_{\xi_{x_{0}}}$ is definite. It follows that the second fundamental form is definite at all points.

We say that an embedded hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ is a convex hypersurface when it is the boundary of a convex body $B \subset \mathbb{R}^{n+1}$. By a convex body we mean an open subset $B$ of $\mathbb{R}^{n+1}$ such that, for any pair of points $p, q \in B$, the line segment joining $p$ and $q$ is contained in $B$.

Theorem 1.8. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be a compact hypersurface satisfying any of the conditions in the preceding result. Then $f$ is a convex hypersurface.

Proof: First we prove that $f$ is an embedding. Since $M^{n}$ is compact, it suffices to show that $f$ is one-to-one. Suppose there exist $x_{1}, x_{2} \in M^{n}$ such that $f\left(x_{1}\right)=f\left(x_{2}\right)$. Choose a unit normal vector field $\xi$ such that $A_{\xi}$ is negative definite and consider the height function $h=h^{\xi\left(x_{1}\right)}: M^{n} \rightarrow \mathbb{R}$ given by

$$
h(x)=\left\langle f(x)-f\left(x_{1}\right), \xi\left(x_{1}\right)\right\rangle .
$$

Then $h\left(x_{1}\right)=0=h\left(x_{2}\right)$. It follows from Corollary 1.3 that $x_{1}$ is a strict local maximum of $h$. We claim that $x_{1}$ is, in fact, the unique strict local maximum of $h$, and hence the unique global maximum of $h$. For if $y \in M^{n}$ is a strict local maximum of $h$ then $\xi\left(x_{1}\right)= \pm \xi(y)$ by the last assertion in Corollary 1.3. whereas (1.7) implies that $A_{\xi\left(x_{1}\right)}$ is negative semi-definite (hence negative definite) at $y$. Thus $\xi(y)=\xi\left(x_{1}\right)$, that is, the Gauss map satisfies $\phi\left(x_{1}\right)=\phi(y)$. Therefore $x_{1}=y$ because $\phi$ is a diffeomorphism, and our claim is proved. From $h\left(x_{1}\right)=h\left(x_{2}\right)$ we obtain $x_{1}=x_{2}$.

Since $f$ is an embedding, it follows from the Jordan-Brower separation theorem that $f(M)$ divides $\mathbb{R}^{n+1}$ into two arcwise-connected components. Both components have $f(M)$ as boundary, and one of them, say, $B$, is bounded. We conclude the proof by showing that the interior $B^{o}$ of $B$ is a convex body.

Consider arbitrary points $p, q \in B^{o}$. There exist points $p=y_{0}, y_{1}, \ldots, y_{r}=q$ in $B^{o}$ such that the segments $\overline{y_{0} y_{1}}, \overline{y_{1} y_{2}}, \ldots, \overline{y_{r-1} y_{r}}$ form a polygonal path entirely contained
in $B^{o}$. We want to prove that the segment $\overline{p q}$ itself is contained in $B^{o}$. Suppose, by contradiction, that there exists some $1<j \leq r$ such that $\overline{p y_{i}} \subset B^{o}, 1 \leq i \leq j-1$, but $\overline{p y_{j}} \not \subset B^{o}$. Let $\beta:[0,1] \rightarrow B^{o}$ be given by

$$
\beta(s)=s y_{j}+(1-s) y_{j-1},
$$

and define $\alpha_{s}:[0,1] \rightarrow \mathbb{R}^{n+1}$ by

$$
\alpha_{s}(t)=t \beta(s)+(1-t) p
$$

Since $f(M)$ is closed, $z_{1}=\alpha_{s_{1}}\left(t_{1}\right) \in f(M)$, where

$$
s_{1}=\sup \left\{s \in[0,1]: \alpha_{s}([0,1]) \cap f(M)=\emptyset\right\}
$$

and

$$
t_{1}=\inf \left\{t \in[0,1]: \alpha_{s_{1}}([0, t]) \cap f(M) \neq \emptyset\right\} .
$$

Let $x_{1} \in M$ be such that $f\left(x_{1}\right)=z_{1}$. Choose a unit normal vector field $\xi$ such that $A_{\xi}$ is negative definite. From a previous argument, the function

$$
h(x)=\left\langle f(x)-f\left(x_{1}\right), \xi\left(x_{1}\right)\right\rangle
$$

has a unique global maximum which is $x_{1}$. On the other hand, by construction, $\xi\left(x_{1}\right)$ points inward, and this allows us to find $\lambda>0$ such that

$$
f\left(x_{1}\right)+\lambda \xi\left(x_{1}\right)=f\left(x_{2}\right) \in f(M)=\partial B
$$

since $f(M)$ is compact. Thus $h\left(x_{2}\right)=\lambda>0$, and this is a contradiction since $x_{1}$ is the maximum of $h$ and $h\left(x_{1}\right)=0$.

### 1.3 The Gauss, Codazzi and Ricci equations

Using the Gauss and Weingarten formulas, we derive the compatibility equations of an isometric immersion. In order to simplify the notation, we use the fact that any immersion $f: M^{n} \rightarrow \tilde{M}^{m}$ is locally an embedding to identify locally $M^{n}$ with $f(M)$ and regard $f$ as the inclusion map.

Let $R$ and $\tilde{R}$ denote the curvature tensors of $M^{n}$ and $\tilde{M}^{m}$, respectively. We first compute the tangent and normal components of $\tilde{R}(X, Y) Z$ for any vector fields $X, Y, Z \in \mathfrak{X}(M)$. By the Gauss and Weingarten formulas we have

$$
\begin{align*}
\tilde{\nabla}_{X} \tilde{\nabla}_{Y} Z & =\tilde{\nabla}_{X} \nabla_{Y} Z+\tilde{\nabla}_{X} \alpha(Y, Z) \\
& =\nabla_{X} \nabla_{Y} Z+\alpha\left(X, \nabla_{Y} Z\right)-A_{\alpha(Y, Z)} X+\nabla_{X}^{\perp} \alpha(Y, Z), \tag{1.9}
\end{align*}
$$

and similarly,

$$
\begin{equation*}
\tilde{\nabla}_{Y} \tilde{\nabla}_{X} Z=\nabla_{Y} \nabla_{X} Z+\alpha\left(Y, \nabla_{X} Z\right)-A_{\alpha(X, Z)} Y+\nabla_{Y}^{\perp} \alpha(X, Z) \tag{1.10}
\end{equation*}
$$

Also

$$
\begin{equation*}
\tilde{\nabla}_{[X, Y]} Z=\nabla_{[X, Y]} Z+\alpha([X, Y], Z) . \tag{1.11}
\end{equation*}
$$

Subtracting (1.10) and (1.11) from (1.9) and taking tangent components yield

$$
R(X, Y) Z=(\tilde{R}(X, Y) Z)^{T}+A_{\alpha(Y, Z)} X-A_{\alpha(X, Z)} Y
$$

known as the Gauss equation. Taking the inner product of both sides of the preceding equation with $W \in \mathfrak{X}(M)$ gives its equivalent form below.

## Gauss equation

$$
\langle R(X, Y) Z, W\rangle=\langle\tilde{R}(X, Y) Z, W\rangle+\langle\alpha(X, W), \alpha(Y, Z)\rangle-\langle\alpha(X, Z), \alpha(Y, W)\rangle
$$

Another equivalent way of writing the Gauss equation is

$$
K(X, Y)=\tilde{K}(X, Y)+\langle\alpha(X, X), \alpha(Y, Y)\rangle-\|\alpha(X, Y)\|^{2}
$$

where $K(X, Y)$ denotes the sectional curvature at $x \in M^{n}$ along the plane spanned by the orthonormal vectors $X, Y \in T_{x} M$, and similarly for $\tilde{K}(X, Y)$.

Computing in a similar way the normal component of $\tilde{R}(X, Y) Z$ gives the Codazzi equation of $f$, which can be written as

## Codazzi equation

$$
(\tilde{R}(X, Y) Z)^{\perp}=\left(\nabla_{X}^{\perp} \alpha\right)(Y, Z)-\left(\nabla_{Y}^{\perp} \alpha\right)(X, Z)
$$

Here

$$
\left(\nabla_{X}^{\perp} \alpha\right)(Y, Z)=\nabla_{X}^{\perp} \alpha(Y, Z)-\alpha\left(\nabla_{X} Y, Z\right)-\alpha\left(Y, \nabla_{X} Z\right)
$$

is the canonical connection on $\operatorname{Hom}^{2}\left(T M, T M ; N_{f} M\right)$.
Let $R^{\perp}$ denote the curvature tensor of the normal bundle $N_{f} M$, that is,

$$
R^{\perp}(X, Y) \xi=\nabla_{X}^{\perp} \nabla_{Y}^{\perp} \xi-\nabla_{Y}^{\perp} \nabla_{X}^{\perp} \xi-\nabla_{[X, Y]}^{\perp} \xi
$$

for all $X, Y \in \mathfrak{X}(M)$ and $\xi \in \Gamma\left(N_{f} M\right)$. Using the Gauss and Weingarten formulas to compute the normal component of $\tilde{R}(X, Y) \xi$ yields the Ricci equation

$$
\begin{equation*}
(\tilde{R}(X, Y) \xi)^{\perp}=R^{\perp}(X, Y) \xi-\alpha\left(X, A_{\xi} Y\right)+\alpha\left(A_{\xi} X, Y\right) \tag{1.12}
\end{equation*}
$$

Taking the inner product of both sides of (1.12) with $\eta \in \Gamma\left(N_{f} M\right)$ and denoting by

$$
\left[A_{\xi}, A_{\eta}\right]=A_{\xi} A_{\eta}-A_{\eta} A_{\xi}
$$

the bracket of $A_{\xi}$ and $A_{\eta}$ yields the following equivalent form of the Ricci equation.

## Ricci equation

$$
\left\langle R^{\perp}(X, Y) \xi, \eta\right\rangle=\langle\tilde{R}(X, Y) \xi, \eta\rangle+\left\langle\left[A_{\xi}, A_{\eta}\right] X, Y\right\rangle .
$$

On the other hand, if we calculate the tangent component of $\tilde{R}(X, Y) \xi$ we obtain

$$
\begin{equation*}
(\tilde{R}(X, Y) \xi)^{T}=\left(\nabla_{Y} A\right)(X, \xi)-\left(\nabla_{X} A\right)(Y, \xi) \tag{1.13}
\end{equation*}
$$

where

$$
\left(\nabla_{Y} A\right)(X, \xi)=\nabla_{Y} A_{\xi} X-A_{\xi} \nabla_{Y} X-A_{\nabla_{\frac{1}{Y} \xi}} X
$$

is the canonical connection on $\operatorname{Hom}^{2}\left(T M, N_{f} M ; T M\right)$. It is easily seen that 1.13 is just an equivalent form of the Codazzi equation.

If $\tilde{M}^{m}=\tilde{M}_{c}^{m}$ denotes a Riemannian manifold with constant sectional curvature $c$, then the Gauss equation becomes

$$
R(X, Y) Z=c(X \wedge Y) Z+A_{\alpha(Y, Z)} X-A_{\alpha(X, Z)} Y
$$

where

$$
(X \wedge Y) Z=\langle Y, Z\rangle X-\langle X, Z\rangle Y
$$

or equivalently,

$$
\begin{equation*}
\langle R(X, Y) Z, W\rangle=c\langle(X \wedge Y) Z, W\rangle+\langle\alpha(X, W), \alpha(Y, Z)\rangle-\langle\alpha(X, Z), \alpha(Y, W)\rangle . \tag{1.14}
\end{equation*}
$$

The Codazzi equation has now the two equivalent versions

$$
\begin{equation*}
\left(\nabla \frac{\perp}{X} \alpha\right)(Y, Z)=\left(\nabla \frac{\perp}{Y} \alpha\right)(X, Z) \tag{1.15}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\nabla_{X} A\right)(Y, \xi)=\left(\nabla_{Y} A\right)(X, \xi) \tag{1.16}
\end{equation*}
$$

whereas the Ricci equation reduces to

$$
\begin{equation*}
R^{\perp}(X, Y) \xi=\alpha\left(X, A_{\xi} Y\right)-\alpha\left(A_{\xi} X, Y\right) \tag{1.17}
\end{equation*}
$$

or equivalently, to

$$
\left\langle R^{\perp}(X, Y) \xi, \eta\right\rangle=\left\langle\left[A_{\xi}, A_{\eta}\right] X, Y\right\rangle
$$

For easy reference, we display together the fundamental equations derived in this section for an isometric immersion $f: M^{n} \rightarrow \tilde{M}_{c}^{m}$.

## Gauss equation

$$
\langle R(X, Y) Z, W\rangle=c\langle(X \wedge Y) Z, W\rangle+\langle\alpha(X, W), \alpha(Y, Z)\rangle-\langle\alpha(X, Z), \alpha(Y, W)\rangle .
$$

## Codazzi equation

$$
\left(\nabla_{X}^{\perp} \alpha\right)(Y, Z)=\left(\nabla_{Y}^{\perp} \alpha\right)(X, Z)
$$

## Ricci equation

$$
\left\langle R^{\perp}(X, Y) \xi, \eta\right\rangle=\left\langle\left[A_{\xi}, A_{\eta}\right] X, Y\right\rangle
$$

Remark 1.9. The theory developed so far can be extended with minor modifications to isometric immersions between semi-Riemannian manifolds, that is, differentiable manifolds endowed with an indefinite metric (cf. O'Neill [277]).

### 1.3.1 The Fundamental theorem of submanifolds

In most of this book we focus on isometric immersions $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ of a Riemannian manifold into one of the simply connected complete space forms $\mathbb{Q}_{c}^{m}$ with constant sectional curvature $c$, that is, Euclidean space $\mathbb{R}^{m}$, the sphere $\mathbb{S}_{c}^{m}$ or the hyperbolic space $\mathbb{H}_{c}^{m}$, according to whether $c=0, c>0$ or $c<0$, respectively. We write simply $\mathbb{S}^{m}$ and $\mathbb{H}^{m}$ when $c=1$ and $c=-1$, respectively.

For these ambient spaces, the compatibility equations derived in the previous section are intrinsic equations relating the curvature tensor of $M^{n}$, the second fundamental form of $f$ and the curvature tensor of the normal connection. Thus, it makes sense, and is a natural question, to ask whether any such data satisfying the compatibility equations on a vector bundle over a given Riemannian manifold $M^{n}$ can be realized as the data associated with an isometric immersion of $M^{n}$ into $\mathbb{Q}_{c}^{m}$.

The following fundamental result states that this is always true locally, and even globally whenever $M^{n}$ is simply connected. Moreover, regardless of the assumption that $M^{n}$ is simply connected, the isometric immersion is unique up to isometries of the ambient space. In other words, any other isometric immersion $g: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ must be congruent to $f$, that is, the composition $g=\tau \circ f$ of $f$ with an isometry $\tau$ : $\mathbb{Q}_{c}^{m} \rightarrow \mathbb{Q}_{c}^{m}$.

Theorem 1.10. (Fundamental theorem of submanifolds)
Existence: Let $M^{n}$ be a simply connected Riemannian manifold, let $\mathcal{E}$ be a Riemannian vector bundle of rank $p$ over $M^{n}$ with compatible connection $\nabla^{\mathcal{E}}$ and curvature tensor $R^{\varepsilon}$, and let $\alpha^{\varepsilon}$ be a symmetric section of $\operatorname{Hom}^{2}(T M, T M ; \mathcal{E})$. For each $\xi \in \Gamma(\mathcal{E})$, define $A_{\xi}^{\varepsilon} \in \Gamma(E n d(T M))$ by

$$
\left\langle A_{\xi}^{\varepsilon} X, Y\right\rangle=\left\langle\alpha^{\varepsilon}(X, Y), \xi\right\rangle .
$$

Assume that $\left(\nabla^{\varepsilon}, \alpha^{\varepsilon}, A^{\varepsilon}, R^{\varepsilon}\right)$ satisfies (1.14), 1.15) and (1.17). Then there exist an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ and a vector bundle isometry $\phi: \mathcal{E} \rightarrow N_{f} M$ such that

$$
\nabla^{\perp} \phi=\phi \nabla^{\varepsilon} \text { and } \alpha^{f}=\phi \circ \alpha^{\varepsilon} .
$$

Uniqueness: Let $f, g: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ be isometric immersions of a Riemannian manifold. Assume that there exists a vector bundle isometry $\phi: N_{f} M \rightarrow N_{g} M$ such that

$$
\phi^{f} \nabla^{\perp}={ }^{g} \nabla^{\perp} \phi \text { and } \phi \circ \alpha^{f}=\alpha^{g} .
$$

Then there exists an isometry $\tau: \mathbb{Q}_{c}^{n+p} \rightarrow \mathbb{Q}_{c}^{n+p}$ such that

$$
\tau \circ f=g \text { and }\left.\tau_{*}\right|_{N_{f} M}=\phi
$$

A proof of Theorem 1.10 is given in Sect. 1.10 .

### 1.4 The basic equations of a hypersurface

In the case of a hypersurface $f: M^{n} \rightarrow \tilde{M}^{n+1}$, the compatibility equations take a rather simpler form. Namely, choosing a local smooth unit normal vector field $\xi$ along $f$ and writing $A=A_{\xi}$, the Gauss equation can be written as

$$
(\tilde{R}(X, Y) Z)^{T}=R(X, Y) Z-(A X \wedge A Y) Z
$$

Equivalently,

$$
\langle R(X, Y) Z, W\rangle=\langle\tilde{R}(X, Y) Z, W\rangle+\langle A X, W\rangle\langle A Y, Z\rangle-\langle A X, Z\rangle\langle A Y, W\rangle
$$

or in terms of sectional curvatures,

$$
K(X, Y)=\tilde{K}(X, Y)+\langle A X, X\rangle\langle A Y, Y\rangle-\langle A X, Y\rangle^{2}
$$

The Codazzi equation becomes

$$
(\tilde{R}(X, Y) \xi)^{T}=\left(\nabla_{Y} A\right) X-\left(\nabla_{X} A\right) Y
$$

where

$$
\left(\nabla_{Y} A\right) X=\nabla_{Y} A X-A \nabla_{Y} X
$$

If $\tilde{M}^{n+1}=\tilde{M}_{c}^{n+1}$, then the equations reduce, respectively, to

## Gauss equation

$$
R(X, Y) Z=c(X \wedge Y) Z+(A X \wedge A Y) Z
$$

## Codazzi equation

$$
\left(\nabla_{Y} A\right) X=\left(\nabla_{X} A\right) Y
$$

### 1.4.1 The Fundamental theorem of hypersurfaces

The statement of the Fundamental theorem of submanifolds also simplifies considerably in the case of hypersurfaces. We first observe that, given an orientable Riemannian manifold $\tilde{M}^{n+1}$ and hypersurfaces $f, g: M^{n} \rightarrow \tilde{M}^{n+1}$, there is always a well-defined vector bundle isometry

$$
\phi: N_{f} M \rightarrow N_{g} M
$$

Namely, fixed an orientation of $\tilde{M}^{n+1}$, for each $x \in M^{n}$ choose an ordered basis $X_{1}, \ldots, X_{n} \in T_{x} M$ and a unit normal vector $\xi_{x} \in N_{f} M(x)$ such that

$$
f_{*} X_{1}, \ldots, f_{*} X_{n}, \xi_{x}
$$

is positively oriented in $T_{f(x)} \tilde{M}$. Let $\eta_{x} \in N_{g} M(x)$ be the unit normal vector such that the ordered basis

$$
g_{*} X_{1}, \ldots, g_{*} X_{n}, \eta_{x}
$$

is positively oriented in $T_{g(x)} \tilde{M}$. It suffices to define $\phi: N_{f} M \rightarrow N_{g} M$ as the bundle map that takes $\xi_{x}$ to $\eta_{x}$ for each $x \in M^{n}$. Clearly $\phi$ and $-\phi$ are the only such maps.

Theorem 1.11. Existence: Let $M^{n}$ be a simply connected Riemannian manifold, and let $A$ be a symmetric section of End(TM) satisfying the Gauss and Codazzi equations. Then there exist an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ and a unit normal vector field $\xi$ such that $A$ coincides with the shape operator $A_{\xi}$ of $f$ with respect to $\xi$.
Uniqueness: Let $f, g: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ be isometric immersions of a Riemannian manifold. Assume that

$$
\phi \circ \alpha^{f}=\alpha^{g}
$$

for one of the vector bundle isometries $\phi: N_{f} M \rightarrow N_{g} M$. Then there exists an isometry $\tau: \mathbb{Q}_{c}^{n+1} \rightarrow \mathbb{Q}_{c}^{n+1}$ such that

$$
\tau \circ f=g \text { and }\left.\tau_{*}\right|_{N_{f} M}=\phi .
$$

### 1.4.2 The principal curvatures

Let $f: M^{n} \rightarrow \tilde{M}^{n+1}$ be a hypersurface and let $\xi_{x}$ denote a unit normal vector at a point $x \in M^{n}$. The principal curvatures $\lambda_{1} \leq \lambda_{2} \leq \cdots \leq \lambda_{n}$ of $f$ at $x$ with respect to $\xi_{x}$ are defined as the eigenvalues of $A_{\xi_{x}}$. Any eigenvector of unit length of $A_{\xi_{x}}$ is called a principal direction of $f$ at $x$.

Assume that $M^{n}$ is orientable, and suppose that an orientation is fixed by a global smooth unit normal vector field $\xi$. Then one can show that the functions $x \in M^{n} \mapsto$ $\lambda_{i}(x)$ are continuous in $M^{n}$ for $1 \leq i \leq n$. In fact, this is true for the eigenvalues of any symmetric smooth tensor on $M^{n}$. Moreover, if $\lambda_{i}$ has constant multiplicity then it is smooth, and so is the distribution $E_{\lambda_{i}}$ given by its eigenspaces. For the proofs of these statements we refer to [73], [268] or [305].

In terms of an orthonormal frame $X_{1}, \ldots X_{n}$ of principal directions of $f$ at $x \in M^{n}$, the Gauss equation reads

$$
K\left(X_{i}, X_{j}\right)=\tilde{K}\left(X_{i}, X_{j}\right)+\lambda_{i} \lambda_{j}
$$

or simply

$$
\begin{equation*}
K\left(X_{i}, X_{j}\right)=c+\lambda_{i} \lambda_{j} \tag{1.18}
\end{equation*}
$$

if $\tilde{M}^{n+1}=\tilde{M}_{c}^{n+1}$.
On an open subset where the principal curvatures have constant multiplicities, the Codazzi equation of an isometric immersion $f: M^{n} \rightarrow \tilde{M}_{c}^{n+1}$ is equivalent to the set of equations

$$
\begin{gather*}
X_{i}\left(\lambda_{j}\right)=0, \quad \lambda_{i}=\lambda_{j}, \\
X_{k}\left(\lambda_{i}\right)\left\langle X_{i}, X_{j}\right\rangle=\left(\lambda_{i}-\lambda_{k}\right)\left\langle\nabla_{X_{i}} X_{j}, X_{k}\right\rangle, \quad \lambda_{i}=\lambda_{j}, \tag{1.19}
\end{gather*}
$$

and

$$
\begin{equation*}
\left(\lambda_{k}-\lambda_{j}\right)\left\langle\nabla_{X_{i}} X_{j}, X_{k}\right\rangle=\left(\lambda_{k}-\lambda_{i}\right)\left\langle\nabla_{X_{j}} X_{i}, X_{k}\right\rangle, \quad \lambda_{i} \neq \lambda_{j} \neq \lambda_{k} \neq \lambda_{i} . \tag{1.20}
\end{equation*}
$$

Given a hypersurface $f: M^{n} \rightarrow \tilde{M}^{n+1}$ with principal curvatures $\lambda_{1} \leq \ldots \leq \lambda_{n}$ at $x \in M^{n}$ with respect to a unit vector $\xi_{x} \in N_{f} M(x)$, the $r^{\text {th }}$-mean curvature $H_{r}$ of $f$ at $x$ is defined by

$$
\binom{n}{r} H_{r}(x)=S_{r}(x)
$$

where $S_{r}: M^{n} \rightarrow \mathbb{R}, 1 \leq r \leq n$, is given by

$$
S_{r}(x)=\sigma_{r}\left(\lambda_{1}(x), \ldots, \lambda_{n}(x)\right)
$$

in terms of the elementary symmetric function $\sigma_{r}: \mathbb{R}^{n} \rightarrow \mathbb{R}, 1 \leq r \leq n$,

$$
\sigma_{r}\left(x_{1}, \ldots, x_{n}\right)=\sum_{i_{1}<\cdots<i_{r}} x_{i_{1}} \cdots x_{i_{r}} .
$$

In particular, $H_{1}$ is the mean curvature of $f$ and $H_{n}$ its Gauss-Kronecker curvature.
In Exercise 1.9, the reader is asked to prove that $H_{r}$ is intrinsic if $r$ is even, and that the Gauss-Kronecker curvature is intrinsic if $n$ is even and intrinsic up to sign if $n$ is odd.

### 1.4.3 Holonomic hypersurfaces

A hypersurface $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ is called holonomic if $M^{n}$ carries a global system of orthogonal coordinates $\left(u_{1}, \ldots, u_{n}\right)$ such that at any point the coordinate vector fields $\partial / \partial u_{j}, 1 \leq j \leq n$, are eigenvectors of its shape operator. It is called locally holonomic if each point $x \in M^{n}$ lies in an open neighborhood $U \subset M^{n}$ where one can define such a system of orthogonal coordinates.

For a holonomic hypersurface $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$, oriented by a smooth unit normal vector field $N$, define $V_{j} \in C^{\infty}(M)$ by

$$
A \partial / \partial u_{j}=\frac{V_{j}}{v_{j}} \partial / \partial u_{j}, \quad 1 \leq j \leq n
$$

where $v_{j}=\left\|\partial / \partial u_{j}\right\|$ and $A=A_{N}$. Thus the induced metric and the second fundamental form of $f$ are given by

$$
\begin{equation*}
d s^{2}=\sum_{i=1}^{n} v_{i}^{2} d u_{i}^{2} \text { and } \alpha\left(\partial / \partial u_{i}, \partial / \partial u_{j}\right)=\delta_{i j} v_{i} V_{i} N \tag{1.21}
\end{equation*}
$$

We call $(v, V)$, with $v=\left(v_{1}, \ldots, v_{n}\right)$ and $V=\left(V_{1}, \ldots, V_{n}\right)$, the pair associated with $f$.

To compute the Gauss and Codazzi equations of $f$ in terms of $(v, V)$ we need the following elementary fact.

Proposition 1.12. Let $\left(u_{1}, \ldots, u_{n}\right)$ be local coordinates on a Riemannian manifold $M^{n}$ with respect to which the metric is given by

$$
d s^{2}=\sum_{i=1}^{n} v_{i}^{2} d u_{i}^{2}
$$

The following assertions hold:
(i) The Levi-Civita connection of $M^{n}$ satisfies

$$
\begin{equation*}
\nabla_{\partial / \partial u_{i}} X_{j}=h_{j i} X_{i}, \quad 1 \leq i \neq j \leq n, \tag{1.22}
\end{equation*}
$$

where $X_{j}=\left(1 / v_{j}\right) \partial / \partial u_{j}$ and

$$
\begin{equation*}
h_{j i}=\frac{1}{v_{j}} \frac{\partial v_{i}}{\partial u_{j}} . \tag{1.23}
\end{equation*}
$$

(ii) The curvature tensor of $M^{n}$ is given by

$$
\begin{equation*}
R\left(\partial / \partial u_{i}, \partial / \partial u_{j}\right) X_{k}=\left(\frac{\partial h_{k j}}{\partial u_{i}}-h_{k i} h_{i j}\right) X_{j}-\left(\frac{\partial h_{k i}}{\partial u_{j}}-h_{k j} h_{j i}\right) X_{i} \text { if } i \neq j \neq k \neq i \tag{1.24}
\end{equation*}
$$

and

$$
\begin{equation*}
-\left\langle R\left(\partial / \partial u_{i}, \partial / \partial u_{j}\right) X_{j}, X_{i}\right\rangle=\frac{\partial h_{i j}}{\partial u_{i}}+\frac{\partial h_{j i}}{\partial u_{j}}+\sum_{k \neq i, j} h_{k i} h_{k j} \text { if } i \neq j . \tag{1.25}
\end{equation*}
$$

Proof: Since $\left[\partial / \partial u_{i}, \partial / \partial u_{j}\right]=0$ for all $1 \leq i \neq j \leq n$, we have

$$
\begin{aligned}
\left\langle\nabla_{\partial / \partial u_{i}} \partial / \partial u_{j}, \partial / \partial u_{k}\right\rangle & =-\left\langle\nabla_{\partial / \partial u_{i}} \partial / \partial u_{k}, \partial / \partial u_{j}\right\rangle \\
& =-\left\langle\nabla_{\partial / \partial u_{k}} \partial / \partial u_{i}, \partial / \partial u_{j}\right\rangle \\
& =\left\langle\nabla_{\partial / \partial u_{k}} \partial / \partial u_{j}, \partial / \partial u_{i}\right\rangle \\
& =\left\langle\nabla_{\partial / \partial u_{j}} \partial / \partial u_{k}, \partial / \partial u_{i}\right\rangle \\
& =-\left\langle\nabla_{\partial / \partial u_{j}} \partial / \partial u_{i}, \partial / \partial u_{k}\right\rangle \\
& =-\left\langle\nabla_{\partial / \partial u_{i}} \partial / \partial u_{j}, \partial / \partial u_{k}\right\rangle .
\end{aligned}
$$

Hence

$$
\left\langle\nabla_{\partial / \partial u_{i}} X_{j}, X_{k}\right\rangle=0 \text { if } i \neq j \neq k \neq i .
$$

Then (1.22) follows from

$$
\begin{aligned}
v_{i} v_{j}\left\langle\nabla_{\partial / \partial u_{i}} X_{j}, X_{i}\right\rangle & =\left\langle\nabla_{\partial / \partial u_{i}} \partial / \partial u_{j}, \partial / \partial u_{i}\right\rangle \\
& =\left\langle\nabla_{\partial / \partial u_{j}} \partial / \partial u_{i}, \partial / \partial u_{i}\right\rangle \\
& =v_{i} \partial v_{i} / \partial u_{j}
\end{aligned}
$$

Now (1.24) and (1.25) follow easily using (1.22).

For a holonomic hypersurface $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$, the Codazzi equation

$$
\nabla_{\partial / \partial u_{i}} A \partial / \partial u_{j}=\nabla_{\partial / \partial u_{j}} A \partial / \partial u_{i}
$$

becomes

$$
\frac{\partial V_{i}}{\partial u_{j}}=h_{j i} V_{j}, \quad 1 \leq i \neq j \leq n
$$

In summary, the Gauss and Codazzi equations for $f$ reduce to the system of partial differential equations

$$
\left\{\begin{array}{l}
\text { (i) } \frac{\partial v_{i}}{\partial u_{j}}=h_{j i} v_{j}  \tag{1.26}\\
\text { (ii) } \frac{\partial h_{i k}}{\partial u_{j}}=h_{i j} h_{j k} \\
\text { (iii) } \frac{\partial h_{i j}}{\partial u_{i}}+\frac{\partial h_{j i}}{\partial u_{j}}+\sum_{k \neq i, j} h_{k i} h_{k j}+V_{i} V_{j}+c v_{i} v_{j}=0 \\
\text { (iv) } \frac{\partial V_{i}}{\partial u_{j}}=h_{j i} V_{j}, \quad 1 \leq i \neq j \neq k \neq i \leq n
\end{array}\right.
$$

Proposition 1.13. If $(v, h, V)$ is a solution of (1.26) on a simply connected open subset $U \subset \mathbb{R}^{n}$, with $v_{i} \neq 0$ everywhere for all $1 \leq i \leq n$, then there exists a holonomic hypersurface $f: U \rightarrow \mathbb{Q}_{c}^{n+1}$ whose induced metric and second fundamental form are given by (1.21).

Proof: Define a metric $d s^{2}$ on $U$ by the first formula in 1.21, and let $A$ be the symmetric tensor on $M^{n}=\left(U, d s^{2}\right)$ given by

$$
A \partial / \partial u_{j}=\frac{V_{j}}{v_{j}} \partial / \partial u_{j}, \quad 1 \leq j \leq n
$$

The Gauss and Codazzi equations for an isometric immersion into $\mathbb{Q}_{c}^{n+1}$ are then satisfied by virtue of (1.26), and the statement follows from Theorem 1.11.

### 1.5 Totally geodesic submanifolds

An isometric immersion $f: M^{n} \rightarrow \tilde{M}^{m}$ is said to be totally geodesic at $x \in M^{n}$ if the second fundamental form $\alpha$ of $f$ vanishes at $x$. If $\alpha$ is identically zero then $f$ is called a totally geodesic isometric immersion.

Clearly, an isometric immersion $f: M^{n} \rightarrow \tilde{M}^{m}$ is totally geodesic if and only if $f \circ \gamma$ is a geodesic of $\tilde{M}^{m}$ for any geodesic $\gamma$ of $M^{n}$. Equivalently, for any $x \in M^{n}$ and any $X \in T_{x} M$ the geodesic in $\tilde{M}^{m}$ through $f(x)$ tangent to $f_{*} X$ coincides with the image by $f$ of the geodesic of $M^{n}$ through $x$ tangent to $X$ in the domain of definition of the latter. Therefore, if $f$ is totally geodesic, then $f(M)$ locally coincides with $\exp _{f(x)} f_{*} T_{x} M$ for all $x \in M^{n}$, where exp denotes the exponential map of $\tilde{M}^{m}$.

Another elementary characterization of the totally geodesic isometric immersions is as follows: an isometric immersion $f: M^{n} \rightarrow \tilde{M}^{m}$ is totally geodesic if and only if $f_{*} T M$ is a parallel subbundle of $f^{*} T M$.

Given a Riemannian manifold $\tilde{M}^{m}$ and a vector subspace $V \subset T_{x} \tilde{M}$ at $x \in \tilde{M}^{m}$, we say that the curvature tensor $\tilde{R}$ of $\tilde{M}^{m}$ preserves $V$ if

$$
\tilde{R}(X, Y) Z \in V \text { for all } X, Y, Z \in V
$$

It follows immediately from the Codazzi equation that $\tilde{R}$ preserves $V$ if $V$ is the tangent space at a point $x$ of a totally geodesic submanifold of $\tilde{M}^{m}$. Moreover, from the observation in the preceding paragraph it follows that $\tilde{R}$ must also preserve the parallel translate of $V$ along a sufficiently small piece of any geodesic through $x$ tangent to $V$. The next result shows that this condition is also sufficient for a vector subspace $V$ of $T_{x} \tilde{M}$ to be the tangent space at $x$ of a totally geodesic submanifold of $\tilde{M}^{m}$.

Theorem 1.14. Given a Riemannian manifold $\tilde{M}$ and a vector subspace $V \subset T_{x} \tilde{M}$ at a point $x \in \tilde{M}$, there exists a totally geodesic submanifold $M$ of $\tilde{M}$ such that $x \in M$ and $T_{x} M=V$ if and only if there exists $\epsilon>0$ such that, for every unit speed geodesic $\gamma$ in $\tilde{M}$ with $\gamma(0)=x$ and $\gamma^{\prime}(0) \in V$, the Riemannian curvature tensor of $\tilde{M}$ preserves the parallel translate of $V$ along $\gamma$ from $x$ to $\gamma(s)$ for every $s \in(0, \epsilon)$.

Theorem 1.14 has the following immediate consequence for locally symmetric spaces.

Corollary 1.15. Let $\tilde{M}^{m}$ be a locally symmetric space and let $V$ be a subspace of $T_{x} \tilde{M}$ that is preserved by the Riemannian curvature tensor of $\tilde{M}^{m}$. Then, for small $\epsilon>0$, $\exp _{x}\left(V \cap B_{\epsilon}(0)\right)$ is a totally geodesic submanifold of $\tilde{M}^{m}$.

A Riemannian manifold $\tilde{M}^{m}, m \geq 3$, is said to satisfy the axiom of $r$-planes, for some fixed $2 \leq r \leq m-1$, if for every $x \in \tilde{M}^{m}$ and every $r$-dimensional vector subspace $V \subset T_{x} \tilde{M}$ there exists a totally geodesic submanifold through $x$ whose tangent space at $x$ is $V$.

Riemannian manifolds that satisfy the axiom of $r$-planes are characterized in the next result.

Theorem 1.16. If a Riemannian manifold $\tilde{M}^{m}$ satisfies the axiom of r-planes for some $2 \leq r \leq m-1$ then it has constant sectional curvature. Conversely, any Riemannian manifold with constant sectional curvature satisfies the axiom of r-planes for all $2 \leq r \leq m-1$.

The proof relies on the following lemma.
Lemma 1.17. At any point $x$ of a Riemannian manifold $\tilde{M}^{m}$ of dimension $m \geq 3$ the following assertions are equivalent:
(i) There exists $2 \leq r \leq m-1$ such that the curvature tensor $\tilde{R}$ preserves every $r$-dimensional subspace $V \subset T_{x} \tilde{M}$.
(ii) $\langle\tilde{R}(X, Y) Z, X\rangle=0$ for all orthonormal vectors $X, Y, Z \in T_{x} \tilde{M}$.
(iii) All sectional curvatures of $\tilde{M}^{m}$ at $x$ are equal.
(iv) $\tilde{R}$ preserves every subspace $V \subset T_{x} \tilde{M}$.

Proof: $(i) \Rightarrow(i i)$. Take orthonormal vectors $X, Y, Z \in T_{x} \tilde{M}$. Then there exists an $r$-dimensional subspace $V \subset T_{x} \tilde{M}$ such that $X, Y \in V$ and $Z \in V^{\perp}$. Since $\tilde{R}$ preserves $V$, then

$$
\langle\tilde{R}(X, Y) Z, X\rangle=-\langle\tilde{R}(X, Y) X, Z\rangle=0
$$

$(i i) \Rightarrow(i i i)$. Given orthonormal vectors $X, Y, Z \in T_{x} \tilde{M}$, the vectors

$$
X, Y^{\prime}=\frac{1}{\sqrt{2}}(Y+Z), Z^{\prime}=\frac{1}{\sqrt{2}}(Y-Z)
$$

are also orthonormal, and hence

$$
0=\left\langle\tilde{R}\left(X, Y^{\prime}\right) Z^{\prime}, X\right\rangle=\frac{1}{2}(\tilde{K}(X, Y)-\tilde{K}(X, Z))
$$

Thus the sectional curvatures of any two planes that intersect orthogonally are equal. To conclude, observe that for any two planes there is always a third one that intersects both orthogonally.
$(i i i) \Rightarrow(i v)$. Since

$$
\tilde{R}(X, Y) Z=c(\langle Y, Z\rangle X-\langle X, Z\rangle Y)
$$

for all $X, Y, Z \in T_{x} M$, then $\tilde{R}$ preserves every subspace $V \subset T_{x} \tilde{M}$.
Proof of Theorem 1.16; Suppose that the Riemannian manifold $\tilde{M}^{m}$ satisfies the axiom of $r$-planes for some $2 \leq r \leq m-1$. By Schur's lemma, it is enough to show that at each $x \in \tilde{M}^{m}$ all the sectional curvatures of $\tilde{M}^{m}$ are equal. By the assumption, for every $r$-dimensional vector subspace $V \subset T_{x} \tilde{M}$ there exists a totally geodesic submanifold $M^{r}$ of $\tilde{M}^{m}$ through $x$ such that $T_{x} M=V$. Hence $\tilde{R}$ preserves $V$ by the Codazzi equation, and the conclusion follows from Lemma 1.17.

Conversely, assume that $\tilde{M}^{m}$ has constant sectional curvature. Given $x \in \tilde{M}^{m}$, $2 \leq r \leq m-1$ and any $r$-dimensional vector subspace $V \subset T_{x} \tilde{M}$, the Riemannian curvature tensor $\tilde{R}$ of $\tilde{M}^{m}$ preserves $V$ by Lemma 1.17, and the conclusion follows from Corollary 1.15.

An explicit description of all totally geodesic submanifolds of $\mathbb{Q}_{c}^{m}$ follows from Proposition 1.20 below.

### 1.6 The relative nullity distribution

Let $f: M^{n} \rightarrow \tilde{M}^{m}$ be an isometric immersion. The relative nullity tangent subspace $\Delta(x)$ of $f$ at $x$ is the kernel of its second fundamental form at $x$, that is,

$$
\Delta(x)=\left\{X \in T_{x} M: \alpha(X, Y)=0 \text { for all } Y \in T_{x} M\right\}
$$

Equivalently,

$$
\Delta(x)=\cap_{\xi \in N_{f} M(x)} \operatorname{ker} A_{\xi} .
$$

The dimension $\nu(x)$ of $\Delta(x)$ is called the index of relative nullity of $f$ at $x$.
A smooth distribution $E$ on a Riemannian manifold $M^{n}$ is totally geodesic if $\nabla_{T} S \in \Gamma(E)$ whenever $T, S \in \Gamma(E)$. A totally geodesic distribution is always integrable and its leaves are totally geodesic submanifolds of $M^{n}$ (see Exercise 1.12).

Proposition 1.18. For an isometric immersion $f: M^{n} \rightarrow \tilde{M}^{m}$, the following assertions hold:
(i) The index of relative nullity $\nu$ is upper semicontinuous. In particular, the subset

$$
M_{0}=\left\{x \in M^{n}: \nu(x)=\nu_{0}\right\}
$$

where $\nu$ attains its minimum value $\nu_{0}$ is open.
(ii) The relative nullity distribution $x \mapsto \Delta(x)$ is smooth on any open subset of $M^{n}$ where $\nu$ is constant.
(iii) If $\tilde{M}^{m}$ has constant sectional curvature $c$ and $U \subset M^{n}$ is an open subset where $\nu$ is constant, then $\Delta$ is a totally geodesic (hence integrable) distribution on $U$ and the restriction of $f$ to each leaf is totally geodesic.

Proof: First notice that

$$
\Delta^{\perp}(x)=\operatorname{span}\left\{A_{\xi} X: X \in T_{x} M, \xi \in N_{f} M(x)\right\}
$$

for any $x \in M^{n}$. Therefore, if $x_{0} \in M^{n}$ is such that $\nu\left(x_{0}\right)=k$, then there exist vectors $X_{1}, \ldots, X_{n-k} \in T_{x_{0}} M$ and $\xi_{1}, \ldots, \xi_{n-k} \in N_{f} M\left(x_{0}\right)$ such that

$$
\Delta^{\perp}\left(x_{0}\right)=\operatorname{span}\left\{A_{\xi_{j}} X_{j}\right\}_{1 \leq j \leq n-k}
$$

Take smooth extensions of $X_{1}, \ldots, X_{n-k}$ and $\xi_{1}, \ldots, \xi_{n-k}$ to a neighborhood of $x_{0}$. By continuity, the vector fields

$$
\left\{A_{\xi_{j}} X_{j}, 1 \leq j \leq n-k\right\}
$$

remain linearly independent in a possibly smaller neighborhood of $x_{0}$. This yields (i) and implies that $\Delta^{\perp}$, and hence $\Delta$, is a smooth distribution on any open subset containing $x_{0}$ where $\nu=k$.

Now we prove (iii). We have

$$
\begin{aligned}
\left(\nabla_{Z}^{\perp} \alpha\right)(X, Y) & =\nabla_{Z}^{\perp} \alpha(X, Y)-\alpha\left(\nabla_{Z} X, Y\right)-\alpha\left(X, \nabla_{Z} Y\right) \\
& =0
\end{aligned}
$$

for all $X, Y \in \Gamma(\Delta)$ and $Z \in \mathfrak{X}(U)$. Using the Codazzi equation, we obtain

$$
0=\left(\nabla_{X}^{\perp} \alpha\right)(Z, Y)=-\alpha\left(Z, \nabla_{X} Y\right)
$$

Thus $\nabla_{X} Y \in \Gamma(\Delta)$. This implies that $\Delta$ is involutive in $U$ with totally geodesic leaves. Finally,

$$
\begin{aligned}
\tilde{\nabla}_{X} f_{*} Y & =f_{*} \nabla_{X} Y+\alpha(X, Y) \\
& =f_{*} \nabla_{X} Y \in f_{*} \Delta
\end{aligned}
$$

for all $X, Y \in \Gamma(\Delta)$, hence the restriction of $f$ to each leaf of $\Delta$ is totally geodesic.

### 1.7 Umbilical submanifolds

An isometric immersion $f: M^{n} \rightarrow \tilde{M}^{m}$ is said to be umbilical at $x \in M^{n}$ if there exists $\eta \in N_{f} M(x)$ such that

$$
\alpha(X, Y)=\langle X, Y\rangle \eta
$$

for all $X, Y \in T_{x} M$. Clearly, in this case $\eta$ is the mean curvature vector $\mathcal{H}(x)$ of $f$ at $x$. Equivalently, $f$ is umbilical at $x$ if

$$
A_{\xi}=\langle\mathcal{H}(x), \xi\rangle I
$$

for every $\xi \in N_{f} M(x)$. A submanifold is called umbilical if it is umbilical at every point.

An isometric immersion $f: M^{n} \rightarrow \tilde{M}^{m}$ is said to have parallel mean curvature vector field if

$$
\nabla_{X}^{\perp} \mathcal{H}=0
$$

for all $x \in M^{n}$ and $X \in T_{x} M$. In particular, if $f$ has parallel mean curvature vector field, then $\|\mathcal{H}\|$ is constant along $M^{n}$. If $f$ is umbilical and has parallel mean curvature vector field, then it is called an extrinsic sphere.

An isometric immersion $f: M^{n} \rightarrow \tilde{M}^{m}$ is said to have flat normal bundle at $x \in M^{n}$ if the curvature tensor of the normal bundle vanishes at $x$. If the latter condition holds at any $x \in M^{n}$, then one just says that $f$ has flat normal bundle.

We first prove the following preliminary fact and then give a complete description of the umbilical (in particular, totally geodesic) submanifolds of space forms.

Proposition 1.19. Let $f: M^{n} \rightarrow \tilde{M}_{c}^{m}, n \geq 2$, be an umbilical isometric immersion. Then $f$ has parallel mean curvature vector field $\mathcal{H}$ and flat normal bundle. Moreover, $M^{n}$ has constant sectional curvature $c+\|\mathcal{H}\|^{2}$.

Proof: We have

$$
\begin{aligned}
\left(\nabla_{X}^{\perp} \alpha\right)(Y, Z) & =\nabla_{X}^{\perp} \alpha(Y, Z)-\alpha\left(\nabla_{X} Y, Z\right)-\alpha\left(Y, \nabla_{X} Z\right) \\
& =X\langle Y, Z\rangle \mathcal{H}+\langle Y, Z\rangle \nabla_{X}^{\perp} \mathcal{H}-\left\langle\nabla_{X} Y, Z\right\rangle \mathcal{H}-\left\langle Y, \nabla_{X} Z\right\rangle \mathcal{H} \\
& =\langle Y, Z\rangle \nabla_{X}^{\perp} \mathcal{H}
\end{aligned}
$$

for all $X, Y, Z \in \mathfrak{X}(M)$. From the Codazzi equation we obtain

$$
\langle Y, Z\rangle \nabla_{X}^{\perp} \mathcal{H}=\langle X, Z\rangle \nabla \nabla_{Y}^{\perp} \mathcal{H} .
$$

Choosing $Y=Z$ orthogonal to $X$, it follows that $\mathcal{H}$ is parallel. On the other hand, the Ricci equation gives

$$
\begin{aligned}
R^{\perp}(X, Y) \xi & =\alpha\left(X, A_{\xi} Y\right)-\alpha\left(A_{\xi} X, Y\right) \\
& =\left\langle X, A_{\xi} Y\right\rangle \mathcal{H}-\left\langle A_{\xi} X, Y\right\rangle \mathcal{H} \\
& =0
\end{aligned}
$$

for all $X, Y \in \mathfrak{X}(M)$ and $\xi \in \Gamma\left(N_{f} M\right)$. The last assertion now follows easily from the Gauss equation.

In the next result, we use the standard model of $\mathbb{Q}_{\tilde{c}}^{m}, \tilde{c} \neq 0$, as

$$
\mathbb{Q}_{\tilde{c}}^{m}=\left\{\left(x_{0}, \ldots, x_{m}\right) \in \mathbb{E}^{m+1}: \epsilon x_{0}^{2}+x_{1}^{2}+\cdots+x_{m}^{2}=1 / \tilde{c}\right\} \quad\left(x_{0}>0 \text { if } \tilde{c}<0\right),
$$

where $\epsilon=\tilde{c} /|\tilde{c}|$ (see Exercise 1.14). Here $\mathbb{E}^{m+1}$ stands for either Euclidean space $\mathbb{R}^{m+1}$ or Lorentzian space $\mathbb{L}^{m+1}$, according to whether $\tilde{c}>0$ or $\tilde{c}<0$, respectively, and $\left(x_{0}, \ldots, x_{m}\right)$ are the standard coordinates on $\mathbb{E}^{m+1}$ with respect to which the flat metric is written as

$$
d s^{2}=\epsilon d x_{0}^{2}+d x_{1}^{2}+\cdots+d x_{m}^{2}
$$

A subspace $W$ of $\mathbb{L}^{m+1}$ is said to be degenerate if $W \cap W^{\perp} \neq\{0\}$ (in which case $W \cap W^{\perp}$ is necessarily one-dimensional). Otherwise, it is called space-like or time-like, depending on whether the induced inner product on $W$ is positive definite or Lorentzian, respectively. Accordingly, a vector $v \in \mathbb{L}^{m+1}$ is said to be light-like (respectively, space-like or time-like) if $\langle v, v\rangle$ is zero (respectively, $\langle v, v\rangle$ is positive or negative).

We point out that in this book an inner product $\langle$,$\rangle on a real vector space V$ is assumed to be a nondegenerate symmetric bilinear form (not necessarily positivedefinite). The signature of the inner product is denoted by ( $p, q$ ), meaning that $p$ (respectively, $q$ ) is the maximal dimension of a subspace restricted to which the inner product is positive definite (respectively, negative definite). The integer $q$ is called the index of the inner product. Thus, a Lorentzian inner product on a vector space of dimension $n$ has index 1 and signature ( $n-1,1$ ). Given an orthogonal basis $v_{1}, \ldots, v_{n}$ of $V$, sometimes it will be convenient to refer to the ordered n-tuple $\left(\epsilon_{1}, \ldots, \epsilon_{n}\right)$, with $\epsilon_{i}=\left\langle v_{i}, v_{i}\right\rangle$ for all $1 \leq i \leq n$, as the signature of $\langle$,$\rangle .$

Proposition 1.20. Let $\bar{x} \in \mathbb{Q}_{\bar{c}}^{m}$ and let $V$ be a proper vector subspace of $T_{\bar{x}} \mathbb{Q}_{\bar{c}}^{m}$ such that $n=\operatorname{dim} V \geq 1$. If $z \in T_{\bar{x}} \mathbb{Q}_{\bar{c}}^{m}$ is orthogonal to $V$, then there exists exactly one $n$-dimensional complete extrinsic sphere $S$ in $\mathbb{Q}_{\tilde{c}}^{m}$ with $\bar{x} \in S$ and $T_{\bar{x}} S=V$ whose mean curvature vector at $\bar{x}$ is $z$. The submanifold $S$ is isometric to $\mathbb{Q}_{c}^{n}$, where $c=\tilde{c}+\|z\|^{2}$, and is totally geodesic if and only if $z=0$. If we denote

$$
a=\tilde{c} \bar{x}-z \text { and } W=\mathbb{R} a \oplus V
$$

then $S$ is explicitly given as follows:
(i) If $\mathbb{Q}_{\bar{c}}^{m}=\mathbb{R}^{m}$, then $S$ is either the affine space

$$
\bar{x}+W, \quad \text { if } c=0
$$

or the sphere

$$
\bar{x}-\frac{1}{c} a+\left\{x \in W:\|x\|^{2}=1 / c\right\}, \text { if } c>0 .
$$

(ii) If $\mathbb{Q}_{\tilde{c}}^{m}=\mathbb{S}_{\tilde{c}}^{m}$, then $S$ is the sphere

$$
S=\mathbb{S}_{\tilde{c}}^{m} \cap(\bar{x}+W)=\bar{x}-\frac{1}{c} a+\left\{x \in W:\|x\|^{2}=1 / c\right\} .
$$

(iii) If $\mathbb{Q}_{\bar{c}}^{m}=\mathbb{H}_{\tilde{c}}^{m}$, then $S=\mathbb{H}_{\tilde{c}}^{m} \cap(\bar{x}+W)$.

If $c>0$ (that is, if $a$, and hence $W$, is space-like), then $S$ is the sphere

$$
S=\bar{x}-\frac{1}{c} a+\left\{x \in W:\|x\|^{2}=1 / c\right\} .
$$

If $c<0$ (that is, if $a$, and hence $W$, is time-like), then $S$ is the hyperbolic space

$$
S=\bar{x}-\frac{1}{c} a+\left\{x \in W:\|x\|^{2}=1 / c,\langle a, x\rangle>0\right\} .
$$

If $c=0$ (that is, if $a$, and hence $W$, is degenerate), then

$$
S=\bar{x}+\left\{-(1 / 2)\|x\|^{2} a+x: x \in V\right\} .
$$

In this case, the map $I: V \rightarrow S$ given by

$$
x \mapsto \bar{x}-\frac{1}{2}\|x\|^{2} a+x
$$

is an isometry of the space-like vector subspace $V$ onto $S$.
Conversely, if $f: M^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{m}, n \geq 2$, is an umbilical isometric immersion then $f(M)$ is an open subset of such an extrinsic sphere $S$.

Proof: We leave as an exercise to the reader to show that each of the submanifolds $S$ in the statement is an extrinsic sphere of $\mathbb{Q}_{\tilde{c}}^{m}$. We give a proof of the converse as an application of Theorem 1.25 .

Let $f: M^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{m}$ be an umbilical isometric immersion. Fix $x \in M^{n}$, and set $\bar{x}=f(x), V=f_{*} T_{x} M$ and $z=\mathcal{H}(x)$. We will show that $f(M)$ is an open subset of the extrinsic sphere $S$ in $\mathbb{Q}_{\bar{c}}^{m}$ determined by $(\bar{x}, V, z)$.

Since $M^{n}$ has constant sectional curvature $c=\tilde{c}+\|z\|^{2}$ by Proposition 1.19, there exists an isometry $\psi: U \subset M^{n} \rightarrow \bar{U}$ of an open simply connected neighborhood $U$ of $x$ onto an open neighborhood $\bar{U} \subset S$ of $\bar{x}$ such that

$$
\psi(x)=\bar{x} \text { and } i_{*}(\bar{x}) \circ \psi_{*}(x)=f_{*}(x)
$$

where $i: S \rightarrow \mathbb{Q}_{\tilde{c}}^{m}$ denotes the inclusion map. We claim that $\left.f\right|_{U}=\tilde{i}=i \circ \psi$.
The mean curvature vector field $\mathcal{H}$ of $f$ being parallel, it has constant length along $M^{n}$, hence it is either identically zero or nowhere vanishing. We argue for the latter case, the former being easier. Since $\mathcal{H}$ is parallel, so is the subbundle $\mathcal{H}^{\perp}$ of $N_{f} M$. The same holds for the subbundle $\mathcal{H}_{\tilde{i}}^{\perp}$ of the normal bundle of $\tilde{i}$, where $\mathcal{H}_{\tilde{i}}=\mathcal{H}_{i} \circ \psi$ and $\mathcal{H}_{i}$ stands for the mean curvature vector field of $i$. Choose an orthonormal basis of $\mathcal{H}^{\perp}(x)=z^{\perp}=\mathcal{H}_{\dot{i}}^{\perp}(x)$. Since the normal bundle of $f$ is flat, we can extend it to parallel orthonormal frames $\xi_{1}, \ldots, \xi_{m-n-1}$ and $\zeta_{1}, \ldots, \zeta_{m-n-1}$ of $\mathcal{H}^{\perp}$ and $\mathcal{H}_{i}^{\perp}$ along $U$, respectively. Define a vector bundle isometry $\phi: N_{\tilde{i}} U \rightarrow N_{\left.f\right|_{U}} U$ by sending $\mathcal{H}_{\tilde{i}}$ to $\mathcal{H}$ and $\zeta_{j}$ to $\xi_{j}, 1 \leq j \leq m-n-1$. Then it is immediate to verify that $\phi$ preserves the normal connections and the second fundamental forms. It follows from Theorem 1.10 that

$$
\left.f\right|_{U}=\Phi \circ \tilde{i}
$$

for some isometry $\Phi$ of $\mathbb{Q}_{\tilde{c}}^{m}$, with $\left.\Phi_{*}\right|_{N_{\bar{\imath}} U}=\phi$. Since $f_{*}(x)$ coincides with $\tilde{i}_{*}(x)$, the differential $\Phi_{*}(\bar{x})$ acts as the identity map $I$ on $V=f_{*} T_{x} M$. From

$$
\left.\Phi_{*}(\bar{x})\right|_{N_{f} M(x)}=\left.\phi\right|_{N_{f} M(x)}=I
$$

and

$$
\Phi(\bar{x})=\Phi(\tilde{i}(x))=f(x)=\bar{x}
$$

we conclude that $\Phi=I$, and hence $\left.f\right|_{U}=\tilde{i}=i \circ \psi$. It follows that $f(U)=\bar{U} \subset S$.
We have shown that for each $x \in M^{n}$ there exist an open neighborhood $U_{x} \subset M^{n}$ of $x$ and an extrinsic sphere $S_{x}$ of dimension $n$ in $\mathbb{Q}_{\tilde{c}}^{m}$ such that $f\left(U_{x}\right)$ is an open subset of $S_{x}$. The proof is completed by applying Exercise 1.20 to the family of extrinsic spheres $S$ in the statement.

A Riemannian manifold $M^{n}, n \geq 3$, is said to satisfy the axiom of $r$-spheres, for some fixed $r \geq 2$, if for every $x \in M^{n}$ and every $r$-dimensional subspace $V \subset T_{x} M$ there exists an extrinsic sphere through $x$ whose tangent space at $x$ is $V$.

It follows from Proposition 1.20 that any Riemannian manifold with constant sectional curvature satisfies the axiom of $r$-spheres for every $r \geq 2$. The following generalization of Theorem 1.16 states that the converse is also true.

Theorem 1.21. If a Riemannian manifold $M^{n}, n \geq 3$, satisfies the axiom of $r$-spheres for some $2 \leq r \leq n-1$, then it has constant sectional curvature.

Proof: It is entirely analogous to the proof of Theorem 1.16 in view of part (ii) of Exercise 1.30 .

### 1.8 Principal normals

Let $f: M^{n} \rightarrow \tilde{M}^{m}$ be an isometric immersion. A vector $\eta \in N_{f} M(x)$ at $x \in M^{n}$ is called a principal normal of $f$ at $x$ if the subspace

$$
E_{\eta}(x)=\left\{T \in T_{x} M: \alpha(T, X)=\langle T, X\rangle \eta \text { for all } X \in T_{x} M\right\}
$$

is nontrivial. A normal vector field $\eta \in \Gamma\left(N_{f} M\right)$ is called a principal normal vector field of $f$ with multiplicity $q>0$ if $E_{\eta}(x)$ has dimension $q$ at any point $x \in M^{n}$.

Notice that

$$
\begin{equation*}
E_{\eta}(x)=\cap_{\gamma \in N_{f} M(x)} \operatorname{ker}\left(A_{\gamma}-\langle\gamma, \eta\rangle I\right) \tag{1.27}
\end{equation*}
$$

In particular, if $m=n+1$ and $\xi \in N_{f} M(x)$ is a unit normal vector at $x$, then $\eta=\lambda \xi$ is a principal normal at $x$ if and only if $\lambda$ is a principal curvature of $f$ at $x$. In this way, principal normals are natural generalizations to submanifolds of higher codimension of principal curvatures of hypersurfaces.

A principal normal vector field $\eta \in \Gamma\left(N_{f} M\right)$ is said to be a Dupin principal normal vector field if $\eta$ is parallel in the normal connection along $E_{\eta}$. Accordingly, a principal curvature $\lambda$ with constant multiplicity of a hypersurface is said to be a Dupin principal curvature if it is constant along the corresponding eigenbundle.

A smooth distribution $E$ on a Riemannian manifold $M^{n}$ is called umbilical if there exists a smooth section $\delta$ of $E^{\perp}$, named the mean curvature vector field of $E$, such that

$$
\left\langle\nabla_{T} S, X\right\rangle=\langle T, S\rangle\langle\delta, X\rangle
$$

for all $T, S \in \Gamma(E)$ and $X \in \Gamma\left(E^{\perp}\right)$.
An umbilical distribution on $M^{n}$ is always integrable and its leaves are umbilical submanifolds of $M^{n}$ (see Exercise 1.12).

The umbilical distribution $E$ is said to be spherical if also

$$
\left(\nabla_{T} \delta\right)_{E^{\perp}}=0
$$

for all $T \in \Gamma(E)$. Then the leaves of $E$ are extrinsic spheres in $M^{n}$ (see Exercise 1.12).
Proposition 1.22. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be an isometric immersion with a principal normal vector field $\eta$ of multiplicity $q$. Then the following assertions hold:
(i) The distribution $x \mapsto E_{\eta}(x)$ is smooth.
(ii) The principal normal vector field $\eta$ is Dupin if and only if $E_{\eta}$ is a spherical distribution and $f$ maps each leaf of $E_{\eta}$ into an extrinsic sphere of $\mathbb{Q}_{c}^{m}$.
(iii) If $q \geq 2$ then $\eta$ is a Dupin principal normal vector field.
(iv) If $\eta$ is a Dupin principal normal vector field and $c=0$, then the map $h: M^{n} \rightarrow \mathbb{R}^{m}$ defined as

$$
h=f+\frac{1}{\|\eta\|^{2}} \eta
$$

is constant along $E_{\eta}$.
Proof: (i) The proof is similar to that of part (ii) of Proposition 1.18 and is left to the reader (see Exercise 1.15).
(ii) Write $\eta=\lambda \zeta$, where $\zeta$ has unit length. Assume that $\eta$ is parallel along $E_{\eta}$ in the normal connection. Then, in particular, $T(\lambda)=0$ for any $T \in \Gamma\left(E_{\eta}\right)$, or equivalently, $\operatorname{grad} \lambda \in \Gamma\left(E_{\eta}^{\perp}\right)$.

Take the $S$-component of the Codazzi equation for $\left(A_{\zeta}, T, X\right)$ for all $S, T \in \Gamma\left(E_{\eta}\right)$ and $X \in \mathfrak{X}(M)$. By this we mean taking the inner product with $S$ of the Codazzi equation

$$
\nabla_{T} A_{\zeta} X-A_{\zeta} \nabla_{T} X-A_{\nabla_{\frac{⿺}{T}} \zeta} X=\nabla_{X} A_{\zeta} T-A_{\zeta} \nabla_{X} T-A_{\nabla_{\frac{1}{X}} \zeta} T
$$

We obtain

$$
\begin{equation*}
\left(A_{\zeta}-\lambda I\right) \nabla_{T} S=-\langle T, S\rangle \operatorname{grad} \lambda \tag{1.28}
\end{equation*}
$$

Similarly, taking the $S$-component of the Codazzi equation for $\left(A_{\xi}, T, X\right)$ for any $\xi \in$ $\Gamma\left(N_{f} M\right)$ with $\xi$ orthogonal to $\eta$ yields

$$
\begin{equation*}
\left\langle A_{\xi} \nabla_{T} S, X\right\rangle=\lambda\langle T, S\rangle\left\langle\nabla_{X}^{\perp} \xi, \zeta\right\rangle \tag{1.29}
\end{equation*}
$$

for all $S, T \in \Gamma\left(E_{\eta}\right)$ and $X \in \mathfrak{X}(M)$.
In view of (1.27), it follows from (1.28) and (1.29) that

$$
\nabla_{T} S \in \Gamma\left(E_{\eta}\right)
$$

for any orthogonal pair $S, T \in \Gamma\left(E_{\eta}\right)$. Using Exercise 1.21, we see that $E_{\eta}$ is an umbilical distribution with mean curvature vector field $\delta$ satisfying

$$
\begin{equation*}
\left(A_{\zeta}-\lambda I\right) \delta=-\operatorname{grad} \lambda \tag{1.30}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle A_{\xi} \delta, X\right\rangle=\lambda\left\langle\nabla_{X}^{\perp} \xi, \zeta\right\rangle . \tag{1.31}
\end{equation*}
$$

Taking the $\delta$-component of the Codazzi equation for $\left(A_{\zeta}, T, X\right)$ gives

$$
\left\langle\nabla_{T}\left(A_{\zeta}-\lambda I\right) X, \delta\right\rangle=\left\langle\left(A_{\zeta}-\lambda I\right) \delta,[T, X]\right\rangle .
$$

Hence

$$
\begin{aligned}
\left\langle\nabla_{T} \delta,\left(A_{\zeta}-\lambda I\right) X\right\rangle & =T\left\langle\left(A_{\zeta}-\lambda I\right) \delta, X\right\rangle-\left\langle\delta, \nabla_{T}\left(A_{\zeta}-\lambda I\right) X\right\rangle \\
& =T\left\langle\left(A_{\zeta}-\lambda I\right) \delta, X\right\rangle-\left\langle\left(A_{\zeta}-\lambda I\right) \delta,[T, X]\right\rangle .
\end{aligned}
$$

It follows using (1.30) that

$$
\begin{equation*}
\left\langle\nabla_{T} \delta,\left(A_{\zeta}-\lambda I\right) X\right\rangle=0 \tag{1.32}
\end{equation*}
$$

Taking the $\delta$-component of the Codazzi equation for $\left(A_{\xi}, T, X\right)$ yields

$$
\left\langle\delta, \nabla_{T} A_{\xi} X\right\rangle=\left\langle A_{\nabla_{\frac{1}{T}} \xi} \delta, X\right\rangle+\left\langle A_{\xi} \delta,[T, X]\right\rangle .
$$

Therefore

$$
\begin{aligned}
\left\langle\nabla_{T} \delta, A_{\xi} X\right\rangle & =T\left\langle A_{\xi} \delta, X\right\rangle-\left\langle\delta, \nabla_{T} A_{\xi} X\right\rangle \\
& =T\left\langle A_{\xi} \delta, X\right\rangle-\left\langle A_{\nabla_{T} \xi} \delta, X\right\rangle-\left\langle A_{\xi} \delta,[T, X]\right\rangle .
\end{aligned}
$$

Using (1.31) and the Ricci equation we have

$$
\begin{align*}
\left\langle\nabla_{T} \delta, A_{\xi} X\right\rangle & =\lambda\left\langle R^{\perp}(T, X) \xi, \zeta\right\rangle \\
& =\lambda\left\langle\left[A_{\xi}, A_{\zeta}\right] T, X\right\rangle \\
& =0 \tag{1.33}
\end{align*}
$$

for all $T \in \Gamma\left(E_{\eta}\right)$ and $X \in \mathfrak{X}(M)$. Since

$$
E_{\eta}^{\perp}(x)=\operatorname{span}\left\{\cup_{\gamma \in N_{f} M(x)} \operatorname{Im}\left(A_{\gamma}-\langle\gamma, \eta\rangle I\right)\right\}
$$

for all $x \in M^{n}$, it follows from (1.32) and (1.33) that $\nabla_{T} \delta \in \Gamma\left(E_{\eta}\right)$ for any $T \in \Gamma\left(E_{\eta}\right)$. Thus $E_{\eta}$ is spherical.

Taking derivatives in the ambient space, we have

$$
\begin{align*}
\tilde{\nabla}_{T} f_{*} S & =f_{*}\left(\nabla_{T} S\right)_{E_{\eta}}+f_{*}\left(\nabla_{T} S\right)_{E_{\eta}^{\perp}}+\alpha(T, S) \\
& =f_{*}\left(\nabla_{T} S\right)_{E_{\eta}}+\langle T, S\rangle f_{*} \delta+\langle T, S\rangle \eta \\
& =f_{*}\left(\nabla_{T} S\right)_{E_{\eta}}+\langle T, S\rangle \sigma, \tag{1.34}
\end{align*}
$$

where $\sigma=f_{*} \delta+\eta$. Using that

$$
\nabla \frac{\perp}{T} \eta=0=\alpha(T, \delta),
$$

we obtain

$$
\begin{align*}
\tilde{\nabla}_{T} \sigma & =f_{*} \nabla_{T} \delta-f_{*} A_{\eta} T \\
& =-\|\delta\|^{2} f_{*} T-\|\eta\|^{2} f_{*} T \\
& =-\|\sigma\|^{2} f_{*} T \tag{1.35}
\end{align*}
$$

It follows from (1.34) and 1.35 that the restriction of $f$ to each leaf of $E_{\eta}$ is an extrinsic sphere in $\mathbb{Q}_{c}^{m}$. The converse is left to the reader.
(iii) Taking the $S$-component of the Codazzi equation for $\left(A_{\zeta}, T, S\right)$ with $S, T \in \Gamma\left(E_{\eta}\right)$, $\|S\|=1$ and $\langle S, T\rangle=0$ gives $T(\lambda)=0$, whereas the Codazzi equation for $\left(A_{\xi}, T, S\right)$ with $\xi$ orthogonal to $\eta$ yields $\nabla \frac{1}{T} \zeta=0$.
(iv) Since $\eta$ is parallel in the normal connection along $E_{\eta}$, then

$$
T\langle\eta, \eta\rangle=2\left\langle\nabla \frac{\perp}{T} \eta, \eta\right\rangle=0
$$

for all $T \in \Gamma\left(E_{\eta}\right)$. Therefore

$$
\begin{aligned}
h_{*} T & =f_{*} T+\frac{1}{\|\eta\|^{2}} \tilde{\nabla}_{T} \eta \\
& =f_{*} T-\frac{1}{\|\eta\|^{2}} f_{*} A_{\eta} T \\
& =0
\end{aligned}
$$

and this concludes the proof.
The last result of this section shows that the existence of a principal normal vector field with multiplicity greater than $n / 2$ of an isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ of a compact manifold imposes restrictions on the topology of $M^{n}$.

Theorem 1.23. Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be an isometric immersion of a compact Riemannian manifold. If there exists a principal normal vector field $\eta$ of $f$ with multiplicity $k>n / 2$ then $M^{n}$ has the homotopy type of a $C W$-complex with no cells of dimension $n-k<r<k$. In particular, the homology groups of $M^{n}$ satisfy

$$
H_{r}(M ; G)=0, \quad n-k<r<k,
$$

for any coefficient group $G$.
Proof: By Exercise 1.27, there exists $v \in \mathbb{R}^{m}$ such that the function $h^{v}: M^{n} \rightarrow \mathbb{R}$ given by

$$
h^{v}(x)=\langle f(x), v\rangle
$$

is a Morse function. On the other hand, by Corollary 1.3 we have

$$
\text { Hess } h^{v}(X, Y)=\langle\alpha(X, Y), v\rangle
$$

for all $x \in M^{n}$ and $X, Y \in T_{x} M$. Moreover, a point $x \in M^{n}$ is critical for $h^{v}$ if and only if $v \in N_{f} M(x)$. For any $x \in M^{n}$ we have

$$
\begin{aligned}
\text { Hess } h^{v}(Z, X) & =\langle\alpha(Z, X), v\rangle \\
& =\langle\eta(x), v\rangle\langle Z, X\rangle
\end{aligned}
$$

if $Z \in E_{\eta}(x)$ and $X \in T_{x} M$. Since $h^{v}$ has only nondegenerate critical points, for any critical point $x \in M^{n}$ we have

$$
c_{x}=\langle\eta(x), v\rangle \neq 0,
$$

and the index of $x$ is at least $k$ if $c_{x}<0$ and at most $n-k$ if $c_{x}>0$. By a well-known result in Morse theory (see Theorem 3.5 in [247]), $M^{n}$ has the homotopy type of a $C W$-complex with no cells of dimension $n-k<r<k$.

### 1.9 Submanifolds with flat normal bundle

An important class of submanifolds consists of those which have flat normal bundle. As shown below, in case the ambient space has constant sectional curvature, the basic equations for a submanifold in this class are very similar to those of a hypersurface.

Flatness of the normal bundle of a submanifold in a space with constant sectional curvature has the following useful characterization.

Proposition 1.24. An isometric immersion $f: M^{n} \rightarrow \tilde{M}_{c}^{m}$ has flat normal bundle at $x \in M^{n}$ if and only if the shape operators

$$
\left\{A_{\xi}: \xi \in N_{f} M(x)\right\}
$$

are simultaneously diagonalizable, or equivalently, if and only if there exists an orthonormal basis $X_{1}, \ldots, X_{n}$ of $T_{x} M$ such that

$$
\alpha\left(X_{i}, X_{j}\right)=0, \quad 1 \leq i \neq j \leq n .
$$

Proof: By the Ricci equation

$$
\left\langle R^{\perp}(X, Y) \xi, \eta\right\rangle=\left\langle\left[A_{\xi}, A_{\eta}\right] X, Y\right\rangle
$$

the normal curvature tensor $R^{\perp}$ vanishes at $x \in M^{n}$ if and only if all shape operators $A_{\xi}, \xi \in N_{f} M(x)$, commute.

By the above result, at each $x \in M^{n}$ where $R^{\perp}(x)=0$ the tangent space $T_{x} M$ decomposes orthogonally as

$$
T_{x} M=E_{1}(x) \oplus \cdots \oplus E_{s}(x),
$$

the decomposition having the property that for each $\xi \in N_{f} M(x)$ there exist real numbers $\lambda_{i}(\xi), 1 \leq i \leq s=s(x)$, such that

$$
\left.A_{\xi}\right|_{E_{i}(x)}=\lambda_{i}(\xi) I
$$

and the maps $\xi \mapsto \lambda_{i}(\xi)$ are pairwise distinct. Since such maps are linear, there exist unique pairwise distinct vectors $\eta_{i}(x) \in N_{f} M(x), 1 \leq i \leq s$, called the principal normals of $f$ at $x$, such that

$$
\lambda_{i}(\xi)=\left\langle\eta_{i}(x), \xi\right\rangle, \quad 1 \leq i \leq s
$$

Therefore

$$
E_{i}(x)=E_{\eta_{i}(x)}=\left\{X \in T_{x} M: \alpha(X, Y)=\langle X, Y\rangle \eta_{i}(x) \text { for all } Y \in T_{x} M\right\}
$$

and the second fundamental form of $f$ has the simple representation

$$
\begin{equation*}
\alpha(X, Y)=\sum_{i=1}^{s}\left\langle X^{i}, Y^{i}\right\rangle \eta_{i}(x), \tag{1.36}
\end{equation*}
$$

where $X \mapsto X^{i}$ is the orthogonal projection onto $E_{i}(x)$. Equivalently,

$$
\begin{equation*}
A_{\xi} X=\sum_{i=1}^{s}\left\langle\xi, \eta_{i}(x)\right\rangle X^{i} \tag{1.37}
\end{equation*}
$$

for all $X \in T_{x} M$ and $\xi \in N_{f} M(x)$. The Gauss equation takes the form

$$
\begin{equation*}
R(X, Y)=\sum_{i, j=1}^{s}\left(c+\left\langle\eta_{i}(x), \eta_{j}(x)\right\rangle\right) X^{i} \wedge Y^{j} \tag{1.38}
\end{equation*}
$$

for all $X, Y \in T_{x} M$. In particular, the sectional curvature $K(X, Y)$ of $M^{n}$ is

$$
\begin{equation*}
K(X, Y)=c+\left\langle\eta_{i}(x), \eta_{j}(x)\right\rangle \tag{1.39}
\end{equation*}
$$

along a plane spanned by $X \in E_{i}(x)$ and $Y \in E_{j}(x), 1 \leq i \neq j \leq s$, whereas

$$
\begin{equation*}
K(X, Y)=c+\left\|\eta_{i}(x)\right\|^{2} \tag{1.40}
\end{equation*}
$$

if $X, Y \in E_{i}(x)$.
If $f: M^{n} \rightarrow \tilde{M}_{c}^{m}$ has flat normal bundle at every point $x \in M^{n}$, then the function

$$
x \in M^{n} \mapsto s(x) \in\{1, \ldots, n\}
$$

is lower semi-continuous. Hence, if $M_{s}$ denotes the interior of the subset where it assumes the value $s$, then $\cup_{s=1}^{n} M_{s}$ is open and dense in $M^{n}$. On each $M_{s}$, the maps $x \mapsto \eta_{i}(x), 1 \leq i \leq s$, define smooth normal vector fields, called the principal normal vector fields of $f$, and each map $x \mapsto E_{i}(x)$ gives rise to a smooth distribution.

The Codazzi equation on $M_{s}$ is equivalent to the following:
(i) The vector field $\eta_{i}$ is parallel in the normal connection along $E_{\eta_{i}}$ if rank $E_{\eta_{i}} \geq 2$.
(ii) If $X_{i} \in \Gamma\left(E_{\eta_{i}}\right)$ and $X_{j}, Y_{j} \in \Gamma\left(E_{\eta_{j}}\right), 1 \leq i \neq j \leq s$, then

$$
\begin{equation*}
\left\langle X_{j}, Y_{j}\right\rangle \nabla_{X_{i}}^{\perp} \eta_{j}=\left\langle\nabla_{X_{j}} Y_{j}, X_{i}\right\rangle\left(\eta_{j}-\eta_{i}\right) . \tag{1.41}
\end{equation*}
$$

(iii) If $X_{i} \in \Gamma\left(E_{\eta_{i}}\right), X_{j} \in \Gamma\left(E_{\eta_{j}}\right)$ and $X_{k} \in \Gamma\left(E_{\eta_{k}}\right), 1 \leq i \neq j \neq k \neq i \leq s$, then

$$
\begin{equation*}
\left\langle\nabla_{X_{j}} X_{i}, X_{k}\right\rangle\left(\eta_{i}-\eta_{k}\right)=\left\langle\nabla_{X_{i}} X_{j}, X_{k}\right\rangle\left(\eta_{j}-\eta_{k}\right) \tag{1.42}
\end{equation*}
$$

### 1.10 Proof of the Fundamental theorem

In this section we give a proof of the Fundamental theorem of submanifolds 1.10, Since we will also need a Lorentzian version of this theorem in our study of conformal immersions, and this also leads to an easy proof of the theorem for the hyperbolic space, at no extra effort we prove a general version for isometric immersions of Riemannian manifolds into semi-Riemannian manifolds of constant sectional curvature.

We denote by $\mathbb{Q}_{c, \mu}^{m}$ a semi-Riemannian manifold with a metric of constant sectional curvature $c$ and index $\mu$. If $c=0$ we write simply $\mathbb{R}_{\mu}^{m}$, and we also use the symbol $\mathbb{S}_{c, \mu}^{m}$ when $c>0$.

Theorem 1.25. Existence: Let $M^{n}$ be a simply connected Riemannian manifold, let $\mathcal{E}$ be a semi-Riemannian vector bundle of rank $p$ and index $\mu$ over $M^{n}$ with compatible connection $\nabla^{\varepsilon}$ and curvature tensor $R^{\varepsilon}$, and let $\alpha^{\varepsilon}$ be a symmetric section of $\operatorname{Hom}^{2}(T M, T M ; \mathcal{E})$. For each $\xi \in \Gamma(\mathcal{E})$ define $A_{\xi}^{\varepsilon} \in \Gamma(\operatorname{End}(T M))$ by

$$
\left\langle A_{\xi}^{\varepsilon} X, Y\right\rangle=\left\langle\alpha^{\varepsilon}(X, Y), \xi\right\rangle \text {. }
$$

Assume that $\left(\nabla^{\varepsilon}, \alpha^{\varepsilon}, A^{\varepsilon}, R^{\varepsilon}\right)$ satisfies (1.14), (1.15) and (1.17). Then there exist an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c, \mu}^{n+p}$ and a vector bundle isometry $\phi: \mathcal{E} \rightarrow N_{f} M$ such that

$$
\alpha^{f}=\phi \circ \alpha^{\varepsilon} \text { and } \nabla^{\perp} \phi=\phi \nabla^{\varepsilon} \text {. }
$$

Uniqueness: Let $f, g: M^{n} \rightarrow \mathbb{Q}_{c, \mu}^{n+p}$ be isometric immersions of a Riemannian manifold. Assume that there exists a vector bundle isometry $\phi: N_{f} M \rightarrow N_{g} M$ such that

$$
\begin{equation*}
\phi \circ \alpha^{f}=\alpha^{g} \quad \text { and } \phi^{f} \nabla^{\perp}={ }^{g} \nabla^{\perp} \phi . \tag{1.43}
\end{equation*}
$$

Then there exists an isometry $\tau: \mathbb{Q}_{c, \mu}^{n+p} \rightarrow \mathbb{Q}_{c, \mu}^{n+p}$ such that

$$
\tau \circ f=g \text { and }\left.\tau_{*}\right|_{N_{f} M}=\phi .
$$

Proof: We first prove the theorem for $c=0$, starting with existence.
Consider the Whitney sum $\overline{\mathcal{E}}=T M \oplus \mathcal{E}$ endowed with the orthogonal sum of the metrics in $T M$ and $\mathcal{E}$. Define

$$
\nabla_{X}^{\bar{\varepsilon}} Y=\nabla_{X} Y+\alpha^{\varepsilon}(X, Y) \text { and } \nabla_{X}^{\bar{\varepsilon}} \xi=-A_{\xi}^{\varepsilon} X+\nabla_{X}^{\varepsilon} \xi
$$

for all $X, Y \in \mathfrak{X}(M)$ and $\xi \in \Gamma(\mathcal{E})$, where $\nabla$ is the Levi-Civita connection on $T M$. It is easy to see that $\nabla^{\bar{\varepsilon}}$ is a compatible connection on $\bar{\varepsilon}$. Moreover, using that 1.14, (1.15) and (1.17) hold with $c=0$, it is straightforward to verify that the curvature tensor of $\bar{\varepsilon}$ vanishes identically. Therefore, $M^{n}$ being simply connected, there exists by Corollary A. 5 a vector bundle isometry $\bar{\phi}: \bar{\varepsilon} \rightarrow M^{n} \times \mathbb{R}_{\mu}^{n+p}$ such that

$$
\bar{\nabla} \bar{\phi}=\bar{\phi} \nabla^{\bar{\varepsilon}}
$$

where $\bar{\nabla}$ denotes the canonical connection on the trivial bundle $M^{n} \times \mathbb{R}_{\mu}^{n+p}$.
Define a one-form $\omega \in \Gamma\left(T^{*} M \otimes \mathbb{R}_{\mu}^{n+p}\right)$ by $\omega=\left.\bar{\phi}\right|_{T M}$. Then $\omega$ is closed, for

$$
\begin{aligned}
d \omega(X, Y) & =\bar{\nabla}_{X} \omega(Y)-\bar{\nabla}_{Y} \omega(X)-\omega([X, Y]) \\
& =\bar{\nabla}_{X} \bar{\phi} Y-\bar{\nabla}_{Y} \bar{\phi} X-\bar{\phi}[X, Y] \\
& =\bar{\phi} \nabla_{X}^{\bar{\varepsilon}} Y-\bar{\phi} \nabla_{Y}^{\bar{\varepsilon}} X-\bar{\phi}[X, Y] \\
& =\bar{\phi}\left(\nabla_{X} Y-\nabla_{Y} X-[X, Y]+\alpha^{\varepsilon}(X, Y)-\alpha^{\varepsilon}(Y, X)\right) \\
& =0 .
\end{aligned}
$$

Since $M^{n}$ is simply connected, there exists a map $f: M^{n} \rightarrow \mathbb{R}_{\mu}^{n+p}$ such that $f_{*}=\omega$. It follows from

$$
\begin{aligned}
\left\langle f_{*} X, f_{*} Y\right\rangle & =\langle\omega X, \omega Y\rangle \\
& =\langle\bar{\phi} X, \bar{\phi} Y\rangle \\
& =\langle X, Y\rangle
\end{aligned}
$$

that $f$ is an isometric immersion. Define $\phi=\left.\bar{\phi}\right|_{\varepsilon}$. On one hand,

$$
\begin{aligned}
\bar{\phi} \nabla_{X}^{\bar{c}} Y & =\bar{\phi} \nabla_{X} Y+\bar{\phi} \alpha^{\varepsilon}(X, Y) \\
& =f_{*} \nabla_{X} Y+\phi \alpha^{\varepsilon}(X, Y)
\end{aligned}
$$

On the other hand, identifying $f^{*} T \mathbb{R}_{\mu}^{n+p}$ with $M^{n} \times \mathbb{R}_{\mu}^{n+p}$,

$$
\begin{aligned}
\bar{\phi} \nabla_{X}^{\bar{c}} Y & =\bar{\nabla}_{X} \bar{\phi} Y \\
& =\bar{\nabla}_{X} f_{*} Y \\
& =f_{*} \nabla_{X} Y+\alpha^{f}(X, Y)
\end{aligned}
$$

Hence $\phi \alpha^{\varepsilon}(X, Y)=\alpha^{f}(X, Y)$, which easily implies that $A_{\xi}^{\varepsilon}=A_{\phi \xi}^{f}$. Thus

$$
\begin{aligned}
\bar{\phi} \nabla_{X}^{\bar{\varepsilon}} \xi & =\bar{\phi}\left(-A_{\xi}^{\varepsilon} X\right)+\bar{\phi} \nabla_{X}^{\varepsilon} \xi \\
& =-f_{*} A_{\xi}^{\varepsilon} X+\bar{\phi} \nabla_{X}^{\varepsilon} \xi \\
& =-f_{*} A_{\phi \xi}^{f} X+\phi \nabla_{X}^{\varepsilon} \xi
\end{aligned}
$$

Then $\phi \nabla_{X}^{\varepsilon} \xi=\nabla \frac{\perp}{X} \phi \xi$ follows by comparing this equation with

$$
\begin{aligned}
\bar{\phi} \nabla_{X}^{\bar{\varepsilon}} \xi & =\bar{\nabla}_{X} \bar{\phi} \xi \\
& =\bar{\nabla}_{X} \phi \xi \\
& =-f_{*} A_{\phi \xi}^{f} X+\nabla_{X}^{\perp} \phi \xi .
\end{aligned}
$$

Now we prove uniqueness. Define $\tilde{\phi}: f^{*} T \mathbb{R}_{\mu}^{n+p} \rightarrow g^{*} T \mathbb{R}_{\mu}^{n+p}$ by

$$
\tilde{\phi} \circ f_{*}=g_{*} \text { and }\left.\tilde{\phi}\right|_{N_{f} M}=\phi
$$

We claim that $\tilde{\phi}$ is parallel with respect to the pulled-back connections ${ }^{f} \tilde{\nabla}$ and ${ }^{g} \tilde{\nabla}$ on $f^{*} T \mathbb{R}_{\mu}^{n+p}$ and $g^{*} T \mathbb{R}_{\mu}^{n+p}$, respectively, of the flat connection on $\mathbb{R}_{\mu}^{n+p}$. In fact, by the first formula in (1.43) and the Gauss formulas for $f$ and $g$ we have

$$
\begin{aligned}
{ }^{g} \tilde{\nabla}_{X} \tilde{\phi} f_{*} Y & ={ }^{g} \tilde{\nabla}_{X} g_{*} Y \\
& =g_{*} \nabla_{X} Y+\alpha^{g}(X, Y) \\
& =\tilde{\phi} f_{*} \nabla_{X} Y+\phi \alpha^{f}(X, Y) \\
& =\tilde{\phi}\left(f_{*} \nabla_{X} Y+\alpha^{f}(X, Y)\right) \\
& =\tilde{\phi}^{f} \tilde{\nabla}_{X} f_{*} Y
\end{aligned}
$$

for all $X, Y \in \mathfrak{X}(M)$. On the other hand, using the second formula in (1.43) and the Weingarten formulas for $f$ and $g$ we obtain

$$
\begin{aligned}
{ }^{g} \tilde{\nabla}_{X} \tilde{\phi} \xi & ={ }^{g} \tilde{\nabla}_{X} \phi \xi \\
& =-g_{*} A_{\phi \xi}^{g} X+{ }^{g} \nabla_{X}^{\perp} \phi \xi \\
& =-\tilde{\phi} f_{*} A_{\xi}^{f} X+\phi^{f} \nabla_{X}^{\perp} \xi \\
& =\tilde{\phi}\left(-f_{*} A_{\xi}^{f} X+{ }^{f} \nabla_{X}^{\perp} \xi\right) \\
& =\tilde{\phi}^{f} \tilde{\nabla}_{X} \xi
\end{aligned}
$$

for all $X \in \mathfrak{X}(M)$ and $\xi \in \Gamma\left(N_{f} M\right)$, and the claim follows.
Therefore $\tilde{\phi}$ defines a linear orthogonal map $B$ on $\mathbb{R}_{\mu}^{n+p}$. From $B \circ f_{*}=\tilde{\phi} \circ f_{*}=g_{*}$ it follows that there exists an isometry $\tau$ of $\mathbb{R}_{\mu}^{n+p}$ such that $\tau \circ f=g$ and $\tau_{*}=B$. In particular,

$$
\left.\tau_{*}\right|_{N_{f} M}=\left.B\right|_{N_{f} M}=\left.\tilde{\phi}\right|_{N_{f} M}=\phi .
$$

We now consider the nonflat case $c \neq 0$. We use the fact that $\mathbb{Q}_{c, \mu}^{N}, c \neq 0$, admits a canonical isometric embedding $i: \mathbb{Q}_{c, \mu}^{N} \rightarrow \mathbb{R}_{\mu+\sigma(c)}^{N+1}$ (see Exercise 1.14) whose image is the hyperquadric

$$
\mathbb{Q}_{c, \mu}^{N}=\left\{X \in \mathbb{R}_{\mu+\sigma(c)}^{N+1}:\langle X, X\rangle=1 / c\right\} .
$$

Here $\sigma(c)=1$ if $c<0$ and $\sigma(c)=0$ if $c>0$.
To prove existence, set $c=\epsilon / r^{2}$, with $\epsilon \in\{1,-1\}$. Let $\tilde{\varepsilon}$ be the Whitney sum of $\mathcal{E}$ and a line bundle $\Upsilon$ over $M^{n}$. Endow $\tilde{\mathcal{E}}$ with the semi-Riemannian metric that makes the decomposition $\tilde{\varepsilon}=\mathcal{E} \oplus \Upsilon$ orthogonal and has index $\epsilon$ on $\Upsilon$.

Choose a unit section $\nu$ of $\Upsilon$ and define a compatible connection $\nabla^{\tilde{\varepsilon}}$ on $\tilde{\varepsilon}$ by

$$
\begin{equation*}
\nabla_{X}^{\tilde{\varepsilon}} \nu=0, \quad \nabla_{X}^{\tilde{\varepsilon}} \xi=\nabla_{X}^{\varepsilon} \xi \text { for } \xi \in \Gamma(\mathcal{\varepsilon}), \tag{1.44}
\end{equation*}
$$

and a symmetric section $\alpha^{\tilde{\varepsilon}}$ of $\operatorname{Hom}^{2}(T M, T M ; \tilde{\varepsilon})$ by

$$
\begin{equation*}
\alpha^{\tilde{\varepsilon}}(X, Y)=\alpha^{\varepsilon}(X, Y)-\frac{\epsilon}{r}\langle X, Y\rangle \nu . \tag{1.45}
\end{equation*}
$$

Then it is straightforward to check that $\left(\tilde{\varepsilon}, \nabla^{\tilde{\varepsilon}}, \alpha^{\tilde{\varepsilon}}\right)$ satisfies the Gauss, Codazzi and Ricci equations for an isometric immersion $\tilde{f}: M^{n} \rightarrow \mathbb{R}_{\mu+\sigma(c)}^{n+p+1}$. By the flat case of the
theorem already proved, there exist an isometric immersion $\tilde{f}: M^{n} \rightarrow \mathbb{R}_{\mu+\sigma(c)}^{n+p+1}$ and a vector bundle isometry $\tilde{\phi}: \tilde{\mathcal{E}} \rightarrow N_{\tilde{f}} M$ such that

$$
\alpha^{\tilde{f}}=\tilde{\phi} \circ \alpha^{\tilde{\varepsilon}} \text { and } \tilde{\nabla}^{\perp} \tilde{\phi}=\tilde{\phi} \nabla^{\tilde{\varepsilon}}
$$

where $\alpha^{\tilde{f}}$ and $\tilde{\nabla}^{\perp}$ denote the second fundamental form and the normal connection of $\tilde{f}$, respectively.

We claim that $\tilde{f}(M) \subset \mathbb{Q}_{c, \mu}^{n+p}$, after composing $\tilde{f}$ with a translation in $\mathbb{R}_{\mu+\sigma(c)}^{n+p+1}$ if necessary. In fact, from

$$
\begin{aligned}
\tilde{\nabla}_{X} \tilde{\phi}(\nu) & =-\tilde{f}_{*} A_{\tilde{\phi}(\nu)}^{\tilde{f}} X+\tilde{\nabla}_{X}^{\perp} \tilde{\phi}(\nu) \\
& =-\tilde{f}_{*} A_{\nu}^{\tilde{\varepsilon}} X+\tilde{\phi}\left(\nabla_{X}^{\tilde{\varepsilon}} \nu\right) \\
& =\frac{1}{r} \tilde{f}_{*} X,
\end{aligned}
$$

it follows that $\tilde{f}-r \tilde{\phi}(\nu)=O \in \mathbb{R}_{\mu+\sigma(c)}^{n+p+1}$ is a constant vector. Hence

$$
\langle\tilde{f}-O, \tilde{f}-O\rangle=\langle r \tilde{\phi}(\nu), r \tilde{\phi}(\nu)\rangle=\epsilon r^{2}
$$

This completes the proof of our claim. We conclude that there exists an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c, \mu}^{n+p}$ such that $\tilde{f}=i \circ f$. Moreover, $\tilde{\phi}(\nu)=(1 / r) \tilde{f}$. Finally, it is now easy to see that the restriction $\phi$ of $\tilde{\phi}$ to $\mathcal{E}$ is a vector bundle isometry onto $N_{f} M$ satisfying the conditions in the statement.
For the uniqueness, set $\tilde{f}=i \circ f$ and $\tilde{g}=i \circ g$. Extend $\phi$ to an isometry $\tilde{\phi}: N_{\tilde{f}} M \rightarrow N_{\tilde{g}} M$ by setting

$$
\tilde{\phi}(\tilde{f})=\tilde{g}
$$

Using (1.43), (1.44) and (1.45), it follows that

$$
\tilde{\phi} \circ \alpha^{\tilde{f}}=\alpha^{\tilde{g}} \text { and } \tilde{\phi}^{\tilde{f}} \nabla^{\perp}=\tilde{g} \nabla^{\perp} \tilde{\phi}
$$

By the uniqueness part of the theorem for the flat case, there is an isometry

$$
\tau: \mathbb{R}_{\mu+\sigma(c)}^{n+p+1} \rightarrow \mathbb{R}_{\mu+\sigma(c)}^{n+p+1}
$$

such that

$$
\tau \circ \tilde{f}=\tilde{g} \text { and }\left.\tau_{*}\right|_{N_{\tilde{f}} M}=\tilde{\phi}
$$

Write $\tau Z=B Z+V$, with $B=\tau_{*}$ orthogonal. Then

$$
B \tilde{f}+V=\tau \tilde{f}=\tilde{g}=\tilde{\phi} \tilde{f}=\tau_{*} \tilde{f}=B \tilde{f}
$$

hence $V=0$. It follows that $\tau$ leaves $\mathbb{Q}_{c, \mu}^{n+p}$ invariant.

### 1.11 Appendix: Burstin-Mayer-Allendoerfer theory

There is an alternative approach to the basic equations and the Fundamental theorem of submanifolds that has proved to be very useful in several situations. In fact, the Burstin-Mayer-Allendoerfer theory naturally extends the Frenet equations for curves to submanifolds of arbitrary dimension and codimension, under similar regularity conditions. The main result is that, for a submanifold of a space form, the tensors determined by the generalized Frenet equations are a complete set of invariants.

The $k^{t h}$-normal space $N_{k}(x)$ of an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ at $x \in M^{n}$ for $k \geq 1$ is defined as

$$
N_{k}(x)=\operatorname{span}\left\{\alpha^{k+1}\left(X_{1}, \ldots, X_{k+1}\right): X_{1}, \ldots, X_{k+1} \in T_{x} M\right\} .
$$

Here $\alpha^{2}=\alpha^{f}$ and $\alpha^{s}: \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \rightarrow \Gamma\left(N_{f} M\right), s \geq 3$ is the symmetric tensor called the $s^{\text {th }}$-fundamental form and defined inductively by

$$
\alpha^{s}\left(X_{1}, \ldots, X_{s}\right)=\left(\nabla_{X_{s}}^{\perp} \cdots \nabla_{X_{3}}^{\perp} \alpha^{2}\left(X_{2}, X_{1}\right)\right)^{\perp}
$$

where ()$^{\perp}$ denotes taking the projection onto the normal subspace $\left(N_{1} \oplus \cdots \oplus N_{s-2}\right)^{\perp}$.
We assume that the immersion $f$ is a regular isometric immersion (sometimes called nicely curved), which means that all $N_{k}$ 's have constant dimension for each $k$ and therefore form normal subbundles. Geometrically, this roughly means that at each point the submanifold bends in the same number of directions. Notice that, for any submanifold, this condition is satisfied along connected components of an open dense subset.

The Frenet equations for a regular isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ are given by

$$
\tilde{\nabla}_{X} \xi=-A_{\xi}^{s} X+D_{X}^{s} \xi+\mathrm{S}_{X}^{s} \xi
$$

if $\xi \in \Gamma\left(N_{s}\right)$ and $X \in \mathfrak{X}(M), s \geq 1$, in terms of the maps

$$
\begin{aligned}
& A^{s}: \mathfrak{X}(M) \times \Gamma\left(N_{s}\right) \rightarrow \Gamma\left(N_{s-1}\right) \text { defined by } A_{\xi}^{s} X=-\pi_{s-1}\left(\tilde{\nabla}_{X} \xi\right), \\
& D^{s}: \mathfrak{X}(M) \times \Gamma\left(N_{s}\right) \rightarrow \Gamma\left(N_{s}\right) \text { defined by } D_{X}^{s} \xi=\pi_{s}\left(\nabla_{X}^{\perp} \xi\right) \text {, } \\
& \mathrm{S}^{s}: \mathfrak{X}(M) \times \Gamma\left(N_{s}\right) \rightarrow \Gamma\left(N_{s+1}\right) \text { defined by } \mathrm{S}_{X}^{s} \xi=\pi_{s+1}\left(\nabla \frac{1}{X} \xi\right) \text {, }
\end{aligned}
$$

where $\tilde{\nabla}$ denotes the connection in the induced bundle $f^{*}\left(T \mathbb{Q}_{c}^{m}\right)=N_{0} \oplus N_{f} M$. In addition, $\pi_{s}: N_{f} M \rightarrow N_{s}, s \geq 1$, stands for the orthogonal projection and $\pi_{0}$ is the orthogonal projection onto $N_{0}=f_{*} T M$. Notice that $A_{\xi}^{1}$ is the standard Weingarten operator and that $D^{s}$ is a connection in $N_{s}$ compatible with the metric. An important fact is that $A^{s}$ and $\mathrm{S}^{s}$ are tensors that are completely determined by the higher fundamental forms, for

$$
\mathrm{S}_{X}^{s}\left(\alpha^{s+1}\left(X_{1}, \ldots, X_{s+1}\right)\right)=\alpha^{s+2}\left(X, X_{1}, \ldots, X_{s+1}\right)
$$

and

$$
\left\langle A_{\xi}^{s} X, \eta\right\rangle=\left\langle\xi, S_{X}^{s-1} \eta\right\rangle
$$

for any $\xi \in \Gamma\left(N_{s}\right)$ and $\eta \in \Gamma\left(N_{s-1}\right)$.
We briefly summarize the basic results of the theory and refer to Spivak [317] for the many details.

Let $f, \tilde{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{N}$ be two regular isometric immersions. If there exist vector bundle isometries $\phi_{k}: N_{k}^{f} \rightarrow N_{k}^{\tilde{f}}$ for all $k \geq 1$, which preserve the fundamental forms $\alpha^{k+1}$ and the induced normal connections $D^{k}$, then there exists an isometry $\tau$ of $\mathbb{Q}_{c}^{N}$ such that $\tilde{f}=\tau \circ f$ and $\phi_{k}=\left.\tau_{*}\right|_{N_{k}^{f}}$. Moreover, there is a set of equations, which are given below and called the Generalized Gauss and Codazzi equations, that relate the higher fundamental forms and the induced connections. It turns out that the set of connections $D^{k}$ in $N_{k}$ is the unique one for which the higher order fundamental forms satisfy the Codazzi equation. Furthermore, the Generalized Gauss and Codazzi equations are the integrability conditions that assure the existence of an isometric immersion with a set of prescribed data.

The Generalized Gauss equation.

$$
A_{\mathrm{S}_{Y} \xi}^{s+1} X-A_{\mathrm{S}_{X} \xi}^{s+1} Y=D_{X}^{s} D_{Y}^{s} \xi-D_{Y}^{s} D_{X}^{s} \xi-\mathrm{S}_{X}^{s-1} A_{\xi}^{s} Y+\mathrm{S}_{Y}^{s-1} A_{\xi}^{s} X-D_{[X, Y]}^{s} \xi
$$

for all $X, Y \in \mathfrak{X}(M)$ and $\xi \in \Gamma\left(N_{s}\right)$.
The Generalized Codazzi equation.

$$
D_{X}^{s+1}\left(\mathrm{~S}_{Y}^{s} \xi\right)-D_{Y}^{s+1}\left(\mathrm{~S}_{X}^{s} \xi\right)+\mathrm{S}_{X}^{s} D_{Y}^{s} \xi-\mathrm{S}_{Y}^{s} D_{X}^{s} \xi-\mathrm{S}_{[X, Y]}^{s} \xi=0
$$

for all $X, Y \in \mathfrak{X}(M)$ and $\xi \in \Gamma\left(N_{s}\right)$.

### 1.12 Notes

The classical Burstin-Janet-Cartan theorem states that an analytic Riemannian metric always locally admits an analytical isometric embedding into $\mathbb{R}^{\frac{1}{2} n(n+1)}$. The first rigorous proof after the claim by Janet was given by Burstin. A completely different proof using differential systems is due to Cartan, and both types of proofs are presented in Spivak [317].

A fundamental theorem due to Nash [265] states that every Riemannian manifold can be isometrically embedded in Euclidean space for some sufficiently large codimension. Improvements of Nash's result were given by Gromov-Rokhlin [203] and by Gromov [201]. For a thorough discussion of the subject we refer to Han-Hong [213]. On the other hand, except for several special cases, little is known about the lowest codimension which makes an isometric embedding or just an isometric immersion possible. This fundamental basic problem is considered many times in this book.

It follows from Proposition 1.7 and Exercise 1.10 that a Riemannian manifold $M^{n}$ with positive sectional curvature that admits an isometric immersion in Euclidean space as a hypersurface must be diffeomorphic to the round sphere $\mathbb{S}^{n}$ with the diffeomorphism being given by the Gauss map. The convexity Theorem 1.8 for compact hypersurfaces goes back to Hadamard, who treated the surface case.

Strengthening previous theorems due to Weinstein [343] and Moore [259], it was shown by Florit-Ziller [193], with the aid of results based on the Ricci flow by Wilking [344], that a compact Riemannian manifold $M^{n}$ of dimension $n \geq 3$ with positive sectional curvature that admits an isometric immersion into $\mathbb{R}^{n+2}$ must also be diffeomorphic to the round sphere $\mathbb{S}^{n}$. The case in which $M^{n}$ is only assumed to have nonnegative sectional curvature is much more subtle. In that case, improving earlier results by Baldin-Mercuri [24], [25], it was shown by Florit-Ziller [193] that two further possible cases may occur for a Riemannian manifold $M^{n}$ with dimension $n \geq 4$ that admits an isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{n+2}$, besides being diffeomorphic to the round sphere $\mathbb{S}^{n}$. Namely, either $M^{n}$ is isometric to a Riemannian product $\mathbb{S}^{k} \times \mathbb{S}^{n-k}$ for some $2 \leq k \leq n-2$, in which case $f$ is the product embedding of two convex Euclidean hypersurfaces, or $M^{n}$ is isometric to $\left(\mathbb{S}^{n} \times \mathbb{R}\right) / \Gamma$, with $\mathbb{S}^{n} \times \mathbb{R}$ endowed with the product metric and $\Gamma$ isomorphic to $\mathbb{Z}$ acting isometrically. In the latter case, as a manifold, $M^{n}$ is diffeomorphic to $\mathbb{S}^{n} \times \mathbb{S}^{1}$ if orientable, or to the nonorientable quotient $\left(\mathbb{S}^{n} \times \mathbb{S}^{1}\right) / \mathbb{Z}_{2}$ otherwise. In particular, it suffices that the sectional curvatures of $M^{n}$ be positive at one point for $M^{n}$ to be diffeomorphic to $\mathbb{S}^{n}$. If $n=3$, the manifold $M^{3}$ might yet be diffeomorphic to a lens space $L_{p, q}$, but it is still an open problem whether an example with this diffeomorphism type exists. The use of results based on the Ricci flow in this context was made possible by the observation in Exercise 1.11, due to Weinstein [343], according to which if $f: M^{n} \rightarrow \mathbb{R}^{n+2}$ is an isometric immersion and the sectional curvatures of $M^{n}$ at a point $x \in M^{n}$ along any plane $\sigma \subset T_{x} M$ are nonnegative (respectively, positive), then $M^{n}$ has nonnegative (respectively, positive) curvature operator at $x$.

As for complete hypersurfaces in Euclidean space with nonnegative sectional curvature, it was shown by Sacksteder [307] that they must be convex as long as there exists a point where all sectional curvatures are positive. The case of a round sphere as ambient space was considered by do Carmo-Warner [61]. For related results see [5], [6], [19], [39], [60, [245] and [346].

Theorem 1.14 is basically due to Cartan; see Berndt-Console-Olmos [34] or PawelReckziegel [287] for a proof of this result, and Pawel-Reckziegel [288] for an extension to the case of extrinsic spheres.

The characterization of Riemannian manifolds that satisfy the axiom of $r$-planes given by Theorem 1.16 is due to Cartan [70]. The generalization in Theorem 1.21 was obtained by Leung-Nomizu [235]. There are several other results of a similar nature in the literature; for instance, see [338].

The terminology extrinsic sphere in a Riemannian manifold was introduced by Nomizu-Yano [271], who gave an interesting characterization of such submanifolds.

After earlier work due to Thomas [328] and Fialkow [179], it was shown by Ryan [305] that for an Einstein hypersurface in Euclidean space either the Ricci curvature $\rho$ is zero and the submanifold is flat or $\rho>0$ and the submanifold is an open subset of a round hypersphere; see Exercise 3.11. Moreover, if the hypersurface is complete, then it is either a round hypersphere or a cylinder over a complete plane curve. In fact, the global assertion in the flat case will be proved, as a particular case, in Theorem 7.15.

Cheng-Yau [85] showed that the only complete hypersurfaces in $\mathbb{R}^{n+1}$ with con-
stant scalar curvature and nonnegative sectional curvature are cylinders $\mathbb{S}^{p} \times \mathbb{R}^{n-p}$, $0 \leq p \leq n$. They also considered the case of nonflat ambient space forms. Ros [304] proved that the round sphere is the only compact hypersurface with constant scalar curvature embedded in Euclidean space. In an appendix to the same paper, Korevaar showed that the result remains true if any of the symmetric functions of the principal curvatures are assumed to be constant.

Principal normal vector fields were introduced by Otsuki [285], where some of its properties in Proposition 1.22 were derived; see also Reckziegel [296]. Euclidean submanifolds that carry a principal normal vector field and satisfy some additional condition were considered by Dajczer-Florit-Tojeiro in [104], [105] and [106]. In Chapter 9 it will be shown (see Proposition 9.5) that Euclidean submanifolds with dimension $n$ that carry a principal normal vector field of multiplicity $q$ are, locally, envelopes of $(n-q)$-parameter congruences of $n$-spheres. The topological restrictions in Theorem 1.23 for the existence of a principal normal vector field with multiplicity $k>n / 2$ on a compact $n$-dimensional submanifold of Euclidean space are due to Moore [256], where they have been used to derive topological restrictions for the existence of an isometric immersion of a compact conformally flat Riemannian manifold with dimension $n \geq 4$ into $\mathbb{R}^{n+p}, p \leq n / 2-1$ (see Corollary 16.6).

A procedure to construct all local $n$-dimensional submanifolds with flat normal bundle of the Euclidean space or the sphere, starting with $n$ smooth functions on an open simply connected subset of $\mathbb{R}^{n}$ whose Hessian operators commute, was given by Dajczer-Florit-Tojeiro [106], generalizing a previous result by Ferapontov [169] for the surface case.

The proof of the Fundamental theorem of submanifolds given in this book appears in Lira-Tojeiro-Vitório [238] and was inspired by the one given by Jacobowitz [224]. For other proofs of this result, we refer to [165], [223], [315], [317] and [323]. The article [238] also contains a version of the Fundamental theorem of submanifolds for the case in which the ambient space is a product of space forms. The key point is that for such product spaces, unlike the case of an arbitrary Riemannian manifold, the Gauss, Codazzi and Ricci equations of a submanifold make sense intrinsically. It was shown by Piccione-Tausk [291] that this is the case for any Riemannian manifold that is "sufficiently homogeneous", a condition that can be formulated in terms of the notion of a $G$-structure on the manifold. We refer to [291] and the references therein for versions of the Fundamental theorem of submanifolds for other ambient spaces.

The isometric immersion of a flat Klein bottle given in Exercise 1.4 below is due to Tompkins [336]. The examples of umbilical surfaces in $\mathbb{H}_{k}^{3} \times \mathbb{R}^{2}$ and $\mathbb{H}_{k_{1}}^{3} \times \mathbb{H}_{k_{2}}^{3}$ in Exercise 1.13 have been taken from [283], where umbilical surfaces of any Riemannian product $\mathbb{Q}_{c_{1}}^{n_{1}} \times \mathbb{Q}_{c_{2}}^{n_{2}}$, with $c_{1}+c_{2} \neq 0$, were classified, making use of previous results in [243] for the higher dimensional case. The content of Exercises 1.36 and 1.37 can be found in Fabricius-Bjerre [168]; see also Perepelkin [290] in regard to Exercise 1.36. The result given in Exercise 1.42 is due to Aminov [18]. Finally, the original reference for the Burstin-Mayer-Allendoerfer theory is the work of Allendoerfer [15], and a modern treatment can be found in [317.

### 1.13 Exercises

Exercise 1.1. Let $f: M^{n} \rightarrow \tilde{M}^{m}$ be an isometric immersion and let $\tilde{\nabla}$ denote the connection on $f^{*} T \tilde{M}$ induced by the Levi-Civita connection of $\tilde{M}^{m}$. Verify that

$$
\nabla_{X} Y=f_{*}^{-1}\left(\tilde{\nabla}_{X} f_{*} Y\right)^{T}
$$

defines a compatible torsion-free connection on $T M$, which therefore coincides with the Levi-Civita connection of $M^{n}$.

Exercise 1.2. Given an isometric immersion $f: M^{n} \rightarrow \tilde{M}^{m}$, show that $N_{f} M$ is a Riemannian vector bundle with the metric induced from the metric of $\tilde{M}^{m}$ and that $\nabla^{\perp}$ is a compatible connection on $N_{f} M$.

Exercise 1.3. Verify that the map $\varphi: \mathbb{S}^{n} \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{\frac{1}{2} n(n+1)+n+1}$ given by

$$
\varphi\left(x_{0}, \ldots, x_{n}\right)=\left(\frac{1}{\sqrt{2}} x_{0}^{2}, \ldots, \frac{1}{\sqrt{2}} x_{n}^{2}, x_{0} x_{1}, x_{0} x_{2}, \ldots, x_{n-1} x_{n}\right)
$$

induces an isometric immersion of the real projective space $\mathbb{R} \mathbb{P}^{n}$ into $\mathbb{R}^{\frac{1}{2} n(n+1)+n+1}$.
Exercise 1.4. Show that the map $F: \mathbb{R}^{2} \rightarrow \mathbb{R}^{4}$ given by

$$
F(u, v)=\left(\cos v \cos u, \cos v \sin u, 2 \sin v \cos \frac{u}{2}, 2 \sin v \sin \frac{u}{2}\right)
$$

induces an immersion of a Klein bottle into $\mathbb{R}^{4}$ with flat induced metric. Show that the image $F\left(\mathbb{R}^{2}\right)$ intersects itself along the circle

$$
\left\{\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \in \mathbb{R}^{4}: x_{1}^{2}+x_{2}^{2}=1, x_{3}=0=x_{4}\right\}
$$

Hint: Verify that

$$
F(u, v+2 \pi)=F(u, v)=F(u+2 \pi, 2 \pi-v) .
$$

Exercise 1.5. Let $f: M^{n} \rightarrow \mathbb{S}^{m}$ be an isometric immersion. The cone over $f$ is the immersion $F: N^{n+1}=\mathbb{R}_{+} \times M^{n} \rightarrow \mathbb{R}^{m+1}$ defined by

$$
F(t, x)=t f(x) .
$$

(i) Compute the second fundamental form of $F$ in terms of that of $f$.
(ii) Show that, if $T_{x} M$ is regarded as a subspace of $T_{(t, x)} N$ in the natural way, then the sectional curvatures of $M^{n}$ and $N^{n+1}$ along a plane $\sigma \subset T_{x} M \subset T_{(t, x)} N$ are related by

$$
K_{M}(\sigma)-1=t^{2} K_{N}(\sigma)
$$

Exercise 1.6. Given isometric immersions $j: M \rightarrow N$ and $F: N \rightarrow P$, set $f=F \circ j$. Show that

$$
N_{f} M(x)=F_{*} N_{j} M(x) \oplus N_{F} N(j(x))
$$

for any $x \in M$, and that the second fundamental forms and normal connections of $j$, $F$ and $f$ are related by

$$
\begin{gathered}
\alpha^{f}(X, Y)=F_{*} \alpha^{j}(X, Y)+\alpha^{F}\left(j_{*} X, j_{*} Y\right), \\
{ }^{f} \nabla_{X}^{\perp} F_{*} \xi=F_{*}^{j} \nabla_{X}^{\perp} \xi+\alpha^{F}\left(j_{*} X, \xi\right)
\end{gathered}
$$

and

$$
{ }^{f} \nabla \stackrel{\perp}{X} \zeta=-F_{*}\left(A_{\zeta}^{F} j_{*} X\right)_{N_{j} M}+{ }^{F} \nabla_{j_{*} X}^{\perp} \zeta
$$

for all $X, Y \in T_{x} M, \xi \in \Gamma\left(N_{j} M\right)$ and $\zeta \in \Gamma\left(N_{F} N\right)$.
Exercise 1.7. Prove the following assertions:
(i) Given $r_{1}, \ldots, r_{n}>0$ with $r_{1}^{2}+\cdots+r_{n}^{2}=1$, the map $f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{2 n}$ defined by

$$
f\left(t_{1}, \ldots, t_{n}\right)=\left(r_{1} \cos \frac{t_{1}}{r_{1}}, r_{1} \sin \frac{t_{1}}{r_{1}}, \ldots, r_{n} \cos \frac{t_{n}}{r_{n}}, r_{n} \sin \frac{t_{n}}{r_{n}}\right)
$$

induces an isometric embedding of $S^{1}\left(r_{1}\right) \times \cdots \times S^{1}\left(r_{n}\right)$ into the unit sphere $\mathbb{S}^{2 n-1}$.
(ii) Given $r_{1}, \ldots, r_{n}>0$ with $-r_{1}^{2}+\cdots+r_{n}^{2}=-1$, the map $f: \mathbb{R}^{n} \rightarrow \mathbb{L}^{2 n}$ defined by

$$
f\left(t_{1}, \ldots, t_{n}\right)=\left(r_{1} \cosh \frac{t_{1}}{r_{1}}, r_{1} \sinh \frac{t_{1}}{r_{1}}, r_{2} \cos \frac{t_{2}}{r_{2}}, r_{2} \sin \frac{t_{2}}{r_{2}}, \ldots, r_{n} \cos \frac{t_{n}}{r_{n}}, r_{n} \sin \frac{t_{n}}{r_{n}}\right)
$$

induces an isometric embedding of $\mathbb{H}^{1}\left(r_{1}\right) \times S^{1}\left(r_{2}\right) \times \cdots \times S^{1}\left(r_{n}\right)$ into $\mathbb{H}^{2 n-1} \subset \mathbb{L}^{2 n}$. (iii) Both isometric immersions have parallel mean curvature vector field.

Exercise 1.8. Let $f: M^{n} \rightarrow \tilde{M}^{m}$ be an isometric immersion, and let $\gamma:[0,1] \rightarrow M^{n}$ be a smooth curve such that $f \circ \gamma$ is a geodesic in $\tilde{M}^{m}$. Show that $\gamma$ is a geodesic in $M^{n}$ and, for each plane $\sigma \subset T_{\gamma(t)} M$ such that $\gamma^{\prime}(t) \in \sigma$, prove the Synge inequality

$$
K(\sigma) \leq \tilde{K}(\sigma)
$$

Exercise 1.9. Let $f: M^{n} \rightarrow \tilde{M}_{c}^{n+1}$ have principal curvatures $k_{1}, \ldots, k_{n}$ at $x \in M^{n}$.
(i) Show that the set of $\binom{n}{2}$ numbers $\left\{k_{i} k_{j}: i<j\right\}$ is intrinsic, that is, does not depend on the isometric immersion $f$.
(ii) Conclude that $H_{r}$ is intrinsic if $r$ is even, and that the Gauss-Kronecker curvature is intrinsic if $n$ is even and intrinsic up to sign if $n$ is odd.

Hint: Let

$$
\Lambda^{2} T_{x} M=\left\{X \wedge Y: X, Y \in T_{x} M\right\}
$$

be the second exterior power of $T_{x} M$ and let $\rho: \Lambda^{2} T_{x} M \rightarrow \Lambda^{2} T_{x} M$ be the curvature operator of $M^{n}$ at $x$, given by

$$
\langle\rho(X \wedge Y), Z \wedge W\rangle=\langle R(X, Y) W, Z\rangle
$$

for all $X, Y, Z, W \in T_{x} M$. Show that the Gauss equation can be written as

$$
\rho(X \wedge Y)=c(X \wedge Y)+A X \wedge A Y
$$

where $A$ is the shape operator of $f$. Conclude that

$$
\left\{c+k_{i} k_{j}: i<j\right\}
$$

is the set of eigenvalues of $\rho$.
Exercise 1.10. Prove that any of the conditions in Proposition 1.7 is equivalent to $M^{n}$ having positive sectional curvatures at any point.

Exercise 1.11. Let $f: M^{n} \rightarrow \mathbb{R}^{n+2}$ be an isometric immersion. Assume that at some point $x \in M^{n}$ the sectional curvatures of $M^{n}$ along any plane $\sigma \subset T_{x} M$ are nonnegative (respectively, positive).
(i) Show that there exists an orthonormal basis $\xi, \eta$ of $N_{f} M(x)$ such that the shape operators $A_{\xi}$ and $A_{\eta}$ are nonnegative definite (respectively, positive definite).
(ii) Conclude that the curvature operator $\rho: \Lambda^{2} T_{x} M \rightarrow \Lambda^{2} T_{x} M$ of $M^{n}$ at $x$ is nonnegative definite (respectively, positive definite).

Hint: Show that the second fundamental form $\alpha: T_{x} M \times T_{x} M \rightarrow N_{f} M(x)$ of $f$ at $x$ satisfies

$$
\langle\alpha(X, X), \alpha(Y, Y)\rangle \geq 0
$$

for all $X, Y \in T_{x} M$, with strict inequality if the sectional curvatures of $M^{n}$ at $x$ along any plane $\sigma \subset T_{x} M$ are positive. Conclude that, in the intersection of the connected subset

$$
\left\{\alpha(X, X): X \in T_{x} M\right\} \subset N_{f} M(x)
$$

with the unit circle in $N_{f} M(x)$, all points have distance at most $\pi / 2$ to each other, hence this intersection lies in the first quadrant with respect to some orthonormal basis $\xi, \eta$ of $N_{f} M(x)$, while it lies in its interior if all sectional curvatures are positive.

Exercise 1.12. Show that an umbilical distribution $E$ on a Riemannian manifold $M^{n}$ is always integrable and that its leaves are umbilical submanifolds of $M^{n}$. If, in addition, $E$ is spherical, show that its leaves are extrinsic spheres in $M^{n}$.

Exercise 1.13. Prove that the following maps define umbilical isometric immersions:
(i) $F: \mathbb{R}^{2} \rightarrow \mathbb{H}_{k}^{3} \times \mathbb{R}^{2} \subset \mathbb{L}^{4} \times \mathbb{R}^{2}=\mathbb{L}^{6}$ given by

$$
F(s, t)=\left(a_{1} \cosh \frac{s}{c}, a_{1} \sinh \frac{s}{c}, a_{2} \cos \frac{t}{c}, a_{2} \sin \frac{t}{c}, b_{1} \frac{s}{c}, b_{2} \frac{t}{c}\right),
$$

where

$$
-a_{1}^{2}+a_{2}^{2}=\frac{1}{k} \text { and } a_{1}^{2}+b_{1}^{2}=c^{2}=a_{2}^{2}+b_{2}^{2} .
$$

(ii) $F: \mathbb{R}^{2} \rightarrow \mathbb{H}_{k_{1}}^{3} \times \mathbb{H}_{k_{2}}^{3} \subset \mathbb{L}^{4} \times \mathbb{L}^{4}$ given by
$F(s, t)=\left(a_{1} \cosh \frac{s}{c}, a_{1} \sinh \frac{s}{c}, a_{2} \cos \frac{t}{c}, a_{2} \sin \frac{t}{c}, a_{3} \cosh \frac{t}{d}, a_{3} \sinh \frac{t}{d}, a_{4} \cos \frac{s}{d} a_{4} \sin \frac{s}{d}\right)$,
where

$$
-a_{1}^{2}+a_{2}^{2}=\frac{1}{k_{1}},-a_{3}^{2}+a_{4}^{2}=\frac{1}{k_{2}}, \frac{a_{1}^{2}}{c^{2}}+\frac{a_{4}^{2}}{d^{2}}=1 \text { and } \frac{a_{2}^{2}}{c^{2}}+\frac{a_{3}^{2}}{d^{2}}=1 .
$$

Prove that the mean curvature vector field of both isometric immersions has constant length but is not parallel.

Exercise 1.14. If $\mu=0$ and $c>0$, or if $\mu \geq 1$ and $c \neq 0$, show that the hyperquadric

$$
\mathbb{Q}_{c, \tilde{\mu}}^{m}=\left\{X \in \mathbb{R}_{\mu}^{m+1}:\langle X, X\rangle=1 / c\right\}
$$

is an umbilical hypersurface of $\mathbb{R}_{\mu}^{m+1}$ that induces a semi-Riemannian metric of constant sectional curvature $c$ and index $\tilde{\mu}=\mu-\sigma(c)$, where $\sigma(c)=1$ if $c<0$ and $\sigma(c)=0$ otherwise.

Exercise 1.15. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be an isometric immersion with a principal normal vector field $\eta$ of multiplicity $q$. Show that the distribution $x \mapsto E_{\eta}(x)$ is smooth.

Exercise 1.16. Let $f: M^{n} \rightarrow \tilde{M}_{c}^{m}$ be an isometric immersion. If $\eta \in N_{f} M(x)$ is a principal normal of $f$ at $x \in M^{n}$, show that

$$
R^{\perp}(T, X) \xi=0
$$

for all $T \in E_{\eta}(x), X \in T_{x} M$ and $\xi \in N_{f} M(x)$. If $n=2$ conclude that $R^{\perp}(x)=0$.
Exercise 1.17. Let $\Phi \in \Gamma(\operatorname{End}(T M))$ be a symmetric Codazzi tensor on a Riemannian manifold $M^{n}$, that is, a symmetric tensor that satisfies the Codazzi-type equation

$$
\begin{equation*}
\left(\nabla_{X} \Phi\right) Y=\left(\nabla_{Y} \Phi\right) X \tag{1.46}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}(M)$. Let $\lambda \in C^{\infty}(M)$ be an eigenvalue of $\Phi$ such that $E_{\lambda}=\operatorname{ker}(\lambda I-\Phi)$ has constant rank $r$. Show that the following facts hold:
(i) $E_{\lambda}$ is an umbilical distribution with mean curvature normal $\eta$ given by

$$
(\lambda I-\Phi) \eta=(\operatorname{grad} \lambda)_{E_{\lambda}^{\perp}} .
$$

(ii) If $r \geq 2$, then $\lambda$ is constant along $E_{\lambda}$.
(iii) If $\lambda$ is constant along $E_{\lambda}$, then $E_{\lambda}$ is spherical.

Hint for $(i)$ : Taking the inner product with $S \in \Gamma\left(E_{\lambda}\right)$ of both sides of 1.46) for $Y=T \in \Gamma\left(E_{\lambda}\right)$ and $X \in \mathfrak{X}(M)$ yields

$$
\begin{equation*}
(\lambda I-\Phi) \nabla_{T} S=\langle T, S\rangle \operatorname{grad} \lambda-T(\lambda) S \tag{1.47}
\end{equation*}
$$

Since $\lambda I-\Phi$ vanishes on $E_{\lambda}$, then

$$
(\lambda I-\Phi) \nabla_{T} S=(\lambda I-\Phi)\left(\nabla_{T} S\right)_{E_{\lambda}^{\perp}} \in \Gamma\left(E_{\lambda}^{\perp}\right),
$$

and the result follows from (1.47).
Hint for (ii): Since the left-hand side of 1.47 is in $\Gamma\left(E_{\lambda}^{\perp}\right)$, it follows that

$$
\langle T, S\rangle(\operatorname{grad} \lambda)_{E_{\lambda}}=T(\lambda) S
$$

Hint for $(i i i):$ Since $(\operatorname{grad} \lambda)_{E_{\lambda}}=0$,

$$
\begin{aligned}
\left\langle\nabla_{T} \eta,(\lambda I-\Phi) X\right\rangle & =T\langle(\lambda I-\Phi) \eta, X\rangle-\left\langle\eta, \nabla_{T}(\lambda I-\Phi) X\right\rangle \\
& =T X(\lambda)-\lambda\left\langle\eta, \nabla_{T} X\right\rangle+\left\langle\nabla_{T} \Phi X, \eta\right\rangle .
\end{aligned}
$$

Now use

$$
\nabla_{T} \Phi X=\nabla_{X} \Phi T-\Phi \nabla_{X} T+\Phi \nabla_{T} X
$$

to obtain

$$
\left\langle\nabla_{T} \Phi X, \eta\right\rangle=\left\langle(\lambda I-\Phi) \eta, \nabla_{X} T\right\rangle+\left\langle\Phi \eta, \nabla_{T} X\right\rangle .
$$

Conclude that

$$
\begin{aligned}
\left\langle\nabla_{T} \eta,(\lambda I-\Phi) X\right\rangle & =T X(\lambda)-\langle(\lambda I-\Phi) \eta,[T, X]\rangle \\
& =T X(\lambda)-[T, X](\lambda) \\
& =0 .
\end{aligned}
$$

Exercise 1.18. Let $M^{n}$ be a Riemannian manifold of dimension $n \geq 3$. Assume that $\Phi \in \Gamma(\operatorname{End}(T M))$ is a symmetric tensor with only two eigenvalues $\lambda$ and $\mu$ everywhere of multiplicities $n-1$ and 1 , respectively. Show that $\Phi$ is a Codazzi tensor if and only if the following conditions are satisfied:
(i) $\lambda$ is constant along $E_{\lambda}=\operatorname{ker}(\lambda I-\Phi)$.
(ii) $E_{\lambda}$ is an umbilical distribution with mean curvature vector field

$$
\eta=\frac{1}{\lambda-\mu} \operatorname{grad} \lambda
$$

(iii) The mean curvature vector field (geodesic curvature vector field) of the onedimensional distribution $E_{\mu}=\operatorname{ker}(\mu I-\Phi)$ is

$$
\zeta=\frac{1}{\mu-\lambda}(\operatorname{grad} \mu)_{E_{\lambda}}
$$

Exercise 1.19. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}, n \geq 3$, be an isometric immersion of a simply connected Riemannian manifold. Given $\tilde{c} \in \mathbb{R}$, assume that $f$ has a principal curvature $\lambda$ of (constant) multiplicity either $n-1$ or $n$ satisfying

$$
\rho=c-\tilde{c}+\lambda^{2} \geq 0
$$

which is always the case if $c \geq \tilde{c}$. Show that there exists an isometric immersion $\tilde{f}: M^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+1}$, which is unique up to congruence if $\rho>0$.
Hint for the case in which $\lambda$ has multiplicity $n-1$ : Let $\mu$ be the principal curvature of multiplicity 1, and let $E_{\lambda}$ and $E_{\mu}$ be the corresponding eigenbundles. If $\lambda=0$, then $M^{n}$ has constant curvature $c$, hence it admits an umbilical isometric immersion into $\mathbb{Q}_{\tilde{c}}^{n+1}$ if $\rho=c-\tilde{c} \geq 0$. From now on, assume that $\lambda \neq 0$. By the assumption, there exist $\tilde{\lambda}, \tilde{\mu} \in C^{\infty}(M)$ such that

$$
c-\tilde{c}+\lambda^{2}=\tilde{\lambda}^{2} \text { and } c-\tilde{c}+\lambda \mu=\lambda \tilde{\mu} .
$$

${\underset{\sim}{~ M o r e o v e r, ~ t h e ~ f i r s t ~ o f ~ t h e ~ p r e c e d i n g ~ e q u a t i o n s ~ i m p l i e s ~ t h a t ~} \tilde{\lambda} \neq 0 \text { everywhere, and hence }}_{\sim}$ $\tilde{\lambda}$ and $\tilde{\mu}$ are unique if $\tilde{\lambda}$ is chosen to be positive. From both equations, we obtain

$$
\lambda^{2}-\tilde{\lambda}^{2}=\lambda \mu-\tilde{\lambda} \tilde{\mu}, \quad \lambda \operatorname{grad} \lambda=\tilde{\lambda} \operatorname{grad} \tilde{\lambda}
$$

and

$$
\mu \operatorname{grad} \lambda+\lambda \operatorname{grad} \mu=\tilde{\mu} \operatorname{grad} \tilde{\lambda}+\tilde{\lambda} \operatorname{grad} \tilde{\mu} .
$$

It follows that

$$
\begin{equation*}
\frac{1}{\tilde{\lambda}-\tilde{\mu}} \operatorname{grad} \tilde{\lambda}=\frac{1}{\lambda-\mu} \operatorname{grad} \lambda \tag{1.48}
\end{equation*}
$$

and similarly, that

$$
\begin{equation*}
\frac{1}{\tilde{\mu}-\tilde{\lambda}}(\operatorname{grad} \tilde{\mu})_{E_{\lambda}}=\frac{1}{\mu-\lambda}(\operatorname{grad} \mu)_{E_{\lambda}} \tag{1.49}
\end{equation*}
$$

Let $\tilde{A}$ be the endomorphism of $T M$ with eigenvalues $\tilde{\lambda}, \tilde{\mu}$ and corresponding eigenbundles $E_{\lambda}$ and $E_{\mu}$, respectively. Since

$$
c+\lambda^{2}=\tilde{c}+\tilde{\lambda}^{2} \text { and } c+\lambda \mu=\tilde{c}+\tilde{\lambda} \tilde{\mu}
$$

the Gauss equations for an isometric immersion $\tilde{f}: M^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+1}$ are satisfied by $\tilde{A}$. By Exercise 1.18 and Eqs. (1.48) and (1.49), it also satisfies the Codazzi equation.

Exercise 1.20. Let $\mathcal{F}$ be a family of $n$-dimensional submanifolds of a differentiable manifold $\tilde{M}^{m}, n<m$, with the property that the intersection of any two of its elements is either empty or a submanifold of $\tilde{M}^{m}$ with dimension less than $n$. Let $f: M^{n} \rightarrow \tilde{M}^{m}$ be an immersion such that, for each $x \in M^{n}$, there exist a neighborhood $V_{x}$ of $x$ and $S_{x} \in \mathcal{F}$ such that $f\left(V_{x}\right) \subset S_{x}$. Prove that $f(M)$ is an open subset of some $S \in \mathcal{F}$.
Hint: Given $x, y \in M^{n}$, if $V_{x} \cap V_{y} \neq \emptyset$, then $V_{x} \cap V_{y}$ is an open subset of $M^{n}$ such that

$$
f\left(V_{x} \cap V_{y}\right) \subset S_{x} \cap S_{y}
$$

If $S_{x} \neq S_{y}$, then $S_{x} \cap S_{y}$ is either empty or a submanifold of $\tilde{M}^{m}$ with dimension less than $n$ by the assumption, and we reach a contradiction with the fact that $f$ is an immersion. Therefore $S_{x}=S_{y}$ whenever $V_{x} \cap V_{y} \neq \emptyset$. Fix $x_{0} \in M^{n}$ and consider the subset

$$
A=\left\{x \in M^{n}: S_{x}=S_{x_{0}}\right\} .
$$

Clearly $A$ is nonempty, for $x_{0} \in A$. If $x \in A$, then $f\left(V_{x}\right)$ is an open subset of $S_{x}=S_{x_{0}}$, thus for any $y \in V_{x}$ we have $y \in V_{y} \cap V_{x}$, hence $S_{y}=S_{x}=S_{x_{0}}$. Therefore $V_{x} \subset A$, and we conclude that $A$ is open. On the other hand, if $\left\{x_{k}\right\}_{k \in \mathbb{N}}$ is a sequence of elements of $A$ that converges to a point $x \in M^{n}$, then there exists $N \in \mathbb{N}$ such that $x_{k} \in V_{x}$ for all $k \geq N$. Thus $x_{N} \in V_{x} \cap V_{x_{N}}$, which implies that $S_{x}=S_{x_{N}}=S_{x_{0}}$. It follows that $x \in A$, and hence $A$ is a closed subset of $M^{n}$. We conclude that $A=M^{n}$ by the connectedness of $M^{n}$, hence $f(M)$ is an open subset of $S_{x_{0}}$.
Exercise 1.21. Let $\beta: V \times V \rightarrow W$ be a bilinear map, where $V$ is a finite-dimensional vector space endowed with a positive definite inner product. Assume that $\beta(X, Y)=0$ for any pair $(X, Y) \in V \times V$ with $\langle X, Y\rangle=0$. Show that there exists a vector $\eta \in W$ such that

$$
\beta(X, Y)=\langle X, Y\rangle \eta
$$

for all $X, Y \in V$.
Exercise 1.22. Prove that each connected component of the fixed point set of an isometry of a Riemannian manifold is a totally geodesic submanifold.

Exercise 1.23. An isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ is said to be ruled if $M^{n}$ admits a codimension one foliation such that the restriction of $f$ to each leaf (ruling) is totally geodesic. Show that the following hypersurfaces are complete and ruled.
(i) Assume that the Frenet curvatures of the curve $c: \mathbb{R} \rightarrow \mathbb{R}^{n+1}$ do not vanish at any point and let $c^{\prime}=e_{1}, e_{2}, \ldots, e_{n+1}$ be the Frenet frame of $c$. Then let $F: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n+1}$ be the hypersurface given by

$$
F\left(s, t_{2}, \ldots, t_{n}\right)=c(s)+\sum_{j=2}^{n} t_{j} e_{j+1}(s)
$$

Similarly, for a curve $c: \mathbb{R} \rightarrow \mathbb{H}^{n+1}$, let $F: \mathbb{R}^{n} \rightarrow \mathbb{H}^{n+1}$ be defined by

$$
F\left(s, t_{2}, \ldots, t_{n}\right)=\exp _{c(s)}\left(\sum_{j=2}^{n} t_{j} e_{j+1}(s)\right) .
$$

(ii) Let $G: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n+1}$ the graph defined by

$$
G\left(x_{1}, \ldots, x_{n}\right)=\left(x_{1}, \ldots, x_{n}, \sum_{j=1}^{n-1} x_{j} \phi_{j}\left(x_{n}\right)\right)
$$

where $\phi_{j} \in C^{\infty}(\mathbb{R}), 1 \leq j \leq n-1$.
Exercise 1.24. The generalized Gauss map $G: M^{n} \rightarrow G_{n, m}$ of a given immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ assigns to each point $x \in M^{n}$ the point $f_{*}\left(T_{x} M\right)$ in the Grassmannian of $n$-planes in $\mathbb{R}^{m}$. Show that, endowing $M^{n}$ with the metric induced by $f$, the kernel of $G_{*}$ at $x \in M^{n}$ is the relative nullity subspace $\Delta(x) \subset T_{x} M$.

Exercise 1.25. If $f, g: M^{n} \rightarrow \mathbb{R}^{m}$ are immersions with the same generalized Gauss map, show that the following assertions hold:
(i) There exists $\Phi \in \Gamma(\operatorname{End}(T M))$ such that

$$
g_{*}=f_{*} \circ \Phi
$$

(ii) The tensor $\Phi$ is a Codazzi tensor on $M^{n}$ with respect to the metric induced by $f$.
(iii) The second fundamental form of $f$ commutes with $\Phi$, that is,

$$
\alpha^{f}(X, \Phi Y)=\alpha^{f}(\Phi X, Y)
$$

for all $X, Y \in \mathfrak{X}(M)$.
(iv) Conversely, if $M^{n}$ is simply connected and $\Phi \in \Gamma(\operatorname{End}(T M))$ has rank $n$ and satisfies (ii) and (iii), then there exists an immersion $g: M^{n} \rightarrow \mathbb{R}^{m}$ such that $g_{*}=f_{*} \circ \Phi$.
(v) The Levi-Civita connections of the metrics induced by $f$ and $g$ are related by

$$
\Phi \nabla_{X} Y=\nabla_{X} \Phi Y
$$

for all $X, Y \in \mathfrak{X}(M)$.
(vi) The second fundamental forms of $f$ and $g$ are related by

$$
\alpha^{g}(X, Y)=\alpha^{f}(\Phi X, Y)
$$

for all $X, Y \in \mathfrak{X}(M)$.
Hint for parts (ii), (iii) and (iv): Regard $\omega=f_{*} \circ \Phi$ as a one-form on $M^{n}$ with values in $\mathbb{R}^{m}$.

Exercise 1.26. Let $f: M^{n} \rightarrow \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ be an isometric immersion of an oriented Riemannian manifold. Then the unit normal vector field of $f$ induces a mapping $\nu: M^{n} \rightarrow \mathbb{S}^{n+1}$ called the spherical Gauss map of $f$.
(i) Prove that $\nu$ is an immersion provided that the second fundamental form $A$ of $f$ is everywhere nonsingular.
(ii) Compute the metric induced by $\nu$ and prove that its second fundamental form is $A^{-1}$.

Exercise 1.27. Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be an isometric immersion and let

$$
N_{f}^{1} M=\left\{(x, w) \in N_{f} M:\|w\|=1\right\}
$$

be its unit normal bundle. The generalized spherical Gauss map $\phi: N_{f}^{1} M \rightarrow \mathbb{S}^{m-1}$ is defined by

$$
\phi(x, w)=w .
$$

(i) Show that

$$
\phi_{*}(x, w) v=v
$$

for any vertical vector $v \in T_{(x, w)} N_{f}^{1} M$ and that

$$
\phi_{*}(x, w) Z^{h}=-f_{*} A_{w} Z,
$$

where $Z^{h}$ is the horizontal lift of $Z \in T_{x} M$.
(ii) Show that the height function $h^{w}$ has a degenerate critical point if and only if $w$ is a critical value of $\phi$.
(iii) Conclude that $h^{w}$ is a Morse function for almost all $w \in \mathbb{S}^{m-1}$.

Exercise 1.28. Let $f: M^{n} \rightarrow \mathbb{Q}_{\epsilon}^{n+1}, \epsilon \in\{-1,0,1\}$, be an oriented hypersurface and let $N$ be its unit normal vector field. If $\epsilon \in\{-1,1\}$, consider the standard model of $\mathbb{Q}_{\epsilon}^{n+1}$ as an umbilical hypersurface of either Euclidean space $\mathbb{R}^{n+2}$ or Lorentzian space $\mathbb{L}^{n+2}$, according to whether $\epsilon=1$ or $\epsilon=-1$, respectively. For each $t>0$, define $f_{t}(x) \in \mathbb{Q}_{\epsilon}^{n+1}$ to be the point on the geodesic starting from $f(x)$ in the direction $N_{x}$ at geodesic distance $t$ from $f(x)$, that is,

$$
\begin{aligned}
& f_{t}(x)=f(x)+t N_{x} \text { if } \epsilon=0 \\
& f_{t}(x)=\cos t f(x)+\sin t N_{x} \quad \text { if } \epsilon=1, \\
& f_{t}(x)=\cosh t f(x)+\sinh t N_{x} \quad \text { if } \epsilon=-1 .
\end{aligned}
$$

(i) Show that a point $x \in M^{n}$ is regular for the parallel hypersurface $f_{t}$ if and only if the endomorphism $P_{t}$ is nonsingular at $x$, where

$$
\begin{aligned}
& P_{t}=I-t A \text { if } \epsilon=0, \\
& P_{t}=\cot t I-A \text { if } \epsilon=1 \\
& P_{t}=\operatorname{coth} t I-A \text { if } \epsilon=-1 .
\end{aligned}
$$

(ii) Verify that, at regular points, a unit normal vector field to $f_{t}$ is given by

$$
\begin{aligned}
& N_{t}=N \text { if } \epsilon=0, \\
& N_{t}=-\sin t f+\cos t N \text { if } \epsilon=1, \\
& N_{t}=\sinh t f+\cosh t N \text { if } \epsilon=-1 .
\end{aligned}
$$

(iii) Show that the shape operator of $f_{t}$ with respect to $N_{t}$ is

$$
\begin{aligned}
& A_{t}=(I-t A)^{-1} A \text { if } \epsilon=0, \\
& A_{t}=(\cot t I-A)^{-1}(I+\cot t A) \text { if } \epsilon=1, \\
& A_{t}=(\operatorname{coth} t I-A)^{-1}(-I+\operatorname{coth} t A) \text { if } \epsilon=-1 .
\end{aligned}
$$

(iv) Show that $f_{t}$ has constant mean curvature for each $t$ if and only if $f$ has constant principal curvatures.
(v) If $f: M^{2} \rightarrow \mathbb{S}^{3}$ has constant Gauss curvature $K=1+r^{2}>1$ (respectively, constant mean curvature $H=r$ ) and $t=\tan ^{-1}(1 / r)$ (respectively, $t=\cot ^{-1}(r / 2)$ ), show that, on the open subset of regular points, the surface $f_{t}$ has constant mean curvature $\left(1-r^{2}\right) / r$ (respectively, constant mean curvature $-r$ ).
(vi) If $f: M^{2} \rightarrow \mathbb{H}^{3}$ has constant Gauss curvature $K=-1+r^{2}>2$ (respectively, constant mean curvature $H=r>2$ ) and $t=\tanh ^{-1}(1 / r)$ (respectively, $t=$ $\tanh ^{-1}(2 / r)$ ), show that, on the open subset of regular points, the surface $f_{t}$ has constant mean curvature $\left(1+r^{2}\right) / r$ (respectively, constant mean curvature $-r$ ).

Exercise 1.29. Give an example of a nontotally geodesic isometric immersion of $\mathbb{S}^{n}$ into $\mathbb{S}^{2 n+1}$.
Hint: See part (i) of Exercise 1.7.
Exercise 1.30. Show that if $f: M^{n} \rightarrow \tilde{M}^{m}$ is an extrinsic sphere, then the following facts hold:
(i) The second fundamental form $\alpha$ of $f$ is parallel, that is,

$$
\left(\nabla_{X}^{\perp} \alpha\right)(Y, Z)=0
$$

for all $X, Y, Z \in \mathfrak{X}(M)$.
(ii)

$$
R(X, Y) Z=\tilde{R}(X, Y) Z+\|\mathcal{H}\|^{2}(X \wedge Y) Z
$$

for all $x \in M^{n}$ and $X, Y, Z \in T_{x} M$.
(iii)

$$
\tilde{R}(X, Y) \xi=R^{\perp}(X, Y) \xi
$$

for all $x \in M^{n}, X, Y \in T_{x} M$ and $\xi \in N_{f} M(x)$.

Exercise 1.31. Let $f: M^{n} \rightarrow \tilde{M}^{m}$ be an isometric immersion with parallel second fundamental form. Prove that $f$ has parallel mean curvature vector field. If, in addition, $\tilde{M}^{m}=\tilde{M}_{c}^{m}$, show that the following conditions hold:
(i) $\nabla R=0$, that is, $M^{n}$ is locally symmetric.
(ii) $\nabla^{\perp} R^{\perp}=0$.

Exercise 1.32. Let $h: M^{n} \rightarrow \mathbb{R}^{m}$ and $g: L^{k} \rightarrow M^{n}, k \geq 2$, be isometric immersions. Show that if $g$ is umbilical and

$$
\alpha^{h}\left(g_{*} X, Z\right)=0
$$

for all $X \in \mathfrak{X}(L)$ and $Z \in \Gamma\left(N_{g} L\right)$, then $g$ is an extrinsic sphere.
Hint: Since $g$ is umbilical, the Codazzi equation for $g$ yields

$$
\begin{equation*}
\langle Y, T\rangle \nabla_{X}^{\perp} \mathcal{H}^{g}-\langle X, T\rangle \nabla_{Y}^{\perp} \mathcal{H}^{g}=\left(R\left(g_{*} X, g_{*} Y\right) g_{*} T\right)_{N_{g} L} \tag{1.50}
\end{equation*}
$$

for all $X, Y, T \in \mathfrak{X}(L)$, where $R$ is the curvature tensor of $M^{n}$ and $\mathcal{H}^{g}$ is the mean curvature vector field of $g$. The Gauss equation of $h$ and the assumption on $\alpha^{h}$ give

$$
\begin{aligned}
R\left(g_{*} X, g_{*} Y\right) Z & =A_{\alpha^{h}\left(g_{*} Y, Z\right)}^{g_{*} X-} A_{\alpha^{h}\left(g_{*} X, Z\right)}^{h} g_{*} Y \\
& =0
\end{aligned}
$$

for all $X, Y \in \mathfrak{X}(L)$ and $Z \in \Gamma\left(N_{g} L\right)$, and hence the right-hand side of 1.50 vanishes. Choosing $Y=T$ orthogonal to $X$ implies that $\mathcal{H}^{g}$ is parallel in the normal connection.

Exercise 1.33. Let $f: M^{n} \rightarrow \tilde{M}^{m}$ be an isometric immersion and let $N$ be a normal subbundle. Show that $N$ is a parallel subbundle if and only if for every normal section $\xi$ that is parallel along a curve $\gamma:[a, b] \rightarrow M^{n}$, and such that $\xi(\gamma(a)) \in N$, one has $\xi(\gamma(t)) \in N$ for all $t \in[a, b]$.
Exercise 1.34. Let $f: M^{n} \rightarrow \tilde{M}_{c}^{m}$ be an isometric immersion and let $\xi \in \Gamma\left(N_{f} M\right)$ be a parallel normal vector field such that the shape operator $A_{\xi}$ has $n$ distinct eigenvalues. Show that $f$ has flat normal bundle.
Exercise 1.35. Let $f: M^{n} \rightarrow \tilde{M}_{c}^{m}$ be an isometric immersion. Assume that at some $x \in M^{n}$ there exists a subspace $W(x) \subset N_{f} M(x)$ such that

$$
R^{\perp}(X, Y) \xi=0
$$

for all $X, Y \in T_{x} M$ and $\xi \in W(x)$. Show that there exist unique pairwise distinct vectors $\eta_{i}(x) \in W(x), 1 \leq i \leq s$, called the principal normals of $f$ at $x$ with respect to $W(x)$, and an orthogonal decomposition

$$
T_{x} M=E_{1}(x) \oplus \cdots \oplus E_{s}(x)
$$

such that

$$
A_{\xi} X=\sum_{i=1}^{s}\left\langle\xi, \eta_{i}(x)\right\rangle X^{i}
$$

for all $\xi \in W(x)$, where $X \in T_{x} M \mapsto X^{i}$ is the orthogonal projection onto $E_{i}(x)$.

Exercise 1.36. Let $f: M^{n} \rightarrow \mathbb{Q}_{\epsilon}^{m}, \epsilon \in\{-1,0,1\}$, be an isometric immersion with flat normal bundle and let $\eta \in \Gamma\left(N_{f} M\right)$ be a parallel unit vector field. If $\epsilon \in\{-1,1\}$, consider the standard model of $\mathbb{Q}_{\epsilon}^{m}$ as an umbilical hypersurface of either Euclidean space $\mathbb{R}^{m+1}$ or Lorentzian space $\mathbb{L}^{m+1}$, according to whether $\epsilon=1$ or $\epsilon=-1$, respectively. For each $t>0$, define the parallel submanifold $f_{t}(x) \in \mathbb{Q}_{\epsilon}^{m}$ by

$$
\begin{aligned}
& f_{t}(x)=f(x)+t \eta(x) \text { if } \epsilon=0, \\
& f_{t}(x)=\cos t f(x)+\sin t \eta(x) \text { if } \epsilon=1, \\
& f_{t}(x)=\cosh t f(x)+\sinh t \eta(x) \text { if } \epsilon=-1 .
\end{aligned}
$$

At regular points show that also $f_{t}$ has flat normal bundle.
Exercise 1.37. Let $f: M^{n} \rightarrow \mathbb{R}^{m+1}$ be an isometric immersion with flat normal bundle and let $\eta \in \Gamma\left(N_{f} M\right)$ be a parallel unit vector field such that the shape operator $A_{\eta}$ is nonsingular at any point. Show that the map $\eta: M^{n} \rightarrow \mathbb{S}^{m}$ is an immersion with flat normal bundle.

Exercise 1.38. An isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ with flat normal bundle is called holonomic if $M^{n}$ carries global orthogonal coordinates $\left(u_{1}, \ldots, u_{n}\right)$ such that the coordinate vector fields are everywhere eigenvectors of all shape operators of $f$.
For a holonomic isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$, let $\xi_{1}, \ldots, \xi_{p}$ be an orthonormal frame of $N_{f} M$ of parallel vector fields. Set $v_{j}=\left\|\partial / \partial u_{j}\right\|$ and define $V_{j r} \in C^{\infty}(M)$ by

$$
A_{\xi_{r}} \partial / \partial u_{j}=\frac{V_{j r}}{v_{j}} \partial / \partial u_{j}, 1 \leq j \leq n, 1 \leq r \leq p
$$

For $h_{i j}$ as in 1.23), set $h=\left(h_{i j}\right), v=\left(v_{1}, \ldots, v_{n}\right)$ and $V=\left(V_{i r}\right) \in M_{n \times p}(\mathbb{R})$.
(i) Show that the triple $(v, h, V)$, called the triple associated with $f$ with respect to $\left(u_{1}, \ldots, u_{n}\right)$ and $\xi_{1}, \ldots, \xi_{p}$, satisfies the system of partial differential equations

$$
\left\{\begin{array}{l}
\text { (i) } \frac{\partial v_{i}}{\partial u_{j}}=h_{j i} v_{j}  \tag{1.51}\\
\text { (ii) } \frac{\partial h_{i k}}{\partial u_{j}}=h_{i j} h_{j k} \\
\text { (iii) } \frac{\partial h_{i j}}{\partial u_{i}}+\frac{\partial h_{j i}}{\partial u_{j}}+\sum_{k} h_{k i} h_{k j}+\sum_{r} V_{i r} V_{j r}+c v_{i} v_{j}=0 \\
\text { (iv) } \frac{\partial V_{i r}}{\partial u_{j}}=h_{j i} V_{j r}, \quad 1 \leq i \neq j \neq k \neq i \leq n .
\end{array}\right.
$$

(ii) Prove that, conversely, if $(v, h, V)$ is a solution of 1.51) on an open simply connected subset $U \subset \mathbb{R}^{n}$ such that $v_{i} \neq 0$ at any point, then there exists a holonomic immersion $f: U \rightarrow \mathbb{Q}_{c}^{n+p}$ and an orthonormal frame $\xi_{1}, \ldots, \xi_{p}$ of $N_{f} U$ of parallel normal vector fields with respect to which the triple associated with $f$ is $(v, h, V)$.

Hint for $(i)$ : Equation $(i)$ is merely the definition of $h_{i j}$, and the remaining ones follow using (1.22) by computing the Gauss and Codazzi equations.
Hint for (ii): Endow $U$ with the metric $d s^{2}=\sum_{i} v_{i}^{2} d u_{i}^{2}$ and set $M^{n}=\left(U, d s^{2}\right)$. Consider the trivial vector bundle $E=M^{n} \times \mathbb{R}^{p}$ endowed with a flat connection $\nabla^{\prime}$ and let $e_{1}, \ldots, e_{p}$ be a parallel orthonormal frame of $E$. Define $\alpha: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \Gamma(E)$ by

$$
\alpha\left(\partial / \partial u_{i}, \partial / \partial u_{j}\right)=\sum_{r=1}^{p} \delta_{i j} v_{i} V_{i r} e_{r} .
$$

Then the Gauss and Codazzi equations follow from (1.51) and the Ricci equation is satisfied because $\nabla^{\prime}$ is flat and $\alpha$ is orthogonally diagonalizable. Now use the Fundamental theorem of submanifolds.

Exercise 1.39. (i) Let $M^{n}$ be a differentiable manifold whose tangent bundle $T M$ splits as the Whitney sum of integrable subbundles $E_{j}$ with ranks $n_{j}, 1 \leq j \leq k$,

$$
T M=\oplus_{j=1}^{k} E_{j} .
$$

Write

$$
E_{j}^{\perp}=\oplus_{i=1, i \neq j}^{k} E_{i}, \quad 1 \leq j \leq k .
$$

Show that the following assertions are equivalent:
(a) Each point $x \in M^{n}$ lies in an open neighborhood $U \subset M^{n}$ where one can define local coordinates $\left(u_{1}, \ldots, u_{n}\right)$ such that

$$
E_{1}=\operatorname{span}\left\{\partial / \partial u_{1}, \ldots, \partial / \partial u_{n_{1}}\right\}, \ldots, E_{k}=\operatorname{span}\left\{\partial / \partial u_{n-n_{k}+1}, \ldots, \partial / \partial u_{n}\right\}
$$

(b) The distribution $E_{j}^{\perp}$ is integrable for all $1 \leq j \leq k$.
(c) The distribution $E_{i} \oplus E_{j}$ is integrable for all $1 \leq i \neq j \leq k$.
(ii) Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be an isometric immersion with flat normal bundle carrying principal normal vector fields $\eta_{1}, \ldots, \eta_{k}$ with corresponding eigendistributions $E_{\eta_{1}}, \ldots, E_{\eta_{k}}$ of constant ranks $n_{1}, \ldots, n_{k}$, respectively. Show that $f$ is locally holonomic if and only if $E_{\eta_{j}}^{\perp}$ is an integrable distribution for all $1 \leq j \leq k$.

Hint for $(i)$ : Assume that the distribution $E_{j}^{\perp}$ is integrable for all $1 \leq j \leq k$. By Frobenius theorem, each point $x \in M^{n}$ lies in an open neighborhood $U \subset M^{n}$ where one can define local coordinates $\left(u_{1}^{j}, \ldots, u_{n}^{j}\right)$ such that

$$
E_{j}^{\perp}=\operatorname{span}\left\{\partial / \partial u_{n-n_{j}+1}^{j}, \ldots, \partial / \partial^{j} u_{n}\right\} .
$$

Then $f^{j}=\left(u_{1}^{j}, \ldots, u_{n_{j}}^{j}\right): U \rightarrow \mathbb{R}^{n_{j}}$ is a submersion with $\operatorname{ker} f_{*}^{j}=\left.E_{j}^{\perp}\right|_{U}$. One can thus choose $U$ so that $f=\left(f^{1}, \ldots, f^{k}\right): U \rightarrow \mathbb{R}^{n}$ defines a coordinate system with

$$
f(U)=\left\{x \in \mathbb{R}^{n}:\left|x_{i}\right| \leq \epsilon \text { for all } 1 \leq i \leq k\right\}
$$

which has the desired property, for $f_{*}^{j}(y) X_{i}=0$ whenever $y \in U, X_{i} \in E_{i}(q)$ and $1 \leq i \neq j \leq k$.

Exercise 1.40. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be an isometric immersion with flat normal bundle carrying principal normal vector fields $\eta_{1}, \ldots, \eta_{k}$ with corresponding eigendistributions $E_{\eta_{1}}, \ldots, E_{\eta_{k}}$ of constant ranks $n_{1}, \ldots, n_{k}$, respectively. Given $1 \leq \ell \leq k$, show that $E_{\eta_{\ell}}^{\perp}$ is integrable if the vectors $\eta_{i}-\eta_{\ell}$ and $\eta_{j}-\eta_{\ell}$ are everywhere linearly independent for any pair of indices $1 \leq i \neq j \leq k$ with $i, j \neq \ell$.
Hint: Use the Codazzi equations (1.42).
Exercise 1.41. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be an isometric immersion with flat normal bundle carrying principal normal vector fields $\eta_{1}, \ldots, \eta_{k}$ with corresponding eigendistributions $E_{\eta_{1}}, \ldots, E_{\eta_{k}}$ of constant ranks $n_{1}, \ldots, n_{k}$, respectively. At $x \in M^{n}$, define

$$
S_{f}(x)=\operatorname{span}\left\{\eta_{i}(x)-\eta_{j}(x): 1 \leq i, j \leq k\right\} .
$$

(i) Prove that $\operatorname{dim} S_{f}(x) \leq k-1$ and that

$$
\operatorname{dim} N_{1}(x)-1 \leq \operatorname{dim} S_{f}(x) \leq \operatorname{dim} N_{1}(x)
$$

(ii) If $\operatorname{dim} S_{f}(x)=\operatorname{dim} N_{1}(x)-1$ everywhere, show that a unit vector field $\zeta \in \Gamma\left(N_{1}\right)$ orthogonal to $S_{f}$ is an umbilical vector field (if $g: M^{n} \rightarrow \tilde{M}^{m}$ is an isometric immersion, then $\zeta \in \Gamma\left(N_{g} M\right)$ is said to be an umbilical vector field if $A_{\zeta}^{g}=\lambda I$ for some $\lambda \in C^{\infty}(M)$, where $I$ is the identity endomorphism).
(iii) Show that $f$ is locally holonomic if $\operatorname{dim} S_{f}(x)=k-1$ for every $x \in M^{n}$.
(iv) Conclude that $f$ is locally holonomic if $\operatorname{dim} N_{1}(x)=n$ for every $x \in M^{n}$ or if $\operatorname{dim} N_{1}(x)=n-1$ for every $x \in M^{n}$ and there does not exist an umbilical vector field $\zeta \in \Gamma\left(N_{1}\right)$.

Hint for (ii): Use 1.37).
Hint for (iii): Use Exercise 1.40 and part (ii) of Exercise 1.39 .
Exercise 1.42. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be an isometric immersion with flat normal bundle such that for any $x \in M^{n}$ and any plane $\sigma \subset T_{x} M$ the extrinsic curvature

$$
K_{f}(\sigma)=K_{M}(\sigma)-c
$$

along $\sigma$ is negative. Show that $f$ admits $n$ pairwise distinct principal normal vector fields and that $f$ is locally holonomic.

Hint: Using (1.40) and the assumption that the extrinsic curvatures of $f$ are negative, show that for any $x \in M^{n}$ and any principal normal $\eta$ of $f$ at $x$ the eigenspace $E_{\eta}(x)$ must be one-dimensional. Conclude that there must exist $n$ distinct principal normals $\eta_{1}(x), \ldots, \eta_{n}(x)$ at any $x \in M^{n}$. To show that

$$
\left\langle\nabla_{X_{j}} X_{i}, X_{k}\right\rangle=0
$$

for all $X_{i} \in \Gamma\left(E_{\eta_{i}}\right), X_{j} \in \Gamma\left(E_{\eta_{j}}\right)$ and $X_{k} \in \Gamma\left(E_{\eta_{k}}\right)$ with $1 \leq i \neq j \neq k \neq i \leq n$, assume otherwise and use (1.42) to prove that

$$
\begin{equation*}
\eta_{i}=(1-\lambda) \eta_{k}+\lambda \eta_{j} \tag{1.52}
\end{equation*}
$$

for some nowhere vanishing $\lambda \in C^{\infty}(M)$. Show that (1.52) implies that

$$
\begin{equation*}
\left\langle\eta_{i}, \eta_{j}\right\rangle^{2}-\left\|\eta_{i}\right\|^{2}\left\|\eta_{j}\right\|^{2}=\left\langle\eta_{j}, \eta_{k}\right\rangle\left(\left\langle\eta_{i}, \eta_{j}\right\rangle-\left\|\eta_{i}\right\|^{2}\right)+\left\langle\eta_{i}, \eta_{k}\right\rangle\left(\left\langle\eta_{i}, \eta_{j}\right\rangle-\left\|\eta_{j}\right\|^{2}\right) \tag{1.53}
\end{equation*}
$$

Use (1.39) and the assumption that the extrinsic curvatures of $f$ are negative to show that the right-hand side of $(1.53)$ is positive. Obtain a contradiction by noticing that the left-hand side of (1.53) is nonpositive by the Cauchy-Schwarz inequality.

## Chapter 2

## Reduction of codimension

The study of isometric immersions becomes increasingly difficult for higher values of the codimension. Therefore, it is important to investigate whether the codimension of an isometric immersion into a space of constant sectional curvature can be reduced. That an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ admits a reduction of codimension to $q<p$ means that there exists a totally geodesic submanifold $\mathbb{Q}_{c}^{n+q}$ in $\mathbb{Q}_{c}^{n+p}$ such that $f(M) \subset \mathbb{Q}_{c}^{n+q}$. The possibility of reducing the codimension fits into the fundamental problem of determining the least possible codimension of an isometric immersion of a given Riemannian manifold into a space of constant sectional curvature.

The starting point for the results of this chapter is the basic fact that the codimension of $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ can be reduced to $q<p$ whenever its normal bundle has a parallel subbundle of rank $p-q$ such that the shape operators with respect to any of its sections vanish everywhere.

The main result of the chapter provides necessary and sufficient conditions for an isometric immersion to admit a reduction of codimension, under a certain regularity assumption, in terms of the normal curvature tensor and the mean curvature vector field. Sufficient conditions are also discussed in terms of the s-nullities and the type number of the isometric immersion. Both concepts play a key role in the study of rigidity aspects of submanifolds in Chapter 4.

As an application of the results discussed in earlier sections, at the end of the chapter we present the classification of constant curvature submanifolds with flat normal bundle and parallel mean curvature vector field of space forms.

### 2.1 Basic facts

The first normal space $N_{1}(x)$ of an isometric immersion $f: M^{n} \rightarrow \tilde{M}^{m}$ at $x \in M^{n}$ is the subspace of its normal space $N_{f} M(x)$ spanned by the image of its second fundamental form $\alpha$ at $x$, that is,

$$
N_{1}(x)=\operatorname{span}\left\{\alpha(X, Y): X, Y \in T_{x} M\right\} .
$$

Notice that the orthogonal complement of $N_{1}(x)$ in $N_{f} M(x)$ is

$$
N_{1}^{\perp}(x)=\left\{\xi \in N_{f} M(x): A_{\xi}=0\right\} .
$$

The following is the basic result on reduction of codimension.
Proposition 2.1. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ be an isometric immersion. Suppose that there exists a parallel subbundle $L$ of the normal bundle $N_{f} M$ with rank $q<p$ such that $N_{1}(x) \subset L(x)$ for all $x \in M^{n}$. Then the codimension of $f$ can be reduced to $q$.

Proof: Case $c=0$. Fix an arbitrary point $x_{0}$ in $M^{n}$. We show next that

$$
f(M) \subset f\left(x_{0}\right)+f_{*} T_{x_{0}} M \oplus L\left(x_{0}\right) .
$$

First observe that $L^{\perp}$ is a parallel subbundle of $f^{*} T \mathbb{R}^{n+p}$. In fact, since $L^{\perp}(x) \subset N_{1}^{\perp}(x)$ for all $x \in M^{n}$ and $L^{\perp}$ is parallel with respect to the normal connection, then

$$
\tilde{\nabla}_{X} \xi=-f_{*} A_{\xi} X+\nabla_{X}^{\perp} \xi=\nabla_{X}^{\perp} \xi \in \Gamma\left(L^{\perp}\right)
$$

for all $\xi \in \Gamma\left(L^{\perp}\right)$. Here $\tilde{\nabla}$ stands for the connection on $f^{*} T \mathbb{R}^{n+p}$. Thus any $\eta \in L^{\perp}\left(x_{0}\right)$ also belongs to $L^{\perp}(x) \subset N_{f} M(x)$ for all $x \in M^{n}$. Given any $x \in M^{n}$ and $X \in T_{x} M$, it follows that

$$
X\left\langle f-f\left(x_{0}\right), \eta\right\rangle=0 .
$$

Hence the function $x \mapsto\left\langle f(x)-f\left(x_{0}\right), \eta\right\rangle$ vanishes on $M^{n}$ for any $\eta \in L^{\perp}\left(x_{0}\right)$, and this completes the proof in this case.
Case $c>0$. Consider the isometric immersion $\tilde{f}: M^{n} \rightarrow \mathbb{R}^{n+p+1}$ given by $\tilde{f}=i \circ f$, where $i: \mathbb{S}_{c}^{n+p} \rightarrow \mathbb{R}^{n+p+1}$ denotes the canonical inclusion. Then

$$
N_{\tilde{f}} M(x)=i_{*} N_{f} M(x) \oplus \operatorname{span}\{\tilde{f}(x)\}
$$

for any $x \in M^{n}$. The first normal spaces $N_{1}(x)$ and $N_{1}^{\tilde{f}}(x)$ of $f$ and $\tilde{f}$, respectively, at $x$ are related by

$$
N_{1}^{\tilde{f}}(x) \subset i_{*} N_{1}(x) \oplus \operatorname{span}\{\tilde{f}(x)\}
$$

and hence

$$
N_{1}^{\tilde{f}}(x) \subset \tilde{L}(x)=i_{*} L(x) \oplus \operatorname{span}\{\tilde{f}(x)\}
$$

Note that the orthogonal complement $\tilde{L}^{\perp}(x)$ of $\tilde{L}(x)$ in $N_{\tilde{f}} M(x)$ is $i_{*} L^{\perp}(x)$. Given $\xi \in \Gamma\left(L^{\perp}\right)$, since $L^{\perp} \subset N_{1}^{\perp}$ we obtain

$$
\begin{aligned}
\tilde{\nabla}_{X} i_{*} \xi & =i_{*} \bar{\nabla}_{X} \xi+\alpha^{i}\left(f_{*} X, \xi\right) \\
& =i_{*} \nabla \nabla_{X}^{\perp} \xi \in \Gamma\left(i_{*} L^{\perp}\right)=\Gamma\left(\tilde{L}^{\perp}\right)
\end{aligned}
$$

for all $X \in \mathfrak{X}(M)$. Here $\bar{\nabla}$ stands for the connection on $f^{*} T \mathbb{S}_{c}^{n+p}$. Hence $\tilde{L}^{\perp}$ is a parallel subbundle of $\tilde{f}^{*} T \mathbb{R}^{n+p+1}$.

By the previous case, for any fixed $x_{0} \in M^{n}$ this implies that

$$
\tilde{f}(M) \subset \tilde{f}\left(x_{0}\right)+\tilde{f}_{*} T_{x_{0}} M \oplus \tilde{L}\left(x_{0}\right)=\tilde{f}_{*} T_{x_{0}} M \oplus i_{*} L\left(x_{0}\right) \oplus \operatorname{span}\left\{\tilde{f}\left(x_{0}\right)\right\}
$$

which is a $(n+q+1)$-dimensional linear subspace of $\mathbb{R}^{n+p+1}$ that we denote by $\mathbb{R}^{n+q+1}$. Thus

$$
\tilde{f}(M) \subset \mathbb{S}_{c}^{n+p} \cap \mathbb{R}^{n+q+1}=\mathbb{S}_{c}^{n+q} .
$$

Case $c<0$. This is analogous to the previous case, by considering the isometric immersion $\tilde{f}: M^{n} \rightarrow \mathbb{L}^{n+p+1}$ given by $\tilde{f}=i \circ f$, where $i: \mathbb{H}_{c}^{n+p} \rightarrow \mathbb{L}^{n+p+1}$ is an umbilical inclusion. Details are left as an exercise.

An isometric immersion $f: M^{n} \rightarrow \tilde{M}^{m}$ is said to be 1-regular if the dimension of its first normal spaces $N_{1}(x)$ is constant along $M^{n}$, in which case these normal subspaces form a subbundle of $N_{f} M$ (see Exercise 2.1) called the first normal bundle of $f$ and denoted by $N_{1}=N_{1}^{f}$.

The substantial codimension of an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ is the smallest number to which the codimension of $f$ can be reduced. If the codimension of $f$ cannot be reduced, then $f$ is said to be substantial. A particular case of Proposition 2.1 is the following result.

Corollary 2.2. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ be a 1-regular isometric immersion such that the first normal bundle $N_{1}$ is a parallel subbundle of $N_{f} M$ with rank $q<p$. Then $f$ has substantial codimension $q$.

The assumption of 1-regularity in Corollary 2.2 is necessary as shown by the next simple example.

Example 2.3. Let $\gamma: \mathbb{R} \rightarrow \mathbb{R}^{3}$ be the smooth curve given by

$$
\gamma(t)= \begin{cases}\left(t, e^{-1 / t^{2}}, 0\right) & \text { for } t>0 \\ (0,0,0) & \text { for } t=0 \\ \left(t, 0, e^{-1 / t^{2}}\right) & \text { for } t<0\end{cases}
$$

Then $\gamma$ has a parallel first normal bundle of rank one on $(-\infty, 0)$ and $(0,+\infty)$. The restriction of $\gamma$ to both intervals has substantial codimension one, but the substantial codimension of $\gamma$ is two.

### 2.2 The parallelism of the first normal bundle

The main result of this section provides necessary and sufficient conditions for the parallelism of the first normal bundle of a 1-regular isometric immersion into a space form. Sufficient conditions for the parallelism in terms of its $s$-nullities or its type number are also given.

### 2.2.1 The main result

First we give a simple proof of a particular case.
Proposition 2.4. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be a 1-regular isometric immersion with flat normal bundle and parallel mean curvature vector field. Then the first normal bundle $N_{1}$ is parallel and has rank $k \leq \min \{n, m-n\}$.

Proof: At each $x \in M^{n}$, it follows from (1.36) that the first normal space $N_{1}(x)$ of $f$ at $x$ is the subspace of $N_{f} M(x)$ spanned by the principal normals $\eta_{1}(x), \ldots, \eta_{s(x)}(x)$ of $f$ at $x$. In particular, this implies that $k \leq s(x) \leq n$. To show that $N_{1}$ is parallel, it suffices to do the same for $N_{1}^{\perp}$. For that, it is enough to prove that, on any open subset where $s=s(x)$ is constant, we have

$$
\begin{equation*}
\left\langle\nabla \stackrel{\perp}{X_{i}} \delta, \eta_{j}\right\rangle=0, \quad 1 \leq i, j \leq s \tag{2.1}
\end{equation*}
$$

for all $X_{i} \in \Gamma\left(E_{\eta_{i}}\right)$ and $\delta \in \Gamma\left(N_{1}^{\perp}\right)$. If $i \neq j$, this is an immediate consequence of 1.41) regardless of the assumption on the mean curvature vector field. On the other hand, since the mean curvature vector field is given by

$$
\mathcal{H}=\frac{1}{n} \sum_{i=1}^{s} d_{i} \eta_{i}
$$

where $d_{i}=\operatorname{rank} E_{\eta_{i}}$, using (2.1) for $i \neq j$ and the fact that $\mathcal{H}$ is parallel yields

$$
0=n\left\langle\nabla \frac{\perp}{X_{i}} \delta, \mathcal{H}\right\rangle=d_{i}\left\langle\nabla \stackrel{\perp}{X_{i}} \delta, \eta_{i}\right\rangle .
$$

The above result has the following useful application.
Proposition 2.5. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be a 1-regular isometric immersion with flat normal bundle and parallel mean curvature vector field. Let $k \leq n$ denote the rank of $N_{1}$. Then $f(M)$ is contained in a totally geodesic submanifold $\mathbb{Q}_{c}^{n+k} \subset \mathbb{Q}_{c}^{m}$. Moreover, if $k=n$ then $f(M)$ is contained in an umbilical submanifold $\mathbb{Q}_{\tilde{c}}^{2 n-1} \subset \mathbb{Q}_{c}^{2 n} \subset \mathbb{Q}_{c}^{m}$.

Proof: It follows from Proposition 2.4 that $N_{1}$ is parallel, thus $f(M)$ is contained in a totally geodesic submanifold $\mathbb{Q}_{c}^{n+k} \subset \mathbb{Q}_{c}^{m}$ by Corollary 2.2. In particular, if $k=n$ we can regard $f$ as an isometric immersion into $\mathbb{Q}_{c}^{2 n}$.

It remains to show that $f(M)$ is contained in an umbilical submanifold $\mathbb{Q}_{\bar{c}}^{2 n-1} \subset$ $\mathbb{Q}_{c}^{2 n}$. Choose $c^{\prime}<c$ and consider $g=i \circ f: M^{n} \rightarrow \mathbb{Q}_{c^{\prime}}^{2 n+1}$, where $i: \mathbb{Q}_{c}^{2 n} \rightarrow \mathbb{Q}_{c^{\prime}}^{2 n+1}$ denotes an umbilical inclusion. We leave as an exercise to the reader to verify that the isometric immersion $g$ has also flat normal bundle and parallel mean curvature vector field, and that $N_{1}^{g}$ has constant rank $n$. Therefore, applying to $g$ the assertion just proved, we see that $g(M)$ is contained in a totally geodesic $\mathbb{Q}_{c^{\prime}}^{2 n} \subset \mathbb{Q}_{c^{\prime}}^{2 n+1}$. Thus $g(M)$ is contained in the intersection $\mathbb{Q}_{c^{\prime}}^{2 n} \cap i\left(\mathbb{Q}_{c}^{2 n}\right)$, which is the image by $i$ of an umbilical hypersurface $\mathbb{Q}_{\tilde{c}}^{2 n-1} \subset \mathbb{Q}_{c}^{2 n}$. Hence $f(M) \subset \mathbb{Q}_{\tilde{c}}^{2 n-1}$.

The next result shows that the first statement of Proposition 2.4 holds under much weaker conditions, and these turn out to be also necessary ones.

Theorem 2.6. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be a 1-regular isometric immersion. Then the first normal bundle $N_{1}$ is parallel if and only if
(i) $\left.\nabla^{\perp} R^{\perp}\right|_{N_{1}^{\perp}}=0$,
(ii) $\nabla^{\perp} \mathcal{H} \in \Gamma\left(N_{1}\right)$.

Proof: If $N_{1}$ is parallel, then $\nabla^{\perp} \mathcal{H} \in \Gamma\left(N_{1}\right)$, for $\mathcal{H} \in \Gamma\left(N_{1}\right)$. To prove that part ( $i$ ) holds, first notice that the Ricci equation implies that

$$
\begin{equation*}
R^{\perp}(X, Y) \eta=0 \tag{2.2}
\end{equation*}
$$

for all $\eta \in \Gamma\left(N_{1}^{\perp}\right)$ and $X, Y \in \mathfrak{X}(M)$. Since $N_{1}^{\perp}$ is also parallel, then

$$
R^{\perp}(X, Y) \nabla \frac{1}{Z} \eta=0
$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Thus

$$
\begin{align*}
\left(\nabla_{Z}^{\perp} R^{\perp}\right)(X, Y, \eta) & =\nabla_{Z}^{\perp} R^{\perp}(X, Y) \eta-R^{\perp}\left(\nabla_{Z} X, Y\right) \eta-R^{\perp}\left(X, \nabla_{Z} Y\right) \eta-R^{\perp}(X, Y) \nabla_{Z}^{\perp} \eta \\
& =0 \tag{2.3}
\end{align*}
$$

for all $\eta \in \Gamma\left(N_{1}^{\perp}\right)$.
Suppose now that ( $i$ ) and (ii) hold. It suffices to prove that $N_{1}^{\perp}$ is parallel. Given $\eta \in \Gamma\left(N_{1}^{\perp}\right)$, we show that $\nabla \frac{\perp}{X} \eta \in \Gamma\left(N_{1}^{\perp}\right)$ for all $X \in \mathfrak{X}(M)$. As above we have (2.2). From part (i) and (2.3) we obtain

$$
R^{\perp}(X, Y) \nabla \frac{1}{Z} \eta=0
$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Using the Ricci equation again, it follows that

$$
\left[A_{\nabla \frac{1}{Z} \eta}, A_{\nabla \frac{1}{W} \eta}\right]=0
$$

for all $Z, W \in \mathfrak{X}(M)$. For any $x \in M^{n}$, this implies the existence of an orthonormal basis $Z_{1}, \ldots, Z_{n}$ of $T_{x} M$ that simultaneously diagonalizes the family of endomorphisms

$$
\left\{A_{\nabla \frac{1}{Z} \eta}: Z \in T_{x} M\right\} .
$$

It suffices to show that

$$
\left\langle\alpha\left(Z_{i}, Z_{j}\right), \nabla \nabla_{Z_{k}}^{\perp} \eta\right\rangle=0, \quad 1 \leq i, j, k \leq n .
$$

From the choice of the basis $Z_{1}, \ldots, Z_{n}$, we have

$$
\left\langle\alpha\left(Z_{i}, Z_{j}\right), \nabla_{Z_{k}}^{\perp} \eta\right\rangle=\left\langle A_{\nabla \frac{Z_{k}}{} \eta} Z_{i}, Z_{j}\right\rangle=0 \text { if } i \neq j .
$$

Suppose that $i=j \neq k$. Since $\eta \in \Gamma\left(N_{1}^{\perp}\right)$, the Codazzi equation yields

$$
A_{\nabla \frac{\partial_{j}}{} \eta} Z_{k}=A_{\nabla \frac{\bar{Z}_{k}}{} \eta} Z_{j} .
$$

Therefore

$$
\begin{aligned}
\left\langle\alpha\left(Z_{j}, Z_{j}\right), \nabla_{Z_{k}} \eta\right\rangle & =\left\langle A_{\nabla_{\frac{Z_{k}}{} \eta} \eta} Z_{j}, Z_{j}\right\rangle \\
& =\left\langle A_{\nabla \frac{1}{Z_{j}} \eta} Z_{k}, Z_{j}\right\rangle \\
& =0 .
\end{aligned}
$$

Finally, the assumption $\nabla^{\perp} \mathcal{H} \in \Gamma\left(N_{1}\right)$ and the above imply that

$$
\begin{aligned}
\left\langle\alpha\left(Z_{j}, Z_{j}\right), \nabla \frac{1}{Z_{j}} \eta\right\rangle & =\left\langle n \mathcal{H}, \nabla_{Z_{z}}^{\perp} \eta\right\rangle \\
& =n Z_{j}\langle\mathcal{H}, \eta\rangle \\
& =0 .
\end{aligned}
$$

This completes the proof that conditions (i) and (ii) are sufficient.

### 2.2.2 The s-nullities

Let $U, V$ and $W$ be real vector spaces of finite dimension, and let $\beta: V \times U \rightarrow W$ be a bilinear form. The nullity subspace $\mathcal{N}(\beta) \subset U$ of $\beta$ is defined by

$$
\mathcal{N}(\beta)=\{Y \in U: \beta(X, Y)=0 \text { for all } X \in V\}
$$

and its image subspace $\mathcal{S}(\beta) \subset W$ by

$$
\mathcal{S}(\beta)=\operatorname{span}\{\beta(X, Y): X \in V \text { and } Y \in U\} .
$$

Assume that $W$ has a positive definite inner product and that $\beta: V \times V \rightarrow W$ is a symmetric bilinear form. For an $s$-dimensional subspace $U^{s} \subset W$, we denote by $\beta_{U^{s}}: V \times V \rightarrow U^{s}$ the map given by

$$
\beta_{U^{s}}(X, Y)=\pi_{U^{s}} \circ \beta(X, Y),
$$

where $\pi_{U^{s}}$ stands for the orthogonal projection $\pi_{U^{s}}: W \rightarrow U^{s}$. The s-nullity $\nu_{s}$ of the bilinear form $\beta$ is defined by

$$
\nu_{s}=\max _{U^{s} \subset W}\left\{\operatorname{dim} \mathcal{N}\left(\beta_{U^{s}}\right)\right\}
$$

for each integer $1 \leq s \leq \operatorname{dim} W$.
For an isometric immersion $f: M^{n} \rightarrow \tilde{M}^{m}$, the s-nullity $\nu_{s}(x)$ at $x \in M^{n}$ is defined as the $s$-nullity of its second fundamental form $\alpha$ at $x$.

If the subspaces $U^{s}$ in the definition of $\nu_{s}(x)$ are restricted to subspaces of $N_{1}(x)$, then one obtains the s-nullity of $f$ on the first normal space, which we denote by $\nu_{s}^{*}(x)$. Notice that $\nu_{k}^{*}(x)$ for $k=\operatorname{dim} N_{1}(x)$ is the usual index of relative nullity.

The next result provides sufficient conditions in terms of the $s$-nullities for the parallelism of the first normal bundle.

Proposition 2.7. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be a 1-regular isometric immersion such that rank $N_{1}=q \leq n-1$. If $\nu_{s}^{*}(x)<n-s$ for all $1 \leq s \leq q$ at any point $x \in M^{n}$, then $N_{1}$ is parallel.
Proof: Given $\eta \in \Gamma\left(N_{1}^{\perp}\right)$, define $\phi_{\eta}: \mathfrak{X}(M) \rightarrow \Gamma\left(N_{1}\right)$ by

$$
\phi_{\eta}(X)=\pi\left(\nabla_{X}^{\perp} \eta\right)
$$

where $\pi: N_{f} M \rightarrow N_{1}$ is the orthogonal projection. The Codazzi equation yields

$$
A_{\nabla \frac{1}{X} \eta} Y=A_{\nabla \frac{1}{Y} \eta} X
$$

for all $X, Y \in \mathfrak{X}(M)$. In particular,

$$
\begin{equation*}
\left\langle\alpha\left(\operatorname{ker} \phi_{\eta}(x), T_{x} M\right), \operatorname{Im}\left(\phi_{\eta}(x)\right)\right\rangle=0 \tag{2.4}
\end{equation*}
$$

at any $x \in M^{n}$. If

$$
r=\operatorname{dim} \operatorname{Im}\left(\phi_{\eta}(x)\right) \neq 0
$$

then (2.4) implies that

$$
\nu_{r}^{*}(x) \geq \operatorname{dim} \operatorname{ker} \phi_{\eta} \geq n-r,
$$

contradicting the assumption. Therefore $\phi_{\eta}$ is identically zero, and thus $N_{1}^{\perp}$ is parallel.

### 2.2.3 The type number

Let $V$ be an $n$-dimensional real vector space, and let $T_{1}, \ldots, T_{r}$ be endomorphisms of $V$. The type number of $\left\{T_{1}, \ldots, T_{r}\right\}$ is defined as the largest integer $\tau$ for which there exist $\tau$ vectors $X_{1}, \ldots, X_{\tau}$ in $V$ such that the $\tau r$ vectors

$$
\left\{T_{i} X_{j}, 1 \leq i \leq r, 1 \leq j \leq \tau\right\}
$$

are linearly independent. Observe that the type number of $\{T\}$ is equal to $\operatorname{rank} T$.
Let $V$ and $W$ be real vector spaces of finite dimension with positive definite inner products and let $\beta: V \times V \rightarrow W$ be a bilinear form. For any given $\xi \in W$, let $B_{\xi}: V \rightarrow V$ be defined by

$$
\left\langle B_{\xi} X, Y\right\rangle=\langle\beta(X, Y), \xi\rangle
$$

The left type number of $\beta$ is defined as the type number of $\left\{B_{\xi_{1}}, \ldots, B_{\xi_{r}}\right\}$, where $\xi_{1}, \ldots, \xi_{r}$ is any basis of $W$. The right type number is defined in a similar way. Notice that the left (right) type number of $\beta$ does not depend on the basis of $W$. Moreover, the left and right type numbers coincide if $\beta$ is symmetric.

The type number $\tau(x)$ of an isometric immersion $f: M^{n} \rightarrow \tilde{M}^{m}$ at a point $x \in M^{n}$ is the type number of its second fundamental form $\alpha$ at $x$.

Observe that, if $\tau(x) \geq 1$, then the first normal space $N_{1}(x)$ of $f$ at $x$ coincides with $N_{f} M(x)$. As in the definition of the $s$-nullity, one defines the type number $\tau^{*}(x)$ of $f$ on $N_{1}(x)$ by taking only a basis of $N_{1}(x)$ in the definition of $\tau(x)$.

As one would expect, the $s$-nullities and the type number of a symmetric bilinear form are related. This is made precise in the next result.

Proposition 2.8. Let $\beta: V \times V \rightarrow W$ be a symmetric bilinear form. If $\tau \geq r$, then $\nu_{s} \leq n-r s$ for $1 \leq s \leq \operatorname{dim} W$.

Proof: Suppose that $r \geq 1$, for if $r=0$ the result holds trivially. Take linearly independent vectors $\xi_{1}, \ldots, \xi_{s} \in W$, and let

$$
U^{s}=\operatorname{span}\left\{\xi_{1}, \ldots, \xi_{s}\right\}
$$

Since $\tau \geq r$, there exist $X_{1}, \ldots, X_{r} \in V$ such that

$$
L=\operatorname{span}\left\{B_{\xi_{i}} X_{j}, 1 \leq i \leq s, 1 \leq j \leq r\right\}
$$

has dimension $r s$. Thus $\operatorname{dim} L^{\perp}=n-r s$, and the result follows from the fact that

$$
\mathcal{N}\left(\pi_{U^{s}} \circ \beta\right)=\left\{Y \in V: B_{\xi_{j}} Y=0,1 \leq j \leq s\right\} \subset L^{\perp}
$$

Proposition 2.9. If $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ is a 1-regular isometric immersion with type number $\tau^{*}(x) \geq 2$ for all $x \in M^{n}$, then the first normal bundle $N_{1}$ is parallel.

Proof: Immediate from Proposition 2.7 and Proposition 2.8.
Notice that the assumption $\tau^{*} \geq 2$ imposes more restrictions on the dimensions of the spaces involved than the assumption $\nu_{s}^{*}<n-s, 1 \leq s \leq \operatorname{dim} N_{1}$. For instance, let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ be an isometric immersion and let $N_{1}(x)=N_{f} M(x)$ at $x \in M^{n}$. If $\tau(x) \geq 2$, then $n \geq 2 p$, whereas $\nu_{s}(x)<n-s, 1 \leq s \leq p$, only requires $n \geq p+1$.

### 2.3 An application

As an application of the results on reduction of codimension, as well as of the basic facts on submanifolds with flat normal bundle in Section 1.9 , we present next the classification of isometric immersions with flat normal bundle and parallel mean curvature vector field of space forms into space forms.

Theorem 2.10. Let $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{m}$ be an isometric immersion with flat normal bundle and parallel mean curvature vector field. Then one of the following possibilities holds:
(i) The immersion $f$ is umbilical.
(ii) $\tilde{c}=0=c$ and there exists $0 \leq r \leq n-1$ such that $n=s+r, m=2 s+r$ and $f(M)$ is an open subset of

$$
\mathbb{S}^{1}\left(r_{1}\right) \times \cdots \times \mathbb{S}^{1}\left(r_{s}\right) \times \mathbb{R}^{r} \subset \mathbb{R}^{m}
$$

(iii) $\tilde{c}>c=0, m=2 n-1$ and $f(M)$ is an open subset of

$$
\mathbb{S}^{1}\left(r_{1}\right) \times \cdots \times \mathbb{S}^{1}\left(r_{n}\right) \subset \mathbb{S}_{\tilde{c}}^{2 n-1} \subset \mathbb{R}^{2 n}
$$

where $r_{1}^{2}+\cdots+r_{n}^{2}=1 / \tilde{c}$.
(iv) $\tilde{c}<c=0, m=2 n-1$ and $f(M)$ is an open subset of

$$
\mathbb{H}^{1}\left(r_{1}\right) \times \mathbb{S}^{1}\left(r_{2}\right) \times \cdots \times \mathbb{S}^{1}\left(r_{n}\right) \subset \mathbb{H}_{\tilde{c}}^{2 n-1} \subset \mathbb{L}^{2 n}
$$

where $-r_{1}^{2}+r_{2}^{2}+\cdots+r_{n}^{2}=1 / \tilde{c}$.
(v) $f=i \circ \tilde{f}$, where $\tilde{f}$ is as in (ii), (iii) or (iv) and $i$ is an umbilical inclusion.

Proof: Assume first that $\tilde{c}=c$. Let $r=\nu_{0}$ be the minimum value of the index of relative nullity $\nu$ of $f$ on $M^{n}$ and let $V$ be a maximal connected open subset of $M^{n}$ where $\nu=r$. Clearly, if $r=n$ then $f$ is totally geodesic; hence we may assume that $0 \leq r \leq n-1$. If $r \geq 1$ (respectively, $r=0$ ), let $\eta_{0}, \ldots, \eta_{s}$ (respectively, $\eta_{1}, \ldots, \eta_{s}$ ) be the distinct principal normal vector fields of $f$ on $V$, with $\eta_{0}=0$. Thus $E_{\eta_{0}}$ is the relative nullity distribution $\Delta$ of $f$.

It follows from the Gauss equation (1.38) that $\eta_{1}, \ldots, \eta_{s}$ have multiplicity one and that

$$
\left\langle\eta_{i}, \eta_{j}\right\rangle=0, \quad 1 \leq i \neq j \leq s
$$

Therefore $s=n-r$, and we can write

$$
\eta_{i}=\lambda_{i} \xi_{i}, \quad 1 \leq i \leq s
$$

where $\lambda_{1}, \ldots, \lambda_{s}$ are smooth positive functions and $\xi_{1}, \ldots, \xi_{s}$ is a smooth orthonormal frame of the first normal bundle $N_{1}$ on $V$. Let $X_{1}, \ldots, X_{s}$ be an orthonormal frame of $\Delta^{\perp}$ such that

$$
\alpha\left(X_{i}, X_{i}\right)=\eta_{i}, \quad 1 \leq i \leq s .
$$

The Codazzi equations (1.41) and (1.42) yield

$$
\begin{gather*}
\left\langle\nabla_{X_{i}} X_{k}, X_{j}\right\rangle=\lambda_{i} \delta_{i j} X_{k}\left(1 / \lambda_{i}\right), \quad 0 \leq k \leq s, \quad 1 \leq i, j \leq s, \quad i, j \neq k,  \tag{2.5}\\
\nabla \frac{X_{i}}{\perp} \xi_{j}=\lambda_{i} X_{i}\left(1 / \lambda_{j}\right) \xi_{i}, \quad 1 \leq i \neq j \leq s \tag{2.6}
\end{gather*}
$$

and

$$
\begin{equation*}
\left\langle\nabla_{X_{0}} X_{i}, X_{k}\right\rangle=0=\nabla_{X_{0}}^{\perp} \xi_{i}, \quad 1 \leq i, k \leq s \tag{2.7}
\end{equation*}
$$

for any $X_{0} \in \Gamma(\Delta)$. The mean curvature vector field is given by

$$
n \mathcal{H}=\sum_{i=1}^{s} \lambda_{i} \xi_{i} .
$$

Using (2.7) we obtain

$$
\begin{align*}
X_{0}\left(\lambda_{i}\right) & =n X_{0}\left\langle\mathcal{H}, \xi_{i}\right\rangle \\
& =n\left\langle\mathcal{H}, \nabla_{X_{0}}^{\perp} \xi_{i}\right\rangle \\
& =0 \tag{2.8}
\end{align*}
$$

whereas (2.6) gives

$$
\begin{equation*}
0=n\left\langle\nabla \stackrel{X_{i}}{\perp} \mathcal{H}, \xi_{j}\right\rangle=\left(1+\frac{\lambda_{i}^{2}}{\lambda_{j}^{2}}\right) X_{i}\left(\lambda_{j}\right), \quad i \neq j, \tag{2.9}
\end{equation*}
$$

and

$$
\begin{equation*}
0=n\left\langle\nabla_{X_{i}}^{\perp} \mathcal{H}, \xi_{i}\right\rangle=X_{i}\left(\lambda_{i}\right)-\sum_{j \neq i} \frac{\lambda_{i}}{\lambda_{j}} X_{i}\left(\lambda_{j}\right) . \tag{2.10}
\end{equation*}
$$

It follows from (2.8), (2.9) and (2.10) that $\lambda_{1}, \ldots, \lambda_{s}$ are constant on $V$. If $V$ was a proper subset of $M^{n}$, then $\lambda_{1}, \ldots, \lambda_{s}$ would assume the same values on the boundary of $V$, and hence remain positive on an open connected neighborhood of $V$, contradicting the fact that $V$ is a maximal connected open subset where $\nu=r$. We conclude that $V=M^{n}$. In particular, this implies that $f$ is 1-regular. From Proposition 2.5 we see that $f(M)$ is contained in a totally geodesic $\mathbb{Q}_{c}^{2 s+r} \subset \mathbb{Q}_{c}^{m}$, and from now on we regard $f$ as an isometric immersion into $\mathbb{Q}_{c}^{2 s+r}$.

In view of (2.8), equation (2.5) for $k=0$ says that $\nabla_{X_{i}} X_{0} \in \Gamma(\Delta)$ for any $X_{0} \in \Gamma(\Delta)$. Together with the fact that $\Delta$ is totally geodesic, this implies that $\Delta$ is in fact a parallel subbundle of $T M$. On the other hand, since $\lambda_{i}$ is constant for $1 \leq i \leq s$, we also see from (2.5) that the distributions $L_{i}$ on $M^{n}$ of rank one spanned by $X_{i}$, $1 \leq i \leq s$, are parallel. Thus $c=0$ and there exists locally an isometry

$$
\psi: U=V \times W \subset \mathbb{R}^{n} \rightarrow M^{n}
$$

where $V \subset \mathbb{R}^{r}$ is an open subset and $W=\prod_{j=1}^{s} I_{j}$ is a product of open intervals $I_{j} \subset \mathbb{R}$ for $1 \leq j \leq s$, such that $\psi_{*} \mathbb{R}^{r}=\Delta$ and $\psi_{*} \partial / \partial u_{j}=X_{j}$.

That $\Delta$ is parallel implies that $f_{*} \Delta$ is a parallel subbundle of $f^{*} T \mathbb{R}^{2 s+r}$, for

$$
\begin{aligned}
\tilde{\nabla}_{X} f_{*} X_{0} & =f_{*} \nabla_{X} X_{0}+\alpha^{f}\left(X, X_{0}\right) \\
& =f_{*} \nabla_{X} X_{0} \in \Gamma\left(f_{*} \Delta\right)
\end{aligned}
$$

for all $X_{0} \in \Gamma(\Delta)$ and $X \in \mathfrak{X}(M)$. Therefore $f_{*} \Delta$ defines a constant subspace $\mathbb{R}^{r}$ of $\mathbb{R}^{2 s+r}$. Consider the orthogonal decomposition $\mathbb{R}^{2 s+r}=\mathbb{R}^{r} \times \mathbb{R}^{2 s}$ and denote by $\pi_{1}$ and $\pi_{2}$ the orthogonal projections onto $\mathbb{R}^{r}$ and $\mathbb{R}^{2 s}$, respectively. The fact that $f_{*} \Delta=\mathbb{R}^{r}$ is constant implies that $\pi_{1} f_{*} T=f_{*} T$, and hence $\pi_{2} f_{*} T=0$, for any $T \in \Gamma(\Delta)$. This means that $\left.f \circ \psi\right|_{V \times W}$ splits as

$$
f \circ \psi=j \times g
$$

where $j: V \rightarrow \mathbb{R}^{r}$ is the inclusion and $g: W \rightarrow \mathbb{R}^{2 s}$ is an isometric immersion.
We now show that

$$
g(W) \subset \mathbb{S}^{1}\left(r_{1}\right) \times \cdots \times \mathbb{S}^{1}\left(r_{s}\right) \subset \mathbb{R}^{2 s}
$$

for some positive real numbers $r_{1}, \ldots, r_{s}$. Since $X_{1}, \ldots, X_{s}$ are parallel and

$$
\alpha^{f}\left(X_{i}, X_{j}\right)=0 \text { for } 1 \leq i \neq j \leq s
$$

we have

$$
\begin{aligned}
\frac{\partial^{2} g}{\partial x_{i} \partial x_{j}} & =\tilde{\nabla}_{\partial / \partial x_{i}} g_{*} \partial / \partial x_{j} \\
& =\tilde{\nabla}_{X_{i}} f_{*} X_{j} \\
& =f_{*} \nabla_{X_{i}} X_{j}+\alpha^{f}\left(X_{i}, X_{j}\right) \\
& =0 .
\end{aligned}
$$

Hence

$$
g\left(x_{1}, \ldots, x_{s}\right)=g_{1}\left(x_{1}\right)+\cdots+g_{s}\left(x_{s}\right)
$$

for some smooth functions $g_{j}: I_{j} \rightarrow \mathbb{R}^{2 s}, 1 \leq j \leq s$. From

$$
\left\langle g_{*} \partial / \partial x_{i}, g_{*} \partial / \partial x_{j}\right\rangle=\delta_{i j}, \quad 1 \leq i, j \leq s
$$

it follows that $g_{i}$ is a unit-speed curve for all $1 \leq i \leq s$, and that the subspaces

$$
W_{i}=\operatorname{span}\left\{g_{i}{ }^{\prime}\left(x_{i}\right)=g_{*} \partial / \partial x_{i}\right\}
$$

are pairwise orthogonal. Since $\operatorname{dim} W_{i} \geq 2$ for $1 \leq i \leq s$, for otherwise $X_{i}$ would belong to $\Delta$, we must have $\operatorname{dim} W_{i}=2$ for $1 \leq i \leq s$, because $\sum_{i=1}^{s} \operatorname{dim} W_{i}=2 s$. Therefore the subspaces $W_{1}, \ldots, W_{s}$ determine an orthogonal decomposition $\mathbb{R}^{2 s}=\mathbb{R}^{2} \times \cdots \times \mathbb{R}^{2}$ into $s$ copies of $\mathbb{R}^{2}$, with respect to which $g$ splits as

$$
g=g_{1} \times \cdots \times g_{s}
$$

where each $g_{i}: \mathbb{R} \rightarrow \mathbb{R}^{2}, 1 \leq i \leq s$, is a unit-speed curve. Notice that the mean curvature vector field of $g$ is

$$
\mathcal{H}^{g}\left(x_{1}, \ldots, x_{s}\right)=\frac{1}{s} \sum_{j=1}^{s} g_{j}^{\prime \prime}\left(x_{j}\right) .
$$

Since $f$, and hence $g$, has parallel mean curvature vector field, then

$$
g_{j}^{\prime \prime \prime}\left(x_{j}\right)=\mu_{j}\left(x_{j}\right) g_{j}^{\prime}\left(x_{j}\right)
$$

for all $x_{j} \in I_{j}$ and some functions $\mu_{j} \in C^{\infty}\left(I_{j}\right), 1 \leq j \leq s$. Thus $g_{j}$ is a circle for $1 \leq j \leq s$ (see Exercise 2.7).

We have shown that for each $x \in M^{n}$ there exists an open neighborhood $V_{x} \subset M^{n}$ of $x$ such that $f\left(V_{x}\right)$ is contained in a product

$$
\mathbb{S}^{1}\left(r_{1}\right) \times \cdots \times \mathbb{S}^{1}\left(r_{s}\right) \times \mathbb{R}^{r} \subset \mathbb{R}^{m}
$$

with respect to some orthogonal decomposition $\mathbb{R}^{m}=\mathbb{R}^{2} \times \cdots \times \mathbb{R}^{2} \times \mathbb{R}^{r}$. Since the family of all such products clearly satisfies the conditions in Exercise 1.20, it follows that $f(M)$ is an open subset of an element of that family.

Suppose now that $\tilde{c}>c$, and consider the composition $F=k \circ f$ of $f$ with an umbilical inclusion $k: \mathbb{Q}_{\tilde{c}}^{m} \rightarrow \mathbb{Q}_{c}^{m+1}$. Then $F$ also has flat normal bundle. If $\eta_{1}, \ldots, \eta_{s}$ are the distinct principal normals of $f$ at some point $x \in M^{n}$, then

$$
\tilde{\eta}_{i}=k_{*} \eta_{i}+\sqrt{\tilde{c}-c} \zeta, \quad 1 \leq i \leq s
$$

are the principal normals of $F$ at $x$, where $\zeta$ is one of the unit normal vectors to $k$ at $f(x)$. In particular, $\tilde{\eta}_{i} \neq 0$ for all $1 \leq i \leq s$.

By the case $c=\tilde{c}$ just considered, we see that $c=0$ and $s=n$ everywhere, and that $F(M)$ is contained in a totally geodesic $\mathbb{R}^{2 n} \subset \mathbb{R}^{m+1}$, and hence in a umbilical $\mathbb{S}_{\tilde{c}}^{2 n-1} \subset \mathbb{R}^{2 n}$. Moreover, regarding $F$ as an isometric immersion into $\mathbb{R}^{2 n}$, we see that $F(M)$ is an open subset of

$$
\mathbb{S}^{1}\left(r_{1}\right) \times \cdots \times \mathbb{S}^{1}\left(r_{n}\right) \subset \mathbb{R}^{2 n}
$$

From $F(M) \subset \mathbb{S}_{\tilde{c}}^{2 n-1} \subset \mathbb{R}^{2 n}$ it follows that

$$
r_{1}^{2}+\cdots+r_{n}^{2}=1 / \tilde{c}
$$

Finally, assume that $\tilde{c}<c$ and consider the composition $F=k \circ f$ of $f$ with an umbilical inclusion $k: \mathbb{Q}_{\tilde{c}}^{m} \rightarrow \mathbb{Q}_{c, 1}^{m+1}$ into a Lorentzian space form of constant sectional curvature $c$. As in the previous case, the principal normals $\tilde{\eta}_{1}, \ldots, \tilde{\eta}_{s}$ of $F$ at any $x \in M^{n}$ are related to the principal normals $\eta_{1}, \ldots, \eta_{s}$ of $f$ at $x$ by

$$
\tilde{\eta}_{i}=k_{*} \eta_{i}+\sqrt{c-\tilde{c}} \zeta, \quad 1 \leq i \leq s,
$$

where $\zeta$ is one of the unit normal vector to $k$ at $f(x)$. As before, the Gauss equation yields

$$
\left\langle\tilde{\eta}_{i}, \tilde{\eta}_{j}\right\rangle=0, \quad 1 \leq i \neq j \leq s .
$$

There are now two distinct cases to consider, according to whether the first normal space $N_{1}^{F}(x)$ of $F$ at $x$, that is, the subspace of $N_{F} M(x)$ spanned by $\tilde{\eta}_{1}, \ldots, \tilde{\eta}_{s}$, is degenerate or not. Since the principal normals of $F$ are pairwise orthogonal, the first possibility happens precisely when exactly one of them, say, $\tilde{\eta}_{s}$, is light-like, that is, when $\left\langle\tilde{\eta}_{s}, \tilde{\eta}_{s}\right\rangle=0$.

Assume first that there exists some $x \in M^{n}$ such that $N_{1}^{F}(x)$ is nondegenerate, that is,

$$
\left\langle\eta_{i}, \eta_{i}\right\rangle \neq 0, \quad 1 \leq i \leq s .
$$

Let $V$ be a maximal connected open neighborhood of $x$ where $N_{1}^{F}$ remains nondegenerate. The Gauss equation (1.38) implies that $s=n$, that all principal normal vector fields $\tilde{\eta}_{1}, \ldots, \tilde{\eta}_{n}$ have multiplicity one, and that

$$
\left\langle\tilde{\eta}_{i}, \tilde{\eta}_{j}\right\rangle=0, \quad 1 \leq i \neq j \leq n
$$

everywhere on $V$. Write $\tilde{\eta}_{i}=\lambda_{i} \xi_{i}, 1 \leq i \leq n$, where $\lambda_{1}, \ldots, \lambda_{n}$ are smooth positive functions and $\xi_{1}, \ldots, \xi_{n}$ is a smooth orthonormal frame of $N_{1}^{F}$ on $V$. Setting

$$
\epsilon_{i}=\left\langle\xi_{i}, \xi_{i}\right\rangle, \quad 1 \leq i \leq n,
$$

we may assume that $\epsilon_{1}=-1$ and $\epsilon_{i}=1$ for $2 \leq i \leq n$. Let $X_{1}, \ldots, X_{n}$ be an orthonormal tangent frame such that

$$
\alpha^{F}\left(X_{i}, X_{i}\right)=\tilde{\eta}_{i}, \quad 1 \leq i \leq n .
$$

Equations (2.9) and (2.10) now read as

$$
\begin{equation*}
0=n\left\langle\nabla_{X_{i}}^{\perp} \mathcal{H}, \xi_{j}\right\rangle=\left(-\epsilon_{j} \lambda_{j}^{2}-\epsilon_{i} \lambda_{i}^{2}\right) X_{i}\left(1 / \lambda_{j}\right), \quad i \neq j \tag{2.11}
\end{equation*}
$$

and

$$
\begin{equation*}
0=n\left\langle\nabla_{X_{i}}^{\perp} \mathcal{H}, \xi_{i}\right\rangle=\epsilon_{i} X_{i}\left(\lambda_{i}\right)-\epsilon_{i} \sum_{j \neq i} \frac{\lambda_{i}}{\lambda_{j}} X_{i}\left(\lambda_{j}\right) . \tag{2.12}
\end{equation*}
$$

On the other hand, from

$$
\alpha^{F}\left(X_{i}, X_{j}\right)=\delta_{i j} \lambda_{i} \xi_{i}
$$

it follows that the normal vector field

$$
\delta=\sum_{j=1}^{n} \frac{1}{\lambda_{j}} \xi_{j}
$$

satisfies $A_{\delta}^{F}=I$, and that any umbilical normal vector field must be a multiple of $\delta$. From

$$
A_{\zeta}^{F}=-\sqrt{c-\tilde{c}} I
$$

we obtain

$$
\zeta=-\sqrt{c-\tilde{c}} \delta
$$

Hence

$$
\frac{1}{\lambda_{1}^{2}}-\sum_{j=2}^{n} \frac{1}{\lambda_{j}^{2}}=\frac{1}{c-\tilde{c}} .
$$

In particular, $\lambda_{j} \neq \lambda_{1}$ for all $2<j \leq n$, and we conclude from (2.11) and (2.12) that $\lambda_{j}$ is constant for all $1 \leq j \leq n$. Therefore, if $V$ was a proper subset of $M^{n}$, then $\lambda_{1}, \ldots, \lambda_{n}$ would assume the same values on the boundary of $V$, and hence remain positive on an open connected neighborhood of $V$, contradicting the fact that $V$ is a maximal connected open subset where $N_{1}^{F}$ is nondegenerate. It follows that $V=M^{n}$.

Arguing as in the case $\tilde{c}>c$, we conclude that $c=0$, and that $F(M)$ is contained in a totally geodesic $\mathbb{L}^{2 n} \subset \mathbb{L}^{m+1}$, and hence in a umbilical

$$
\mathbb{H}_{\tilde{c}}^{2 n-1}=\mathbb{L}^{2 n} \cap \mathbb{H}_{\tilde{c}}^{m} \subset \mathbb{L}^{2 n}
$$

Moreover, regarding $F$ as an isometric immersion into $\mathbb{L}^{2 n}$, we see that $F(M)$ is an open subset of

$$
\mathbb{H}^{1}\left(r_{1}\right) \times \mathbb{S}^{1}\left(r_{2}\right) \times \cdots \times \mathbb{S}^{1}\left(r_{n}\right) \subset \mathbb{L}^{2 n}
$$

and the fact that $F(M) \subset \mathbb{H}_{\tilde{c}}^{2 n-1} \subset \mathbb{L}^{2 n}$ implies that

$$
-r_{1}^{2}+r_{2}^{2}+\cdots+r_{n}^{2}=1 / \tilde{c}
$$

It remains to consider the case in which $N_{1}^{F}$ is everywhere degenerate. Let $V$ be a maximal connected open subset where $s$ assumes its maximum value. We may assume that $\left\langle\tilde{\eta}_{s}, \tilde{\eta}_{s}\right\rangle=0$ everywhere. In particular, this implies that $\left\|\eta_{s}\right\|^{2}=c-\tilde{c}$. On the other hand, from

$$
0=\left\langle\tilde{\eta}_{s}, \tilde{\eta}_{i}\right\rangle=\left\langle\eta_{s}, \eta_{i}\right\rangle-c+\tilde{c}, \quad 1 \leq i \leq s-1
$$

we obtain

$$
\begin{aligned}
\left.A_{\eta_{s}}^{f}\right|_{E_{\eta_{i}}} & =\left\langle\eta_{s}, \eta_{i}\right\rangle I \\
& =(c-\tilde{c}) I .
\end{aligned}
$$

Therefore, writing $\eta_{s}=\sqrt{c-\tilde{c}} \delta$, where $\delta$ has unit length, we have

$$
A_{\delta}^{f}=\sqrt{c-\tilde{c}} I .
$$

It also follows from the Gauss equation that $\eta_{i}$ has multiplicity one for $1 \leq i \leq s-1$ and, writing $\zeta_{i}=\eta_{i}-\eta_{s}$ for $1 \leq i \leq s-1$, we have

$$
\begin{aligned}
\left\langle\zeta_{i}, \zeta_{j}\right\rangle & =\left\langle\eta_{i}-\eta_{s}, \eta_{j}-\eta_{s}\right\rangle \\
& =\left\langle\eta_{i}, \eta_{j}\right\rangle-\left\langle\eta_{i}, \eta_{s}\right\rangle-\left\langle\eta_{s}, \eta_{j}\right\rangle+\left\|\eta_{s}\right\|^{2} \\
& =0
\end{aligned}
$$

because $\left\|\eta_{s}\right\|^{2}=c-\tilde{c}$ and all the remaining terms on the right-hand side are also equal to $c-\tilde{c}$ by the Gauss equation. Moreover,

$$
\left\langle\zeta_{i}, \eta_{s}\right\rangle=\left\langle\eta_{i}, \eta_{s}\right\rangle-\left\|\eta_{s}\right\|^{2}=0, \quad 1 \leq i \leq s-1 .
$$

Write $\zeta_{i}=\mu_{i} \xi_{i}, 1 \leq i \leq s-1$. Then

$$
\eta_{i}=\eta_{s}-\left(\eta_{s}-\eta_{i}\right)=\eta_{s}+\mu_{i} \xi_{i}, \quad 1 \leq i \leq s-1
$$

Therefore

$$
\mathcal{H}=\sqrt{c-\tilde{c}} \delta+\frac{1}{n} \sum_{i=1}^{s-1} \mu_{i} \xi_{i}
$$

The Codazzi equation for $\delta$ yields

$$
\begin{equation*}
A_{\nabla_{\frac{1}{Y} \delta} \delta} Z=A_{\nabla_{\frac{1}{Z} \delta}} Y \tag{2.13}
\end{equation*}
$$

for all $Y, Z \in \mathfrak{X}(M)$. If $X_{1}, \ldots, X_{s-1}$ is an orthonormal frame of $E_{\eta_{s}}^{\perp}$ along $V$ such that

$$
\alpha^{f}\left(X_{i}, X_{i}\right)=\eta_{i}, \quad 1 \leq i \leq s-1,
$$

applying 2.13) to $Y=X_{i}$ and $Z=X_{j}, 1 \leq i \neq j \leq s-1$, we obtain

$$
\left\langle\nabla{\stackrel{\rightharpoonup}{X_{i}}}_{\perp} \delta, \xi_{j}\right\rangle=0
$$

On the other hand, for $Y \in \Gamma\left(E_{\eta_{s}}\right)$ and $Z=X_{j}$ it yields

$$
\left\langle\nabla_{Y}^{\perp} \delta, \xi_{j}\right\rangle=0, \quad 1 \leq j \leq s-1 .
$$

Now, using that the mean curvature vector field of $f$ is parallel in the normal connection, we have

$$
\begin{aligned}
0 & =\left\langle\nabla_{X_{i}}^{\perp} \mathcal{H}, \delta\right\rangle \\
& =X_{i}\langle\mathcal{H}, \delta\rangle-\left\langle\mathcal{H}, \nabla_{X_{i}}^{\perp} \delta\right\rangle \\
& =-\frac{\mu_{i}}{n}\left\langle\nabla{ }_{X_{i}}^{\perp} \delta, \xi_{i}\right\rangle .
\end{aligned}
$$

Hence

$$
\left\langle\nabla_{X_{i}}^{\perp} \delta, \xi_{i}\right\rangle=0, \quad 1 \leq i \leq s-1
$$

It follows that $\delta$ is parallel in the normal connection, and hence $f(V)$ is contained in an umbilical hypersurface $\mathbb{Q}_{c}^{m-1}$ of $\mathbb{Q}_{\bar{c}}^{m}$ by Exercise 2.9. In other words, $f=i \circ g$, where $g: M_{c}^{n} \rightarrow \mathbb{Q}_{c}^{m-1}$ is an isometric immersion and $i: \mathbb{Q}_{c}^{m-1} \rightarrow \mathbb{Q}_{\bar{c}}^{m}$ is an umbilical inclusion. Applying to $\left.g\right|_{V}$ the conclusion in the case $c=\tilde{c}$ already considered, we see that $c=0$ and that

$$
\left.g\right|_{V}=j \circ h,
$$

where $h: V \rightarrow \mathbb{R}^{2(s-1)+r}$ and $j: \mathbb{R}^{2(s-1)+r} \rightarrow \mathbb{R}^{m-1}$ are isometric immersions such that $n=r+s-1, j$ is totally geodesic and $h(V)$ is an open subset of

$$
\mathbb{S}^{1}\left(r_{1}\right) \times \cdots \times \mathbb{S}^{1}\left(r_{s-1}\right) \times \mathbb{R}^{r} \subset \mathbb{R}^{2(s-1)+r}
$$

for some positive real numbers $r_{1}, \ldots, r_{s-1}$. In particular, the lengths of the principal normal vector fields of $g$ are constant on $V$. If $V$ was a proper subset of $M^{n}$, then they would have the same values on the boundary of $V$, and hence the number of distinct principal normals would still be $s$ on an open connected neighborhood of $V$. This contradicts the maximality of $V$ with respect to this property and shows that $V=M^{n}$.

Corollary 2.11. Let $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{m}$ be a minimal isometric immersion with flat normal bundle. Then one of the following possibilities holds:
(i) $\tilde{c}=c$ and the immersion $f$ is totally geodesic.
(ii) $c=0<\tilde{c}$ and $f=i \circ \tilde{f}$, where $\tilde{f}(M)$ is an open subset of a Clifford torus

$$
\mathbb{S}^{1}(r) \times \cdots \times \mathbb{S}^{1}(r) \subset \mathbb{S}_{\tilde{c}}^{2 n-1} \subset \mathbb{R}^{2 n}
$$

with $r=1 / \sqrt{n \tilde{c}}$, and $i: \mathbb{S}_{\tilde{c}}^{2 n-1} \rightarrow \mathbb{S}_{\tilde{c}}^{m}$ is a totally geodesic inclusion.

### 2.4 Notes

Proposition 2.1 is frequently referred to as Erbacher's theorem, although this elementary fact has been used since long before; for instance, see Allendoerfer [16]. A version of this result for submanifolds of a symmetric space was proved by Di ScalaVittone [162]. See also [244] for the case of submanifolds of a product of two space forms, and [243] for the special case of submanifolds of $\mathbb{Q}_{c}^{n} \times \mathbb{R}$.

Theorem 2.6 on reduction of codimension, as well as the more general result given in Exercise 2.5, are due to Dajczer [92]. Exercise 2.17 was also taken from [92]. Under the stronger assumption that the normal bundle is flat, Theorem 2.6 was known to Lagrange [232]. Versions of Theorem 2.6 for submanifolds of $\mathbb{Q}_{c}^{n} \times \mathbb{R}$ and, more generally, for submanifolds of any product of two space forms, were proved by Mendonça-Tojeiro [243], [244].

The notions of type number and $s$-nullity were introduced by Allendoerfer [16] and do Carmo-Dajczer [59], respectively. Proposition 2.7 was proved by Dajczer-Rodríguez [127] and Proposition 2.9 by Allendoerfer [16]. Theorem 2.10 was obtained by DajczerTojeiro [134]. Its Corollary 2.11 for minimal immersions and $n=2 m-1$ was previously proved by Moore [253]. An extension of Theorem 2.10 to the case of Einstein submanifolds was obtained by Onti [280].

The structure of submanifolds that carry a nonparallel first normal bundle of low rank is discussed in Chapter 12, For other results on the subject of this chapter, we refer to Dajczer [92], Dajczer-Rodríguez [127] and Dajczer-Tojeiro [148].

### 2.5 Exercises

Exercise 2.1. Let $f: M^{n} \rightarrow \tilde{M}^{m}$ be a 1-regular isometric immersion. Show that $N_{1}$ and $N_{1}^{\perp}$ are vector subbundles of $N_{f} M$.

Exercise 2.2. Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be a 1-regular isometric immersion of a Riemannian manifold without flat points. If rank $N_{1}=1$, show that $f$ has substantial codimension one.

Exercise 2.3. Let $\gamma: \mathbb{R} \rightarrow \mathbb{R}^{3}$ be a smooth curve whose curvature and torsion are nowhere vanishing. Determine the dimension of the first normal spaces of the immersion $f: \mathbb{S}_{1}^{n} \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+3}$ given by

$$
f\left(t_{0}, t_{1}, \ldots, t_{n}\right)=\left(\gamma\left(t_{0}\right), t_{1}, \ldots, t_{n}\right)
$$

Exercise 2.4. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be an isometric immersion with parallel second fundamental form. Prove that $f$ is 1-regular and admits a reduction of codimension to $k=\operatorname{rank} N_{1}$.

Exercise 2.5. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be a regular isometric immersion. Prove that the $k^{\text {th }}$-normal subbundle $N_{k}, k \geq 1$, is parallel if and only if
(i) $\left.\left(\nabla^{\perp}\right)^{k} R^{\perp}\right|_{N_{k}^{\perp}}=0$,
(ii) $\left(\nabla^{\perp}\right)^{k} \mathcal{H} \subset N_{k}$.

Exercise 2.6. Provide a direct proof of Proposition 2.9.
Exercise 2.7. Let $\gamma: I \rightarrow \mathbb{R}^{m}$ be a unit-speed curve. Assume that $\gamma^{\prime \prime}(t) \neq 0$ and $\gamma^{\prime \prime \prime}(t)=\lambda(t) \gamma^{\prime}(t)$ for any $t \in I$ and some $\lambda \in C^{\infty}(I)$. Show that $\gamma(I)$ is a circle in some two-dimensional affine subspace of $\mathbb{R}^{m}$.

Exercise 2.8. Let $\gamma: I \rightarrow \mathbb{R}^{m}$ be a unit-speed curve. Assume that there exists a parallel normal subbundle $\mathcal{L}$ of $N_{\gamma} I$ of rank $s$ such that $\gamma^{\prime \prime \prime}(t) \in \mathcal{L}(t)^{\perp}$ but $\gamma^{\prime \prime}(t) \notin \mathcal{L}(t)^{\perp}$ for any $t \in I$. Prove that $\gamma(I)$ is contained in a hypersphere $\mathbb{S}^{m-s}$ of an affine subspace $H$ of $\mathbb{R}^{m}$ of dimension $m-s+1$, and that $\mathcal{L}(t)$ is spanned by the normal space $\mathbb{R}^{s-1}$ of $H$ and the position vector of $\gamma$ with respect to the center of $\mathbb{S}^{m-s}$.
Hint: Since $\gamma^{\prime \prime}(t) \notin \mathcal{L}(t)^{\perp}$ for any $t \in I$, the orthogonal projection $\left(\gamma^{\prime \prime}(t)\right)_{\mathcal{L}(t)}$ of $\gamma^{\prime \prime}(t)$ onto $\mathcal{L}(t)$ is nowhere vanishing. Let $\zeta(t)$ be a unit vector field along $\gamma$ in the direction of $\left(\gamma^{\prime \prime}(t)\right)_{\mathcal{L}(t)}$. For any section $\xi$ of the orthogonal complement $\{\zeta\}^{\perp}$ of $\{\zeta\}$ in $\mathcal{L}$, using that $\gamma^{\prime \prime \prime}(t) \in \mathcal{L}(t)^{\perp}$ for any $t \in I$ and that $\mathcal{L}$ is parallel in the normal connection of $\gamma$ gives

$$
\left\langle\xi^{\prime}, \zeta\right\rangle=\left\langle\xi^{\prime}, \gamma^{\prime \prime}\right\rangle=-\left\langle\gamma^{\prime \prime \prime}, \xi\right\rangle=0 .
$$

It follows that $\{\zeta\}^{\perp}$ is also parallel in the normal connection of $\gamma$, and hence $\{\zeta\}^{\perp}$ is a constant subspace $\mathbb{R}^{s-1}$ of $\mathbb{R}^{m}$. Thus $\gamma(I)$ is contained in an affine subspace $\mathcal{H}$ normal to $\mathbb{R}^{s-1}$ in $\mathbb{R}^{m}$. Moreover, we have $\zeta^{\prime}=\lambda \gamma^{\prime}$, with $\lambda=\left\langle\zeta^{\prime}, \gamma^{\prime}\right\rangle=-\left\langle\zeta, \gamma^{\prime \prime}\right\rangle$. Now,

$$
\left\langle\zeta, \gamma^{\prime \prime}\right\rangle^{\prime}=\left\langle\zeta^{\prime}, \gamma^{\prime \prime}\right\rangle+\left\langle\zeta, \gamma^{\prime \prime \prime}\right\rangle=0,
$$

hence $\lambda$ is a nonzero constant $1 / r \in \mathbb{R}$. Thus $\gamma-r \zeta$ is a constant vector of $H$.
Exercise 2.9. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be an isometric immersion and let $\eta \in \Gamma\left(N_{f} M\right)$ be an umbilical unit vector field, that is, $A_{\eta}=\lambda I$ for some $\lambda \in C^{\infty}(M)$. If $\eta$ is parallel in the normal connection, show that $\lambda$ is constant and that $f(M)$ is contained in an umbilical hypersurface $\mathbb{Q}_{\tilde{c}}^{m-1}$ of $\mathbb{Q}_{c}^{m}$ with constant curvature $\tilde{c}=c+\lambda^{2}$.

Exercise 2.10. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be an isometric immersion with flat normal bundle. Assume that $f$ has exactly two principal normal vector fields $\eta_{1}$ and $\eta_{2}$ that are everywhere linearly independent and parallel along the corresponding eigenbundles, that is,

$$
\nabla_{X_{i}}^{\perp} \eta_{i}=0 \text { for all } X_{i} \in \Gamma\left(E_{\eta_{i}}\right), \quad 1 \leq i \leq 2 .
$$

Prove that $f=i \circ g$, where the submanifold $g: M^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+1}, \tilde{c}>c$, is a cyclide of Dupin and $i: \mathbb{Q}_{\tilde{c}}^{n+1} \rightarrow \mathbb{Q}_{c}^{m}$ is an umbilical inclusion. (A hypersurface $g: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ is called a cyclide of Dupin if it has exactly two distinct principal curvatures everywhere, both of which are constant along the corresponding eigenbundles.)
Hint: First use the Codazzi equations (1.41) and the assumptions on the principal normal vector fields $\eta_{1}$ and $\eta_{2}$ to show that $N_{1}$ is a parallel subbundle of rank two. Then prove that a unit vector field $\zeta \in \Gamma\left(N_{1}\right)$ orthogonal to $\eta_{1}-\eta_{2}$ is a parallel umbilical vector field and use the preceding exercise.

Exercise 2.11. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be a 1-regular $k$-Dupin submanifold, that is, $f$ is an isometric immersion with flat normal bundle that has exactly $k$ distinct principal normal vector fields $\eta_{1}, \ldots, \eta_{k}$, with $\eta_{i}$ parallel in the normal connection along $E_{\eta_{i}}$ for all $1 \leq i \leq k$. If the subspaces

$$
S_{f}(x)=\operatorname{span}\left\{\eta_{i}(x)-\eta_{j}(x): 1 \leq i, j \leq k\right\}
$$

have constant dimension $s$ on $M^{n}$, prove that $f(M)$ is contained in an umbilical submanifold $\mathbb{Q}_{\tilde{c}}^{n+s}$ of $\mathbb{Q}_{c}^{m}$.
Hint: First use the Codazzi equations (1.41) and the assumptions on the principal normal vector fields to show that $N_{1}$ is parallel. Then notice that $N_{1}$ has rank either $s$ or $s+1$ by part $(i)$ of Exercise 1.41. In the first case, conclude that $f(M)$ is contained in a totally geodesic submanifold $\mathbb{Q}_{c}^{n+s}$ of $\mathbb{Q}_{c}^{m}$. In the latter, prove that a unit vector field $\zeta \in \Gamma\left(N_{1}\right)$ orthogonal to $S_{f}$ is parallel in the normal connection and use part (ii) of Exercise 1.41 and Exercise 2.9 to conclude that $f(M)$ is contained in a umbilical submanifold $\mathbb{Q}_{\bar{c}}^{n+s}$ of $\mathbb{Q}_{c}^{m}$.

Exercise 2.12. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be an isometric immersion with flat normal bundle. If $f$ has parallel mean curvature vector field and $\operatorname{dim} N_{1}(x)=n$ for all $x \in M^{n}$, show that there exists a parallel umbilical normal vector field along $f$, and use Exercise 2.9 to give another proof of the last assertion in Proposition 2.5.
Hint: Let $\eta_{1}, \ldots, \eta_{n}$ be the principal normal vector fields of $f$. Use part ( $i$ ) of Exercise 1.41 to conclude that $S_{f}(x)$ has dimension $n-1$ at every $x \in M^{n}$. Let $\zeta$ be a unit vector field spanning the orthogonal complement of $S_{f}$ in $N_{1}$. Use part (ii) of Exercise 1.41 to show that $\zeta$ is an umbilical vector field. Prove that $\zeta$ is parallel in the normal connection by first deriving from the Codazzi equations 1.41) that

$$
\left\langle\zeta, \nabla_{X_{i}}^{\perp} \eta_{j}\right\rangle=0, \quad 1 \leq i \neq j \leq n,
$$

and then using that the mean curvature vector field $\mathcal{H}=\frac{1}{n} \sum_{i=1} d_{i} \eta_{i}, d_{i}=\operatorname{rank} E_{\eta_{i}}$, is parallel in the normal connection to show that also

$$
\left\langle\zeta, \nabla_{X_{i}}^{\perp} \eta_{i}\right\rangle=0, \quad 1 \leq i \leq n .
$$

Conclude that $\nabla \frac{\stackrel{1}{X_{i}}}{} \delta$ is orthogonal to $S_{f}$ for any $X_{i} \in \Gamma\left(E_{\eta_{i}}\right), 1 \leq i \leq n$, and hence must vanish.

Exercise 2.13. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}, p \leq n-1$, be an isometric immersion and let $\eta \in \Gamma\left(N_{f} M\right)$ be a unit umbilical vector field. Assume that $\nu_{s}<n-s$ for all $1 \leq s \leq p-1$. Show that $\eta$ is parallel in the normal connection.
Hint: At $x \in M^{n}$ consider the linear map $\phi: T_{x} M \rightarrow N_{f} M(x)$ defined by

$$
\phi(X)=\nabla_{X}^{\perp} \eta .
$$

Then $\operatorname{dim} \operatorname{Im} \phi=r \leq p-1$, and $\operatorname{dim} \operatorname{ker} \phi=n-r$. If $A_{\eta}=\lambda I, \lambda \neq 0$, use the Codazzi equation to show that $Y(\lambda)=0$ for all $Y \in \operatorname{ker} \phi$. Now take $X \in \operatorname{ker} \phi^{\perp}$, define

$$
\delta=\phi(X)-X(\log \lambda) \eta
$$

and use the Codazzi equation again to verify that $A_{\delta} Y=0$ for all $Y \in \operatorname{ker} \phi$. Conclude that $r=0$.

Exercise 2.14. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ be an isometric immersion, and let $L$ be a parallel subbundle of $N_{f} M$ with rank $k$. Suppose that $L$ is an umbilical subbundle of $N_{f} M$, that is, that there exists $\xi \in \Gamma(L)$ such that

$$
A_{\eta}=\langle\eta, \xi\rangle I
$$

for all $\eta \in \Gamma(L)$. Show that $f(M)$ is contained in an $(n+p-k)$-dimensional umbilical submanifold of $\mathbb{Q}_{c}^{n+p}$ having $L$ as its normal bundle along $f$.

Exercise 2.15. Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be an isometric immersion, and let $\eta \in \Gamma\left(N_{f} M\right)$ be a parallel vector field such that the shape operator $A_{\eta}$ is nowhere singular. Assume that the support function

$$
x \in M^{n} \mapsto\langle f(x), \eta(x)\rangle
$$

is constant. Prove that $f(M)$ is contained in a hypersphere of $\mathbb{R}^{m}$ centered at the origin.

Exercise 2.16. Given an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$, at each point $x \in M^{n}$ consider the normal subspace

$$
U_{1}(x)=\left\{\xi \in N_{f} M(x): A_{\xi}=\lambda(\xi) I\right\} .
$$

Assume that the subspaces $U_{1}(x)$ have constant dimension, and thus form a smooth umbilical subbundle $U_{1}$ of the normal bundle. Show that $U_{1}$ is parallel in the normal connection if and only if
(i) $\left.\nabla^{\perp} R^{\perp}\right|_{U_{1}}=0$,
(ii) $\nabla^{\perp} \mathcal{H} \in \Gamma\left(U_{1}^{\perp}\right)$.

Exercise 2.17. Given an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$, consider the normal subspace at $x \in M^{n}$ given by

$$
T(x)=\left\{\xi \in N_{f} M(x): R^{\perp}(X, Y) \xi=0 \text { for all } X, Y \in T_{x} M\right\}^{\perp} .
$$

Show that if $\operatorname{dim} T(x)>\frac{1}{2} n(n-1)+1$, then either
(i) $N_{1}(x)=T(x)$ or
(ii) $N_{1}(x)=T(x) \oplus \operatorname{span}\{\eta\}$, where $\eta$ is an umbilical direction.

Moreover, prove the following assertions:
(iii) If $\operatorname{dim} T(x)=\frac{1}{2} n(n+1)$ then (ii) holds.
(iv) If $\mathcal{H}(x)=0$ and $\operatorname{dim} N_{1}(x)>\frac{1}{2} n(n-1)$ then (i) holds.

## Chapter 3

## Minimal submanifolds

The theory of minimal submanifolds is one of the most beautiful and developed subjects of differential geometry. The aim of this chapter is to introduce a few of its general aspects.

The equation that defines minimal submanifolds turns out to be the EulerLagrange equation for the volume functional; hence such submanifolds have a natural variational characterization as critical points of that functional. This is a consequence of the first variational formula, which is discussed at the beginning of this chapter.

Minimal submanifolds of Euclidean space are characterized by having harmonic coordinate functions. We illustrate the strong implications of this fact. Minimal submanifolds of the sphere, in turn, are characterized by having eigenfunctions of the Laplace operator as coordinate functions. We briefly discuss how this can be used to construct nice examples of minimal isometric immersions of spheres into spheres.

The Ricci tensor of a submanifold of a space form is computed in terms of its second fundamental form, and this is used to derive an obstruction to the existence of minimal isometric immersions into a space of constant sectional curvature. We then present a strong rigidity result for minimal hypersurfaces of space forms within the class of minimal isometric immersions.

The Ricci condition gives necessary and sufficient conditions for some neighborhood of a point with nonpositive Gauss curvature of a two-dimensional Riemannian manifold to admit a minimal isometric immersion in $\mathbb{R}^{3}$. The chapter ends with a generalization of the Ricci condition for hypersurfaces with arbitrary dimension of space forms, which characterizes the Riemannian metrics that arise as the induced metrics of minimal immersions with codimension one into space forms.

### 3.1 The first variational formula

In order to obtain a variational characterization of the minimal submanifolds we start with the following result.

Proposition 3.1. Let $F: I \times M^{n} \rightarrow \tilde{M}^{m}$ be a smooth variation of an isometric immersion $f: M^{n} \rightarrow \tilde{M}^{m}$ and let $T=f_{*} Z+\eta$ be the decomposition of the variational
vector field into its tangent and normal components. Then

$$
\left.\frac{d}{d t}\right|_{t=0} d V_{t}=(-n\langle\mathcal{H}, \eta\rangle+\operatorname{div} Z) d V_{0},
$$

where $d V_{t}$ is the volume element of the metric induced by $f_{t}$ and the divergence of $Z$ is

$$
\operatorname{div} Z=\operatorname{tr}\left(X \mapsto \nabla_{X} Z\right)
$$

Proof: For a fixed $x \in M^{n}$, let $X_{1}, \ldots, X_{n}$ be an orthonormal basis of $T_{x} M$ and set

$$
g_{i j}(t)=\left\langle f_{t_{*}} X_{i}, f_{t_{*}} X_{j}\right\rangle, \quad 1 \leq i, j \leq n
$$

Then

$$
d V_{t}=\sqrt{g(t)} d V_{0}
$$

where $g(t)=\operatorname{det}\left(g_{i j}(t)\right)$. By formula (1.4),

$$
g_{i i}^{\prime}(0)=-2\left\langle\alpha\left(X_{i}, X_{i}\right), \eta\right\rangle+2\left\langle\nabla_{X_{i}} Z, X_{i}\right\rangle, \quad 1 \leq i \leq n .
$$

Hence

$$
\begin{aligned}
\left.\frac{d}{d t}\right|_{t=0} d V_{t} & =\left.\frac{d}{d t}\right|_{t=0} \sqrt{g(t)} d V_{0} \\
& =\frac{1}{2} g^{\prime}(0) d V_{0} \\
& =\frac{1}{2} \operatorname{tr}\left(g_{i j}^{\prime}(0)\right) d V_{0} \\
& =(-n\langle\mathcal{H}, \eta\rangle+\operatorname{div} Z) d V_{0}
\end{aligned}
$$

The volume of an immersion $f: M^{n} \rightarrow \tilde{M}^{m}$ of a compact oriented manifold is defined as

$$
V(f)=\int_{M} d V
$$

where $d V$ is the volume element of the induced metric.
Corollary 3.2. Let $f: M^{n} \rightarrow \tilde{M}^{m}$ be an immersion of a compact oriented manifold, possibly with boundary, and let $F: I \times M^{n} \rightarrow \tilde{M}^{m}$ be a smooth variation of $f$ such that $\left.f_{t}\right|_{\partial M}=\left.f\right|_{\partial M}$ for all $t \in I$. Let $T=f_{*} Z+\eta$ be the variational vector field of $F$. Then

$$
\begin{equation*}
\left.\frac{d}{d t}\right|_{t=0} V\left(f_{t}\right)=-\int_{M} n\langle\mathcal{H}, \eta\rangle d V_{0} . \tag{3.1}
\end{equation*}
$$

Proof: Proposition 3.1 and the divergence theorem yield

$$
\begin{aligned}
\left.\frac{d}{d t}\right|_{t=0} V\left(f_{t}\right) & =\left.\frac{d}{d t}\right|_{t=0} \int_{M} d V_{t} \\
& =\left.\int_{M} \frac{d}{d t}\right|_{t=0} d V_{t} \\
& =-\int_{M} n\langle\mathcal{H}, \eta\rangle d V_{0}+\int_{M}(\operatorname{div} Z) d V_{0} \\
& =-\int_{M} n\langle\mathcal{H}, \eta\rangle d V_{0} .
\end{aligned}
$$

Remarks 3.3. (i) Let $M^{n}$ be a compact manifold and let $\mathcal{F}(M, \tilde{M})$ denote the space of immersions of $M^{n}$ into $\tilde{M}^{m}$. Then $\mathcal{F}(M, \tilde{M})$ has the structure of an infinite-dimensional smooth manifold. Given $f \in \mathcal{F}(M, \tilde{M})$, a smooth curve $f_{t}$ in $\mathcal{F}(M, \tilde{M})$ with $f_{0}=f$ corresponds to a smooth variation of $f$, and the tangent vector

$$
T=\left.\frac{d f_{t}}{d t}\right|_{t=0}
$$

naturally corresponds to the variational vector field. In this way, the tangent space of $\mathcal{F}(M, \tilde{M})$ at $f$ can be identified with the space of sections of $f^{*} T \tilde{M}$. Integrating the inner product of the variational vector fields over $M^{n}$ gives an inner product in this tangent space. Consider the functional $V$ on $\mathcal{F}(M, \tilde{M})$ that assigns to each $f \in \mathcal{F}(M, \tilde{M})$ its volume $V(f)$. Then (3.1) says that the gradient $\operatorname{grad} V$ of $V$ at $f$ is $-n \mathcal{H}$. In particular, deforming $f$ along $\mathcal{H}$ decreases its volume most rapidly.
(ii) For normal variations, formula (3.1) remains valid without the condition on the boundary.
(iii) If $M^{n}$ is noncompact or nonorientable, then (3.1) can be used for compactly supported variations and local volume functionals. A smooth variation $F: I \times M^{n} \rightarrow \tilde{M}^{m}$ of $f$ is said to be compactly supported if there exists a relatively compact subset $U \subset M^{n}$ such that

$$
f_{t}(M \backslash U)=f(M \backslash U)
$$

for all $t \in I$.
Corollary 3.2 yields a variational characterization of minimal immersions.
Corollary 3.4. An isometric immersion $f: M^{n} \rightarrow \tilde{M}^{m}$ is minimal if and only if

$$
\left.\frac{d}{d t}\right|_{t=0} V\left(f_{t}\right)=0
$$

for every compactly supported smooth variation of $f$.

### 3.2 Euclidean minimal submanifolds

If $M^{n}$ is a Riemannian manifold, the Laplacian $\Delta h$ of $h \in C^{\infty}(M)$ at $x \in M^{n}$ is defined as

$$
\Delta h(x)=\operatorname{tr} \operatorname{Hess} h(x)=\operatorname{div} \operatorname{grad} h(x) .
$$

For an isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ and $x \in M^{n}$, by $\Delta f(x)$ we mean the vector

$$
\left(\Delta f_{1}(x), \ldots, \Delta f_{m}(x)\right)
$$

where $f=\left(f_{1}, \ldots, f_{m}\right)$. Taking traces in (1.7) gives the following result.

Proposition 3.5. If $f: M^{n} \rightarrow \mathbb{R}^{m}$ is an isometric immersion, then

$$
\Delta f(x)=n \mathcal{H}(x)
$$

where $\mathcal{H}(x)$ is the mean curvature vector of $f$ at $x$.
If $M^{n}$ is a Riemannian manifold, then $h \in C^{\infty}(M)$ is called harmonic if $\Delta h=0$. By Proposition 3.5, an isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ is minimal if and only if any height function is harmonic on $M^{n}$. This fact has strong consequences, as illustrated below.

Recall that the convex hull $\mathcal{C}(X)$ of a subset $X \subset \mathbb{R}^{m}$ is the smallest closed convex set containing $X$. If we denote by $H_{v, w}$, for each pair of vectors $v, w \in \mathbb{R}^{m}$, the half-space given by

$$
H_{v, w}=\left\{v+y \in \mathbb{R}^{m}:\langle y, w\rangle \leq 0\right\},
$$

then

$$
\mathcal{C}(X)=\cap\left\{H_{v, w} \subset \mathbb{R}^{m}: X \subset H_{v, w}\right\} .
$$

The proof of the next result relies on the well-known maximum principle due to Hopf, which implies that a harmonic function on a Riemannian manifold $M^{n}$ with boundary $\partial M$ that attains a local maximum at a point in $M \backslash \partial M$ must be constant.

Proposition 3.6. Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be a minimal immersion of a compact manifold $M^{n}$ with boundary $\partial M$. Then $f(M) \subset \mathcal{C}(f(\partial M))$. Moreover, if $f(M)$ lies in no proper affine subspace, then $f(M \backslash \partial M)$ is contained in the interior $\mathcal{C}(f(\partial M))^{\circ}$ of $\mathcal{C}(f(\partial M))$.

Proof: For any pair of vectors $v, w \in \mathbb{R}^{m}$, define $h_{v, w} \in C^{\infty}\left(\mathbb{R}^{m}\right)$ by

$$
h_{v, w}(y)=\langle y-v, w\rangle .
$$

Then $y \in H_{v, w}$ if and only if $h_{v, w}(y) \leq 0$. Therefore, for any $X \subset \mathbb{R}^{m}$ we have

$$
y \in \mathcal{C}(X) \text { if and only if } h_{v, w}(y) \leq 0 \text { whenever } h_{v, w} \leq 0 \text { on } X,
$$

whereas

$$
y \in \mathcal{C}(X)^{o} \text { if and only if } h_{v, w}(y)<0 \text { whenever } h_{v, w} \leq 0 \text { on } X .
$$

The statements then follow from the fact that, by Hopf's maximum principle, $h_{v, w} \circ f \leq 0$ on $\partial M$ implies $h_{v, w} \circ f \leq 0$ on $M^{n}$, whereas $h_{v, w} \circ f \leq 0$ on $\partial M$ implies $h_{v, w} \circ f<0$ on $M \backslash \partial M$, unless $h_{v, w}$ is identically zero.

The following result is also an immediate consequence of Corollary 1.6.
Corollary 3.7. There exists no minimal isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ of a compact Riemannian manifold without boundary.

### 3.3 Minimal submanifolds of the sphere

The following result shows that the minimal immersions of an $n$-dimensional differentiable manifold into the Euclidean unit sphere $\mathbb{S}^{m}$ are precisely those whose coordinate functions in $\mathbb{R}^{m+1}$ are eigenfunctions with eigenvalue $-n$ of the Laplace operator in the induced metric.

Proposition 3.8. Let $f: M^{n} \rightarrow \mathbb{S}_{c}^{m}$ be an isometric immersion. Set $F=i \circ f$, where $i: \mathbb{S}_{c}^{m} \rightarrow \mathbb{R}^{m+1}$ is the inclusion map. Then $f$ is minimal if and only if $\Delta F=-n c F$.

Proof: The second fundamental forms of $f$ and $F$ are related by

$$
\alpha^{F}(X, Y)=i_{*} \alpha^{f}(X, Y)-c\langle X, Y\rangle F
$$

for all $x \in M^{n}$ and all $X, Y \in T_{x} M$. Taking traces and using Proposition 3.5 yield

$$
\Delta F=i_{*} n \mathcal{H}^{f}-n c F,
$$

and the conclusion follows.
The next result states that any isometric immersion of a Riemannian manifold $M^{n}$ into Euclidean space $\mathbb{R}^{m+1}$ whose coordinate functions are eigenfunctions of the Laplace operator with the same nonzero eigenvalue arises as in the previous result for a minimal isometric immersion of $M^{n}$ into some sphere $\mathbb{S}_{c}^{m} \subset \mathbb{R}^{m+1}$.

Theorem 3.9. Let $F: M^{n} \rightarrow \mathbb{R}^{m+1}$ be an isometric immersion such that $\Delta F=-n c F$ for some constant $c \neq 0$. Then $c>0$ and there exists a minimal isometric immersion $f: M^{n} \rightarrow \mathbb{S}_{c}^{m}$ such that $F=i \circ f$.

Proof: Since $\Delta F=n \mathcal{H}^{F}$ by Proposition 3.5, the assumption implies that the position vector field $F$ is normal to $F$. Hence

$$
X\langle F, F\rangle=2\left\langle F_{*} X, F\right\rangle=0
$$

for any $X \in T M$, and thus $\langle F, F\rangle=r^{2}$ for some constant $r$. From

$$
\begin{aligned}
0=\frac{1}{2} \Delta\|F\|^{2} & =\langle F, \Delta F\rangle+n \\
& =n\left(1-c r^{2}\right)
\end{aligned}
$$

(see Exercise 3.6) we obtain $c=1 / r^{2}$. It follows that there exists an isometric immersion $f: M^{n} \rightarrow \mathbb{S}_{c}^{m}$ such that $F=i \circ f$, and minimality of $f$ is a consequence of Proposition 3.8.

### 3.3.1 Standard minimal immersions of spheres into spheres

Theorem 3.9 can be used to construct nice examples of minimal isometric immersions of spheres into spheres, as described next.

Denote by $\mathcal{H}(d)$ the vector space of harmonic homogeneous polynomials of degree $d$ on $\mathbb{R}^{m+1}$. It can be shown (see Corollaire C.I. 3 in [33]) that its dimension is $n+1$, where

$$
n=n(d)=(2 d+m-1) \frac{(d+m-2)!}{d!(m-1)!}-1 .
$$

Let $\Delta_{\mathbb{S}^{m}}$ denote the Laplacian on $\mathbb{S}^{m}$. It follows from Exercise 3.7 that

$$
W^{d}=\left\{\left.\varphi\right|_{\mathbb{S}^{m}}: \varphi \in \mathcal{H}(d)\right\}
$$

is contained in the eigenspace of $-\Delta_{\mathbb{S}^{m}}$ associated with the eigenvalue

$$
\lambda(d)=d(m+d-1) .
$$

In fact, $W^{d}$ coincides with such eigenspace (see Proposition C.I.1 in [33]).
Introduce on $W^{d}$ the inner product

$$
\langle\varphi, \psi\rangle=\int_{\mathbb{S}^{m}} \varphi \psi d V
$$

where $d V$ is the volume element of $\mathbb{S}^{m}$. Let $f_{0}, \ldots, f_{n}$ be an orthonormal basis of $W^{d}$ with respect to $\langle$,$\rangle and define F: \mathbb{S}^{m} \rightarrow \mathbb{R}^{n+1}$ by

$$
F=\left(f_{0}, \ldots, f_{n}\right)
$$

We prove next that $F$ is an immersion and that the metric on $\mathbb{S}^{m}$ induced by $F$ is a constant multiple of the standard metric.

By the change of variables formula, the inner product $\langle$,$\rangle on W^{d}$ is invariant by the action of $G=O(m+1)$ on $W^{d}$ given by

$$
(g \varphi)(x)=\varphi(g(x))
$$

for all $g \in G, \varphi \in W^{d}$ and $x \in \mathbb{S}^{m}$. Therefore $f_{0} \circ g, \ldots, f_{n} \circ g$ is also an orthonormal basis of $W^{d}$ for any $g \in G$; hence there exists $\tilde{g} \in O(n+1)$ such that

$$
F \circ g=\tilde{g} \circ F .
$$

Let $\langle,\rangle^{\sim}=F^{*}\langle,\rangle_{\mathbb{R}^{n+1}}$ denote the pull-back of the Euclidean metric, that is,

$$
\langle X, Y\rangle_{x}^{\sim}=\left\langle F_{*}(x) X, F_{*}(x) Y\right\rangle_{\mathbb{R}^{n+1}}
$$

for all $x \in \mathbb{S}^{m}$ and $X, Y \in T_{x} \mathbb{S}^{m}$. Then

$$
\begin{align*}
g^{*}\langle,\rangle^{\sim} & =g^{*} F^{*}\langle,\rangle_{\mathbb{R}^{n+1}}=(F \circ g)^{*}\langle,\rangle_{\mathbb{R}^{n+1}}=(\tilde{g} \circ F)^{*}\langle,\rangle_{\mathbb{R}^{n+1}}=F^{*} \tilde{g}^{*}\langle,\rangle_{\mathbb{R}^{n+1}} \\
& =F^{*}\langle,\rangle_{\mathbb{R}^{n+1}}=\langle,\rangle^{\sim} . \tag{3.2}
\end{align*}
$$

Now, for any fixed $x \in \mathbb{S}^{m}$, the isotropy subgroup $G_{x}$ of the $G$-action on $\mathbb{S}^{m}$ (that is, the subgroup of $G$ of all elements that fix $x$ ) is isomorphic to $S O(m)$ and acts irreducibly on $T_{x} \mathbb{S}^{m}$ (that is, no proper subspace of $T_{x} \mathbb{S}^{m}$ is invariant by $G_{x}$ ). Moreover, it follows from (3.2) that $\langle,\rangle_{x}^{\sim}$ is invariant by $G_{x}=S O(m)$ for all $x \in \mathbb{S}^{m}$. But if a symmetric bilinear form on a Euclidean space $\mathbb{R}^{m}$ is invariant by a subgroup of $O(m)$ that acts irreducibly on $\mathbb{R}^{m}$, then it must be a multiple of the standard inner product on $\mathbb{R}^{m}$ (see Theorem 1 of Appendix 5 in [230], vol. I). Thus there exists $\tilde{c}(x) \in \mathbb{R}$ such that

$$
\langle,\rangle_{x}^{\sim}=\tilde{c}(x)\langle,\rangle_{x},
$$

where $\langle,\rangle_{x}$ stands for the standard inner product on $T_{x} \mathbb{S}^{m}$. Given $x, y \in \mathbb{S}^{m}$, let $g \in G$ be such that $y=g(x)$. Then, for all $X, Y \in T_{x} \mathbb{S}^{m}$, the invariance of $\langle,\rangle^{\sim}$ with respect to the action of $G$ on $\mathbb{S}^{m}$ gives

$$
\begin{aligned}
\tilde{c}(y)\langle X, Y\rangle_{x} & =\tilde{c}(y)\left\langle g_{*} X, g_{*} Y\right\rangle_{y} \\
& =\left\langle g_{*} X, g_{*} Y\right\rangle_{y}^{\sim} \\
& =\langle X, Y\rangle_{x}^{\sim} \\
& =\tilde{c}(x)\langle X, Y\rangle_{x},
\end{aligned}
$$

hence $\tilde{c}(y)=\tilde{c}(x)=\tilde{c} \in \mathbb{R}$. Therefore $F$ induces an isometric immersion of $\mathbb{S}_{1 / \tilde{c}}^{m}$ into $\mathbb{R}^{n+1}$, and

$$
\tilde{\Delta} F=-\frac{\lambda(d)}{\tilde{c}} F
$$

with respect to the Laplacian $\tilde{\Delta}=(1 / \tilde{c}) \Delta_{\mathbb{S}^{m}}$ of $\mathbb{S}_{1 / \tilde{c}}^{m}$. By Theorem 3.9, there exists a minimal isometric immersion

$$
f: \mathbb{S}_{1 / \tilde{c}}^{m} \rightarrow \mathbb{S}_{c}^{n(d)}, \quad c=\lambda(d) / m \tilde{c}
$$

such that $F=i \circ f$, where $i: \mathbb{S}_{c}^{n(d)} \rightarrow \mathbb{R}^{n(d)+1}$ is the standard inclusion.
Equivalently, an orthonormal basis of $W^{d}$ gives rise to a minimal isometric immersion

$$
f: \mathbb{S}_{k(d)}^{m} \rightarrow \mathbb{S}^{n(d)} \text { with } k(d)=\frac{m}{d(m+d-1)}
$$

For instance, for $m=2=d$ we have $k(d)=1 / 3$ and $n(d)=4$, hence $f: \mathbb{S}_{1 / 3}^{2} \rightarrow \mathbb{S}^{4}$, given by

$$
\begin{equation*}
f(x, y, z)=\left(\frac{1}{\sqrt{3}} x y, \frac{1}{\sqrt{3}} x z, \frac{1}{\sqrt{3}} y z, \frac{1}{2 \sqrt{3}}\left(x^{2}-y^{2}\right), \frac{1}{6}\left(x^{2}+y^{2}-2 z^{2}\right)\right), \tag{3.3}
\end{equation*}
$$

provides a minimal isometric immersion of $\mathbb{S}_{1 / 3}^{2}$ into $\mathbb{S}^{4}$, called the Veronese surface. It induces a minimal isometric embedding of the real projective plane of constant sectional curvature $1 / 3$ into $\mathbb{S}^{4}$.

### 3.4 The Ricci tensor of a submanifold

Given an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$, we compute the Ricci tensor of $M^{n}$, defined by

$$
\operatorname{Ric}(X, Y)=\operatorname{tr} Z \mapsto R(Z, X) Y,
$$

in terms of the second fundamental form of $f$. Choosing an orthonormal tangent frame $X_{1}, \ldots, X_{n}$, the Gauss equation of $f$ yields

$$
\begin{align*}
\operatorname{Ric}(X, Y) & =\sum_{i=1}^{n}\left\langle R\left(X_{i}, X\right) Y, X_{i}\right\rangle \\
& =(n-1) c\langle X, Y\rangle+\sum_{i=1}^{n}\left(\left\langle\alpha\left(X_{i}, X_{i}\right), \alpha(X, Y)\right\rangle-\left\langle\alpha\left(X, X_{i}\right), \alpha\left(Y, X_{i}\right)\right\rangle\right) \\
& =(n-1) c\langle X, Y\rangle+n\langle\alpha(X, Y), \mathcal{H}\rangle-I I I(X, Y) \tag{3.4}
\end{align*}
$$

where

$$
I I I(X, Y)=\sum_{i=1}^{n}\left\langle\alpha\left(X, X_{i}\right), \alpha\left(Y, X_{i}\right)\right\rangle
$$

is known as the third fundamental form of $f$. Note that

$$
\operatorname{III}(X, X)=\sum_{i=1}^{n}\left\|\alpha\left(X, X_{i}\right)\right\|^{2} \geq 0
$$

for any $X \in \mathfrak{X}(M)$, and that $I I I(X, X)=0$ at $x \in M^{n}$ if and only if $X(x)$ belongs to the relative nullity subspace $\Delta(x)$ of $f$ at $x$.

If $T, S \in \Gamma(\operatorname{End}(T M))$ denote the symmetric tensors given by

$$
\begin{equation*}
\operatorname{Ric}(X, Y)=\langle T X, Y\rangle \text { and } \operatorname{III}(X, Y)=\langle S X, Y\rangle \tag{3.5}
\end{equation*}
$$

then (3.4) can be written as

$$
\begin{equation*}
T=(n-1) c I+n A_{\mathscr{H}}-S . \tag{3.6}
\end{equation*}
$$

In terms of an orthonormal normal frame $\xi_{1}, \ldots, \xi_{m-n}$ we have

$$
\begin{aligned}
\operatorname{III}(X, Y) & =\sum_{i=1}^{n} \sum_{r=1}^{m-n}\left\langle\alpha\left(X, X_{i}\right), \xi_{r}\right\rangle\left\langle\alpha\left(Y, X_{i}\right), \xi_{r}\right\rangle \\
& =\sum_{i=1}^{n} \sum_{r=1}^{m-n}\left\langle A_{\xi_{r}} X, X_{i}\right\rangle\left\langle A_{\xi_{r}} Y, X_{i}\right\rangle \\
& =\sum_{r=1}^{m-n}\left\langle A_{\xi_{r}}^{2} X, Y\right\rangle,
\end{aligned}
$$

hence

$$
S=\sum_{r=1}^{m-n} A_{\xi_{r}}^{2} .
$$

In particular, if $m=n+1$ and we write $\mathcal{H}=H \xi$ for a unit normal vector field $\xi$, then (3.6) reduces to

$$
\begin{equation*}
T=(n-1) c I+n H A-A^{2} \tag{3.7}
\end{equation*}
$$

where we write $A=A_{\xi}$.
By (3.4), the Ricci curvature in the direction of a unit vector $X \in T M$, defined as

$$
\operatorname{Ric}(X)=\frac{1}{n-1} \operatorname{Ric}(X, X)
$$

is given by

$$
\begin{equation*}
\operatorname{Ric}(X)=c+\frac{n}{n-1}\left\langle A_{\mathcal{H}} X, X\right\rangle-\frac{1}{n-1} I I I(X, X) \tag{3.8}
\end{equation*}
$$

Taking traces in (3.8) yields

$$
\begin{equation*}
s=c+\frac{n}{n-1}\|\mathcal{H}\|^{2}-\frac{1}{n(n-1)}\|\alpha\|^{2}, \tag{3.9}
\end{equation*}
$$

where $s$ is the scalar curvature defined as

$$
s=\frac{1}{n} \sum_{i=1}^{n} \operatorname{Ric}\left(X_{i}\right)
$$

and $\|\alpha\|$ denotes the norm of the second fundamental form, given by

$$
\|\alpha\|^{2}=\sum_{i, j=1}^{n}\left\|\alpha\left(X_{i}, X_{j}\right)\right\|^{2}
$$

Formula (3.8) yields the following obstruction for the existence of a minimal isometric immersion into any manifold of constant sectional curvature.
Proposition 3.10. If an isometric immersion $f: M^{n} \rightarrow \tilde{M}_{c}^{m}$ is minimal at $x \in M^{n}$ then $\operatorname{Ric}(X) \leq c$ for every unit vector $X \in T_{x} M$. Moreover, equality holds for every $X \in T_{x} M$ if and only if $f$ is totally geodesic at $x$.

### 3.5 Rigidity of minimal hypersurfaces

This section presents a strong rigidity property of minimal hypersurfaces within the class of minimal isometric immersions.

For a Riemannian manifold $M^{n}$ we denote by $\mu_{c}(x)$ the dimension of the c-nullity subspace at $x \in M^{n}$, given by

$$
\begin{equation*}
\Gamma_{c}(x)=\left\{X \in T_{x} M: R(X, Y)=c(X \wedge Y) \text { for all } Y \in T_{x} M\right\} \tag{3.10}
\end{equation*}
$$

If $c=0$ we write simply $\Gamma(x)$ instead of $\Gamma_{0}(x)$, and refer to $\mu(x)=\mu_{0}(x)$ as the nullity of $M^{n}$ at $x$.

Theorem 3.11. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}, n \geq 3$, be a minimal hypersurface. If there exists a point $x_{0} \in M^{n}$ such that $\mu_{c}\left(x_{0}\right) \leq n-3$, then any minimal isometric immersion $g: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ is congruent to $i \circ f$, where $i: \mathbb{Q}_{c}^{n+1} \rightarrow \mathbb{Q}_{c}^{n+p}$ is a totally geodesic inclusion.

The proof relies on the following algebraic lemma.
Lemma 3.12. Let $\gamma: V \times V \rightarrow \mathbb{R}$ and $\alpha: V \times V \rightarrow W$ be traceless symmetric bilinear forms, where $V$ and $W$ are vector spaces of dimensions $n \geq 3$ and $p$, respectively, endowed with positive definite inner products. Assume that

$$
\operatorname{span}\{\alpha(X, Y): X, Y \in V\}=W
$$

and that $\operatorname{dim} \mathcal{N}(\gamma) \leq n-3$, where

$$
\mathcal{N}(\gamma)=\{Y \in V: \gamma(X, Y)=0 \text { for all } X \in V\}
$$

If

$$
\begin{equation*}
\gamma(X, X) \gamma(Y, Y)-\gamma^{2}(X, Y)=\langle\alpha(X, X), \alpha(Y, Y)\rangle-\|\alpha(X, Y)\|^{2} \tag{3.11}
\end{equation*}
$$

for all $X, Y \in V$, then $p=1$ and $\alpha= \pm \gamma$.
Proof: Let $Y_{1}, \ldots, Y_{n}$ be an orthonormal basis of $V$. By assumption,

$$
\sum_{j=1}^{n} \gamma\left(Y_{j}, Y_{j}\right)=0=\sum_{j=1}^{n} \alpha\left(Y_{j}, Y_{j}\right)
$$

It follows using (3.11) that

$$
\begin{equation*}
\sum_{j=1}^{n} \gamma^{2}\left(X, Y_{j}\right)=\sum_{j=1}^{n}\left\|\alpha\left(X, Y_{j}\right)\right\|^{2} \tag{3.12}
\end{equation*}
$$

In particular, this implies that $\mathcal{N}(\gamma)=\mathcal{N}(\alpha)$. Let $X_{1}, \ldots, X_{m}$ be an orthonormal basis of $\mathcal{N}(\gamma)^{\perp}$ which diagonalizes $\gamma$, and set

$$
\lambda_{i}=\gamma\left(X_{i}, X_{i}\right), \quad 1 \leq i \leq m .
$$

Then (3.12) yields

$$
\lambda_{i}^{2}=\sum_{j=1}^{m}\left\|\alpha_{i j}\right\|^{2}, \quad 1 \leq i \leq m
$$

where

$$
\alpha_{i j}=\alpha\left(X_{i}, X_{j}\right), \quad 1 \leq i, j \leq m .
$$

On the other hand, from $(\sqrt{3.11})$ it follows that

$$
\begin{equation*}
\lambda_{i} \lambda_{j}=\left\langle\alpha_{i i}, \alpha_{j j}\right\rangle-\left\|\alpha_{i j}\right\|^{2} . \tag{3.13}
\end{equation*}
$$

The two preceding equations and the Cauchy-Schwarz inequality yield

$$
\begin{align*}
\left(\left\langle\alpha_{i i}, \alpha_{j j}\right\rangle-\left\|\alpha_{i j}\right\|^{2}\right)^{2} & =\lambda_{i}^{2} \lambda_{j}^{2}  \tag{3.14}\\
& =\sum_{k=1}^{m}\left\|\alpha_{i k}\right\|^{2} \sum_{\ell=1}^{m}\left\|\alpha_{j \ell}\right\|^{2} \\
& \geq\left(\left\|\alpha_{i i}\right\|^{2}+\left\|\alpha_{i j}\right\|^{2}\right)\left(\left\|\alpha_{j j}\right\|^{2}+\left\|\alpha_{i j}\right\|^{2}\right) \\
& \geq\left(\left\langle\alpha_{i i}, \alpha_{j j}\right\rangle+\left\|\alpha_{i j}\right\|^{2}\right)^{2} . \tag{3.15}
\end{align*}
$$

If $\alpha_{i j} \neq 0$ for some $1 \leq i \neq j \leq m$, it follows that $\left\langle\alpha_{i i}, \alpha_{j j}\right\rangle \leq 0$, hence

$$
\lambda_{i} \lambda_{j} \leq-\left\|\alpha_{i j}\right\|^{2}<0
$$

by (3.13). Since $m \geq 3$ by assumption, the products $\lambda_{i} \lambda_{j}, 1 \leq i \neq j \leq m$, cannot be all negative, hence there must exist indices $1 \leq i \neq j \leq m$ for which $\alpha_{i j}=0$. For such a pair of indices, we must have equality in the first inequality in 3.14) by the Cauchy-Schwarz inequality, which implies that $\alpha_{i i}$ and $\alpha_{j j}$ are linearly dependent and

$$
\alpha_{i k}=\alpha_{j k}=0, \quad 1 \leq k \leq m, k \neq i, j .
$$

Applying the same conclusion for each such pair $(i, k)$, it follows that

$$
\alpha_{i j}=0, \quad 1 \leq i \neq j \leq m
$$

and that all the vectors $\alpha_{i i}, 1 \leq i \leq m$, are linearly dependent. Hence $p=1$, and since $n \geq 3, \alpha= \pm \gamma$ by (3.13).

Proof of Theorem 3.11: Since $f$ is minimal, by Exercise 3.8 the nullity subspace $\Gamma_{c}\left(x_{0}\right)$ coincides with the relative nullity subspace of $f$ at $x_{0}$. Therefore, it follows from Lemma 3.12 and the assumption $\mu_{c}\left(x_{0}\right) \leq n-3$ that $\operatorname{dim} N_{1}^{g}\left(x_{0}\right)=1$ and that the second fundamental forms of $f$ and $g$ coincide at $x_{0}$, after choosing the appropriate orientations.

Since $\mu_{c}(x) \leq n-3$ also in an open simply connected neighborhood $U$ of $x_{0}$, the first normal space $N_{1}^{g}(x)$ has dimension 1 for all $x \in U$. If $\zeta \in \Gamma\left(N_{g} U\right)$ is a unit normal vector field spanning $N_{1}^{g}$ along $U$, we can then assume that the shape operator $A$ of $f$ with respect to a unit normal vector field $N$ coincides with $A_{\zeta}^{g}$. It follows from Proposition 2.7 that $\zeta$ is parallel in the normal connection. Then $\zeta^{\perp}$ is a parallel subbundle of $N_{g} U$, which is also flat, for $\zeta^{\perp}=N_{1}^{g \perp}$. The same holds for the subbundle $\bar{N}^{\perp}$ of the normal bundle of $i \circ f$, where $\bar{N}=i_{*} N$. Choose orthonormal frames $\xi_{1}, \ldots, \xi_{p-1}$ and $\eta_{1}, \ldots, \eta_{p-1}$ of $\bar{N}^{\perp}$ and $\zeta^{\perp}$, respectively, along $U$. Define a vector bundle isometry $\phi$ between $N_{i \circ f} U$ and $N_{g} U$ by $\phi(\bar{N})=\zeta$ and $\phi\left(\xi_{j}\right)=\eta_{j}, 1 \leq j \leq p-1$. Then it is clear that $\phi$ preserves the normal connections and the second fundamental forms. We conclude from Theorem 1.25 that

$$
\left.g\right|_{U}=\left.\Phi \circ i \circ f\right|_{U}
$$

for some isometry $\Phi$ of $\mathbb{Q}_{c}^{n+p}$. Since minimal immersions are real analytic, we must have $g=\Phi \circ i \circ f$.

Remark 3.13. Theorem 3.11 does not hold if $\mu_{c}(x) \geq n-2$ at any point of $M^{n}$. In fact, if $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ is a minimal isometric immersion of a simply connected manifold whose second fundamental form $A$ has exactly two nonzero principal curvatures everywhere, then there exists a one-parameter family $f_{\theta}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}, \theta \in[0, \pi)$, of noncongruent minimal isometric immersions. See Exercise 3.9.

### 3.6 The Ricci condition

Let $M^{2}$ be a Riemannian manifold and let $x \in M^{2}$ be a point where the Gauss curvature $K$ is negative. The Ricci condition states that a necessary and sufficient condition for some neighborhood of $x$ to admit a minimal isometric immersion in $\mathbb{R}^{3}$ is that the metric $\langle\langle\rangle\rangle=,\sqrt{-K}\langle$,$\rangle be flat. This condition is equivalent to the metric$ $-K\langle$,$\rangle having constant Gauss curvature equal to one. In this section we give a proof$ of an extension of the latter version of the Ricci condition to hypersurfaces in space forms.

Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ be a minimal isometric immersion with shape operator $A$. By (3.4) we have

$$
\operatorname{Ric}(X, Y)=(n-1) c\langle X, Y\rangle-\langle A X, A Y\rangle
$$

for all $X, Y \in \mathfrak{X}(M)$. Therefore

$$
\langle\langle,\rangle\rangle=-\operatorname{Ric}(,,)+(n-1) c\langle,\rangle
$$

defines a new Riemannian metric on $M^{n}$ if $A$ is everywhere invertible. The Levi-Civita connection $\hat{\nabla}$ of $\langle\langle\rangle$,$\rangle satisfies$

$$
\begin{aligned}
2\left\langle\left\langle\hat{\nabla}_{X} Y, Z\right\rangle\right\rangle= & X\langle\langle Y, Z\rangle\rangle+Y\langle\langle X, Z\rangle\rangle-Z\langle\langle X, Y\rangle\rangle+\langle\langle[Z, Y], X\rangle\rangle+\langle\langle[X, Y], Z\rangle\rangle \\
& +\langle\langle[Z, X], Y\rangle\rangle \\
= & X\langle A Y, A Z\rangle+Y\langle A X, A Z\rangle-Z\langle A X, A Y\rangle+\langle A[Z, Y], A X\rangle \\
& +\langle A[X, Y], A Z\rangle+\langle A[Z, X], A Y\rangle \\
= & \left\langle\nabla_{X} A Y, A Z\right\rangle+\left\langle A Y, \nabla_{X} A Z\right\rangle+\left\langle\nabla_{Y} A X, A Z\right\rangle+\left\langle A X, \nabla_{Y} A Z\right\rangle \\
& -\left\langle\nabla_{Z} A X, A Y\right\rangle-\left\langle A X, \nabla_{Z} A Y\right\rangle+\langle A[Z, Y], A X\rangle+\langle A[X, Y], A Z\rangle \\
& +\langle A[Z, X], A Y\rangle \\
= & 2\left\langle\nabla_{X} A Y, A Z\right\rangle-\left\langle\left(\nabla_{X} A\right) Y-\left(\nabla_{Y} A\right) X, A Z\right\rangle \\
& +\left\langle\left(\nabla_{X} A\right) Z-\left(\nabla_{Z} A\right) X, A Y\right\rangle+\left\langle\left(\nabla_{Y} A\right) Z-\left(\nabla_{Z} A\right) Y, A X\right\rangle .
\end{aligned}
$$

Hence the Codazzi equation implies that

$$
\begin{equation*}
\left\langle\left\langle\hat{\nabla}_{X} Y, Z\right\rangle\right\rangle=\left\langle\nabla_{X} A Y, A Z\right\rangle \tag{3.16}
\end{equation*}
$$

for all $X, Y, Z \in \mathfrak{X}(M)$.
We now compute

$$
\langle\langle\hat{R}(X, Y) Y, X\rangle\rangle=\left\langle\left\langle\hat{\nabla}_{X} \hat{\nabla}_{Y} Y, X\right\rangle\right\rangle-\left\langle\left\langle\hat{\nabla}_{Y} \hat{\nabla}_{X} Y, X\right\rangle\right\rangle-\left\langle\left\langle\hat{\nabla}_{[X, Y]} Y, X\right\rangle\right\rangle .
$$

Using (3.16) and

$$
\begin{equation*}
\langle\langle X, Y\rangle\rangle=\langle A X, A Y\rangle \tag{3.17}
\end{equation*}
$$

we obtain

$$
\begin{aligned}
\left\langle\left\langle\hat{\nabla}_{X} \hat{\nabla}_{Y} Y, X\right\rangle\right\rangle & =X\left\langle\nabla_{Y} A Y, A X\right\rangle-\left\langle\nabla_{Y} A Y, \nabla_{X} A X\right\rangle \\
& =\left\langle\nabla_{X} \nabla_{Y} A Y, A X\right\rangle \\
\left\langle\left\langle\hat{\nabla}_{Y} \hat{\nabla}_{X} Y, X\right\rangle\right\rangle & =\left\langle\nabla_{Y} \nabla_{X} A Y, A X\right\rangle \\
\left\langle\left\langle\hat{\nabla}_{[X, Y]} Y, X\right\rangle\right\rangle & =\left\langle\nabla_{[X, Y]} A Y, A X\right\rangle .
\end{aligned}
$$

Hence

$$
\langle\langle\hat{R}(X, Y) Y, X\rangle\rangle=\langle R(X, Y) A Y, A X\rangle .
$$

Then the Gauss equation

$$
\langle R(X, Y) A Y, A X\rangle=c\langle(A X \wedge A Y) Y, X\rangle+\|A X \wedge A Y\|^{2}
$$

implies that the sectional curvature of the new metric satisfies

$$
\begin{equation*}
\hat{K}(X, Y)=1+c \frac{\langle(A X \wedge A Y) Y, X\rangle}{\|A X \wedge A Y\|^{2}} \tag{3.18}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}(M)$.
The next result shows that the preceding conditions are also sufficient for a simply connected Riemannian manifold to admit a minimal isometric immersion into $\mathbb{Q}_{c}^{n+1}$.

Theorem 3.14. Let $M^{n}$ be a simply connected Riemannian manifold such that

$$
\operatorname{Ric}(,)-(n-1) c\langle,\rangle
$$

is everywhere negative definite, so that

$$
\langle\langle,\rangle\rangle=-\operatorname{Ric}(,)+(n-1) c\langle,\rangle
$$

defines a new Riemannian metric on $M^{n}$. Assume that $B \in \Gamma(E n d(T M))$, defined by

$$
\langle\langle X, Y\rangle\rangle=\langle B X, Y\rangle,
$$

admits a smooth symmetric square root $A$, that is, (3.17) holds. Assume also that the Levi-Civita connection and sectional curvature of the new metric satisfy (3.16) and (3.18), respectively. Then $M^{n}$ admits a minimal isometric immersion into $\mathbb{Q}_{c}^{n+1}$.

Proof: The proof consists in showing that $A$ satisfies the Gauss and Codazzi equations for an isometric immersion into $\mathbb{Q}_{c}^{n+1}$. Notice that it is not required to be traceless, which is a consequence of the remaining assumptions.

Using (3.16), (3.17) and the fact that $A$ is invertible, the equation

$$
\left\langle\left\langle\hat{\nabla}_{X} Y, Z\right\rangle\right\rangle-\left\langle\left\langle\hat{\nabla}_{Y} X, Z\right\rangle\right\rangle-\langle\langle[X, Y], Z\rangle\rangle=0
$$

reduces to the Codazzi equation for $A$.
As shown before the statement of Theorem 3.14. Eqs. (3.16) and (3.17) also yield

$$
\langle\langle\hat{R}(X, Y) Y, X\rangle\rangle=\langle R(X, Y) A Y, A X\rangle .
$$

Then (3.18) reads as

$$
\langle R(X, Y) A Y, A X\rangle=c\langle(A X \wedge A Y) Y, X\rangle+\|A X \wedge A Y\|^{2},
$$

which is equivalent to the Gauss equation.
By Theorem 1.11, there exist an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ and a unit normal vector field $\xi$ such that $A$ coincides with the shape operator $A_{\xi}$ of $f$ with respect to $\xi$. It now follows from (3.4) applied to $f$ that

$$
\operatorname{Ric}(X, Y)-(n-1) c\langle X, Y\rangle=n H\langle A X, Y\rangle-\langle A X, A Y\rangle,
$$

and hence $H=0$.
Corollary 3.15. Let $M^{2}$ be a Riemannian manifold, let $c$ be a real number and let $x \in M^{2}$ be a point where the Gauss curvature satisfies $K<c$. A necessary and sufficient condition for a neighborhood of $x$ to be isometrically and minimally immersed in $\mathbb{Q}_{c}^{3}$ is that the Gauss curvature $\hat{K}$ of the metric $(-K+c)\langle$,$\rangle satisfies$

$$
\hat{K}=1+\frac{c}{K-c} .
$$

After the immersion given by Theorem 3.14 has been obtained, condition (3.18) becomes

$$
\begin{equation*}
\hat{K}(X, Y)=1+c(K(X, Y)-c) \frac{\|X \wedge Y\|^{2}}{\|A X \wedge A Y\|^{2}} . \tag{3.19}
\end{equation*}
$$

The following example shows that condition (3.18) cannot be replaced by (3.19).
Example 3.16. A Riemannian manifold $M^{n}$ is an Einstein manifold if the Ricci tensor satisfies

$$
\operatorname{Ric}(X, Y)=\rho\langle X, Y\rangle
$$

for all $X, Y \in \mathfrak{X}(M)$ and some constant $\rho \in \mathbb{R}$. If $M^{n}, n=2 m+1$, is an Einstein manifold with

$$
\begin{equation*}
\operatorname{Ric}(,)=(n-2) c\langle,\rangle, \tag{3.20}
\end{equation*}
$$

where $c$ is a positive number, then $\operatorname{Ric}()-,(n-1) c\langle$,$\rangle is negative definite and$

$$
\langle\langle X, Y\rangle\rangle=c\langle X, Y\rangle .
$$

Thus $M^{n}$ cannot be isometrically and minimally immersed in $\mathbb{Q}_{c}^{n+1}$, since the odd dimensional identity matrix has no traceless square root.

### 3.7 Notes

It was shown in Section 3.1 that a minimal immersion $f: M^{n} \rightarrow \tilde{M}^{m}$ represents a critical point for the area function on the space of all immersions of $M^{n}$ into $\tilde{M}^{m}$. It is therefore natural to ask whether $f$ is actually a local minimum for the area function, which requires the computation of its second derivative. This leads to the second variation formula, which relates the second derivative of the area function to geometric invariants of the immersion. We refer the reader to the books of Lawson [234] and Xin [352] for a discussion of this topic, as well as for many other results on minimal submanifolds.

Theorem 3.9 is due to Takahashi 320 . He also observed that if $M^{n}$ is an isotropyirreducible Riemannian homogeneous space, that is, if the isotropy group of a point acts irreducibly on the tangent space, then an orthonormal basis of each eigenspace of the Laplacian operator on $M^{n}$ gives rise to a minimal isometric immersion into a round sphere. These are called the standard minimal immersions.

In particular, if $M^{n}=\mathbb{S}^{n}$ one obtains the sequence of minimal isometric immersions described in Section 3.3.1, one for each nonzero eigenvalue $\lambda=d(n+d-1)$ of the Laplacian, which corresponds to harmonic homogeneous polynomials of degree $d$. For odd $g$, the standard minimal isometric immersion is an embedding. For even $g$, all components of the immersion are invariant under the antipodal map. In fact, in this case it gives rise to a minimal isometric embedding of the real projective space.

Calabi [49] proved that every minimal isometric immersion of a two-dimensional round sphere into $\mathbb{S}^{m}$ is congruent to one of these standard immersions. For $m \geq 3$, do Carmo-Wallach [62] showed that if $f: \mathbb{S}_{k}^{m} \rightarrow \mathbb{S}^{n}$ is a minimal isometric immersion then $k=k(d)$ for some $d$, and if $f$ is substantial then $n \leq n(d)$, where $k(d)$ and $n(d)$ are as in Section 3.3.1. They also showed that if $d \leq 3$ and $f$ is substantial, then $f$ must be congruent to the standard minimal isometric immersion of $\mathbb{S}_{k(d)}^{m}$ into $\mathbb{S}^{n(d)}$. Moreover, if $m \geq 3$ and $d \geq 4$, they proved that the set of substantial minimal isometric immersions of $\mathbb{S}_{k(d)}^{m}$ into $\mathbb{S}^{n}$ (up to congruence of the ambient space) is parametrized by a compact convex body in a finite dimensional vector space. The problem of finding the exact dimension of this convex body was studied by Toth [337]. The immersions corresponding to interior points of the convex body are all embedded spheres or embedded real projective spaces. The question of which spherical space forms admit minimal isometric immersions or embeddings into spheres was addressed by De Turck-Ziller [160], who proved this to be the case for every homogeneous spherical space form.

Theorem 3.11 on the rigidity of minimal isometric immersions was obtained by Barbosa-Dajczer-Jorge [26].

The Ricci condition for minimal surfaces in $\mathbb{R}^{3}$ was given by Ricci 301 and extended to surfaces in $\mathbb{S}^{3}$ by Lawson [233]. The extension to hypersurfaces of the Ricci condition given by Theorem 3.14 is due to do Carmo-Dajczer [54]. The case of Euclidean hypersurfaces was also considered by Chern-Osserman [87]. A necessary condition for the existence of minimal isometric immersions into Euclidean space in codimension one was given by Barbosa-do Carmo [30]. For arbitrary codimension, Chen [79] has shown that a necessary condition for a Riemannian manifold $M^{n}$ to
admit a minimal isometric immersion into $\mathbb{R}^{n+p}$ is for the sectional curvature of $M^{n}$ to satisfy $K(\sigma) \geq(n(n-1) / 2) s(x)$ for all $x \in M^{n}$ and any plane $\sigma \in T_{x} M$, where $s(x)$ is the scalar curvature of $M^{n}$ at $x$. Other necessary conditions for the case in which the codimension is arbitrary were found by Chen [82], [83] and Vlachos [339]. For the case in which the ambient space is the sphere see also Hasanis-Vlachos [212].

Chen's condition above was derived as a consequence of a general inequality that holds for any isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$. Namely, let $\delta_{M}$ denote the intrinsic quantity defined by

$$
\delta_{M}(x)=n(n-1) s(x)-2 \inf \left\{K(\sigma): \sigma \subset T_{x} M\right\}
$$

It was shown by Chen [79, [80 that the inequality

$$
\delta_{M} \leq \frac{n-2}{2(n-1)}\left(n^{2}\|\mathcal{H}\|^{2}+\left(n^{2}-1\right) c\right)
$$

holds at any point of $M^{n}$. Notice that the equality holds at $x$ in the above inequality if $c=0=\mathcal{H}(x)$ and $\nu(x)=n-2$. It was shown by Dajczer-Florit [95] that any nonminimal isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{n+p}, n \geq 4$, for which the equality is attained at every point of $M^{n}$ is a rotation hypersurface over a surface $h: L^{2} \rightarrow \mathbb{R}^{2+p}$ satisfying some conditions.

Ruled minimal submanifolds in space forms were shown to be generalized helicoids by Barbosa-Dajczer-Jorge [27]. In particular, in Euclidean space the examples in part (ii) of Exercise 3.1 comprise all possible ones. The case of ruled submanifolds with mean curvature vector field of constant length was treated by Barbosa-Delgado [29].

Minimal isometric immersions with codimension two of Einstein manifolds into space forms were classified by Matsuyama [242]. By formula 3.4, any such isometric immersion into Euclidean space has the property that its third fundamental form is a constant multiple of the metric. Isometric immersions with codimension two into Euclidean space with this property were classified by Freitas [194].

The result on minimal hypersurfaces contained in Exercise 3.4 is due to PinlZiller [295]. Extensions to higher codimension, as well as other results concerning the existence of asymptotic directions of minimal submanifolds, have been given by Dajczer-Rodríguez [125].

The result in Exercise 3.9 was observed in Dajczer-Gromoll 109 and analyzed in Dajczer-Vlachos [153]. The classification of Einstein hypersurfaces in Euclidean space given in part (iii) of Exercise 3.11] was obtained by Fialkow [179]; see also Ryan [305]. For a generalization of the Einstein condition for compact Euclidean hypersurfaces see Vlachos 341 . Exercise 3.12 on the local holonomicity of isometric immersions with flat normal bundle of Einstein submanifolds into space forms was taken from Dajczer-Onti-Vlachos [123]. The result in Exercise 3.18 is due to Costa [90].

### 3.8 Exercises

Exercise 3.1. Show that the following maps are minimal immersions.
(i) The Clifford torus $f: \mathbb{S}^{1} \times \cdots \times \mathbb{S}^{1} \rightarrow \mathbb{S}^{2 n-1} \subset \mathbb{R}^{2 n}$ of dimension $n$ given by

$$
f\left(t_{1}, \ldots, t_{n}\right)=\frac{1}{\sqrt{n}}\left(\cos \sqrt{n} t_{1}, \sin \sqrt{n} t_{1}, \ldots, \cos \sqrt{n} t_{n}, \sin \sqrt{n} t_{n}\right)
$$

(ii) The generalized helicoid $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+k+1}, n \geq k$, defined by

$$
f\left(s, t_{1}, \ldots, t_{n}\right)=s b v_{0}+\sum_{i=1}^{k} t_{i} e_{i}(s)+\sum_{i=1}^{n-k} t_{k+i} v_{2 k+i}
$$

where $v_{0}, \ldots, v_{n+k}$ is the canonical basis of $\mathbb{R}^{n+k+1}$,

$$
e_{i}(s)=\cos \left(a_{i} s\right) v_{2 i-1}+\sin \left(a_{i} s\right) v_{2 i}
$$

and $b \neq 0 \neq a_{i} \in \mathbb{R}, i=1, \ldots, k$.
(iii) The spherical ruled minimal surfaces $f_{\alpha}: \mathbb{R}^{2} \rightarrow \mathbb{S}^{3}, \alpha>0$, given by

$$
f_{\alpha}(x, y)=(\cos \alpha x \cos y, \sin \alpha x \cos y, \cos x \sin y, \sin x \sin y)
$$

## Exercise 3.2.

(i) Let $f_{1}: M_{1}^{n_{1}} \rightarrow \mathbb{S}^{m_{1}} \subset \mathbb{R}^{m_{1}+1}, \ldots, f_{k+1}: M_{k+1}^{n_{k+1}} \rightarrow \mathbb{S}^{m_{k+1}} \subset \mathbb{R}^{m_{k+1}+1}$ be minimal isometric immersions. Set $M^{n}=M_{1}^{n_{1}} \times \cdots \times M_{k+1}^{n_{k+1}}$. Define $f: M^{n} \rightarrow \mathbb{R}^{m+k+1}$, $m=\sum_{j=1}^{k+1} m_{j}$, by

$$
f\left(x_{1}, \ldots, x_{k+1}\right)=\left(\sqrt{n_{1} / n} f_{1}\left(x_{1}\right), \ldots, \sqrt{n_{k+1} / n} f_{k+1}\left(x_{k+1}\right)\right)
$$

for all $\left(x_{1}, \ldots, x_{k+1}\right) \in M^{n}$. Show that $f$ induces an immersion into $\mathbb{S}^{m+k}$, which is a minimal isometric immersion if $M^{n}$ is endowed with the induced metric.
(ii) Conclude that if $M^{n}=\mathbb{S}^{n_{1}} \times \cdots \times \mathbb{S}^{n_{k+1}}$ and $f: M^{n} \rightarrow \mathbb{R}^{n+k+1}$ is defined by

$$
f\left(x_{1}, \ldots, x_{k+1}\right)=\left(\sqrt{n_{1} / n} i_{1}\left(x_{1}\right), \ldots, \sqrt{n_{k+1} / n} i_{k+1}\left(x_{k+1}\right)\right)
$$

where $i_{j}: \mathbb{S}^{n_{j}} \rightarrow \mathbb{R}^{n_{j}+1}, 1 \leq j \leq k+1$, are inclusions, then $f$ induces an immersion into $\mathbb{S}^{n+k}$, which is a minimal isometric immersion if $M^{n}$ is endowed with the induced metric, called a generalized Clifford torus.

Exercise 3.3. Prove directly that the map $f$ given by (3.3) induces a minimal isometric embedding of the real projective plane of constant sectional curvature $1 / 3$ into $\mathbb{S}^{4}$. Show that its normal curvature tensor satisfies

$$
\left\langle R^{\perp}(X, Y) \xi, \eta\right\rangle=2 / 3
$$

where $X, Y$ and $\xi, \eta$ are positively oriented orthonormal tangent and normal frames, respectively.

Exercise 3.4. A nonzero vector $X \in T_{x} M$ is called asymptotic for an isometric immersion $f: M^{n} \rightarrow \tilde{M}^{n+1}$ if its second fundamental form satisfies $\alpha(X, X)=0$. Show that $f$ is minimal at $x \in M^{n}$ if and only if there are $n$ orthogonal asymptotic tangent vectors at $x$.
Hint: Use induction on the dimension $n$.
Exercise 3.5. Let $M^{n}$ be a Riemannian manifold and let $h \in C^{\infty}(M)$.
(i) Prove the identity

$$
\frac{1}{2} \Delta h^{2}=h \Delta h+\|\operatorname{grad} h\|^{2}
$$

(ii) Use the identity in part ( $i$ ) and the divergence theorem to prove Hopf's theorem, namely, if $M^{n}$ is compact and $\Delta h \geq 0$, then $h$ is constant.

Exercise 3.6. Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be an isometric immersion. Show that

$$
\begin{equation*}
\frac{1}{2} \Delta\|f\|^{2}=\langle f, \Delta f\rangle+n \tag{3.21}
\end{equation*}
$$

Use (3.21) and Hopf's theorem in part (ii) of Exercise 3.5 to give another proof of Corollary 3.7.

Exercise 3.7. Let $\Delta$ and $\Delta_{\mathbb{S}^{m}}$ denote the Laplacians on $\mathbb{R}^{m+1}$ and $\mathbb{S}^{m}$, respectively. For $\varphi \in C^{\infty}\left(\mathbb{R}^{m+1}\right)$ show that

$$
\left.\Delta \varphi\right|_{\mathbb{S}^{m}}=\Delta_{\mathbb{S}^{m}}\left(\left.\varphi\right|_{\mathbb{S}^{m}}\right)+\left.\frac{\partial^{2} \varphi}{\partial r^{2}}\right|_{\mathbb{S}^{m}}+\left.m \frac{\partial \varphi}{\partial r}\right|_{\mathbb{S}^{m}}
$$

where $\partial / \partial r$ denotes radial derivative.
Exercise 3.8. Let $f: M^{n} \rightarrow \tilde{M}_{c}^{m}$ be a minimal isometric immersion. Prove that

$$
\Gamma_{c}(x)=\Delta(x)
$$

at any $x \in M^{n}$.
Exercise 3.9. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be a minimal simply connected submanifold with constant index of relative nullity $\nu=n-2$. For any constant $\theta \in[0, \pi)$ consider the tensor field $R(\theta) \in \Gamma(\operatorname{End}(T M))$ which is the identity along the relative nullity distribution $\Delta$ and a rotation through $\theta$ on $\Delta^{\perp}$. Prove that the traceless bilinear form $\alpha_{\theta}: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \Gamma\left(N_{f} M\right)$ defined by

$$
\alpha_{\theta}(X, Y)=\alpha(R(\theta) X, Y)
$$

satisfies the Gauss, Codazzi and Ricci equation with respect to the normal connection of $f$. Conclude from the Fundamental theorem of submanifolds that there exists a one parameter associated family $f_{\theta}: M^{n} \rightarrow \mathbb{Q}_{c}^{m}, \theta \in[0, \pi)$, of minimal isometric immersions with $f_{0}=f$.

Exercise 3.10. Let $f: M^{n} \rightarrow \tilde{M}_{c}^{n+1}$ be an isometric immersion. Given $x \in M^{n}$, show that the following conditions are equivalent:
(i) The sectional curvature $K(\sigma)$ along any two-plane $\sigma \subset T_{x} M$ satisfies the inequality $K(\sigma)>c$ (respectively, $K(\sigma) \geq c$ ).
(ii) $\operatorname{Ric}(X)>c$ (respectively, $\operatorname{Ric}(X) \geq c)$ for any unit vector $X \in T_{x} M$.

Hint: Use equation (3.8).

## Exercise 3.11.

(i) Show that any three-dimensional Einstein manifold has constant sectional curvature.
(ii) Show that the Riemannian product $M_{c_{1}}^{p} \times M_{c_{2}}^{n-p}$ is an Einstein manifold if and only if

$$
(p-1) c_{1}=(n-p-1) c_{2} .
$$

(iii) If $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, is an isometric immersion of an Einstein manifold, show that either $M^{n}$ is flat or $f$ is umbilical (and hence $f(M)$ is an open subset of $\mathbb{S}_{c}^{n} \subset \mathbb{R}^{n+1}$ for some $c>0$ ).

Hint for (iii): Let $T \in \Gamma(\operatorname{End}(T M))$ be given by (3.5). Then $T=\rho I$ for some $\rho \in \mathbb{R}$. Use (3.7) to show that the principal curvatures $\lambda_{1}, \ldots, \lambda_{n}$ of $f$ satisfy

$$
\lambda_{j}^{2}-r \lambda_{j}+\rho=0, \quad 1 \leq j \leq n
$$

where $r=n H$. If $\rho=0$, conclude that at most one principal curvature does not vanish. If $\rho>0$, show that all principal curvatures coincide. Assuming $\rho<0$, reach a contradiction as follows. Write

$$
\lambda_{1}=\cdots=\lambda_{p}=\nu \text { and } \lambda_{p+1}=\cdots=\lambda_{n}=\mu
$$

for some $1 \leq p \leq n$, with $\mu \neq \nu$ everywhere. Prove that

$$
(p-1) \nu^{2}+(n-p-1) \rho=0
$$

which implies that $p>1$ and

$$
\nu^{2}=-\frac{n-p-1}{p-1} \rho .
$$

Conclude that $\nu, \mu$ and $p$ are all constant on $M^{n}$, and then use the Codazzi equation to show that both $E_{\nu}$ and $E_{\mu}$ are parallel distributions on $M^{n}$. Obtain a contradiction by showing that this implies that the sectional curvature $K(X, Y)$ of $M^{n}$ along a plane spanned by unit vectors $X \in E_{\nu}$ and $Y \in E_{\mu}$ vanishes, whereas the Gauss equation gives

$$
K(X, Y)=\nu \mu=\rho<0
$$

Exercise 3.12. If $M^{n}$ is an Einstein manifold, show that any isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ with flat normal bundle and a constant number of pairwise distinct principal normals $\eta_{1}, \ldots \eta_{s}$ at any point is locally holonomic.
Hint: By assumption, the endomorphism $T \in \Gamma(\operatorname{End}(T M))$ associated with the Ricci tensor of $M^{n}$ satisfies $T=\lambda I$ for some $\lambda \in \mathbb{R}$. Show that the vector fields

$$
\hat{\eta}_{i}=\eta_{i}-\frac{n}{2} H, \quad 1 \leq i \leq s,
$$

satisfy

$$
\begin{equation*}
\left\|\hat{r}_{i}\right\|^{2}=\frac{n^{2}}{4}\|H\|^{2}+c(n-1)-\lambda . \tag{3.22}
\end{equation*}
$$

Assume that there exist $i \neq j \neq \ell \neq i$ and $\mu \in C^{\infty}(M)$ such that

$$
\eta_{i}-\eta_{j}=\mu\left(\eta_{i}-\eta_{\ell}\right)
$$

Derive from this equation that

$$
(1-\mu) \hat{\eta}_{i}=\hat{\eta}_{j}-\mu \hat{\eta}_{l}
$$

and hence

$$
\left\|\hat{\eta}_{i}\right\|^{2}-2 \mu\left\|\hat{\eta}_{i}\right\|^{2}+\mu^{2}\left\|\hat{\eta}_{i}\right\|^{2}=\left\|\hat{\eta}_{j}\right\|^{2}-2 \mu\left\langle\hat{\eta}_{j}, \hat{\eta}_{\ell}\right\rangle+\mu^{2}\left\|\hat{\eta}_{\ell}\right\|^{2} .
$$

Now use the preceding equation and the fact that all the vectors $\hat{\eta}_{j}$ have the same length by (3.22) to obtain

$$
\left\langle\hat{\eta}_{j}, \hat{\eta}_{\ell}\right\rangle=\left\|\hat{\eta}_{j}\right\|\left\|\hat{\eta}_{\ell}\right\|,
$$

and hence $\hat{\eta}_{j}=\hat{\eta}_{\ell}$, a contradiction. Conclude that the vector fields $\eta_{i}-\eta_{j}$ and $\eta_{i}-\eta_{\ell}$ are linearly independent for all $i \neq j \neq \ell \neq i$, and then use part (ii) of Exercise 1.39 and Exercise 1.40

Exercise 3.13. Show that, in Theorem 3.11, neither the assumption on the codimension of $f$ nor that on $\mu_{c}$ can be weakened.

Exercise 3.14. Let $g: L^{\ell} \rightarrow M^{n}$ and $f: M^{n} \rightarrow \tilde{M}_{c}^{m}$ be isometric immersions such that $f \circ g$ is totally geodesic. Show that $\operatorname{Ric}\left(g_{*} X\right) \leq c$ for all $X \in T L$, with equality if and only if $g_{*} X$ belongs to the relative nullity subspace of $f$.

Exercise 3.15. Give a proof of Corollary 3.15 ,
Exercise 3.16. Give an example of an Einstein manifold $M^{n}, n=2 m$, that satisfies condition 3.20 and admits a minimal isometric immersion into $\mathbb{S}_{c}^{n+1}$.

Exercise 3.17. Let $f: M^{3} \rightarrow \tilde{M}_{c}^{3+p}$ be an isometric immersion. Given $x \in M^{3}$, suppose that $H(x)=0$ and $R^{\perp}(x)=0$. Show that there exists an orthonormal basis $Y_{1}, Y_{2}, Y_{3}$ of $T_{x} M$ such that the sectional curvature of $M^{3}$ satisfies

$$
K\left(Y_{i}, Y_{j}\right) \leq c, \quad 1 \leq i, j \leq 3
$$

Hint: If $X_{1}, X_{2}, X_{3}$ is an orthonormal basis that diagonalizes the second fundamental form, take $Y_{1}$ to be the asymptotic vector

$$
Y_{1}=\frac{1}{\sqrt{3}}\left(X_{1}+X_{2}+X_{3}\right) .
$$

Exercise 3.18. Let $f: M^{n} \rightarrow \tilde{M}_{c}^{n+2}, n \geq 3$, be an isometric immersion of an Einstein manifold with Ric $=\rho\langle\rangle,, \rho \geq(n-1) c$. Prove that $f$ has flat normal bundle at a point $x \in M^{n}$ unless $\rho=(n-1) c$ and all sectional curvatures at $x$ are equal to $c$.
Hint: Let $x \in M^{n}$. Use (3.6) to show that $\mathcal{H}(x) \neq 0$ unless $\rho=(n-1) c$ and $f$ is totally geodesic at $x$. Since in the latter case $f$ has trivially flat normal bundle at $x$, one can assume that $\mathcal{H}(x) \neq 0$ if $\rho=(n-1) c$. Set $H=\|\mathcal{H}(x)\|$ and let $\xi_{1}=(1 / H) \mathcal{H}, \xi_{2}$ be an orthonormal basis of $N_{f} M(x)$. Show that

$$
\begin{equation*}
A_{1}^{2}-n H A_{1}+A_{2}^{2}=\alpha I \tag{3.23}
\end{equation*}
$$

where $A_{j}=A_{\xi_{j}}, j=1,2$, and $\alpha=(n-1) c-\rho \leq 0$. Write the preceding equation as

$$
\left(A_{1}-(n H / 2) I\right)^{2}+A_{2}^{2}=\left((n H / 2)^{2}+\alpha\right) I .
$$

Use (3.23) to show that all eigenvalues of $A_{1}$ are positive if $\rho>(n-1) c$, and nonnegative if $\rho=(n-1) c$. Deduce from this that $A_{1}-(n H / 2) I$ does not admit any pair of eigenvalues of the form $\pm \mu$ with $\mu \neq 0$ if $\rho>(n-1) c$, and conclude in this case that $\left[A_{1}, A_{2}\right]=0$ from the following elementary fact observed in 90): If $V$ is a vector space of dimension $n \geq 3$ endowed with an inner product and $A, B$ are symmetric endomorphisms of $V$ such that

$$
A^{2}+B^{2}=r I, \quad r \geq 0,
$$

and $A$ does not admit any pair of eigenvalues of the form $\pm \mu$ with $\mu \neq 0$, then $[A, B]=0$.

Suppose now that $\rho=(n-1) c$ and that $\left[A_{1}, A_{2}\right] \neq 0$. Deduce from the preceding fact that $A_{1}-(n H / 2) I$ admits a pair of eigenvalues of the form $\pm \mu$ with $\mu \neq 0$, and hence that $A_{1}$ admits two eigenvalues $\lambda_{1}, \lambda_{2}$ such that $\lambda_{1}+\lambda_{2}=n H$. Using this and (3.23) with $\alpha=0$, show that if $e_{1}$ and $e_{2}$ are eigenvectors of $A_{1}$ correspondent to $\lambda_{1}$ and $\lambda_{2}$, respectively, then $\left\{e_{1}, e_{2}\right\}^{\perp}$ belongs to the relative nullity subspace of $f$ at $x$. Finally, conclude from this that all sectional curvatures of $M^{n}$ at $x$ are equal to $c$.

## Chapter 4

## Local rigidity of submanifolds

One of the basic problems in submanifold theory addressed in this book concerns the uniqueness of isometric immersions $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ of Riemannian manifolds into space forms. Clearly, since $g=\tau \circ f$ is also an isometric immersion for any isometry $\tau: \mathbb{Q}_{c}^{m} \rightarrow \mathbb{Q}_{c}^{m}$, uniqueness should be understood to be up to congruences by isometries of the ambient space.

The usual terminology for uniqueness of an isometric immersion in the above sense is isometric rigidity. Therefore, an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ is said to be isometrically rigid, or simply rigid, if any other isometric immersion $g: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ is congruent to it by an isometry of the ambient space $\mathbb{Q}_{c}^{m}$. Otherwise, the isometric immersion $f$ is said to be isometrically deformable, or simply deformable.

Since an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ is given in local coordinates by a solution of the nonlinear system of partial differential equations (1.2), and this system is overdetermined if $m<(1 / 2) n(n+1)$, it is natural to expect $f$ to be rigid under "generic" conditions if $m$ is small when compared with the bound in the preceding inequality.

The main results of this chapter, namely, Allendoerfer and do Carmo-Dajczer theorems, provide sufficient algebraic conditions on the second fundamental form of an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ in order to assure that $f$ is isometrically rigid. Both results are of local nature and in large part consequences of the Gauss equation. Roughly speaking, they say that an isometric immersion in very low codimension is rigid unless its second fundamental form is "degenerate" enough.

Let $f, \tilde{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be isometric immersions with second fundamental forms $\alpha$ and $\tilde{\alpha}$, respectively. According to the Fundamental theorem of submanifolds, to show that $f$ and $\tilde{f}$ are congruent by an isometry of the ambient space, it suffices to prove the assertions in the following three steps:
(i) There is a linear isometry $T: N_{f} M(x) \rightarrow N_{\tilde{f}} M(x)$ at each $x \in M^{n}$ satisfying

$$
\tilde{\alpha}(x)=T \circ \alpha(x) .
$$

(ii) These isometries form a smooth vector bundle isometry $T: N_{f} M \rightarrow N_{\tilde{f}} M$.
(iii) The vector bundle isometry $T$ is parallel, that is, it preserves the normal connections.

A linear isometry that satisfies the condition in $(i)$ is unique whenever the first normal space $N_{1}(x)$ of $f$ at $x$ coincides with $N_{f} M(x)$. If this condition is satisfied at every point $x \in M^{n}$, then one obtains condition (ii) almost for free. Moreover, condition (iii) is also automatically satisfied under this assumption.

Allendoerfer's theorem uses the type number of $\alpha$, whereas the algebraic conditions on $\alpha$ in do Carmo-Dajczer's theorem are given in terms of its s-nullities. Both results are proved using the theory of flat bilinear forms, which play a central role in this book. In particular, the main tool in the proof of do Carmo-Dajczer's theorem is a lemma on symmetric flat bilinear forms whose most general version is stated and proved in the appendix to this chapter. A counterexample showing that it cannot be improved with respect to the dimension of the target vector space is given.

### 4.1 Flat bilinear forms

Let $W^{p, q}$ be a real vector space of dimension $p+q$ endowed with an inner product $\langle$,$\rangle of signature (p, q)$. Let $V$ and $U$ be finite dimensional vector spaces. A bilinear form $\beta: V \times U \rightarrow W^{p, q}$ is said to be flat if

$$
\langle\beta(X, Y), \beta(Z, T)\rangle-\langle\beta(X, T), \beta(Z, Y)\rangle=0
$$

for all $X, Z \in V$ and $Y, T \in U$. It is called null if

$$
\langle\beta(X, Y), \beta(Z, T)\rangle=0
$$

for all $X, Z \in V$ and $Y, T \in U$. Thus a null bilinear form is necessarily flat.
To see how the above concepts are related to the rigidity problem for submanifolds of space forms, consider a pair $f, \tilde{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ of isometric immersions with second fundamental forms $\alpha$ and $\tilde{\alpha}$, respectively. For each $x \in M^{n}$, set

$$
W(x)=N_{f} M(x) \oplus N_{\tilde{f}} M(x)
$$

and endow $W(x)$ with the inner product of signature $(p, p)$ given by

$$
\langle\langle(\xi, \tilde{\xi}),(\eta, \tilde{\eta})\rangle\rangle_{W(x)}=\langle\xi, \eta\rangle_{N_{f} M(x)}-\langle\tilde{\xi}, \tilde{\eta}\rangle_{N_{\tilde{f}} M(x)} .
$$

Define a bilinear form $\left.\beta(x): T_{x} M \times T_{x} M \rightarrow W^{( } x\right) s$ by

$$
\begin{equation*}
\beta(x)=\alpha(x) \oplus \tilde{\alpha}(x) . \tag{4.1}
\end{equation*}
$$

Then the next basic result leads to an approach to carry out step $(i)$ in the introduction of this chapter. It also shows that, once it has been overcome, then one automatically obtains step (ii) provided that the first normal space $N_{1}(x)$ of $f$ at $x$ coincides with $N_{f} M(x)$ for any $x \in M^{n}$.

Proposition 4.1. The following assertions hold:
(i) The bilinear form $\beta(x)$ is flat.
(ii) There exists a linear isometry $T: N_{f} M(x) \rightarrow N_{\tilde{f}} M(x)$ such that $\tilde{\alpha}(x)=T \circ \alpha(x)$ if and only if $\beta(x)$ is null. Moreover, such a $T$ is unique if $N_{1}(x)=N_{f} M(x)$.
(iii) If $\beta(x)$ is null and $N_{1}(x)=N_{f} M(x)$ for all $x \in M^{n}$, then there exists a smooth vector bundle isometry $T: N_{f} M \rightarrow N_{\tilde{f}} M$ such that $\tilde{\alpha}=T \circ \alpha$.

Proof: The assertion in (i) follows immediately from the Gauss equations for $f$ and $\tilde{f}$. As for the first assertion in (ii), that $\beta(x)$ is null is equivalent to

$$
\langle\alpha(X, Y), \alpha(Z, V)\rangle=\langle\tilde{\alpha}(X, Y), \tilde{\alpha}(Z, V)\rangle
$$

for all $X, Y, Z, V \in T_{x} M$. Clearly, this is satisfied if and only if there is a linear isometry $T: N_{1}(x) \rightarrow N_{1}(x)$ such that $\tilde{\alpha}(x)=T \circ \alpha(x)$. It now suffices to define $T$ on $N_{1}^{\perp}(x)$ as any linear isometry onto $\tilde{N}_{1}^{\perp}(x)$. The last assertion in $(i i)$ is clear.

To prove assertion (iii), for each $x \in M^{n}$ let $T: N_{f} M(x) \rightarrow N_{\tilde{f}} M(x)$ be the unique linear isometry given by (ii) such that $\tilde{\alpha}(x)=T \circ \alpha(x)$. To show that these linear isometries fit together to yield a smooth vector bundle isometry $T: N_{f} M \rightarrow N_{\tilde{f}} M$ such that $\tilde{\alpha}=T \circ \alpha$, at any $x \in M^{n}$ take $X_{i}, Y_{i} \in T_{x} M, 1 \leq i \leq p$, such that the vectors $\alpha\left(X_{i}, Y_{i}\right)$ form a basis of $N_{f} M(x)$, and extend them to smooth vector fields $X_{i}, Y_{i}$ in a small neighborhood of $x$ where they still span the normal space at every point. Then the vector fields

$$
T \alpha\left(X_{i}, Y_{i}\right)=\tilde{\alpha}\left(X_{i}, Y_{i}\right), \quad 1 \leq i \leq p,
$$

are also smooth, thus showing smoothness of $T$.
The theory developed in this section will provide, in particular, conditions under which a flat bilinear form is null.

### 4.1.1 Indefinite inner products

It is a standard fact that any vector space $W^{p, q}$ admits an orthonormal basis $Y_{1}, \ldots, Y_{p+q}$, that is, a basis such that

$$
\left\langle Y_{k}, Y_{l}\right\rangle= \pm \delta_{k l}, \quad 1 \leq k, l \leq p+q,
$$

and $q$ is the number of indices $k \in\{1, \ldots, p+q\}$ with $\left\langle Y_{k}, Y_{k}\right\rangle=-1$.
Lesser known is the fact, implied by Proposition 4.2 below, that $W^{p, q}$ also admits a pseudo-orthonormal basis

$$
\begin{equation*}
X_{1}, \ldots, X_{r}, \bar{X}_{1}, \ldots, \bar{X}_{r}, Y_{1}, \ldots, Y_{p+q-2 r}, \tag{4.2}
\end{equation*}
$$

for which we have:
(i) $\left\langle X_{i}, X_{j}\right\rangle=\left\langle\bar{X}_{i}, \bar{X}_{j}\right\rangle=\left\langle X_{i}, Y_{k}\right\rangle=\left\langle\bar{X}_{i}, Y_{k}\right\rangle=0$,
(ii) $\left\langle X_{i}, \bar{X}_{j}\right\rangle=\delta_{i j}$,
(iii) $\left\langle Y_{k}, Y_{l}\right\rangle= \pm \delta_{k l}$,
for all $1 \leq i, j \leq r, 1 \leq k, l \leq p+q-2 r$.
A vector subspace $U \subset W^{p, q}$ is said to be degenerate if the restriction of the inner product of $W^{p, q}$ to $U$ is degenerate, that is, if the subspace $\mathcal{E}=U \cap U^{\perp}$ is nontrivial. Otherwise, the subspace $U$ is called nondegenerate. We denote by rank $U$ the rank of the induced inner product on $U$, defined as

$$
\operatorname{rank} U=\operatorname{dim} U-\operatorname{dim} \mathcal{E}
$$

If rank $U=0$, that is, $U=\mathcal{E}$, then $U$ is called isotropic. The same terminology is used for a vector $X \in W^{p, q}$ such that $\langle X, X\rangle=0$. Thus, if $q=1$, then isotropic vectors are the same as light-like vectors, and we shall continue to use the latter terminology in this case.

We also recall that if $U$ is any subspace of $W^{p, q}$, then

$$
\operatorname{dim} U+\operatorname{dim} U^{\perp}=p+q \text { and } U^{\perp \perp}=U
$$

Proposition 4.2. Let $U \subset W^{p, q}$ be a subspace, and set $\mathcal{E}=U \cap U^{\perp}$. Let $R \subset U$ be a subspace such that $U$ is the direct sum $U=\mathcal{E} \oplus R$. Let $X_{1}, \ldots, X_{r}$ be a basis of $\mathcal{E}$. Then there exist isotropic pairwise orthogonal vectors $\bar{X}_{1}, \ldots, \bar{X}_{r}$ in $R^{\perp}$ such that

$$
\left\langle X_{i}, \bar{X}_{j}\right\rangle=\delta_{i j}, \quad 1 \leq i, j \leq r .
$$

Consequently, $X_{1}, \ldots, X_{r}$ can be extended to a pseudo-orthonormal basis of $W^{p, q}$ as in (4.2). In particular, $r \leq \min \{p, q\}$.

Proof: If $r=0$, the result is trivially true. By induction, suppose that it holds for $r-1$. Define

$$
U_{0}=\operatorname{span}\left\{X_{1}, \ldots, X_{r-1}\right\} \oplus R
$$

Notice that $X_{r} \notin U_{0}$ and that $X_{r} \in U_{0}^{\perp}$. Also,

$$
\varepsilon_{0}=U_{0} \cap U_{0}^{\perp}=\operatorname{span}\left\{X_{1}, \ldots, X_{r-1}\right\} .
$$

Since $X_{r} \notin U_{0}=U_{0}^{\perp \perp}$, there exists $Y \in U_{0}^{\perp}$ such that $\left\langle X_{r}, Y\right\rangle \neq 0$. It follows easily that there exists $\bar{X}_{r} \in P=\operatorname{span}\left\{X_{r}, Y\right\}$ such that $\left\langle\bar{X}_{r}, \bar{X}_{r}\right\rangle=0$ and $\left\langle X_{r}, \bar{X}_{r}\right\rangle=1$. In particular, the plane $P$ is Lorentzian.

Since $P \subset U_{0}^{\perp}$, we have $P^{\perp} \supset U_{0}^{\perp \perp}=U_{0}$. Therefore, by applying the induction hypothesis to $U_{0} \subset P^{\perp}$, we obtain vectors $\bar{X}_{1}, \ldots, \bar{X}_{r-1}$ in $P^{\perp}$ orthogonal to $R$ such that

$$
\left\langle\bar{X}_{i}, \bar{X}_{j}\right\rangle=0 \text { and }\left\langle X_{i}, \bar{X}_{j}\right\rangle=\delta_{i j}, \quad 1 \leq i, j \leq r-1 .
$$

It is clear that $\bar{X}_{1}, \ldots, \bar{X}_{r}$ have the desired properties. Moreover, setting

$$
\mathcal{E}=\operatorname{span}\left\{X_{1}, \ldots, X_{r}\right\}, \quad \widehat{\mathcal{E}}=\operatorname{span}\left\{\bar{X}_{1}, \ldots, \bar{X}_{r}\right\}
$$

$$
Z_{i}^{+}=\frac{1}{\sqrt{2}}\left(X_{i}+\bar{X}_{i}\right) \text { and } Z_{i}^{-}=\frac{1}{\sqrt{2}}\left(X_{i}-\bar{X}_{i}\right),
$$

then $Z_{1}^{+}, \ldots, Z_{r}^{+}, Z_{1}^{-}, \ldots, Z_{r}^{-}$is an orthonormal basis of $\mathcal{E} \oplus \widehat{\mathcal{E}}$, the induced inner product being positive definite on $V^{+}=\operatorname{span}\left\{Z_{1}^{+}, \ldots, Z_{r}^{+}\right\}$and negative definite on $V^{-}=\operatorname{span}\left\{Z_{1}^{-}, \ldots, Z_{r}^{-}\right\}$. This implies the last statement and shows that the subspace $\mathcal{E} \oplus \widehat{\mathcal{E}}$ is nondegenerate. To conclude the proof, take $Y_{1}, \ldots, Y_{p+q-2 r}$ as an orthonormal basis of $(\mathcal{E} \oplus \widehat{\mathcal{E}})^{\perp}$.

Corollary 4.3. Let $U \subset W^{p, q}$ be a degenerate subspace, and set $\mathcal{E}=U \cap U^{\perp} \neq\{0\}$. Then there exists a direct sum decomposition

$$
\begin{equation*}
W^{p, q}=\mathcal{E} \oplus \widehat{\mathcal{E}} \oplus \mathcal{V} \tag{4.3}
\end{equation*}
$$

such that $U \subset \mathcal{E} \oplus \mathcal{V}$, the subspace $\widehat{\mathcal{E}}$ is isotropic and $\mathcal{V}$ is nondegenerate with $\mathcal{V}^{\perp}=\mathcal{E} \oplus \widehat{\mathcal{E}}$.

### 4.1.2 Basic properties of flat bilinear forms

For a bilinear form $\beta: V \times U \rightarrow W$ between finite dimensional real vector spaces, denote

$$
\rho=\max \left\{\operatorname{dim} B_{Z}(U): Z \in V\right\},
$$

where $B_{Z}=\beta(Z):, U \rightarrow W$. We call $X \in V$ a (left) regular element of $\beta$ if

$$
\operatorname{dim} B_{X}(U)=\rho,
$$

and denote by $R E(\beta)$ the subset of $V$ of regular elements of $\beta$.
Proposition 4.4. The subset $R E(\beta)$ is open and dense in $V$.
Proof: Take $X \in R E(\beta)$ and choose $Z_{1}, \ldots, Z_{\rho} \in U$ such that

$$
B_{X}(U)=\operatorname{span}\left\{\beta\left(X, Z_{j}\right): 1 \leq j \leq \rho\right\} .
$$

Then $R E(\beta)$ is open, for the vectors

$$
\beta\left(X+Y, Z_{j}\right), \quad 1 \leq j \leq \rho,
$$

are also linearly independent for any $Y$ in a neighborhood of $0 \in V$.
We now prove that $R E(\beta)$ is dense. Given any $Y \in V$ and $t \in \mathbb{R}$,

$$
\beta\left(Y+t X, Z_{j}\right)=\beta\left(Y, Z_{j}\right)+t \beta\left(X, Z_{j}\right) .
$$

Since the vectors $\beta\left(Y+t X, Z_{j}\right)$ are linearly independent except for a finite number of values of $t$, there exists a sequence $\left\{t_{k}\right\}$ converging to 0 so that $Y+t_{k} X \in R E(\beta)$.

Proposition 4.5. Let $\beta: V \times U \rightarrow W^{p, q}$ be a flat bilinear form. If $B_{X}(U)$ is an isotropic subspace of $W^{p, q}$ for any $X$ in a dense subset of $V$, then $\beta$ is null.

Proof: Under the assumption, by continuity we have

$$
\langle\beta(X, Y), \beta(X, Z)\rangle=0
$$

for all $X \in V$ and $Y, Z \in U$. Using this and flatness of $\beta$ we obtain

$$
0=\langle\beta(X+T, Y), \beta(X+T, Z)\rangle=2\langle\beta(X, Y), \beta(T, Z)\rangle
$$

for all $X, T \in V$ and $Y, Z \in U$.
The following is a basic property of flat bilinear forms that will be extensively used.

Proposition 4.6. Let $\beta: V \times U \rightarrow W^{p, q}$ be a bilinear form. If $X \in R E(\beta)$, then

$$
\begin{equation*}
\mathcal{S}\left(\left.\beta\right|_{V \times \operatorname{ker} B_{X}}\right) \subset B_{X}(U) . \tag{4.4}
\end{equation*}
$$

Moreover, if $\beta$ is flat then

$$
\begin{equation*}
\mathcal{S}\left(\left.\beta\right|_{V \times \operatorname{ker} B_{X}}\right) \subset B_{X}(U) \cap B_{X}(U)^{\perp} . \tag{4.5}
\end{equation*}
$$

Proof: Given $Y \in V$, there exists $\epsilon>0$ such that

$$
\operatorname{dim} \beta(X+t Y, U)=\rho=\operatorname{dim} B_{X}(U)
$$

if $|t|<\epsilon$. Now, if $n \in \operatorname{ker} B_{X}$, then

$$
\beta(X+t Y, n)=t \beta(Y, n) .
$$

Thus

$$
\beta(Y, n) \in \beta(X+t Y, U) \text { for } t \neq 0
$$

Since the vector subspaces $\beta(X+t Y, U)$ vary continuously with $t \in \mathbb{R}$, this also holds for $t=0$, that is, $\beta(Y, n) \in B_{X}(U)$. Hence $\beta(V, n) \subset B_{X}(U)$.

Assume further that $\beta$ is flat. Then

$$
\langle\beta(Y, n), \beta(X, Z)\rangle=\langle\beta(Y, Z), \beta(X, n)\rangle=0
$$

for all $Y \in V$ and $Z \in U$. Thus $\beta(V, n) \subset B_{X}(U)^{\perp}$.
Corollary 4.7. Let $\beta: V \times U \rightarrow W^{p, q}$ be a bilinear form and let $X \in R E(\beta)$. Then

$$
\mathcal{S}(\beta)=\mathcal{S}\left(\left.\beta\right|_{V \times S}\right)
$$

if $S \subset U$ is a subspace such that $U=\operatorname{ker} B_{X} \oplus S$.
We conclude this section with the following fact, which will be used in the proof of the main result of Chapter 12 .

Proposition 4.8. Let $\beta: V \times U^{n} \rightarrow W^{p, q}$ be a bilinear form. Then

$$
1 \leq \rho \leq \operatorname{dim} \mathcal{S}(\beta)-k+1 \text { and } \operatorname{dim} \mathcal{N}\left(\left.\beta\right|_{L^{k} \times U}\right) \geq n-k(\rho-1)-1
$$

where $L^{k} \subset V$ is a subspace of minimal dimension $k$ such that

$$
\mathcal{S}\left(\left.\beta\right|_{L^{k} \times U}\right)=\mathcal{S}(\beta) .
$$

Proof: Suppose first that $L^{k}$ is spanned by $X_{1}, \ldots, X_{k} \in R E(\beta)$. Then

$$
\mathcal{S}(\beta)=\mathcal{S}\left(\left.\beta\right|_{L^{k} \times U}\right)=B_{X_{1}}(U)+\cdots+B_{X_{k}}(U),
$$

and the assumption on $L^{k}$ implies that

$$
B_{X_{j}}(U) \not \subset B_{X_{1}}(U)+\cdots+B_{X_{j-1}}(U)
$$

for any $1 \leq j \leq k$. This already implies the first inequality. For the second, notice that

$$
\mathcal{N}\left(\left.\beta\right|_{L^{k} \times U}\right)=\operatorname{ker} B_{X_{1}} \cap \cdots \cap \operatorname{ker} B_{X_{k}}
$$

and that

$$
B_{X_{i}}\left(\operatorname{ker} B_{X_{j}}\right) \subset B_{X_{j}}(U), \quad 1 \leq i \neq j \leq k
$$

by (4.4). Moreover, again by the assumption on $L^{k}$ we must have

$$
\begin{aligned}
\operatorname{dim} B_{X_{i}}\left(\operatorname{ker} B_{X_{j}}\right) & \leq \operatorname{dim} B_{X_{j}}(U)-1 \\
& =\rho-1
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\operatorname{dim} \mathcal{N}\left(\left.\beta\right|_{L^{k} \times U}\right) & \geq n-\rho-(k-1)(\rho-1) \\
& =n-k(\rho-1)-1 .
\end{aligned}
$$

To obtain the proof for an arbitrary $L^{k}$ as in the statement, observe that there exists a sequence $L_{j}^{k} \rightarrow L^{k}$ such that each $L_{j}^{k}$ still satisfies the assumption and is spanned by vectors in $R E(\beta)$.

### 4.1.3 The Chern-Kuiper inequalities

Let $f: M^{n} \rightarrow \tilde{M}^{m}$ be an isometric immersion. Given $x \in M^{n}$, the subspace $\Gamma(x)=\left\{X \in T_{x} M:\langle R(X, Y) Z, W\rangle=\left\langle\tilde{R}\left(f_{*} X, f_{*} Y\right) f_{*} Z, f_{*} W\right\rangle\right.$ for all $\left.Y, Z, W \in T_{x} M\right\}$ is called the nullity subspace, and its dimension $\mu(x)$ the index of nullity of $f$ at $x$. If $\tilde{M}^{m}=\tilde{M}_{c}^{m}$, then $\Gamma(x)$ coincides with the intrinsic subspace $\Gamma_{c}(x)$ defined in 3.10).

The first application of the theory of flat bilinear forms is to establish the following inequalities relating $\mu(x)$ and the index of relative nullity $\nu(x)$.

Theorem 4.9. If $f: M^{n} \rightarrow \tilde{M}^{n+p}$ is an isometric immersion and $x \in M^{n}$, then

$$
\begin{equation*}
\nu(x) \leq \mu(x) \leq \nu(x)+p \tag{4.6}
\end{equation*}
$$

Proof: The first inequality is obvious, for $\Delta(x) \subset \Gamma(x)$ by the Gauss equation. In order to prove the second, let $L$ denote the orthogonal complement of $\Delta(x)$ in $\Gamma(x)$. Given $X \in L$, since $X \notin \Delta(x)$ there exists $Y \in T_{x} M$ such that $\alpha(X, Y) \neq 0$. Using that $X \in \Gamma(x)$, the Gauss equation yields

$$
\langle\alpha(X, X), \alpha(Y, Y)\rangle=\|\alpha(X, Y)\|^{2}
$$

hence $\alpha(X, X) \neq 0$. Thus $\beta=\left.\alpha\right|_{L \times L}$ satisfies $\mathcal{N}(\beta)=0$. Then the lemma below for not necessarily symmetric bilinear forms and target vector spaces with positive definite inner products gives

$$
p \geq \operatorname{dim} L=\operatorname{dim} \Gamma(x)-\operatorname{dim} \Delta(x)
$$

as we wished.
Lemma 4.10. Let $\beta: V \times V \rightarrow W$ be a flat bilinear form with respect to a positive definite inner product. Then

$$
\operatorname{dim} \mathcal{N}(\beta) \geq \operatorname{dim} V-\operatorname{dim} W
$$

Proof: Take any $X \in R E(\beta)$. We assert that $\mathcal{N}(\beta)=\operatorname{ker} B_{X}$. Clearly, $\mathcal{N}(\beta) \subset \operatorname{ker} B_{X}$. On the other hand, if $n \in \operatorname{ker} B_{X}$, then for any $Y \in V$ we see from 4.5) that

$$
\beta(Y, n) \in B_{X}(V) \cap B_{X}(V)^{\perp}=\{0\}
$$

since the inner product on $W$ is positive definite. Thus $n \in \mathcal{N}(\beta)$, that is, ker $B_{X} \subset$ $\mathcal{N}(\beta)$. Therefore

$$
\begin{aligned}
\operatorname{dim} \mathcal{N}(\beta) & =\operatorname{dim} \operatorname{ker} B_{X} \\
& =\operatorname{dim} V-\operatorname{dim} B_{X}(V) \\
& \geq \operatorname{dim} V-\operatorname{dim} W
\end{aligned}
$$

as we wished.
Proposition 4.11. Let $M^{n}$ be a compact Riemannian manifold such that $\mu(x) \geq \ell$ for some integer $\ell \geq 1$ and any $x \in M^{n}$. If $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ is an isometric immersion, then $p \geq \ell$.

Proof: Since $M^{n}$ is compact, by Corollary 1.6 there exists a point $x_{0} \in M^{n}$ such that $\nu\left(x_{0}\right)=0$. Then the second inequality in (4.6) gives $p \geq \mu\left(x_{0}\right) \geq \ell$.

Corollary 4.12. Let $M^{n}$ be a compact flat Riemannian manifold. If $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ is an isometric immersion, then $p \geq n$.

### 4.1.4 The Beez-Killing theorem

The next application of the theory of flat bilinear forms is a proof of the following rigidity result for hypersurfaces of dimension $n \geq 3$.

Theorem 4.13. A hypersurface $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ with type number $\tau \geq 3$ at any point is rigid.

Proof: Let $\tilde{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ be another isometric immersion. Denote by $\alpha$ and $\tilde{\alpha}$ the second fundamental forms of $f$ and $\tilde{f}$, respectively. Defining

$$
\beta(x)=\alpha(x) \oplus \tilde{\alpha}(x)
$$

at $x \in M^{n}$ as in 4.1, the assumption that $\tau(x) \geq 3$ for all $x \in M^{n}$ yields

$$
\operatorname{dim} \mathcal{N}(\beta(x)) \leq n-3
$$

for all $x \in M^{n}$. Hence the version below of Lemma 4.10 for the case of symmetric flat bilinear forms into Lorentzian target vector spaces implies that $\beta(x)$ is null for all $x \in M^{n}$. Then Proposition 4.1 implies that $\tilde{\alpha}=\phi \circ \alpha$ for one of the two vector bundle isometries $\phi: N_{f} M \rightarrow N_{\tilde{f}} M$, and the proof is completed by Theorem 1.11 .

Lemma 4.14. Let $\beta: V^{n} \times V^{n} \rightarrow W^{p, 1}, 1 \leq p \leq n-2$, be a flat symmetric bilinear form such that $\mathcal{S}(\beta)=W^{p, 1}$. Then

$$
\operatorname{dim} \mathcal{N}(\beta) \geq \operatorname{dim} V-\operatorname{dim} W
$$

Proof: If there exists $X \in R E(\beta)$ such that the subspace $B_{X}(V)$ is nondegenerate, then, as in the proof of Lemma 4.10. Proposition 4.6 implies that $\mathcal{N}(\beta)=\operatorname{ker} B_{X}$, and the conclusion follows.

Therefore we can assume that $B_{X}(V)$ is degenerate for all $X \in R E(\beta)$. Given $X \in R E(\beta)$, denote

$$
\mathcal{U}(X)=B_{X}(V) \cap B_{X}(V)^{\perp}
$$

Since $W$ is Lorentzian, then $\mathcal{U}(X)$ is a one-dimensional isotropic subspace of $W$, and from (4.5) we have $\beta(V, n) \subset \mathcal{U}(X)$ for all $n \in \operatorname{ker} B_{X}$. By the assumption that $\mathcal{S}(\beta)=W^{p, 1}$, there exist $Z, T \in V$ such that $\beta(Z, T) \notin \mathcal{U}(X)^{\perp}$. Set

$$
L=\operatorname{ker} B_{X} \cap \operatorname{ker} B_{Z} .
$$

Since the linear map $B_{Z}$ from ker $B_{X}$ to $W$ is $\mathcal{U}(X)$-valued,

$$
\operatorname{dim} L \geq \operatorname{dim} \operatorname{ker} B_{X}-1
$$

If $n \in L$, it follows from the flatness of $\beta$ that

$$
\langle\beta(Y, n), \beta(Z, T)\rangle=\langle\beta(Y, T), \beta(Z, n)\rangle=0
$$

for all $Y \in V$. This, together with the fact that $\beta(Y, n) \in \mathcal{U}(X)$, gives $\beta(Y, n)=0$. Hence $L=\mathcal{N}(\beta)$, and

$$
\begin{aligned}
\operatorname{dim} \mathcal{N}(\beta) & \geq \operatorname{dim} \operatorname{ker} B_{X}-1 \\
& =\operatorname{dim} V-\operatorname{dim} B_{X}(V)-1
\end{aligned}
$$

Finally, observe that $\operatorname{dim} B_{X}(V)+1 \leq \operatorname{dim} W$, for $B_{X}(V)$ is degenerate.
The following consequence of the proof of Theorem 4.13 will be the starting point in the Sbrana-Cartan classification of isometrically deformable hypersurfaces of space forms in Chapter 11.

Corollary 4.15. Let $f, \tilde{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ be nowhere congruent isometric immersions of a Riemannian manifold with no points of constant sectional curvature c. Then $f$ and $\tilde{f}$ carry a common relative nullity distribution of rank $n-2$.

Proof: Since $f$ and $\tilde{f}$ are nowhere congruent, the proof of Theorem 4.13 shows that the bilinear form $\beta$ therein cannot be null on any open subset of $M^{n}$. On the other hand, $\beta$ must be null on any open subset where $\operatorname{dim} \mathcal{N}(\beta) \leq n-3$ by Lemma 4.14. Therefore $\operatorname{dim} \mathcal{N}(\beta) \geq n-2$ on $M^{n}$. Since $\mathcal{N}(\beta)$ is contained in the relative nullity subspaces of both $f$ and $\tilde{f}$, and these cannot have dimension greater than $n-2$ by the assumption that $M^{n}$ has no points with constant sectional curvature $c$, the conclusion follows.

### 4.2 Uniqueness of the normal connection

As discussed in the introduction to this chapter, to prove that two isometric immersions $f, \tilde{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ are congruent by means of the Fundamental theorem of submanifolds, one must first construct a vector bundle isometry $T: N_{f} M \rightarrow N_{\tilde{f}} M$ satisfying $\tilde{\alpha}=T \circ \alpha$, and then prove that $T$ also preserves the normal connections. The following useful result states that this last condition is automatically satisfied whenever the first normal spaces of $f$ coincide everywhere with its normal spaces.
Lemma 4.16. Let $f, \tilde{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be isometric immersions and let $\phi: N_{f} M \rightarrow N_{\tilde{f}} M$ be a vector bundle isomorphism that preserves the metrics and the second fundamental forms. If $N_{1}(x)=N_{f} M(x)$ for every $x \in M^{n}$, then $\phi$ preserves the normal connections.

Proof: Define

$$
\hat{\nabla}_{X} \xi=\phi^{-1}\left(\tilde{\nabla}_{X}^{\perp}(\phi \xi)\right)
$$

for $X \in \mathfrak{X}(M)$ and $\xi \in \Gamma\left(N_{f} M\right)$, where $\tilde{\nabla}^{\perp}$ is the normal connection of $\tilde{f}$. It is easy to see that $\hat{\nabla}$ defines a compatible connection on $N_{f} M$. Moreover,

$$
\begin{aligned}
\left(\hat{\nabla}_{X} \alpha\right)(Y, Z) & =\phi^{-1}\left(\left(\tilde{\nabla}_{X}^{\perp} \tilde{\alpha}\right)(Y, Z)\right) \\
& =\phi^{-1}\left(\left(\tilde{\nabla}_{Y}^{\perp} \tilde{\alpha}\right)(X, Z)\right) \\
& =\left(\hat{\nabla}_{Y} \alpha\right)(X, Z) .
\end{aligned}
$$

Then Proposition 4.17 below implies that $\hat{\nabla}=\nabla^{\perp}$, and hence $\tilde{\nabla}_{X}^{\perp}(\phi \xi)=\phi\left(\nabla \frac{\perp}{X} \xi\right)$.

Proposition 4.17. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be an isometric immersion. Suppose that $N_{1}(x)=N_{f} M(x)$ for every $x \in M^{n}$. Then the normal connection $\nabla^{\perp}$ is the only connection in $N_{f} M$ that is compatible with the metric and satisfies the Codazzi equation.

Proof: Let $\hat{\nabla}$ be a connection on $N_{f} M$ that is compatible with the metric and satisfies the Codazzi equation, that is,

$$
\left(\hat{\nabla}_{X} \alpha\right)(Y, Z)=\left(\hat{\nabla}_{Y} \alpha\right)(X, Z)
$$

For each $X \in \mathfrak{X}(M)$, define a map $K(X): \Gamma\left(N_{f} M\right) \rightarrow \Gamma\left(N_{f} M\right)$ by

$$
K(X) \xi=\nabla_{X}^{\perp} \xi-\hat{\nabla}_{X} \xi
$$

Clearly, $K(X)$ is linear over $C^{\infty}(M)$. Also, $K(X)$ is skew-symmetric, because

$$
\begin{aligned}
\langle K(X) \xi, \eta\rangle & =\left\langle\nabla_{X}^{\perp} \xi-\hat{\nabla}_{X} \xi, \eta\right\rangle \\
& =X\langle\xi, \eta\rangle-\left\langle\xi, \nabla_{X}^{\perp} \eta\right\rangle-X\langle\xi, \eta\rangle+\left\langle\xi, \hat{\nabla}_{X} \eta\right\rangle \\
& =-\langle\xi, K(X) \eta\rangle
\end{aligned}
$$

Since both $\nabla^{\perp}$ and $\hat{\nabla}$ satisfy the Codazzi equation, we obtain

$$
K(X) \alpha(Y, Z)=K(Y) \alpha(X, Z)
$$

Denote

$$
\left\langle K\left(X_{1}\right) \alpha\left(X_{2}, X_{3}\right), \alpha\left(X_{4}, X_{5}\right)\right\rangle=\left(X_{1}, X_{2}, X_{3}, X_{4}, X_{5}\right)
$$

Then

$$
\begin{aligned}
\left(X_{1}, X_{2}, X_{3}, X_{4}, X_{5}\right) & =-\left(X_{1}, X_{4}, X_{5}, X_{2}, X_{3}\right)=-\left(X_{5}, X_{4}, X_{1}, X_{2}, X_{3}\right) \\
& =\left(X_{5}, X_{2}, X_{3}, X_{4}, X_{1}\right)=\left(X_{3}, X_{2}, X_{5}, X_{4}, X_{1}\right) \\
& =-\left(X_{3}, X_{4}, X_{1}, X_{2}, X_{5}\right)=-\left(X_{4}, X_{3}, X_{1}, X_{2}, X_{5}\right) \\
& =\left(X_{4}, X_{2}, X_{5}, X_{3}, X_{1}\right)=\left(X_{2}, X_{4}, X_{5}, X_{3}, X_{1}\right) \\
& =-\left(X_{2}, X_{3}, X_{1}, X_{4}, X_{5}\right)=-\left(X_{1}, X_{2}, X_{3}, X_{4}, X_{5}\right)
\end{aligned}
$$

hence $\left(X_{1}, X_{2}, X_{3}, X_{4}, X_{5}\right)=0$. Since $N_{1}=N_{f} M$, this means that $K(X)=0$ for all $X \in \mathfrak{X}(M)$, that is, $\nabla^{\perp}=\hat{\nabla}$.

### 4.3 The Allendoerfer rigidity result

The Allendoerfer rigidity theorem given in this section is a natural extension to arbitrary codimension of the Beez-Killing result for hypersurfaces.

Let $\alpha, \tilde{\alpha}: V^{n} \times V^{n} \rightarrow U^{p}$ be bilinear forms between finite dimensional vector spaces with positive definite inner products. Endow $W=U^{p} \oplus U^{p}$ with the inner product of signature ( $p, p$ ) given by

$$
\langle\langle(\xi, \tilde{\xi}),(\eta, \tilde{\eta})\rangle\rangle=\langle\xi, \eta\rangle-\langle\tilde{\xi}, \tilde{\eta}\rangle
$$

and define a bilinear form $\beta: V^{n} \times V^{n} \rightarrow W^{p, p}$ by

$$
\beta=\alpha \oplus \tilde{\alpha} .
$$

Proposition 4.18. Assume that the left (or right) type number $\tau$ of $\alpha$ satisfies $\tau \geq 3$. If $\beta$ is flat, then there exists a linear isometry $T: U^{p} \rightarrow U^{p}$ such that $\tilde{\alpha}=T \circ \alpha$.

Proof: By Proposition 4.1, it suffices to show that $\beta$ is null. For any $X \in V$, denote

$$
\mathcal{U}(X)=B_{X}(V) \cap B_{X}(V)^{\perp}
$$

Set

$$
k=\min \{\operatorname{dim} \mathcal{U}(X): X \in R E(\beta)\}
$$

and define

$$
R E^{o}(\beta)=\{X \in R E(\beta): \operatorname{dim} \mathcal{U}(X)=k\} .
$$

We claim that $R E^{o}(\beta)$ is dense in $V$. First observe that $Y_{0} \in R E^{o}(\beta)$ if and only if $Y_{0} \in R E(\beta)$ and there exist $Z_{1}, \ldots, Z_{\rho-k}$, with $\rho=\operatorname{dim} B_{Y_{0}}(V)$, such that $\operatorname{det}\left(c_{i j}\right) \neq 0$, where

$$
c_{i j}=\left\langle\left\langle\beta\left(Y_{0}, Z_{i}\right), \beta\left(Y_{0}, Z_{j}\right)\right\rangle\right\rangle, \quad 1 \leq i, j \leq \rho-k
$$

Let $X \in R E(\beta)$ be arbitrary and take $Y_{0} \in R E^{o}(\beta)$. Now choose $\epsilon>0$ such that

$$
X_{t}=X+t Y_{0} \in R E(\beta) \text { for }|t|<\epsilon
$$

and set

$$
b_{i j}(t)=\left\langle\left\langle\beta\left(X_{t}, Z_{i}\right), \beta\left(X_{t}, Z_{j}\right)\right\rangle\right\rangle .
$$

Then

$$
b_{i j}(t)=b_{i j}(0)+t\left(\left\langle\left\langle\beta\left(X, Z_{i}\right), \beta\left(Y_{0}, Z_{j}\right)\right\rangle\right\rangle+\left\langle\left\langle\beta\left(Y_{0}, Z_{i}\right), \beta\left(X, Z_{j}\right)\right\rangle\right\rangle\right)+t^{2} c_{i j} .
$$

Thus $\operatorname{det}\left(b_{i j}(t)\right)$ is a polynomial in $t$ of degree $2(\rho-k)$ having $\operatorname{det}\left(c_{i j}\right)$ as its leading coefficient. Hence it has only a finite number of zeros. Consequently, there exists $0<\epsilon^{\prime} \leq \epsilon$ such that

$$
\operatorname{det}\left(b_{i j}(t)\right) \neq 0 \text { for } 0<|t|<\epsilon^{\prime},
$$

thus showing that $R E^{o}(\beta)$ is dense in $R E(\beta)$, and the claim follows.
To conclude that $\beta$ is null, it suffices to show that $k=p$. For, if this is the case, then from $\mathcal{U}(X) \subset B_{X}(V), \mathcal{U}(X) \subset B_{X}(V)^{\perp}$ and

$$
\operatorname{dim} B_{X}(V)+\operatorname{dim} B_{X}(V)^{\perp}=\operatorname{dim} W^{p, p}=2 p
$$

we obtain $B_{X}(V)=B_{X}(V)^{\perp}$. Since $R E^{o}(\beta)$ is dense in $V$, the assertion then follows from Proposition 4.5.

Suppose $k \leq p-1$. Let $\xi_{1}, \ldots, \xi_{p}$ be a basis of $U^{p}$ and set

$$
\left\langle A_{\xi_{j}} X, Y\right\rangle=\left\langle\alpha(X, Y), \xi_{j}\right\rangle .
$$

Since $\tau \geq 3$, there exist vectors $X_{1}, X_{2}, X_{3} \in V$ such that

$$
\operatorname{dim} \operatorname{span}\left\{A_{\xi_{j}} X_{i}, \quad 1 \leq i \leq 3, \quad 1 \leq j \leq p\right\}=3 p
$$

Furthermore, we may assume that $X_{1}, X_{2}, X_{3} \in R E^{o}(\beta)$. The subspace

$$
S=\left\{Z \in V: \alpha\left(X_{i}, Z\right)=0, \quad 1 \leq i \leq 3\right\}
$$

satisfies

$$
S=\left(\operatorname{span}\left\{A_{\xi_{j}} X_{i}, \quad 1 \leq i \leq 3, \quad 1 \leq j \leq p\right\}\right)^{\perp} .
$$

Therefore

$$
\begin{equation*}
\operatorname{dim} S=n-3 p \tag{4.7}
\end{equation*}
$$

From

$$
\operatorname{dim} B_{X_{1}}(V) \leq 2 p-\operatorname{dim} \mathcal{U}\left(X_{1}\right)
$$

we obtain

$$
\operatorname{dim} \operatorname{ker} B_{X_{1}} \geq n-2 p+k .
$$

By Proposition 4.6,

$$
B_{X_{2}}\left(\operatorname{ker} B_{X_{1}}\right) \subset \mathcal{U}\left(X_{1}\right) .
$$

Therefore the linear transformation

$$
\bar{B}_{X_{2}}=\left.B_{X_{2}}\right|_{\operatorname{ker} B_{X_{1}}}: \operatorname{ker} B_{X_{1}} \rightarrow \mathcal{U}\left(X_{1}\right)
$$

satisfies

$$
\operatorname{ker} \bar{B}_{X_{2}}=\operatorname{ker} B_{X_{1}} \cap \operatorname{ker} B_{X_{2}}
$$

and

$$
\begin{aligned}
\operatorname{dim} \operatorname{ker} \bar{B}_{X_{2}} & \geq \operatorname{dim} \operatorname{ker} B_{X_{1}}-\operatorname{dim} \mathcal{U}\left(X_{1}\right) \\
& \geq n-2 p .
\end{aligned}
$$

Similarly, the linear transformation

$$
\bar{B}_{X_{3}}=\left.B_{X_{3}}\right|_{\text {ker } \bar{B}_{X_{2}}}: \operatorname{ker} \bar{B}_{X_{2}} \rightarrow \mathcal{U}\left(X_{1}\right) \cap U\left(X_{2}\right)
$$

satisfies

$$
\operatorname{ker} \bar{B}_{X_{3}}=\cap_{j=1}^{3} \operatorname{ker} B_{X_{j}}
$$

and

$$
\begin{aligned}
\operatorname{dim} \operatorname{ker} \bar{B}_{X_{3}} & \geq \operatorname{dim} \operatorname{ker} \bar{B}_{X_{2}}-k \\
& \geq n-2 p-k \\
& \geq n-3 p+1
\end{aligned}
$$

which contradicts 4.7), since $\cap_{j=1}^{3}$ ker $B_{X_{j}} \subset S$.

Theorem 4.19. An isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ with type number $\tau \geq 3$ at any point is rigid.

Proof: Given an isometric immersion $\tilde{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$, define $\beta(x): T_{x} M \times T_{x} M \rightarrow W^{p, p}$ at $x \in M^{n}$ as in (4.1). By Proposition 4.18, $\beta(x)$ is null. That $\tau \geq 3$ implies that $N_{1}(x)=N_{f} M(x)$ for any $x \in M^{n}$. Hence, by part (iii) of Proposition 4.1, there exists a smooth vector bundle isometry $T: N_{f} M \rightarrow N_{\tilde{f}} M$ such that

$$
\tilde{\alpha}=T \circ \alpha .
$$

Moreover, by Lemma 4.16 it preserves the normal connections. This completes the necessary steps to conclude that $f$ and $\tilde{f}$ are congruent by means of Theorem 1.10.

### 4.4 Rigidity in low codimension

In this section we prove a rigidity result for submanifolds with low codimension that is also an extension of the Beez-Killing theorem for hypersurfaces.

### 4.4.1 The Main Lemma

The proof of Theorem 4.23 below relies on the next lemma, which is the most important result on flat symmetric bilinear forms. It has Lemma 4.14 as a special case.

Lemma 4.20. (Main Lemma) Let $\beta: V^{n} \times V^{n} \rightarrow W^{p, q}$ be a symmetric flat bilinear form such that $\mathcal{S}(\beta)=W^{p, q}$. If $p \leq 5$ and $p+q<n$, then

$$
\operatorname{dim} \mathcal{N}(\beta) \geq \operatorname{dim} V-\operatorname{dim} W .
$$

It is shown at the end of this section that the above result is not true if $p, q \geq 6$. See also Exercise 4.4 for a version of the lemma for not necessarily symmetric flat bilinear forms with $1 \leq p=q \leq 2$.

When applying the Main Lemma, the following fact is often useful.
Lemma 4.21. Let $\beta: V^{n} \times V^{n} \rightarrow W^{p, q}$ be a nonzero flat bilinear form such that $\mathcal{S}(\beta)$ is degenerate, and let

$$
W^{p, q}=\mathcal{E} \oplus \widehat{\mathcal{E}} \oplus \mathcal{V}
$$

be a direct sum decomposition as in (4.3) with

$$
\mathcal{E}=\mathcal{S}(\beta) \cap \mathcal{S}(\beta)^{\perp} \neq 0 \text { and } \mathcal{S}(\beta) \subset \mathcal{E} \oplus \mathcal{V}
$$

## Decompose accordingly

$$
\beta=\beta_{1}+\beta_{2}
$$

with $\mathcal{S}\left(\beta_{1}\right)=\mathcal{E}$ and $\mathcal{S}\left(\beta_{2}\right) \subset \mathcal{V}$. Then $\beta_{2}$ is flat and $\mathcal{S}\left(\beta_{2}\right)$ is nondegenerate.

Proof: That $\beta_{2}=\beta-\beta_{1}$ is flat follows from the fact that $\beta_{1}$ is null. Now let

$$
\eta=\sum_{j} \beta_{2}\left(X_{j}, Y_{j}\right) \in \mathcal{S}\left(\beta_{2}\right) \cap \mathcal{S}\left(\beta_{2}\right)^{\perp}
$$

From

$$
\left\langle\beta_{2}(Z, T), \eta\right\rangle=0
$$

for all $Z, T \in V$, we obtain

$$
0=\sum_{j}\left\langle\beta_{2}(Z, T), \beta_{2}\left(X_{j}, Y_{j}\right)\right\rangle=\sum_{j}\left\langle\beta(Z, T), \beta\left(X_{j}, Y_{j}\right)\right\rangle,
$$

which implies that

$$
\sum_{j} \beta\left(X_{j}, Y_{j}\right) \in \mathcal{E}
$$

Therefore $\eta=0$, and hence $\mathcal{S}\left(\beta_{2}\right)$ is nondegenerate.
In view of the last result, the Main Lemma 4.20 for $p, q \geq 1$ can also be stated as follows.

Lemma 4.22. (Main Lemma bis) Let $\beta: V^{n} \times V^{n} \rightarrow W^{p, q}$, with $1 \leq p \leq 5$ and $p+q<n$, be a symmetric flat bilinear form. If $\operatorname{dim} \mathcal{N}(\beta) \leq n-p-q-1$, then there is an orthogonal decomposition

$$
W^{p, q}=W_{1}^{\ell, \ell} \oplus W_{2}^{p-\ell, q-\ell}, \quad 1 \leq \ell \leq p
$$

such that the $W_{j}$-components $\beta_{j}$ of $\beta$ satisfy:
(i) $\beta_{1}$ is nonzero and null.
(ii) $\beta_{2}$ is flat and $\operatorname{dim} \mathcal{N}\left(\beta_{2}\right) \geq \operatorname{dim} V-\operatorname{dim} W_{2}$.

The proof of the Main Lemma is increasingly difficult for increasing values of $\min \{p, q\}$. We leave its proof in full generality for appendix (Sect. 4.5).

### 4.4.2 The do Carmo-Dajczer rigidity result

We are now in a position to state and prove the following rigidity theorem.
Theorem 4.23. An isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}, p \leq 5$, whose s-nullities satisfy $\nu_{s} \leq n-2 s-1$ for all $1 \leq s \leq p$ at any point is rigid.

Proof: Given another isometric immersion $\tilde{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$, define $W^{p, p}(x)$ and $\beta(x)$ at $x \in M^{n}$ as in 4.1). To prove that $\beta(x)$ is null, let $L \subset W^{p, p}(x)$ be the vector subspace

$$
L=\mathcal{S}(\beta(x)) \cap \mathcal{S}(\beta(x))^{\perp}
$$

It suffices to show that $L$ has dimension $p$. Assume otherwise that $s=p-\operatorname{dim} L \geq 1$. The orthogonal projections $P_{1}: W^{p, p}(x) \rightarrow N_{f} M(x)$ and $P_{2}: W^{p, p}(x) \rightarrow N_{\tilde{f}} M(x)$ map $L$ isomorphically onto $P_{1}(L)$ and $P_{2}(L)$, respectively. Hence we have the orthogonal splittings

$$
N_{f} M(x)=U \oplus U^{\perp} \text { and } N_{\tilde{f}} M(x)=\tilde{U} \oplus \tilde{U}^{\perp}
$$

where $U^{\perp}=P_{1}(L)$ and $\tilde{U}^{\perp}=P_{2}(L)$. By Lemma 4.21, the component $\hat{\beta}(x)$ of $\beta(x)$ in $U \oplus \tilde{U}$ is flat and nondegenerate. Since $\mathcal{N}(\hat{\beta}(x)) \subset \mathcal{N}\left(\pi_{U} \circ \alpha(x)\right)$, the Main Lemma 4.20 gives

$$
\begin{aligned}
\operatorname{dim} \mathcal{N}\left(\pi_{U} \circ \alpha(x)\right) & \geq \operatorname{dim} \mathcal{N}(\hat{\beta}(x)) \\
& \geq n-\operatorname{dim} U \oplus \tilde{U} \\
& >n-(2 s+1) .
\end{aligned}
$$

This is a contradiction with the hypothesis on $\nu_{s}$ and proves that $\beta(x)$ is null.
The assumption that $\nu_{1} \leq n-3$ everywhere implies that $N_{1}(x)=N_{f} M(x)$ for any point $x \in M^{n}$. Now the proof is completed in exactly the same way as in the proof of Theorem 4.19.

Notice that the assumption on $\nu_{s}$ for $2 \leq p \leq 5$ is much weaker than the assumption on $\tau$ in Theorem 4.19, because $\tau \geq 3$ implies that $\nu_{s} \leq n-3 s$. Moreover, the assumption for $s=p$ implies that $n \geq 2 p+1$, whereas that in Theorem 4.19 forces $n \geq 3 p$.

### 4.4.3 A counterexample

The following counterexample shows that the Main Lemma is false for target vector spaces $W^{p, p}$ with $p \geq 6$. Hence the proof of Theorem 4.23 does not extend to isometric immersions $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ with codimension $p \geq 6$.

Proposition 4.24. For a given $r \in \mathbb{N}$ with $r \geq 3$, set $2 p=r(r+1)$. Then there exists a flat symmetric bilinear form $\beta: V^{n} \times V^{n} \rightarrow W^{p, p}$ such that $\mathcal{S}(\beta)=W^{p, p}$ and

$$
\operatorname{dim} \mathcal{N}(\beta) \geq n-2 p-\binom{r}{3}
$$

Proof: Denote

$$
L=\{1,2, \ldots, r\}, I=(L \times L) / S(2) \text { and } J=(L \times L \times L) / S(3),
$$

where $S(n)$ is the group of permutations of $n$ elements. Then

$$
\# I=p \text { and } \# J=m=\binom{r+2}{3}
$$

For $a \in L, k=[(i, j)] \in I$ and $s=[(u, v, w)] \in J$, we say that $a \in s$ if $a \in\{u, v, w\}$, and define $*: L \times I \rightarrow J$ by

$$
a * k=[(a, i, j)] .
$$

Then either $a \notin s$ or there is a unique $k \in I$ such that $a * k=s$.
Let $V^{n}=\mathbb{R}^{r} \oplus \mathbb{R}^{m}$, and take bases $Y_{1}, \ldots, Y_{r}$ and $\left\{Z_{s}: s \in J\right\}$ of $\mathbb{R}^{r}$ and $\mathbb{R}^{m}$, respectively. Let

$$
B_{1}=\left\{e_{r}: r \in I\right\} \text { and } B_{2}=\left\{\hat{e}_{r}: r \in I\right\}
$$

be two bases of $\mathbb{R}^{p}$, and consider on $W^{p, p}=\mathbb{R}^{2 p}$ the inner product of signature $(p, p)$ given by

$$
\left\langle e_{r}, e_{s}\right\rangle=\left\langle\hat{e}_{r}, \hat{e}_{s}\right\rangle=0 \text { and }\left\langle e_{r}, \hat{e}_{s}\right\rangle=\delta_{r s} \text { for all } r, s \in I .
$$

Thus the ordered union of the bases $B_{1}$ and $B_{2}$ is a pseudo-orthonormal basis of $W^{p, p}$. Define a symmetric bilinear map $\beta$ as follows:

$$
\begin{array}{ll}
\beta\left(Z_{s}, Z_{r}\right)=0, & r, s \in J, \\
\beta\left(Y_{i}, Y_{j}\right)=\hat{e}_{[(i, j)]}, & \\
\beta, j \in L, \\
\beta\left(Y_{i}, Z_{s}\right)=0, & \text { if } i \notin s, \\
\beta\left(Y_{i}, Z_{s}\right)=e_{k}, & \\
\text { if } i \in s \text { and } i * k=s .
\end{array}
$$

To verify that $\mathcal{N}(\beta)=0$, take $X \in \mathcal{N}(\beta)$ given by

$$
X=\sum_{j=1}^{r} a_{j} Y_{j}+\sum_{s \in J} b_{s} Z_{s} .
$$

Then

$$
\left.a_{i}=\left\langle\beta\left(X, Y_{i}\right), e_{[(i, i)]}\right)\right\rangle=0 \text { and } b_{s}=\left\langle\beta\left(X, Y_{u}\right), \hat{e}_{[(v, w)]}\right\rangle=0
$$

for $s=[(u, v, w)]$. To see that $\beta$ is flat, just observe that

$$
\left\langle\beta\left(Y_{i}, Y_{j}\right), \beta\left(Y_{t}, Z_{s}\right)\right\rangle=\delta_{s[(i, j, t)]}
$$

is symmetric in $i, j, t \in L$.

### 4.5 Appendix 1: Proof of the Main Lemma

In this appendix we prove the following slightly stronger version of the Main Lemma that will be needed in Chapter 12 .

Lemma 4.25. Let $\beta: V^{n} \times V^{n} \rightarrow W^{p, q}$ be a flat symmetric bilinear form. If $p \leq 5$ and $\mathcal{S}(\beta)$ is nondegenerate, then

$$
\operatorname{dim} \mathcal{N}(\beta) \geq n-\operatorname{dim} \operatorname{Im} B_{Y}-\operatorname{dim} \mathcal{S}\left(\left.\beta\right|_{\text {ker } B_{Y} \times V}\right)
$$

for any $Y \in R E(\beta)$.
Proof: First we observe that the subset of non-asymptotic regular elements

$$
R E^{*}(\beta)=\{X \in R E(\beta): \beta(X, X) \neq 0\}
$$

is open and dense in $V$. Given $X \in R E^{*}(\beta)$ and $X=X_{1}, \ldots, X_{r} \in V$ such that

$$
B_{X}(V)=\operatorname{span}\left\{B_{X}\left(X_{j}\right), \quad 1 \leq j \leq r\right\},
$$

where $r=\operatorname{dim} B_{X}(V)$, by Proposition 4.6 and the symmetry of $\beta$ we have

$$
\begin{equation*}
\mathcal{S}(\beta)=\operatorname{span}\left\{\beta\left(X_{i}, X_{j}\right), \quad 1 \leq i \leq j \leq r\right\} . \tag{4.8}
\end{equation*}
$$

We may assume that

$$
\mathcal{U}(X)=B_{X}(V) \cap B_{X}(V)^{\perp} \neq 0
$$

for any $X \in R E(\beta)$. If otherwise, then $N=\operatorname{ker} B_{X}$ satisfies $N \subset \mathcal{N}(\beta)$ by Proposition 4.6. Since $\mathcal{N}(\beta) \subset N$, we conclude that

$$
\operatorname{dim} \mathcal{N}(\beta)=\operatorname{dim} N \geq n-p-q,
$$

as we wished.
Set $\tau=\min \{\operatorname{dim} \mathcal{U}(X): X \in R E(\beta)\}$. We claim that the subset

$$
\mathcal{R}(\beta)=\{X \in R E(\beta): \operatorname{dim} \mathcal{U}(X)=\tau\}
$$

is open and dense in $V$. In fact, if $Y \in \mathcal{R}(\beta)$ then there exist $Z_{1}, \ldots, Z_{r-\tau}$ such that $r=\operatorname{dim} B_{Y}(V)$ and $\operatorname{det}\left(c_{i j}\right) \neq 0$, where

$$
c_{i j}=\left\langle\beta\left(Y, Z_{i}\right), \beta\left(Y, Z_{j}\right)\right\rangle .
$$

Thus

$$
\operatorname{det}\left(\left\langle\beta\left(X, Z_{i}\right), \beta\left(X, Z_{j}\right)\right\rangle\right) \neq 0
$$

for $X$ in a neighborhood of $Y$ in $R E(\beta)$. Since $\tau$ is the minimum, then $\mathcal{R}(\beta)$ is open.
Let $X \in R E(\beta)$ be arbitrary and let $Y \in \mathcal{R}(\beta)$. Take $\epsilon>0$ such that

$$
X_{t}=X+t Y \in R E(\beta) \text { for }|t|<\epsilon
$$

Set

$$
b_{i j}(t)=\left\langle\beta\left(X_{t}, Z_{i}\right), \beta\left(X_{t}, Z_{j}\right)\right\rangle .
$$

Then $\operatorname{det}\left(b_{i j}(t)\right)$ is a polynomial in $t$ of degree $2(r-\tau)$ having $\operatorname{det}\left(c_{i j}\right)$ as its leading coefficient. Thus it has a finite number of zeros. Hence there exists $0<\epsilon^{\prime} \leq \epsilon$ such that

$$
\operatorname{det}\left(b_{i j}(t)\right) \neq 0 \text { for } 0<|t|<\epsilon^{\prime},
$$

showing that $\mathcal{R}(\beta)$ is dense.
We fix $X \in \mathcal{R}(\beta)$ for the remainder of the proof. Setting $\mathcal{U}=\mathcal{U}(X)$, there is an isotropic subspace $\widehat{\mathcal{U}}$ with $\operatorname{dim} \mathcal{U}=\operatorname{dim} \widehat{\mathcal{U}}$ and a decomposition

$$
\begin{equation*}
W^{p, q}=\mathcal{U} \oplus \widehat{\mathcal{U}} \oplus \mathcal{V} \tag{4.9}
\end{equation*}
$$

where $B_{X}(V) \subset \mathcal{U} \oplus \mathcal{V}$ and $\mathcal{V}^{\perp}=\mathcal{U} \oplus \widehat{\mathcal{U}}$. Let

$$
\hat{\beta}: V^{n} \times V^{n} \rightarrow \widehat{U}
$$

be the $\widehat{\mathcal{U}}$-component of $\beta$ according to the decomposition 4.9. Set $\widehat{B}_{X}=\hat{\beta}(X$,$) and$ $\kappa=\operatorname{dim} \widehat{B}_{Y}(V)$ for $Y \in R E(\hat{\beta})$. Since $\mathcal{S}(\beta)$ is nondegenerate, for any vector $\xi \in \mathcal{U}$ there are vectors $Y, Z \in V$ such that

$$
0 \neq\langle\xi, \beta(Y, Z)\rangle=\langle\xi, \hat{\beta}(Y, Z)\rangle
$$

It follows that

$$
\begin{equation*}
\mathcal{S}(\hat{\beta})=\widehat{\mathcal{U}} \tag{4.10}
\end{equation*}
$$

Fact: Given $Y \in \mathcal{R}(\beta) \cap R E(\hat{\beta})$, let $\rho \geq 0$ be defined by

$$
2 \rho=\operatorname{rank}\left(B_{Y}(V) \cap \mathcal{U}\right) \oplus \widehat{B}_{Y}(V)
$$

Then $\rho \leq p-\tau$ and $\operatorname{dim} B_{Y}(N) \leq p-\kappa$.
Let us prove the fact. Set

$$
V^{n}=\mathcal{L} \oplus \widetilde{\mathcal{L}}
$$

where $\widetilde{\mathcal{L}}=\operatorname{ker} \widehat{B}_{Y}$. Then $B_{Y}(\widetilde{\mathcal{L}}) \subset \mathcal{U} \oplus \mathcal{V}$ and

$$
B_{Y}(\mathcal{L}) \cap(\mathcal{U} \oplus \mathcal{V})=0
$$

Hence $\operatorname{dim} B_{Y}(\mathcal{L})=\kappa$. The matrix of inner products of the elements of a basis of $B_{Y}(V)$ associated with the decomposition

$$
\begin{equation*}
B_{Y}(V)=B_{Y}^{0} \oplus B_{Y}(\mathcal{L}) \oplus B_{Y}^{1} \tag{4.11}
\end{equation*}
$$

where

$$
B_{Y}^{0}=B_{Y}(\widetilde{\mathcal{L}}) \cap \mathcal{U}=B_{Y}(V) \cap \mathcal{U}
$$

and $B_{Y}(\widetilde{\mathcal{L}})=B_{Y}^{0} \oplus B_{Y}^{1}$, has the form

$$
\left[\begin{array}{ccc}
0 & A & 0 \\
A^{t} & B & C \\
0 & C^{t} & D
\end{array}\right] .
$$

Since rank $A=\rho$, we obtain

$$
\operatorname{rank} B_{Y}(V) \geq 2 \rho+\operatorname{rank} B_{Y}^{1}
$$

From $B_{Y}^{1} \subset \mathcal{U} \oplus \mathcal{V}$ and $B_{Y}^{1} \cap \mathcal{U}=0$ it follows that

$$
\operatorname{rank} B_{Y}^{1} \geq \operatorname{dim} B_{Y}^{1}-p+\tau
$$

Therefore

$$
\operatorname{dim} B_{Y}(V)-\tau=\operatorname{rank} B_{Y}(V) \geq 2 \rho+\operatorname{dim} B_{Y}^{1}-p+\tau
$$

From (4.11) we obtain

$$
\begin{equation*}
2 \rho \leq \operatorname{dim} B_{Y}(V) \cap \mathcal{U}+\kappa+p-2 \tau \tag{4.12}
\end{equation*}
$$

Clearly, $\rho \geq \operatorname{dim} B_{Y}(V) \cap \mathcal{U}+\kappa-\tau$. It follows using (4.12) that

$$
\begin{equation*}
\operatorname{dim} B_{Y}(V) \cap \mathcal{U} \leq p-\kappa \tag{4.13}
\end{equation*}
$$

Therefore the first statement follows from (4.12) and (4.13), whereas the second one follows from Proposition 4.6 and 4.13).

Fix $Y_{1} \in \mathcal{R}(\beta) \cap R E^{*}(\hat{\beta})$. Then 4.8 and 4.10 yield

$$
\begin{equation*}
\kappa(\kappa+1) \geq 2 \tau \tag{4.14}
\end{equation*}
$$

and

$$
\begin{equation*}
\widehat{\mathcal{U}}=\operatorname{span}\left\{\hat{\beta}\left(Y_{i}, Y_{j}\right): 1 \leq i \leq j \leq \kappa\right\}, \tag{4.15}
\end{equation*}
$$

where

$$
\widehat{B}_{Y_{1}}(V)=\operatorname{span}\left\{\widehat{B}_{Y_{1}}\left(Y_{j}\right): 1 \leq j \leq \kappa\right\}
$$

Given any $n \in N$, we see from (4.5) that $\beta(n, Z) \in \mathcal{U}$ for all $Z \in V$. It follows from (4.15) that

$$
\begin{equation*}
\beta(n, Z)=0 \text { if and only if }\left\langle\beta(n, Z), \hat{\beta}\left(Y_{i}, Y_{j}\right)\right\rangle=0, \quad 1 \leq i, j \leq \kappa . \tag{4.16}
\end{equation*}
$$

We conclude the proof arguing for the most difficult case $p=5$, the cases $p \leq 4$ being similar. First suppose that $4 \leq \tau \leq 5$, and assume that $\kappa=3$, which from (4.14) is its lowest possible value. Thus there exist vectors $Y_{1}, Y_{2}, Y_{3} \in \mathcal{R}(\beta) \cap R E^{*}(\hat{\beta})$ such that

$$
\begin{equation*}
\widehat{\mathcal{U}}=\operatorname{span}\left\{\hat{\beta}\left(Y_{i}, Y_{j}\right): 1 \leq i \leq j \leq 3\right\} . \tag{4.17}
\end{equation*}
$$

Using (4.8) again, we choose $Y_{2}$ so that in 4.17) we may drop the element corresponding to $(i, j)=(3,3)$ when $\tau=5$, and when $\tau=4$ the ones for which $(i, j)=(2,3),(3,3)$. Hence

$$
\widehat{\mathcal{U}}=\widehat{B}_{Y_{1}}(V)+\widehat{B}_{Y_{2}}(V) .
$$

Consider the linear map $B_{1}=\left.B_{Y_{1}}\right|_{N}: N \rightarrow B_{Y_{1}}(N)$. From the above fact, we have $\operatorname{dim} B_{Y_{i}}(N) \leq 3$. Hence $N_{1}=$ ker $B_{1}$ satisfies

$$
\begin{equation*}
\operatorname{dim} N_{1} \geq \operatorname{dim} N-3 \tag{4.18}
\end{equation*}
$$

Flatness gives $\left\langle\beta\left(N_{1}, V\right), \widehat{B}_{Y_{1}}(V)\right\rangle=0$. In particular,

$$
\operatorname{rank} B_{Y_{2}}\left(N_{1}\right) \oplus \widehat{B}_{Y_{1}}(V)=0
$$

Now we use the fact again. If $\tau=5$, then $\rho=0$ and

$$
\begin{equation*}
\operatorname{rank} B_{Y_{2}}\left(N_{1}\right) \oplus \widehat{\mathcal{U}}=0 \tag{4.19}
\end{equation*}
$$

If $\tau=4$, then $\rho \leq 1$. Thus

$$
\begin{equation*}
\operatorname{rank} B_{Y_{2}}\left(N_{1}\right) \oplus \widehat{\mathbb{U}} \leq 2 \tag{4.20}
\end{equation*}
$$

We see from (4.19) and (4.20) that $\operatorname{dim} B_{Y_{2}}\left(N_{1}\right) \leq 1$ for $4 \leq \tau \leq 5$. Set

$$
B_{2}=\left.B_{Y_{2}}\right|_{N_{1}}: N_{1} \rightarrow B_{Y_{2}}\left(N_{1}\right) .
$$

It follows using (4.18) and $\operatorname{dim} N \geq n-q-5+\tau$ that $N_{2}=$ ker $B_{2}$ satisfies

$$
\begin{aligned}
\operatorname{dim} N_{2} & \geq \operatorname{dim} N_{1}-\operatorname{dim} B_{Y_{2}}\left(N_{1}\right) \\
& \geq \operatorname{dim} N-4 \\
& \geq n-q-5 .
\end{aligned}
$$

It follows from 4.16) that $N_{2} \subset \mathcal{N}(\beta)$. In particular, $\operatorname{dim} \mathcal{N}(\beta) \geq n-q-5$ as we wished. Finally, one can easily check that the estimate for $\operatorname{dim} N_{2}$ is even larger if $\kappa>3$, and this concludes the proof for $4 \leq \tau \leq 5$.

The argument for the remaining cases is similar and easier unless $\tau=3$, and $Y_{1}, Y_{2} \in \mathcal{R}(\beta) \cap R E^{*}(\hat{\beta})$ are such that

$$
\widehat{\mathcal{U}}=\operatorname{span}\left\{\hat{\beta}\left(Y_{i}, Y_{j}\right): 1 \leq i, j \leq 2\right\}
$$

and $B_{Y_{1}}(N)=\mathcal{U}$, since in this case our estimate would fail. But this case cannot occur. In fact, in this situation it is not difficult to see that we would have $\mathcal{U}\left(Y_{1}\right) \subset B_{X}(V)$ and $\operatorname{dim} \mathcal{U} \cap \mathcal{U}\left(Y_{1}\right)=1$. It is now easy to see that $\mathcal{U}+\mathcal{U}\left(Y_{1}\right)+\mathcal{U}\left(Y_{2}\right)$ would be an isotropic space of dimension 6 , which, of course, is not possible unless $p \geq 6$.

### 4.6 Notes

Flat bilinear forms were introduced by Moore [255] as an outgrowth of Cartan's theory of exteriorly orthogonal quadratic forms, which correspond to symmetric flat bilinear forms with respect to positive definite inner products. In particular, the symmetric version of Lemma 4.10 is due to Cartan [67]. The version for flat bilinear forms with respect to Lorentzian inner products in Lemma 4.14 was proved in [255]. Its general form in Lemma 4.20 is due to do Carmo-Dajczer [59]. The slightly stronger version in the appendix was taken from Dajczer-Florit [99]. The counterexample in Section 4.4.3 was obtained by Dajczer-Florit [101].

The relation between the intrinsic index of nullity and the extrinsic index of relative nullity given by Theorem 4.9 is due to Chern-Kuiper [86], where they introduced these fundamental concepts in the theory of submanifolds. In fact, it was Otsuki 284] who first gave a proof of the Chern-Kuiper inequality for all dimensions.

The classical rigidity result for Euclidean hypersurfaces, namely, Theorem 4.13, was first stated by Beez [31] but correctly proved by Killing [227] several years later. The case of nonflat ambient spaces is due to Eisenhart [165].

Allendoerfer's rigidity result was given in [16]. Theorem 4.23 is due to do CarmoDajczer [59], where the theory of flat bilinear forms was first applied to the rigidity
problem and the Main Lemma was proved. It is not known whether the statement of Theorem 4.23 is still true for codimension $p \geq 6$. For another related result see Silva [316].

Allendoerfer [16] has shown that for type number $\tau \geq 4$ the Codazzi and Ricci equations of a submanifold are a consequence of the Gauss equation, thus reducing the existence problem to a purely algebraic one. More precisely, he showed that in order to have a (necessarily unique) isometric immersion of a simply connected Riemannian manifold $M^{n}$ into $\mathbb{Q}_{c}^{n+p}, p \leq n / 4$, it is sufficient to construct a "second fundamental form" with type number $\tau \geq 4$ satisfying the Gauss equation. An alternative proof of this result was given by Chern-Osserman [87]. The case of hypersurfaces was done much earlier by Thomas [329].

The argument in the proof of Proposition 4.17 that asserts uniqueness of the normal connection was taken from Nomizu [270]. It has been seen in appendix of Chap. 1 that this result is a special case of the uniqueness of the connections $D^{k}$ in the $k^{t h}$-normal spaces in the classical Burstin-Mayer-Allendoerfer theory.

Berger-Bryant-Griffiths [32 made use of the theory of exterior systems combined with methods of algebraic geometry to obtain rigidity results for local isometric embeddings $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ with low codimension $p$. In particular, they proved that if $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ is an isometric embedding of an open neighborhood of a point at which the second fundamental form belongs to a certain dense Zariski open subset of the set of all such forms, then $f$
(i) depends only on constants if $p \leq \frac{1}{2}(n-1)(n-2)$,
(ii) depends formally on functions of at most $s$ variables if $p=\frac{1}{2}(n-1)(n-2)+s$,
(iii) is unique up to rigid motions if the conditions $p \leq n, n \geq 8$, or

$$
\left\{\begin{array}{l}
p \leq 3, \text { if } n=4 \\
p \leq 4, \text { if } n=5,6 \\
p \leq 6, \text { if } n=7,8
\end{array}\right.
$$

are satisfied.
Gromov [202] observed that "counting parameters" tells us that an isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ with $m<(1 / 2) n(n+1)$ should be rigid, locally or globally, "unless there is some miraculous identity between high derivatives of the (extrinsic) curvatures of $M^{n}$."

Exercises 4.3, 4.4 and 4.9 were taken from do Carmo-Dajczer [55], Dajczer 93], and Barbosa-Dajczer-Jorge [26], respectively.

### 4.7 Exercises

Exercise 4.1. Let $f: M^{n} \rightarrow \tilde{M}^{n+1}$ be a hypersurface with type number $\tau \geq 2$ at $x \in M^{n}$. Prove that the relative nullity subspace $\Delta(x)$ coincides with the nullity subspace $\Gamma(x)$.

Exercise 4.2. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ be an isometric immersion with type number $\tau \geq 3$ at any point of $M^{n}$. Show that for every isometry $\phi$ of $M^{n}$ there is an isometry $T$ of $\mathbb{Q}_{c}^{n+1}$ such that $f \circ \phi=T \circ f$.

Exercise 4.3. Assume that a Riemannian manifold $M^{n}$ of dimension $n \geq 4$ admits isometric immersions $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ and $g: M^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+p}$ with $\tilde{c}>c$ and $p \leq n-3$. Show that at any point $x \in M^{n}$ there exist a principal curvature $\lambda$ of $f$ with multiplicity greater than or equal to $n-p$ and a principal normal $\eta \in N_{g} M(x)$ such that $E_{\eta}=E_{\lambda}$. Hint: Let $G: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p+1}$ be given by $G=i \circ g$, where $i: \mathbb{Q}_{\tilde{c}}^{n+p} \rightarrow \mathbb{Q}_{c}^{n+p+1}$ is an umbilical inclusion. Note that the second fundamental form of $G$ is

$$
\alpha_{G}(X, Y)=i_{*} \alpha_{g}(X, Y)+\sqrt{\tilde{c}-c}\langle X, Y\rangle \zeta
$$

for all $X, Y \in T_{x} M$, where $\zeta$ is one of the unit vectors normal to $i$. For $x \in M^{n}$, define

$$
W=N_{f} M(x) \oplus N_{G} M(x)
$$

and endow $W$ with the Lorentzian inner product $\langle\langle\rangle$,$\rangle defined by$

$$
\langle\langle\xi+\nu, \tilde{\xi}+\tilde{\nu}\rangle\rangle=-\langle\xi, \tilde{\xi}\rangle+\langle\nu, \tilde{\nu}\rangle
$$

for all $\xi, \tilde{\xi} \in N_{f} M(x)$ and $\nu, \tilde{\nu} \in N_{G} M(x)$. Now define $\beta: T_{x} M \times T_{x} M \rightarrow W$ by

$$
\beta(X, Y)=\alpha_{f}(X, Y)+\alpha_{G}(X, Y) .
$$

Show that $\beta$ is flat and that $\mathcal{N}(\beta)=\{0\}$. Then conclude from Lemmas 4.10 and 4.14 that $\mathcal{S}(\beta)$ must be degenerate, that is, there must exist a nonzero light-like vector $e \in \mathcal{S}(\beta) \cap \mathcal{S}(\beta)^{\perp}$. Thus one can write $\mathcal{S}(\beta)=V \oplus \operatorname{span}\{e\}$, with $V$ space-like, and if $\bar{e} \in V^{\perp}$ is a light-like vector such that $\langle\bar{e}, e\rangle=1$, then the bilinear form

$$
\tilde{\beta}=\beta-\langle\beta, \bar{e}\rangle e
$$

takes values in $V$ and is also flat. Conclude from Lemma 4.10 that

$$
\operatorname{dim} \mathcal{N}(\tilde{\beta}) \geq n-\operatorname{dim} V=n-p
$$

Now decompose $e$ as

$$
e=N+\cos \varphi \zeta+\sin \varphi i_{*} \delta
$$

where $N \in N_{f} M(x)$ and $\delta \in N_{g} M(x)$ are unit vectors. Show that

$$
\lambda=\frac{\sqrt{\tilde{c}-c}}{\cos \varphi}
$$

is a principal curvature of $f$ and that

$$
\eta=\sqrt{\tilde{c}-c} \tan \varphi \delta
$$

is a principal normal of $g$ at $x$ such that $E_{\lambda}=\mathcal{N}(\tilde{\beta})=E_{\eta}$.

Exercise 4.4. Let $\beta: V \times V \rightarrow W^{p, p}$ be a (not necessarily symmetric) flat bilinear form such that $\mathcal{S}(\beta)=W^{p, p}$, where $\operatorname{dim} V>p$ and $1 \leq p \leq 2$. Prove that

$$
\begin{equation*}
\operatorname{dim} \mathcal{N}(\beta) \geq \operatorname{dim} V-\operatorname{dim} W \tag{4.21}
\end{equation*}
$$

Hint: If for some $X \in R E(\beta)$ the subspace $B_{X}(V)$ is nondegenerate, then

$$
U(X)=B_{X}(V) \cap B_{X}(V)^{\perp}=\{0\} .
$$

Conclude that $\mathcal{N}(\beta)=\operatorname{ker}\left(B_{X}\right)$, and hence that 4.21) holds. If $B_{X}(V)$ is degenerate for all $X \in R E(\beta)$, assuming that they are all null subspaces, then

$$
\langle\beta(X, Y), \beta(X, Z)\rangle=0
$$

for all $Y, Z \in V$ and $X \in R E(\beta)$. Show that this implies that $\beta$ is null and obtain a contradiction with the assumption $\mathcal{S}(\beta)=W^{p, p}$.

The only remaining case is that in which $p=2$ and there exists $X \in R E(\beta)$ such that $B_{X}(V)$ is not null. Prove that $\operatorname{dim} U(X)=1$ in the following way: if $\operatorname{dim} U(X)=2$, obtain a contradiction from the fact that

$$
\operatorname{dim} B_{X}(V)+\operatorname{dim} B_{X}(V)^{\perp}=4
$$

Therefore $2 \leq \operatorname{dim} B_{X}(V) \leq 3$, which implies dim ker $B_{X} \geq \operatorname{dim} V-3$. Since $\mathcal{S}(\beta)=W$, there exist vectors $u, v \in V$ such that $\langle\beta(u, v), \xi\rangle \neq 0$, where $\xi$ spans $U(X)$. Now define $B$ : ker $B_{X} \rightarrow U(X)$ by $B_{u}(n)=\beta(u, n)$. Using that $\beta$ is flat, show that ker $B \subset \mathcal{N}(\beta)$ and conclude that

$$
\begin{aligned}
\operatorname{dim} \mathcal{N}(\beta) & \geq \operatorname{dim} \operatorname{ker} B \\
& \geq \operatorname{dim} \operatorname{ker} B_{X}-\operatorname{dim} U(X) \\
& \geq \operatorname{dim} V-4 \\
& =\operatorname{dim} V-\operatorname{dim} W .
\end{aligned}
$$

Exercise 4.5. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be a 1-regular isometric immersion. Then the Riemannian connection induced on the first normal bundle $N_{1}$ by the normal connection is unique, in the sense that any other connection on $N_{f} M$ compatible with the metric and that satisfies the Codazzi equation induces on $N_{1}$ the same Riemannian connection.

Exercise 4.6. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be a 1-regular isometric immersion. Endow $N_{1}$ with the induced metric and normal connection still denoted by $\nabla^{\perp}$. Then $\nabla^{\perp}$ is the only connection on $N_{1}$ that is compatible with the metric and satisfies

$$
\pi_{N_{1}}\left[\left(\nabla_{X}^{\perp} \alpha\right)(Y, Z)\right]=\pi_{N_{1}}\left[\left(\nabla_{Y}^{\perp} \alpha\right)(X, Z)\right]
$$

where $\pi_{N_{1}}: N_{f} M \rightarrow N_{1}$ denotes the orthogonal projection.

Exercise 4.7. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ be an isometric immersion whose type number satisfies $\tau(x) \geq 2$ at any $x \in M^{n}$. If $g: M^{n} \rightarrow \mathbb{Q}_{c}^{n+q}$ is any other isometric immersion, show that $\operatorname{dim} N_{1}^{g}(x) \geq p$ for any $x \in M^{n}$. Conclude that there exists no isometric immersion of $M^{n}$ into $\mathbb{Q}_{c}^{n+q}$ if $q<p$.
Hint: Use an argument similar to that in the proof of Theorem 4.19.
Exercise 4.8. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ be a minimal isometric immersion. Assume that there exists $x_{0} \in M^{n}$ such that

$$
\nu_{s}\left(x_{0}\right) \leq n-q-2 s-1
$$

for all $1 \leq s \leq p \leq 5$. Prove that any minimal isometric immersion $g: M^{n} \rightarrow \mathbb{R}^{n+p+q}$ is congruent to $i \circ f$, where $i: \mathbb{R}^{n+p} \rightarrow \mathbb{R}^{n+p+q}$ is the inclusion.
Hint: Let $U \subset M^{n}$ be an open connected neighborhood of $x_{0}$ where $\nu_{s} \leq n-q-2 s-1$ for $1 \leq s \leq p$. By Lemma 4.22, $\alpha_{g}$ decomposes as $\alpha_{g}=\alpha_{f}+\gamma$ on $U$. Moreover, since $\gamma$ is flat and traceless then $\gamma=0$. It follows that $g$ reduces codimension to $p$. Thus $\left.g\right|_{U}$ is congruent to $\left.i \circ f\right|_{U}$, and the statement follows using that minimal submanifolds of Euclidean space are real analytic.

Exercise 4.9. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}, 2 \leq p \leq n-2$, be a minimal substantial isometric immersion. Assume that there is a point $x \in M^{n}$ such that $\mu_{c}(x) \leq n-p-2$. Show that $M^{n}$ cannot be isometrically immersed in $\mathbb{Q}_{c}^{n+1}$.
Hint: Assume that there exists an isometric immersion $g: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$. Making use of Exercise 3.8 and Lemma 4.14, prove that $g$ has to be minimal. Now use Theorem 3.11 to obtain a contradiction.

Exercise 4.10. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+2}$ be a minimal isometric immersion. Assume that there is a point $x \in M^{n}$ such that $\nu_{1}(x) \leq n-3$ and $\nu(x) \leq n-p-3$, where $2 \leq p \leq \min \{n-3,5\}$. Show that any minimal isometric immersion $g: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ is congruent to $i \circ f$, where $i: \mathbb{Q}_{c}^{n+2} \rightarrow \mathbb{Q}_{c}^{n+p}$ is a totally geodesic inclusion.
Hint: Use Lemmas 3.12 and 4.22.

## Chapter 5

## Constant curvature submanifolds

The theory of flat bilinear forms has been developed in the previous chapter aiming at its applications to rigidity aspects of submanifolds in space forms. However, the initial motivation of Cartan's theory of exteriorly orthogonal quadratic forms, which are equivalent to symmetric flat bilinear forms with respect to positive definite inner products, was to study isometric immersions $f: M_{c}^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$. Indeed, the second fundamental form of such an isometric immersion at any point $x \in M_{c}^{n}$ provides the basic example of a symmetric bilinear form which is flat with respect to the positive definite inner product on $N_{f} M(x)$. In fact, Cartan also used his theory to study isometric immersions $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+p}$ with $c<\tilde{c}$, just by looking at the composition $\tilde{f}=i \circ f$ of $f$ with an umbilical inclusion of $\mathbb{Q}_{\tilde{c}}^{n+p}$ into $\mathbb{Q}_{c}^{n+p+1}$.

Flat bilinear forms with respect to inner products that are not necessarily positive definite were introduced by J. D. Moore. Although his primary interest was to study the dual case of isometric immersions $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+p}$ with $c>\tilde{c}$, he also realized the wide range of applications of the theory.

The first section of this chapter provides an account of some basic results on isometric immersions $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+p}$ that can be derived from the theory of flat bilinear forms. As a consequence, in the following sections it is shown that, under appropriate conditions, such submanifolds are holonomic. This allows to show that they are in correspondence with solutions of certain systems of nonlinear partial differential equations.

The last section is devoted to Nikolayevsky's theorem on the nonexistence of an isometric immersion into $\mathbb{Q}_{\tilde{c}}^{2 n-1}$ of a complete non-simply connected Riemannian manifold of dimension $n$ and constant curvature $c<\min \{0, \tilde{c}\}$.

### 5.1 The structure of the second fundamental form

A first application of the theory of flat bilinear forms to the study of isometric immersions between manifolds of constant sectional curvature is shown in the following basic result.

Theorem 5.1. Let $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+p}$ be an isometric immersion. Then the following assertions hold:
(i) If $c<\tilde{c}$ then $p \geq n-1$.
(ii) If $c>\tilde{c}$ and $p \leq n-2$ then for any $x \in M^{n}$ there exists a unit vector $\zeta \in N_{f} M(x)$ and a flat bilinear form

$$
\gamma(x): T_{x} M \times T_{x} M \rightarrow\{\zeta\}^{\perp} \subset N_{f} M(x)
$$

such that the second fundamental form of $f$ at $x$ is given by

$$
\alpha(x)=\gamma(x)+\sqrt{c-\tilde{c}}\langle,\rangle \zeta .
$$

Proof: Let $\tilde{f}=i \circ f$, where $i: \mathbb{Q}_{\tilde{c}}^{n+p} \rightarrow \mathbb{Q}_{c, s}^{n+p+1}$ is an umbilical inclusion, with $s=0$ or $s=1$ depending on whether $c<\tilde{c}$ or $c>\tilde{c}$, respectively. Then, at any $x \in M_{c}^{n}$, the second fundamental form $\tilde{\alpha}(x)$ of $\tilde{f}$ is flat with respect to the inner product on $N_{\tilde{f}} M(x)$, which is either positive definite or Lorentzian according to whether $c<\tilde{c}$ or $c>\tilde{c}$, respectively. We have

$$
\begin{equation*}
\tilde{\alpha}(x)=i_{*} \alpha(x)+\sqrt{|\tilde{c}-c|}\langle,\rangle \xi \tag{5.1}
\end{equation*}
$$

where $\xi$ is one of the unit vectors normal to $i$ at $f(x)$. In particular,

$$
\begin{equation*}
\langle\tilde{\alpha}(x), \xi\rangle=\sqrt{|\tilde{c}-c|}\langle\xi, \xi\rangle\langle,\rangle \tag{5.2}
\end{equation*}
$$

and hence $\tilde{\alpha}(x)$ has trivial kernel.
If $c<\tilde{c}$, then Lemma 4.10 yields $p+1=\operatorname{dim} N_{\tilde{f}} M(x) \geq n$, and part (i) follows. On the other hand, if $c>\tilde{c}$ and $p \leq n-2$, then Lemma 4.14 implies that $\mathcal{S}(\tilde{\alpha}(x))$ must be degenerate, that is, there must exist a unit vector $\zeta \in N_{f} M(x)$ such that

$$
i_{*} \zeta+\xi \in \mathcal{S}(\tilde{\alpha}(x)) \cap \mathcal{S}(\tilde{\alpha}(x))^{\perp}
$$

Hence

$$
0=\left\langle\tilde{\alpha}(x), i_{*} \zeta+\xi\right\rangle=\langle\alpha(x), \zeta\rangle-\sqrt{c-\tilde{c}}\langle,\rangle .
$$

Then, by the Gauss equation of $f$, the bilinear form $\gamma(x): T_{x} M \times T_{x} M \rightarrow\{\zeta\}^{\perp}$ given by

$$
\gamma(x)=\alpha(x)-\sqrt{c-\tilde{c}}\langle,\rangle \zeta
$$

is flat.
Let $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+p}$ be an isometric immersion with $c>\tilde{c}$. A point $x \in M_{c}^{n}$ where the second fundamental form of $f$ is given as in part (ii) of Theorem 5.1 is called a weak-umbilic point for $f$.

Part (ii) of Theorem 5.1 states that all points of $M_{c}^{n}$ are weak-umbilics for an isometric immersion $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+p}$ with $c>\tilde{c}$ and $p \leq n-2$. Notice that at a weak-umbilic point $x \in M_{c}^{n}$ the structure of the second fundamental form is that of a
composition $f=\bar{f} \circ i$, where $i: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+1}$ is an umbilical inclusion and $\bar{f}: U \rightarrow \mathbb{Q}_{\tilde{c}}^{n+p}$ is an isometric immersion of an open subset $U \subset \mathbb{Q}_{\tilde{c}}^{n+1}$ containing $i\left(M_{c}^{n}\right)$. The question of whether any isometric immersion $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+p}$ with $c>\tilde{c}$ and $p \leq n-2$ must actually be such a composition, at least locally, will be considered in Chapter 12 .

To determine the structure of the second fundamental form of an isometric immersion $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+p}$, free of weak-umbilic points if $c>\tilde{c}$, with the lowest possible codimension $p=n-1$ allowed by Theorem 5.1, we need the following result on flat bilinear forms that attain equality in the inequality given in Lemmas 4.10 and 4.14 .

Theorem 5.2. Let $\beta: V^{n} \times V^{n} \rightarrow W^{p}$ be a flat symmetric bilinear form with respect to an inner product $\langle\rangle:, W \times W \rightarrow \mathbb{R}$ which is either positive definite or Lorentzian. Assume that $\mathcal{S}(\beta)=W$ and that

$$
\operatorname{dim} \mathcal{N}(\beta)=\operatorname{dim} V-\operatorname{dim} W
$$

If $\langle$,$\rangle is Lorentzian, assume further that there exists a vector e \in W$ such that the real-valued symmetric bilinear form $\varphi=\langle\beta, e\rangle$ is positive definite. Then there exist an orthogonal decomposition $W=\oplus_{i=1}^{p} W_{i}$ into one-dimensional subspaces, uniquely determined up to permutations, and flat bilinear forms $\beta_{i}: V \times V \rightarrow W_{i}, 1 \leq i \leq p$, such that

$$
\beta=\beta_{1} \oplus \cdots \oplus \beta_{p} .
$$

Thus there exists a basis $X_{1}, \ldots, X_{n}$ of $V$ that diagonalizes $\beta$, that is,

$$
\beta_{r}\left(X_{i}, X_{j}\right)=0 \text { unless } r=i=j
$$

and

$$
W_{i}=\operatorname{span}\left\{\beta\left(X_{i}, X_{i}\right)\right\}, \quad 1 \leq i \leq p
$$

Proof: We give the proof only when $\langle$,$\rangle is positive definite, and refer the reader to part$ (b) of Theorem 2 in [255] for the proof in the case of flat bilinear forms with respect to Lorentzian inner products.

We may assume that $\operatorname{dim} V=\operatorname{dim} W$ and that $N(\beta)=\{0\}$, for otherwise we can work with $\hat{V}=V / N(\beta)$ and the induced bilinear form $\hat{\beta}: \hat{V} \times \hat{V} \rightarrow W$. Then, for any $X \in R E(\beta)$, the map $B_{X}: V \rightarrow W$ is an isomorphism. Fix $X_{0} \in R E(\beta)$ and, for any $Y \in V$, define a linear endomorphism $C(Y)$ of $W$ by

$$
C(Y)=B_{Y} \circ B_{X_{0}}^{-1} .
$$

We claim that $C(Y)$ is symmetric and commutes with $C(\tilde{Y})$ for all $Y, \tilde{Y} \in V$. Given $Z, \tilde{Z} \in W$, let $X, \tilde{X} \in V$ be such that

$$
B_{X_{0}} X=Z \text { and } B_{X_{0}} \tilde{X}=\tilde{Z}
$$

Then the symmetry of $C(Y)$ follows from

$$
\begin{aligned}
\langle C(Y) Z, \tilde{Z}\rangle & =\left\langle C(Y) B_{X_{0}} X, B_{X_{0}} \tilde{X}\right\rangle \\
& =\left\langle\beta(Y, X), \beta\left(X_{0}, \tilde{X}\right)\right\rangle \\
& =\left\langle\beta\left(X_{0}, X\right), \beta(Y, \tilde{X})\right\rangle \\
& =\left\langle B_{X_{0}} X, C(Y) B_{X_{0}} \tilde{X}\right\rangle \\
& =\langle Z, C(Y) \tilde{Z}\rangle .
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\langle C(\tilde{Y}) C(Y) Z, \tilde{Z}\rangle & =\langle C(Y) Z, C(\tilde{Y}) \tilde{Z}\rangle \\
& =\langle\beta(Y, Z), \beta(\tilde{Y}, \tilde{Z})\rangle \\
& =\langle\beta(\tilde{Y}, Z), \beta(Y, \tilde{Z})\rangle \\
& =\langle C(\tilde{Y}) Z, C(Y) \tilde{Z}\rangle \\
& =\langle C(Y) C(\tilde{Y}) Z, \tilde{Z}\rangle
\end{aligned}
$$

and the claim follows.
From the claim, there exists a common orthonormal diagonalizing basis $\xi_{1}, \ldots, \xi_{n}$ of all the $C(Y), Y \in V$, that is, there are linear functionals $\mu_{i}$ on $W$ such that

$$
\begin{equation*}
C(Y) \xi_{i}=\mu_{i}(Y) \xi_{i}, \quad 1 \leq i \leq n \tag{5.3}
\end{equation*}
$$

for all $Y \in V$. Set $W_{i}=\operatorname{span}\left\{\xi_{i}\right\}$ and let $X_{i} \in V$ be such that

$$
B_{X_{0}} X_{i}=\xi_{i}, \quad 1 \leq i \leq n .
$$

Then (5.3) reads as

$$
\beta\left(Y, X_{i}\right)=\mu_{i}(Y) \xi_{i}
$$

for all $Y \in V$, and hence the $W_{i}$-component $\beta_{i}$ of $\beta$ satisfies

$$
\begin{aligned}
\beta_{i}(Y,) & =\left\langle\beta(Y,), \xi_{i}\right\rangle \xi_{i} \\
& =\left\langle\beta(Y,), \beta\left(X_{0}, X_{i}\right)\right\rangle \xi_{i} \\
& =\left\langle\beta\left(Y, X_{i}\right), \beta\left(X_{0},\right)\right\rangle \xi_{i} \\
& =\mu_{i}(Y) \beta_{i}\left(X_{0},\right) .
\end{aligned}
$$

In particular,

$$
\beta_{i}\left(X_{0},\right)=a_{i} \mu_{i}
$$

where $a_{i}=\beta_{i}\left(X_{0}, X_{0}\right)$. Therefore

$$
\begin{aligned}
a_{i} \mu_{i}\left(X_{j}\right) & =\beta_{i}\left(X_{0}, X_{j}\right) \\
& =\left\langle\beta\left(X_{0}, X_{j}\right), \xi_{i}\right\rangle \xi_{i} \\
& =\left\langle\xi_{j}, \xi_{i}\right\rangle \xi_{i} \\
& =\delta_{i j} \xi_{i}, \quad 1 \leq i, j \leq n,
\end{aligned}
$$

and

$$
\begin{equation*}
\beta_{i}=a_{i} \mu_{i} \otimes \mu_{i}, \quad 1 \leq i \leq n . \tag{5.4}
\end{equation*}
$$

Thus $\beta_{i}$ is flat, $\beta\left(X_{i}, X_{i}\right)$ spans $W_{i}$ and $\beta_{r}\left(X_{i}, X_{j}\right)=0$ unless $r=i=j$.
It remains to show that the subspaces $W_{1}, \ldots, W_{n}$ are uniquely determined up to permutations. To prove this, it suffices to show that, for any unit-length vector

$$
\xi=\sum_{i=1}^{n} b_{i} \xi_{i}
$$

such that $\langle\beta, \xi\rangle=\rho \otimes \rho$ for some $\rho \in V^{*}$, there exists $i_{0} \in\{1, \ldots, n\}$ such that $\xi$ coincides with $\xi_{i_{0}}$ up to sign. Write $\rho=\sum_{i=1}^{n} c_{i} \mu_{i}$. By (5.4) we have

$$
\begin{aligned}
\sum_{i=1}^{n} a_{i} b_{i} \mu_{i} \otimes \mu_{i} & =\langle\beta, \xi\rangle \\
& =\rho \otimes \rho \\
& =\sum_{i, j=1}^{n} c_{i} c_{j} \mu_{i} \otimes \mu_{j}
\end{aligned}
$$

hence $c_{i} c_{j}=0$ for $1 \leq i \neq j \leq n$. This implies that $c_{i}=0$ for all but one value of $i \in\{1, \ldots, n\}$, say, $i=i_{0}$. Thus also $b_{i}=0$ for $i \neq i_{0}$, that is, $\xi= \pm \xi_{i_{0}}$, as we wished.

The following result is an immediate consequence of Theorem 5.2.
Proposition 5.3. Let $f: M_{c}^{n} \rightarrow \mathbb{Q}_{c}^{2 n}$ be an isometric immersion. Then, at any point $x \in M_{c}^{n}$ where $\nu(x)=0$, there exists a unique, up to signs, basis $X_{1}, \ldots, X_{n}$ of $T_{x} M$ of unit-length vectors and a unique orthogonal basis $\eta_{1}, \ldots, \eta_{n}$ of $N_{f} M(x)$ such that the second fundamental form $\alpha$ at $x$ satisfies

$$
\begin{equation*}
\alpha\left(X_{i}, X_{j}\right)=\delta_{i j} \eta_{i}, \quad 1 \leq i, j \leq n \tag{5.5}
\end{equation*}
$$

Moreover, the basis $X_{1}, \ldots, X_{n}$ is also orthogonal if and only if $f$ has flat normal bundle at $x$, in which case $\eta_{1}, \ldots, \eta_{n}$ are the principal normals of $f$ at $x$.

Remark 5.4. (i) By the Lorentzian case of Theorem 5.2, the statement of Proposition 5.3 remains true if the ambient space $\mathbb{Q}_{c}^{2 n}$ is replaced by a Lorentzian space form $\mathbb{Q}_{c, 1}^{2 n}$, under the additional assumptions that the first normal space $N_{1}(x)$ of $f$ at $x$ coincides with the whole normal space $N_{f} M(x)$, and that there exists a normal vector $\xi \in N_{f} M(x)$ such that the shape operator $A_{\xi}$ is positive definite. Notice that, since $\nu(x)=0$, from Lemmas 4.10 and 4.14 it follows that $N_{1}(x)$ does not coincide with $N_{f} M(x)$ only if it is a degenerate subspace of $N_{f} M(x)$.
(ii) It was shown by Cartan [67] that the statement of Theorem 55.2, except for the last assertion, is also true for a symmetric bilinear form $\beta: V^{n} \times V^{n} \rightarrow W^{p}$ that is flat with respect to a positive definite inner product $\langle\rangle:, W \times W \rightarrow \mathbb{R}$ if $n-\operatorname{dim} \mathcal{N}(\beta) \leq 3$ and $p$ is arbitrary.

Theorem 5.5. Let $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{2 n-1}, c \neq \tilde{c}$, be an isometric immersion and let $x \in M_{c}^{n}$. If $c>\tilde{c}$, assume that $x$ is not a weak-umbilic point for $f$. Then $f$ has flat normal bundle at $x$.

Proof: As in the proof of Theorem 5.1, consider the composition $\tilde{f}=i \circ f$ of $f$ with an umbilical inclusion $i: \mathbb{Q}_{\tilde{c}}^{2 n-1} \rightarrow \mathbb{Q}_{c, s}^{2 n}$, with $s=0$ or $s=1$ depending on whether $c<\tilde{c}$ or $c>\tilde{c}$, respectively. As already observed in that proof, for any $x \in M_{c}^{n}$ the second fundamental form $\tilde{\alpha}(x)$ of $\tilde{f}$ at $x$ is flat with respect to the inner product on $N_{\tilde{f}} M(x)$, which is either positive definite or Lorentzian, depending on whether $c<\tilde{c}$ or $c>\tilde{c}$, respectively. Moreover, equation (5.2) implies that $\mathcal{N}(\tilde{\alpha}(x))=\{0\}$, and that the extra assumption of Theorem 5.2 in the Lorentzian case is satisfied.

If $c<\tilde{c}$, it follows from $\mathcal{N}(\tilde{\alpha}(x))=\{0\}$ and Lemma 4.10 that $\operatorname{dim} \mathcal{S}(\tilde{\alpha}(x)) \geq n$, and hence that $\mathcal{S}(\tilde{\alpha}(x))=N_{\tilde{f}} M(x)$.

If $c>\tilde{c}$, to conclude that $\operatorname{dim} \mathcal{S}(\tilde{\alpha}(x)) \geq n$, and hence that $\mathcal{S}(\tilde{\alpha}(x))=N_{\tilde{f}} M(x)$, from Lemma 4.10, Lemma 4.14 and the fact that $\mathcal{N}(\tilde{\alpha}(x))=\{0\}$, we have to show that $\mathcal{S}(\tilde{\alpha}(x))$ cannot be a degenerate subspace of $N_{\tilde{f}} M(x)$. But, arguing as in the proof of Theorem 5.1, in that case we would conclude that the second fundamental form $\alpha(x)$ of $f$ would be given as in part (ii) of that result, which has been ruled out by assumption.

It now follows from Theorem 5.2 that there exists a basis $X_{1}, \ldots, X_{n}$ of $T_{x} M$ that diagonalizes $\tilde{\alpha}(x)$. In particular, this basis diagonalizes

$$
\langle\tilde{\alpha}(x), \xi\rangle=\sqrt{|\tilde{c}-c|}\langle\xi, \xi\rangle\langle,\rangle
$$

where $\xi$ is one of the unit vectors normal to $i$ at $f(x)$. Therefore the vectors $X_{1}, \ldots, X_{n}$ must be orthogonal, and since the basis also diagonalizes $\alpha(x)$ by (5.1), we conclude from Proposition 1.24 that the normal curvature tensor of $f$ vanishes at $x$.

### 5.2 Principal coordinates

The next result shows that isometric immersions $f: M_{c}^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ with flat normal bundle and vanishing index of relative nullity are always locally holonomic, and provides conditions under which they are globally holonomic. An isometric immersion $f: M^{n} \rightarrow$ $\mathbb{Q}_{c}^{n+p}$ with flat normal bundle is said to be holonomic (or globally holonomic) if $M^{n}$ carries global orthogonal coordinates $\left(u_{1}, \ldots, u_{n}\right)$ such that the coordinate vector fields are everywhere eigenvectors of all shape operators of $f$.

Proposition 5.6. Let $f: M_{c}^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ be an isometric immersion with flat normal bundle and vanishing index of relative nullity. Then, for each $x_{0} \in M_{c}^{n}$, there exists a diffeomorphism $\psi: U \rightarrow W$ from an open subset $U \subset \mathbb{R}^{n}$ onto an open neighborhood $W \subset M_{c}^{n}$ of $x_{0}$ such that the following assertions hold:
(i) The tangent frame $\left\|\eta_{1}\right\| \psi_{*} \partial / \partial u_{1}, \ldots,\left\|\eta_{n}\right\| \psi_{*} \partial / \partial u_{n}$ is orthonormal and principal, where $\eta_{1}, \ldots, \eta_{n}$ are the principal normal vector fields of $f$.
(ii) The diffeomorphism $\psi$ is an isometry with respect to the standard metric on $U$ and the metric on $W$ given by the third fundamental form III. Thus III is a flat metric on $W$.

Moreover, if $M_{c}^{n}$ is complete and there exists $\delta>0$ such that $\left\|\eta_{i}\right\|>\delta, 1 \leq i \leq n$, then the following assertions are true:
(a) There is a global diffeomorphism $\psi: \mathbb{R}^{n} \rightarrow \hat{M}_{c}^{n}$ onto the universal covering $\hat{M}_{c}^{n}$ of $M_{c}^{n}$ such that ( $i$ ) and (ii) above hold for $\hat{f}=f \circ \pi$, where $\pi: \hat{M}_{c}^{n} \rightarrow M_{c}^{n}$ is the covering map. In particular, $c \leq 0$.
(b) The metric III is complete.

Proof: Given $x_{0} \in M_{c}^{n}$, choose an orthonormal tangent frame $X_{1}, \ldots, X_{n}$ on an open simply connected neighborhood $W \subset M_{c}^{n}$ of $x_{0}$ such that the second fundamental form of $f$ is given by

$$
\alpha\left(X_{i}, X_{j}\right)=\delta_{i j} \eta_{i}, \quad 1 \leq i, j \leq n,
$$

where $\eta_{1}, \ldots, \eta_{n}$ are the principal normal vector fields of $f$ on $W$, which satisfy

$$
\left\langle\eta_{i}, \eta_{j}\right\rangle=0, \quad i \neq j,
$$

by the Gauss equation. Write $\eta_{i}=\lambda_{i} \xi_{i}$, where $\lambda_{i}=\left\|\eta_{i}\right\|, 1 \leq i \leq n$. Then the Codazzi equations (1.41) and (1.42) give

$$
\nabla_{X_{i}} X_{j}=\lambda_{i} X_{j}\left(1 / \lambda_{i}\right) X_{i}, \quad 1 \leq i \neq j \leq n
$$

which implies that

$$
\left[\lambda_{i}^{-1} X_{i}, \lambda_{j}^{-1} X_{j}\right]=0, \quad 1 \leq i \neq j \leq n .
$$

Set $Y_{i}=\lambda_{i}^{-1} X_{i}$, and let $\varphi_{i}$ be the local one-parameter group of diffeomorphisms generated by $Y_{i}$ on $W$. Thus, for any $x \in W$, the map $t \mapsto \varphi_{i}(x, t)$ is the maximal integral curve of $Y_{i}$ with $\varphi_{i}(0, x)=x$. Now define $\psi=\psi_{x_{0}}$ by

$$
\begin{equation*}
\psi\left(t_{1}, \ldots, t_{n}\right)=\varphi_{n}\left(\cdots\left(\varphi_{2}\left(\varphi_{1}\left(x_{0}, t_{1}\right), t_{2}\right), \cdots\right), t_{n}\right) \tag{5.6}
\end{equation*}
$$

for $t_{i}, 1 \leq i \leq n$, small enough. Since $\left[Y_{i}, Y_{j}\right]=0$, the one-parameter groups $\varphi_{i}$ and $\varphi_{j}$ commute. This implies that

$$
\begin{equation*}
\psi_{x_{0}}(t+s)=\varphi_{n}\left(\cdots\left(\varphi_{2}\left(\varphi_{1}\left(\psi_{x_{0}}(s), t_{1}\right), t_{2}\right), \cdots\right), t_{n}\right)=\psi_{\psi_{x_{0}}(s)}(t) \tag{5.7}
\end{equation*}
$$

whenever both sides are defined, where $t=\left(t_{1}, \ldots, t_{n}\right)$ and $s=\left(s_{1}, \ldots, s_{n}\right)$. Thus

$$
\begin{aligned}
\psi_{*}(s) \partial / \partial u_{i} & =\left.\frac{d}{d t}\right|_{t=0} \psi\left(s_{1}, \ldots, s_{i}+t, \ldots, s_{n}\right) \\
& =\left.\frac{d}{d t}\right|_{t=0} \varphi_{i}(\psi(s), t) \\
& =Y_{i}(\psi(s))
\end{aligned}
$$

for any $s=\left(s_{1}, \ldots, s_{n}\right)$ with $s_{i}, 1 \leq i \leq n$, sufficiently small. In particular, $\psi$ is a diffeomorphism satisfying condition $(i)$ of an open ball centered at the origin onto an open neighborhood of $x_{0}$. Since

$$
\begin{align*}
\operatorname{III}\left(X_{i}, X_{j}\right) & =\sum_{k=1}^{n}\left\langle\alpha\left(X_{i}, X_{k}\right), \alpha\left(X_{j}, X_{k}\right)\right\rangle \\
& =\sum_{k=1}^{n}\left\langle\delta_{i k} \lambda_{i} \xi_{i}, \delta_{j k} \lambda_{j} \xi_{j}\right\rangle \\
& =\lambda_{i} \lambda_{j} \delta_{i j}, \tag{5.8}
\end{align*}
$$

it follows that

$$
\operatorname{III}\left(\psi_{*} \partial / \partial u_{i}, \psi_{*} \partial / \partial u_{j}\right)=I I I\left(\lambda_{i}^{-1} X_{i}, \lambda_{j}^{-1} X_{j}\right)=\delta_{i j} .
$$

Thus (ii) holds.
Now suppose that there exists $\delta>0$ such that $\left\|\eta_{i}\right\|>\delta$ for all $1 \leq i \leq n$. We show that there is a global diffeomorphism $\psi: \mathbb{R}^{n} \rightarrow M_{c}^{n}$ satisfying conditions ( $i$ ) and (ii) if $M_{c}^{n}$ is simply connected. First notice that, under the assumption that $\left\|\eta_{i}\right\|>\delta$ for some $\delta>0,1 \leq i \leq n$, the vector fields $Y_{i}, 1 \leq i \leq n$, have bounded lengths, hence their one-parameter groups are defined for all $t \in \mathbb{R}$. We claim that, in this case, the map $\psi$ defined by (5.6) on the entire $\mathbb{R}^{n}$ is a covering map. This, together with the assumption that $M_{c}^{n}$ is simply connected, will imply that $\psi$ is actually a diffeomorphism and conclude the proof.

Given $x \in M_{c}^{n}$, let $\tilde{B}_{2 \epsilon}(0)$ be an open ball of radius $2 \epsilon$ centered at the origin such that $\left.\psi_{x}\right|_{\tilde{B}_{2 \epsilon}(0)}$ is a diffeomorphism onto $B_{2 \epsilon}(x)=\psi_{x}\left(\tilde{B}_{2 \epsilon}(0)\right)$. If

$$
\psi_{x_{0}}^{-1}(x)=\cup_{\alpha \in A} \tilde{x}_{\alpha},
$$

then let $\tilde{B}_{2 \epsilon}\left(\tilde{x}_{\alpha}\right)$ for each $\alpha \in A$ denote the open ball of radius $2 \epsilon$ centered at $\tilde{x}_{\alpha}$. Define a map $\phi_{\alpha}: B_{2 \epsilon}(x) \rightarrow \tilde{B}_{2 \epsilon}\left(\tilde{x}_{\alpha}\right)$ by

$$
\phi_{\alpha}(y)=\tilde{x}_{\alpha}+\psi_{x}^{-1}(y) .
$$

Then from (5.7) we obtain

$$
\begin{aligned}
\psi_{x_{0}}\left(\phi_{\alpha}(y)\right) & =\psi_{x_{0}}\left(\tilde{x}_{\alpha}+\psi_{x}^{-1}(y)\right) \\
& =\psi_{\psi_{x_{0}}\left(\tilde{x}_{\alpha}\right)}\left(\psi_{x}^{-1}(y)\right) \\
& =\psi_{x}\left(\psi_{x}^{-1}(y)\right) \\
& =y
\end{aligned}
$$

for all $y \in B_{2 \epsilon}(x)$. Thus $\psi_{x_{0}}$ is a diffeomorphism from $\tilde{B}_{2 \epsilon}\left(\tilde{x}_{\alpha}\right)$ onto $B_{2 \epsilon}(x)$ having $\phi_{\alpha}$ as its inverse. In particular, this implies that $\tilde{B}_{\epsilon}\left(\tilde{x}_{\alpha}\right)$ and $\tilde{B}_{\epsilon}\left(\tilde{x}_{\beta}\right)$ are disjoint if $\alpha$ and $\beta$ are distinct indices in $A$. Finally, it remains to check that if $\tilde{y} \in \psi_{x_{0}}^{-1}\left(B_{\epsilon}(x)\right)$ then $\tilde{y} \in \tilde{B}_{\epsilon}\left(\tilde{x}_{\alpha}\right)$ for some $\alpha \in A$. This follows from the fact that

$$
\psi_{x_{0}}\left(\tilde{y}-\psi_{x}^{-1}\left(\psi_{x_{0}}(\tilde{y})\right)\right)=\psi_{\psi_{x_{0}}(\tilde{y})}\left(-\psi_{x}^{-1}\left(\psi_{x_{0}}(\tilde{y})\right)\right)=x
$$

For the last equality, observe from (5.7) that for all $x, y \in M_{c}^{n}$ we have $\psi_{x}(t)=y$ if and only if $\psi_{y}(-t)=x$.

Finally, if there exists $\delta>0$ such that $\left\|\eta_{i}\right\|>\delta$ for all $1 \leq i \leq n$, then $I I I \geq \delta^{2}\langle$, in view of (5.8). Hence III is complete if so is $\langle$,$\rangle .$
Remark 5.7. The assertions $(i)$ and $(a)$ in Proposition 5.6 remain true if the ambient space $\mathbb{Q}_{c}^{n+p}$ is replaced by a Lorentzian space form $\mathbb{Q}_{c, 1}^{n+p}$, under the additional assumption that the first normal spaces of $f$ are nondegenerate subspaces of the corresponding normal spaces at any point. Indeed, if this condition is satisfied then the second fundamental form of $f$ is also given by (5.5), hence the arguments in the proofs of those assertions are still valid.

### 5.3 The associated systems of PDEs

The existence of principal coordinates for isometric immersions $f: M_{c}^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ with flat normal bundle and vanishing index of relative nullity allows to show that such isometric immersions are in correspondence with solutions of a certain system of partial differential equations.

Proposition 5.8. Let $f: M_{c}^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ be an isometric immersion with flat normal bundle and vanishing index of relative nullity, and let $\left(u_{1}, \ldots, u_{n}\right)$ be a local principal coordinate system on $M_{c}^{n}$ given by Proposition 5.6. The following assertions hold:
(i) There exist orthonormal frames $\xi_{1}, \ldots, \xi_{n}$ of $N_{1}, \xi_{n+1}, \ldots, \xi_{n+p}$ of $N_{1}^{\perp}$, and smooth functions $v_{1}, \ldots, v_{n}$ and $h_{i \alpha}, 1 \leq i \leq n, n+1 \leq \alpha \leq n+p$, such that the induced metric, second fundamental form and normal connection of $f$ are given, respectively, by

$$
\begin{gather*}
d s^{2}=\sum_{i=1}^{n} v_{i}^{2} d u_{i}^{2}, \\
\alpha\left(\partial / \partial u_{i}, \partial / \partial u_{j}\right)=\delta_{i j} v_{i} \xi_{i} \tag{5.9}
\end{gather*}
$$

and

$$
\begin{equation*}
\nabla \frac{1}{\partial} / \partial u_{i} \xi_{r}=h_{i r} \xi_{i}, \quad 1 \leq r \leq n+p \tag{5.10}
\end{equation*}
$$

with $h_{i j}=\left(1 / v_{i}\right) \partial v_{j} / \partial u_{i}$ for $1 \leq j \leq n$.
(ii) The pair $(v, h)$, with $v=\left(v_{1}, \ldots, v_{n}\right)$ and $h=\left(h_{i r}\right)$, satisfies the system of partial differential equations

$$
\left\{\begin{array}{l}
\text { (i) } \frac{\partial v_{i}}{\partial u_{j}}=h_{j i} v_{j}  \tag{5.11}\\
\text { (ii) } \frac{\partial h_{i j}}{\partial u_{i}}+\frac{\partial h_{j i}}{\partial u_{j}}+\sum_{k} h_{k i} h_{k j}+c v_{i} v_{j}=0 \\
\text { (iii) } \frac{\partial h_{i r}}{\partial u_{j}}=h_{i j} h_{j r} \\
\text { (iv) } \frac{\partial h_{i j}}{\partial u_{j}}+\frac{\partial h_{j i}}{\partial u_{i}}+\sum_{r} h_{i r} h_{j r}=0
\end{array}\right.
$$

where $1 \leq i, j, k \leq n, 1 \leq r \leq n+p, i \neq j$ and $k, r \notin\{i, j\}$.
Conversely, let the pair $(v, h)$ be a solution of (5.11) on a simply connected open subset $U \subset \mathbb{R}^{n}$, with $v_{i} \neq 0,1 \leq i \leq n$, at any point. Then there exists an immersion $f: U \rightarrow \mathbb{Q}_{c}^{n+p}$ with flat normal bundle and vanishing index of relative nullity whose induced metric $d s^{2}=\sum_{i=1}^{n} v_{i}^{2} d u_{i}^{2}$ has constant sectional curvature $c$ and whose second fundamental form and normal connection are given by (5.9) and (5.10), respectively.

Proof: As shown in the proof of Proposition 5.6, there exist orthonormal frames $X_{1}, \ldots, X_{n}$ of $T M, \xi_{1}, \ldots, \xi_{n}$ of $N_{1}$ and smooth functions $\lambda_{1}, \ldots, \lambda_{n}$ such that

$$
\alpha\left(X_{i}, X_{j}\right)=\delta_{i j} \lambda_{i} \xi_{i}, \quad 1 \leq i \neq j \leq n
$$

and $\partial / \partial u_{i}=\left(1 / \lambda_{i}\right) X_{i}$. The Codazzi equation (1.41) gives (5.10) for $1 \leq r \leq n$. In particular,

$$
\begin{equation*}
\left\langle\nabla \frac{\perp}{\partial / \partial u_{i}} \delta, \xi_{j}\right\rangle=0, \quad 1 \leq i \neq j \leq n, \tag{5.12}
\end{equation*}
$$

for $\delta \in \Gamma\left(N_{1}^{\perp}\right)$.
Let $\nabla^{\prime}$ denote the connection on $N_{1}^{\perp}$ induced by $\nabla^{\perp}$, that is, $\nabla_{X}^{\prime} \delta$ is the orthogonal projection of $\nabla_{X}^{\perp} \delta$ onto $N_{1}^{\perp}$ for all $X \in \mathfrak{X}(M)$ and $\delta \in \Gamma\left(N_{1}^{\perp}\right)$. Observe that

$$
\left\langle\nabla_{\partial / \partial u_{j}}^{\prime} \delta, \nabla_{\partial / \partial u_{i}}^{\prime} \zeta\right\rangle=\left\langle\nabla_{\partial}^{\perp} / \partial u_{j} \delta, \nabla_{\partial / \partial u_{i}}^{\perp} \zeta\right\rangle, \quad 1 \leq i \neq j \leq n,
$$

for all $\delta, \zeta \in \Gamma\left(N_{1}^{\perp}\right)$, because the components in $N_{1}$ of $\nabla \frac{\perp}{\partial} / \partial u_{j} \delta$ and $\nabla \frac{\perp}{\partial / \partial u_{i}} \zeta$ are orthogonal by (5.12). Therefore

$$
\begin{aligned}
\left\langle\nabla_{\partial / \partial u_{i}}^{\prime} \nabla_{\partial / \partial u_{j}}^{\prime} \delta, \zeta\right\rangle & =\partial / \partial u_{i}\left\langle\nabla_{\partial / \partial u_{j}}^{\prime} \delta, \zeta\right\rangle-\left\langle\nabla_{\partial / \partial u_{j}}^{\prime} \delta, \nabla_{\partial / \partial u_{i}}^{\prime} \zeta\right\rangle \\
& =\partial / \partial u_{i}\left\langle\nabla^{\frac{1}{\partial} / \partial u_{j}} \delta, \zeta\right\rangle-\left\langle\nabla_{\partial / \partial u_{j}} \delta, \nabla_{\partial / \partial u_{i}} \zeta\right\rangle \\
& =\left\langle\nabla_{\partial / \partial u_{i}}^{\perp} \nabla_{\partial / \partial u_{j}}^{\perp} \delta, \zeta\right\rangle .
\end{aligned}
$$

Hence the curvature tensor $R^{\prime}$ of $\nabla^{\prime}$ satisfies

$$
\begin{aligned}
\left\langle R^{\prime}\left(\partial / \partial u_{i}, \partial / \partial u_{j}\right) \delta, \zeta\right\rangle & =\left\langle\nabla_{\partial / \partial u_{i}}^{\prime} \nabla_{\partial / \partial u_{j}}^{\prime} \delta-\nabla_{\partial / \partial u_{j}}^{\prime} \nabla_{\partial / \partial u_{i}}^{\prime} \delta, \zeta\right\rangle \\
& =\left\langle\nabla_{\partial / \partial u_{i}}^{\perp} \nabla_{\partial / \partial u_{j}}^{\perp} \delta-\nabla_{\partial / \partial u_{j}}^{\perp} \nabla_{\partial / \partial u_{i}}^{\perp} \delta, \zeta\right\rangle \\
& =\left\langle R^{\perp}\left(\partial / \partial u_{i}, \partial / \partial u_{j}\right) \delta, \zeta\right\rangle \\
& =0,
\end{aligned}
$$

where the last equality follows from the Ricci equation of $f$.
Let $\xi_{n+1}, \ldots, \xi_{n+p}$ be a parallel orthonormal frame of $N_{1}^{\perp}$ with respect to $\nabla^{\prime}$. Then $\nabla \frac{\perp}{\partial} / \partial u_{i} \xi_{\alpha}$ has no component in $N_{1}^{\perp}$ for $n+1 \leq \alpha \leq n+p$. Hence (5.10) holds for $n+1 \leq \alpha \leq n+p$, with

$$
h_{i \alpha}=\left\langle\nabla \frac{\perp}{\partial / \partial u_{i}} \xi_{\alpha}, \xi_{i}\right\rangle, \quad n+1 \leq \alpha \leq n+p,
$$

in view of (5.12).

It is now straightforward to verify, as in the computations in Section 1.4.3 for holonomic hypersurfaces, that the Gauss and Codazzi equations of $f$ reduce to the equations (ii) and (iii) of (5.11) with $1 \leq r \leq n$. On the other hand, using (5.10) one obtains that the normal curvature tensor of $f$ satisfies

$$
R^{\perp}\left(\partial / \partial u_{i}, \partial / \partial u_{j}\right) \xi_{r}=\left(\partial h_{j r} / \partial u_{i}-h_{j i} h_{i r}\right) \xi_{j}-\left(\partial h_{i r} / \partial u_{j}-h_{i j} h_{j r}\right) \xi_{i}
$$

if $i \neq j \neq r \neq i$, and

$$
\left\langle R^{\perp}\left(\partial / \partial u_{i}, \partial / \partial u_{j}\right) \xi_{i}, \xi_{j}\right\rangle=\frac{\partial h_{i j}}{\partial u_{j}}+\frac{\partial h_{j i}}{\partial u_{i}}+\sum_{r \neq i, j} h_{i r} h_{j r}, \quad \text { if } i \neq j .
$$

Therefore the Ricci equations of $f$ reduce to the equations (iii) and (iv) of (5.11).
For the converse, consider on $U$ the metric $d s^{2}=\sum_{i} v_{i}^{2} d u_{i}^{2}$. Using equations $(i)$, (ii) and (iii) in (5.11), it follows from (1.24) and (1.25) that this metric has constant sectional curvature $c$. Set $M_{c}^{n}=\left(U, d s^{2}\right)$. To conclude the proof from the Fundamental theorem of submanifolds, consider the trivial vector bundle $E=M_{c}^{n} \times \mathbb{R}^{p}$, and let $e_{1}, \ldots, e_{p}$ be an orthonormal frame of $E$. The compatible vector bundle connection $\nabla^{\prime}$ defined by

$$
\nabla_{\partial / \partial u_{i}}^{\prime} e_{r}=h_{i r} e_{i}
$$

is flat from equations (iii) and (iv) of 5.11. Define $\alpha \in \Gamma\left(\operatorname{Hom}^{2}(T M, T M ; E)\right)$ by

$$
\alpha\left(\partial / \partial u_{i}, \partial / \partial u_{j}\right)=v_{i} \delta_{i j} e_{i} .
$$

That $\alpha$ satisfies the Gauss equations follows from flatness of $\alpha$ and the fact that the metric $d s^{2}=\sum_{i} v_{i}^{2} d u_{i}^{2}$ on $U$ has constant sectional curvature $c$. The Codazzi equation follows from (iii) and the Ricci equation is satisfied because $\nabla^{\prime}$ is flat by (iii) and (iv) and $\alpha$ is orthogonally diagonalizable.

Remark 5.9. Again, the statement of Proposition 5.8 remains true if the ambient space $\mathbb{Q}_{c}^{n+p}$ is replaced by a Lorentzian space form $\mathbb{Q}_{c, 1}^{n+p}$, under the additional assumption that the first normal spaces of $f$ are nondegenerate subspaces of the corresponding normal spaces at any point.

### 5.4 The case of distinct curvatures

In this section we study isometric immersions $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{2 n-1}, c \neq \tilde{c}$, which are free of weak-umbilic points if $c>\tilde{c}$. The idea is to consider the composition $\tilde{f}=i \circ f$ with an umbilical inclusion $i: \mathbb{Q}_{\tilde{c}}^{2 n-1} \rightarrow \mathbb{Q}_{c, s}^{2 n}$, where $s=0$ or $s=1$, depending on whether $c<\tilde{c}$ or $c>\tilde{c}$, respectively, and then to apply the results of the previous sections to $\tilde{f}$. The next result characterizes which isometric immersions $\tilde{f}: M_{c}^{n} \rightarrow \mathbb{Q}_{c, s}^{2 n}$ arise in this way.

Proposition 5.10. Let $\tilde{f}: M_{c}^{n} \rightarrow \mathbb{Q}_{c, s}^{2 n}, s \in\{0,1\}$, be an isometric immersion. Then the following assertions are equivalent:
(i) There exist $\tilde{c} \in \mathbb{R}$, with $\tilde{c}>c$ if $s=0$ and $\tilde{c}<c$ if $s=1$, and an isometric immersion $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{2 n-1}$, free of weak-umbilic points if $\tilde{c}<c$, such that $\tilde{f}=i \circ f$, where $i: \mathbb{Q}_{\tilde{c}}^{2 n-1} \rightarrow \mathbb{Q}_{c, s}^{2 n}$ is an umbilical inclusion.
(ii) The isometric immersion $\tilde{f}$ has flat normal bundle and pairwise orthogonal principal normal vector fields $\eta_{1}, \ldots, \eta_{n}$ such that $\left\langle\eta_{i}, \eta_{i}\right\rangle \neq 0$ at any point of $M_{c}^{n}$ and

$$
\begin{equation*}
\sum_{i=1}^{n} \frac{1}{\left\langle\eta_{i}, \eta_{i}\right\rangle}=\frac{1}{\tilde{c}-c}, \tag{5.13}
\end{equation*}
$$

where $\tilde{c}>c$ if $s=0$ and $\tilde{c}<c$ if $s=1$.
Proof: Assume first that $\tilde{f}=i \circ f$ for some isometric immersion $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{2 n-1}$, where $\tilde{c}>c$ if $s=0$ and $\tilde{c}<c$ if $s=1$. Suppose further that $f$ is free of weak-umbilic points if $\tilde{c}<c$.

By Theorem 5.5, the isometric immersion $f$ has flat normal bundle. It follows from (5.1) that the same holds for $\tilde{f}$, and that $\tilde{f}$ has vanishing index of relative nullity. Moreover, as seen in the proof of Theorem 5.5, the assumption that $f$ is free of weakumbilic points if $\tilde{c}<c$ implies that the first normal spaces of $\tilde{f}$ are nondegenerate at any point of $M_{c}^{n}$. By Proposition 5.3 and Remark 5.4 , $\tilde{f}$ has principal normal vector fields $\eta_{1}, \ldots, \eta_{n}$ such that

$$
\begin{equation*}
\left\langle\eta_{i}, \eta_{i}\right\rangle \neq 0 \text { and }\left\langle\eta_{i}, \eta_{j}\right\rangle=0, \quad 1 \leq i \neq j \leq n, \tag{5.14}
\end{equation*}
$$

at any point of $M_{c}^{n}$. Thus the vector field

$$
\eta=\sum_{i=1}^{n} \frac{1}{\left\langle\eta_{i}, \eta_{i}\right\rangle} \eta_{i}
$$

satisfies $A_{\eta}=I$. On the other hand, if $\zeta=\sum_{i} a_{i} \eta_{i} \in \Gamma\left(N_{f} M\right)$ is an umbilical normal vector field, then

$$
0=\left\langle\zeta, \eta_{i}-\eta_{j}\right\rangle=a_{i}\left\langle\eta_{i}, \eta_{i}\right\rangle-a_{j}\left\langle\eta_{j}, \eta_{j}\right\rangle, \quad 1 \leq i, j \leq n,
$$

hence $\zeta=a \eta$ if $A_{\zeta}=a I$ for some $a \in C^{\infty}(M)$. Since the shape operator $A_{\xi}$ of $f$ with respect to one of the unit vector fields normal to the inclusion $i: \mathbb{Q}_{\tilde{c}}^{2 n-1} \rightarrow \mathbb{Q}_{c, s}^{2 n}$ along $f$ is

$$
A_{\xi}=\sqrt{|\tilde{c}-c|} I,
$$

we conclude that $\xi=\sqrt{|\tilde{c}-c|} \eta$. Hence

$$
(\tilde{c}-c) \sum_{i=1}^{n} \frac{1}{\left\langle\eta_{i}, \eta_{i}\right\rangle}=1,
$$

which implies (5.13).

To prove the converse, we compute

$$
\nabla \stackrel{1}{X}_{j}^{\perp} \eta=\sum_{k=1}^{n} X_{j}\left(1 /\left\langle\eta_{k}, \eta_{k}\right\rangle\right) \eta_{k}+\sum_{k=1}^{n} \frac{1}{\left\langle\eta_{k}, \eta_{k}\right\rangle} \nabla_{X_{j}}^{\perp} \eta_{k}, \quad 1 \leq j \leq n .
$$

The Codazzi equation (1.41) gives

$$
\begin{align*}
\frac{1}{\left\langle\eta_{j}, \eta_{j}\right\rangle}\left\langle\nabla{ }_{X_{j}}^{\perp} \eta_{j}, \eta_{i}\right\rangle & =\frac{1}{\left\langle\eta_{i}, \eta_{i}\right\rangle}\left\langle\nabla_{X_{j}}^{\perp} \eta_{i}, \eta_{i}\right\rangle \\
& =\frac{1}{2\left\langle\eta_{i}, \eta_{i}\right\rangle} X_{j}\left\langle\eta_{i}, \eta_{i}\right\rangle, \quad 1 \leq i \neq j \leq n . \tag{5.15}
\end{align*}
$$

Thus (1.41) and (5.15) yield

$$
\begin{aligned}
\left\langle\nabla_{X_{j}}^{\perp} \eta, \eta_{i}\right\rangle & =X_{j}\left(1 /\left\langle\eta_{i}, \eta_{i}\right\rangle\right)\left\langle\eta_{i}, \eta_{i}\right\rangle+\frac{1}{\left\langle\eta_{i}, \eta_{i}\right\rangle}\left\langle\nabla_{X_{j}}^{\perp} \eta_{i}, \eta_{i}\right\rangle+\frac{1}{\left\langle\eta_{j}, \eta_{j}\right\rangle}\left\langle\nabla_{X_{j}}^{\perp} \eta_{j}, \eta_{i}\right\rangle \\
& =X_{j}\left(1 /\left\langle\eta_{i}, \eta_{i}\right\rangle\right)\left\langle\eta_{i}, \eta_{i}\right\rangle+\frac{1}{\left\langle\eta_{i}, \eta_{i}\right\rangle} X_{j}\left\langle\eta_{i}, \eta_{i}\right\rangle \\
& =0, \quad 1 \leq i \neq j \leq n .
\end{aligned}
$$

On the other hand, using (5.15) we obtain

$$
\begin{aligned}
\left\langle\nabla_{X_{j}}^{\perp} \eta, \eta_{j}\right\rangle & =X_{j}\left(1 /\left\langle\eta_{j}, \eta_{j}\right\rangle\right)\left\langle\eta_{j}, \eta_{j}\right\rangle+\sum_{k \neq j} \frac{1}{\left\langle\eta_{k}, \eta_{k}\right\rangle}\left\langle\nabla_{X_{j}}^{\perp} \eta_{k}, \eta_{j}\right\rangle+\frac{1}{\left\langle\eta_{j}, \eta_{j}\right\rangle}\left\langle\nabla_{X_{j}}^{\perp} \eta_{j}, \eta_{j}\right\rangle \\
& =-\frac{1}{\left\langle\eta_{j}, \eta_{j}\right\rangle} X_{j}\left\langle\eta_{j}, \eta_{j}\right\rangle-\frac{1}{2} \sum_{k \neq j} \frac{\left\langle\eta_{j}, \eta_{j}\right\rangle}{\left\langle\eta_{k}, \eta_{k}\right\rangle^{2}} X_{j}\left\langle\eta_{k}, \eta_{k}\right\rangle+\frac{1}{2\left\langle\eta_{j}, \eta_{j}\right\rangle} X_{j}\left\langle\eta_{j}, \eta_{j}\right\rangle \\
& =\frac{1}{2}\left\langle\eta_{j}, \eta_{j}\right\rangle X_{j}\langle\eta, \eta\rangle .
\end{aligned}
$$

Therefore, since 5.13 holds, then $\eta$ is parallel in the normal connection. It follows from Exercise 2.9 that $\tilde{f}\left(M_{c}^{n}\right)$ is contained in an umbilical hypersurface $\mathbb{Q}_{\tilde{c}}^{2 n-1}$ of $\mathbb{Q}_{c, s}^{2 n}$, with $\tilde{c}>c$ if $s=0$ and $\tilde{c}<c$ if $s=1$. Thus there exists an isometric immersion $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{2 n-1}$ such that $\tilde{f}=i \circ f$.

It remains to show that $f$ is free of weak-umbilic points if $c>\tilde{c}$. Assume otherwise, that is, that at some point $x \in M_{c}^{n}$ there exist $\zeta \in N_{f} M(x)$ and a flat bilinear form

$$
\gamma(x): T_{x} M \times T_{x} M \rightarrow\{\zeta\}^{\perp} \subset N_{f} M(x)
$$

such that the second fundamental form of $f$ at $x$ is given by

$$
\alpha(x)=\gamma(x)+\sqrt{c-\tilde{c}}\langle,\rangle \zeta .
$$

Arguing as in the proof of Theorem 5.1, we see that the vector $i_{*} \zeta+\xi$, where $\xi$ is one of the unit vector fields normal to $i$ along $f$, belongs to $N_{1}^{\tilde{f}}(x) \cap\left(N_{1}^{\tilde{f}}(x)\right)^{\perp}$. Thus $N_{1}^{\tilde{f}}(x)$ is a degenerate subspace of $N_{\tilde{f}} M(x)$. But this contradicts the assumption
that $\tilde{f}$ has pairwise orthogonal principal normal vectors $\eta_{1}(x), \ldots, \eta_{n}(x)$ at $x$ with $\left\langle\eta_{i}(x), \eta_{i}(x)\right\rangle \neq 0$ for $1 \leq i \leq n$.

An orthogonal metric $d s^{2}=\sum_{i=1}^{n} v_{i}^{2} d u_{i}^{2}$ on an open subset $U \subset \mathbb{R}^{n}$ is called a Guichard-Darboux metric of signature $\left(\epsilon_{1}, \ldots, \epsilon_{n}\right)$, with $\epsilon_{i} \in\{-1,1\}$ for $1 \leq i \leq n$, if it has constant sectional curvature and there exists $k \in \mathbb{R}$ such that

$$
\begin{equation*}
\sum_{i=1}^{n} \epsilon_{i} v_{i}^{2}=k \tag{5.16}
\end{equation*}
$$

It follows from (1.24) and (1.25) that an orthogonal metric $d s^{2}=\sum_{i} v_{i}^{2} d u_{i}^{2}$ has constant sectional curvature $c$ if and only if the functions $v_{1}, \ldots, v_{n}$ and

$$
h_{i j}=\frac{1}{v_{i}} \frac{\partial v_{j}}{\partial u_{i}}, \quad 1 \leq i \neq j \leq n,
$$

satisfy the differential equations

$$
\frac{\partial h_{i j}}{\partial u_{i}}+\frac{\partial h_{j i}}{\partial u_{j}}+\sum_{k} h_{k i} h_{k j}+c v_{i} v_{j}=0
$$

and

$$
\frac{\partial h_{i k}}{\partial u_{j}}=h_{i j} h_{j k}
$$

where $1 \leq i \neq j \neq k \neq i \leq n$. Moreover, equation (5.16) holds for some $k \in \mathbb{R}$ if and only if

$$
\epsilon_{i} \frac{\partial v_{i}}{\partial u_{i}}+\sum_{j \neq i} \epsilon_{j} h_{i j} v_{j}=0, \quad 1 \leq i \leq n
$$

which is the derivative of (5.16) with respect to $u_{i}$.
In summary, Guichard-Darboux metrics of constant sectional curvature $c$ and signature $\left(\epsilon_{1}, \ldots, \epsilon_{n}\right)$ on an open subset $U \subset \mathbb{R}^{n}$ are in correspondence with solutions $(v, h)$ on $U$, with $v=\left(v_{1}, \ldots, v_{n}\right)$ and $h=\left(h_{i j}\right)$, of the system of partial differential equations

$$
\left\{\begin{array}{l}
\text { (i) } \frac{\partial v_{i}}{\partial u_{j}}=h_{j i} v_{j}  \tag{5.17}\\
\text { (ii) } \frac{\partial h_{i j}}{\partial u_{i}}+\frac{\partial h_{j i}}{\partial u_{j}}+\sum_{k} h_{k i} h_{k j}+c v_{i} v_{j}=0 \\
\text { (iii) } \frac{\partial h_{i k}}{\partial u_{j}}=h_{i j} h_{j k} \\
\text { (iv) } \epsilon_{i} \frac{\partial v_{i}}{\partial u_{i}}+\sum_{j \neq i} \epsilon_{j} h_{i j} v_{j}=0,
\end{array}\right.
$$

where $1 \leq i \neq j \neq k \neq i \leq n$.
Notice that, if $n=2,\left(\epsilon_{1}, \epsilon_{2}\right)=(1,1)$ and (5.16) holds with, say, $k=1$, then we can write

$$
v_{1}=\cos \phi \text { and } v_{2}=\sin \phi
$$

for some $\phi \in C^{\infty}(U)$. Then system (5.17) reduces to a single PDE, namely,

$$
\begin{equation*}
\frac{\partial^{2} \phi}{\partial u_{1}^{2}}-\frac{\partial^{2} \phi}{\partial u_{2}^{2}}+c \sin \phi \cos \phi=0 \tag{5.18}
\end{equation*}
$$

which is either the sine-Gordon or the wave equation for $\phi$, depending on whether $c \neq 0$ or $c=0$, respectively. Accordingly, if $\left(\epsilon_{1}, \epsilon_{2}\right)=(-1,1)$ and 5.16) holds with, say, $k=-1$, then we can write

$$
v_{1}=\cosh \phi \text { and } v_{2}=\sinh \phi
$$

for some $\phi \in C^{\infty}(U)$, in which case system (5.17) reduces to

$$
\frac{\partial^{2} \phi}{\partial u_{1}^{2}}+\frac{\partial^{2} \phi}{\partial u_{2}^{2}}+c \sinh \phi \cosh \phi=0
$$

which is either the sinh-Gordon or the Laplace equation for $\phi$, depending on whether $c \neq 0$ or $c=0$, respectively.

Corollary 5.11. Let $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{2 n-1}, c \neq \tilde{c}$, be an isometric immersion free of weak umbilic points if $c>\tilde{c}$. Then the following assertions hold:
(i) For each $x_{0} \in M_{c}^{n}$ there exists a diffeomorphism $\psi: U \rightarrow V$ from an open subset $U \subset \mathbb{R}^{n}$ onto an open simply connected neighborhood $V \subset M_{c}^{n}$ which is principal for $f$.
(ii) If $c<\tilde{c}$ and $M_{c}^{n}$ is complete, there exists a global diffeomorphism $\psi: \mathbb{R}^{n} \rightarrow \hat{M}_{c}^{n}$ onto the universal covering $\hat{M}_{c}^{n}$ of $M_{c}^{n}$ which is principal for $\hat{f}=f \circ \pi$, where $\pi: \hat{M}_{c}^{n} \rightarrow M_{c}^{n}$ is the covering map. In particular, $c \leq 0$.
(iii) The metric induced on $U$ (on $\mathbb{R}^{n}$, if $c<\tilde{c}$ and $M_{c}^{n}$ is complete) by $\psi$ is a Guichard-Darboux metric whose signature is either $(1, \ldots, 1)$ or $(-1,1, \ldots, 1)$, depending on whether $c<\tilde{c}$ or $c>\tilde{c}$, respectively.

Conversely, let ds $s^{2}=\sum_{i=1}^{n} v_{i}^{2} d u_{i}^{2}$ be a Guichard-Darboux metric of constant sectional curvature $c$ and signature $\left(\epsilon_{1}, \ldots, \epsilon_{n}\right)$ on a simply connected open subset $U \subset \mathbb{R}^{n}$ so that

$$
\begin{equation*}
\sum_{i=1}^{n} \epsilon_{i} v_{i}^{2}=\frac{1}{\tilde{c}-c} \tag{5.19}
\end{equation*}
$$

for some $\tilde{c} \in \mathbb{R}$, where $\left(\epsilon_{1}, \ldots, \epsilon_{n}\right)=(1, \ldots, 1)$ or $(-1,1, \ldots, 1)$, depending on whether $c<\tilde{c}$ or $c>\tilde{c}$, respectively. Then there exists an immersion $f: U \rightarrow \mathbb{Q}_{\tilde{c}}^{2 n-1}$ which has $d s^{2}$ as induced metric and is free of weak-umbilic points if $c>\tilde{c}$.

Proof: Let $\tilde{f}=i \circ f$, where $i: \mathbb{Q}_{\tilde{c}}^{2 n-1} \rightarrow \mathbb{Q}_{c, s}^{2 n}$ is an umbilical inclusion, with $s=0$ or 1 depending on whether $c<\tilde{c}$ or $c>\tilde{c}$, respectively. By Proposition 5.10, $\tilde{f}$ has flat normal bundle, vanishing index of relative nullity and nondegenerate first normal
spaces at any point of $M_{c}^{n}$ if $c>\tilde{c}$. Therefore, by Proposition 5.6 and Remark 5.7, for each $x_{0} \in M_{c}^{n}$ there exists a diffeomorphism $\psi: U \rightarrow V$ from an open subset $U \subset \mathbb{R}^{n}$ onto an open simply connected neighborhood $V \subset M_{c}^{n}$ of $x_{0}$ such that the frame

$$
\sqrt{\left|\left\langle\eta_{i}, \eta_{i}\right\rangle\right|} \psi_{*} \partial / \partial u_{i}, \quad 1 \leq i \leq n
$$

is orthonormal and principal, where $\eta_{1}, \ldots, \eta_{n}$ are the principal normal vector fields of $\tilde{f}$. It follows from 5.1) that this frame is also principal for $f$, thus part $(i)$ is proved.

By (5.13), if $c<\tilde{c}$ then the principal normal vector fields $\eta_{1}, \ldots, \eta_{n}$ satisfy

$$
\left\|\eta_{i}\right\| \geq \sqrt{\tilde{c}-c}, \quad 1 \leq i \leq n
$$

Therefore the assertion in part (ii) follows from assertion (a) in Proposition 5.6.
To prove part (iii), notice that the metric induced by $\psi$ is

$$
d s^{2}=\sum_{i=1}^{n} v_{i}^{2} d u_{i}^{2}, \quad \text { where } \quad v_{i}=\frac{1}{\sqrt{\left|\left\langle\eta_{i}, \eta_{i}\right\rangle\right|}} .
$$

Therefore, it follows from 5.13 that $d s^{2}$ is a Guichard-Darboux metric of signature $\left(\epsilon_{1}, \ldots, \epsilon_{n}\right)$, where $\left(\epsilon_{1}, \ldots, \epsilon_{n}\right)=(1, \ldots, 1)$ or $(-1,1, \ldots, 1)$, depending on whether $c<\tilde{c}$ or $c>\tilde{c}$, respectively.

Conversely, let $d s^{2}=\sum_{i=1}^{n} v_{i}^{2} d u_{i}^{2}$ be a Guichard-Darboux metric of constant sectional curvature $c$ and signature $\left(\epsilon_{1}, \ldots, \epsilon_{n}\right)$ on a simply connected open subset $U \subset \mathbb{R}^{n}$ such that

$$
\sum_{i=1}^{n} \epsilon_{i} v_{i}^{2}=\frac{1}{\tilde{c}-c}
$$

for some $\tilde{c} \in \mathbb{R}$, where $\left(\epsilon_{1}, \ldots, \epsilon_{n}\right)=(1, \ldots, 1)$ or $(-1,1, \ldots, 1)$, depending on whether $c<\tilde{c}$ or $c>\tilde{c}$, respectively.

Differentiating equation $(i)$ of (5.17) with respect to $u_{i}$ gives

$$
\begin{equation*}
\frac{\partial^{2} v_{i}}{\partial u_{i} \partial u_{j}}=\frac{\partial h_{j i}}{\partial u_{i}} v_{j}+h_{j i} h_{i j} v_{i} . \tag{5.20}
\end{equation*}
$$

On the other hand, differentiating (iv) with respect to $u_{j}$ and using using (i), (iii) and (iv) yield

$$
\begin{align*}
\epsilon_{i} \frac{\partial^{2} v_{i}}{\partial u_{j} \partial u_{i}}= & -\sum_{k \neq i} \epsilon_{k} \frac{\partial h_{i k}}{\partial u_{j}} v_{k}-\sum_{k \neq i} \epsilon_{k} h_{i k} \frac{\partial v_{k}}{\partial u_{j}} \\
= & -\sum_{k \neq i, j} \epsilon_{k} h_{i j} h_{j k} v_{k}-\epsilon_{j} \frac{\partial h_{i j}}{\partial u_{j}} v_{j}-\sum_{k \neq i, j} \epsilon_{k} h_{i k} h_{j k} v_{j} \\
& +\sum_{k \neq i, j} \epsilon_{k} h_{i j} h_{j k} v_{k}+\epsilon_{i} h_{i j} h_{j i} v_{i} \\
= & -\epsilon_{j} \frac{\partial h_{i j}}{\partial u_{j}} v_{j}-\sum_{k \neq i, j} \epsilon_{k} h_{i k} h_{j k} v_{j}+\epsilon_{i} h_{i j} h_{j i} v_{i} . \tag{5.21}
\end{align*}
$$

Comparing (5.20) and 5.21 implies that $h=\left(h_{i j}\right)$ satisfies

$$
\epsilon_{i} \frac{\partial h_{i j}}{\partial u_{j}}+\epsilon_{j} \frac{\partial h_{j i}}{\partial u_{i}}+\sum_{k \neq i, j} \epsilon_{k} h_{i k} h_{j k}=0 .
$$

By Proposition 5.8 (see also Remark 5.9), there exists an immersion $\tilde{f}: U \rightarrow \mathbb{Q}_{c, s}^{2 n}$ with flat normal bundle and vanishing index of relative nullity with $d s^{2}=\sum_{i=1}^{n} v_{i}^{2} d u_{i}^{2}$ as its induced metric and whose second fundamental form and normal connection are given by (5.9) and (5.10), respectively.

Since (5.19) holds, the principal normals $\eta_{1}, \ldots, \eta_{n}$ of $\tilde{f}$ satisfy (5.13). It follows from Proposition 5.10 that there exists an isometric immersion $f:\left(U, d s^{2}\right) \rightarrow \mathbb{Q}_{\tilde{c}}^{2 n-1}$, free of weak-umbilic points if $c>\tilde{c}$, such that $\tilde{f}=i \circ f$, where $i: \mathbb{Q}_{\tilde{c}}^{2 n-1} \rightarrow \mathbb{Q}_{c, s}^{2 n}$ is an umbilical inclusion.

A diffeomorphism $\psi: \Pi_{j=1}^{n} I_{j} \rightarrow V$ of a product of open intervals $I_{j} \subset \mathbb{R}$ onto an open subset $V$ of a Riemannian manifold $M^{n}$ is called a Tschebyscheff net if the parameter curves are parametrized by arc-length.

Corollary 5.12. If $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{2 n-1}$ is an isometric immersion with $c<\tilde{c}$, then for each $x_{0} \in M_{c}^{n}$ there exists a Tschebyscheff net $\psi: \Pi_{j=1}^{n} I_{j} \rightarrow V$ onto an open neighborhood $V \subset M_{c}^{n}$ of $x_{0}$ such that the coordinate vectors $\psi_{*} \partial / \partial w_{i}, 1 \leq i \leq n$, are unit-length asymptotic vectors for $f$. Moreover, if $c \leq 0$ and $M_{c}^{n}$ is complete and simply connected then $\psi$ can be taken as a global Tschebyscheff net $\psi: \mathbb{R}^{n} \rightarrow M_{c}^{n}$.

Proof: Let $u_{1}, \ldots, u_{n}$ be the principal coordinates given by Corollary 5.11, with respect to which the induced metric is $d s^{2}=\sum_{i} v_{i}^{2} d u_{i}^{2}$, with

$$
\begin{equation*}
\sum_{i=1}^{n} v_{i}^{2}=\frac{1}{\tilde{c}-c} \tag{5.22}
\end{equation*}
$$

Choose any invertible $n \times n$-matrix $\epsilon=\left(\epsilon_{i j}\right)$ with $\epsilon_{i j} \in\{-1,1\}$ for all $1 \leq i, j \leq n$, and define $w_{1}, \ldots, w_{n}$ by

$$
u_{j}=\sqrt{\tilde{c}-c} \sum_{k=1}^{n} \epsilon_{k j} w_{k}, \quad 1 \leq j \leq n .
$$

Then

$$
\frac{\partial}{\partial w_{i}}=\sqrt{\tilde{c}-c} \sum_{k=1}^{n} \epsilon_{i k} \frac{\partial}{\partial u_{k}}
$$

and hence $\partial / \partial w_{i}$ are unit-length vectors by $(5.22)$.
Moreover, if $\tilde{f}=i \circ f$, where $i: \mathbb{Q}_{\tilde{c}}^{2 n-1} \rightarrow \mathbb{Q}_{c}^{2 n}$ is an umbilical inclusion, then

$$
\alpha^{\tilde{f}}\left(\partial / \partial u_{i}, \partial / \partial u_{j}\right)=\frac{\delta_{i j}}{\left\langle\eta_{i}, \eta_{i}\right\rangle} \eta_{i},
$$

where $\eta_{1}, \ldots, \eta_{n}$ are the principal normal vector fields of $\tilde{f}$. Therefore

$$
\alpha^{\tilde{f}}\left(\partial / \partial w_{i}, \partial / \partial w_{i}\right)=\eta=\sum_{j=1}^{n} \frac{1}{\left\langle\eta_{j}, \eta_{j}\right\rangle} \eta_{j}, \quad 1 \leq i \leq n,
$$

and, from the proof of Proposition 5.10, $\eta=(\tilde{c}-c)^{-1 / 2} \xi$, where $\xi$ is one of the unit vector fields normal to $i$. By (5.1), this means that

$$
\alpha^{f}\left(\partial / \partial w_{i}, \partial / \partial w_{i}\right)=0, \quad 1 \leq i \leq n,
$$

that is, $\partial / \partial w_{i}, 1 \leq i \leq n$, is an asymptotic vector field for $f$.
Remarks 5.13. (i) It is a classical theorem of Hilbert that there exists no isometric immersion $f: M_{c}^{2} \rightarrow \mathbb{R}^{3}$ if $c<0$ and $M_{c}^{2}$ is complete. More generally, a complete $M_{c}^{2}$ with $c<0$ cannot be isometrically immersed in $\mathbb{Q}_{\tilde{c}}^{3}$ if $c<\tilde{c}$. The standard proofs of this result consist in showing either that there is no global nonvanishing solution $\phi: \mathbb{R}^{2} \rightarrow \mathbb{R}$ of the sine-Gordon equation (5.18) or that there does not exist a global Tschebyscheff net $\psi: \mathbb{R}^{2} \rightarrow \mathbb{H}^{2}$ (see [317], vol. III for proofs of both assertions).
(ii) If $n \geq 3$, according to Theorem 5.1 there exists no isometric immersion of $M_{c}^{n}$ into $\mathbb{Q}_{\tilde{c}}^{n+p}$ if $c<\tilde{c}$ and $p \leq n-2$, even local ones. On the other hand, plenty of local examples exist if $p=n-1$ by the results of this section (see Exercise 5.2 for an explicit example).
(iii) It is a major open problem on the subject whether there exist isometric immersions $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{2 n-1}$ with $n \geq 3, c<\min \{0, \tilde{c}\}$ and $M_{c}^{n}$ complete. The results of this chapter show that the existence of such an isometric immersion with $M_{c}^{n}$ simply connected is equivalent both to the existence of a global Tschebyscheff net on $\mathbb{H}_{c}^{n}$ and to the existence of a global solution $\left(v_{1}, \ldots, v_{n}\right)$ of system (5.17) on $\mathbb{R}^{n}$ with $v_{i}(u) \neq 0$, $1 \leq i \leq n$, at any point $u \in \mathbb{R}^{n}$.
(iv) In the next section it is shown that if $M_{c}^{n}$ complete and there exists an isometric immersion $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{2 n-1}$ with $n \geq 3$ and $c<\min \{0, \tilde{c}\}$, then $M_{c}^{n}$ has to be simply connected.

### 5.5 Complete hyperbolic submanifolds

In this section we provide a proof of the nonexistence of an isometric immersion with flat normal bundle $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+p}, c<\tilde{c}$, of a complete non-simply connected Riemannian manifold of constant sectional curvature $c<0$. Recall that, by Theorem 5.5. flatness of the normal bundle is automatic if $p=n-1$.

Theorem 5.14. If there exists an isometric immersion $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+p}$ with flat normal bundle of a complete $M_{c}^{n}$ with $p \geq n-1$ and $c<\min \{0, \tilde{c}\}$, then $M_{c}^{n}$ is simply connected.

Proof: Let $i: \mathbb{Q}_{\tilde{c}}^{n+p} \rightarrow \mathbb{Q}_{c}^{n+p+1}$ be an umbilical inclusion, and set $\tilde{f}=i \circ f$. Since the second fundamental form of $\tilde{f}$ is

$$
\tilde{\alpha}=i_{*} \alpha+\sqrt{\tilde{c}-c}\langle,\rangle \xi,
$$

where $\xi$ is one of the unit vector fields normal to $i$, also $\tilde{f}$ has flat normal bundle. Moreover, it has vanishing index of relative nullity, for

$$
\langle\tilde{\alpha}, \xi\rangle=\sqrt{\tilde{c}-c}\langle,\rangle .
$$

On the other hand, by Proposition 5.10 the principal normal vector fields $\tilde{\eta}_{1}, \ldots, \tilde{\eta}_{n}$ of $\tilde{f}$ satisfy condition 5.13. Thus

$$
\left\|\tilde{\eta}_{i}\right\| \geq \sqrt{\tilde{c}-c}>0, \quad 1 \leq i \leq n
$$

Hence, by Proposition 5.6, the third fundamental form III of $\tilde{f}$ defines a complete flat metric on $M_{c}^{n}$.

Let $\pi$ : $\hat{M}_{c}^{n} \rightarrow M_{c}^{n}$ be the universal covering map and set $\hat{f}=\tilde{f} \circ \pi$. Suppose that there exists a nontrivial deck transformation $\gamma$ of $\pi$. Then $\gamma$ is an isometry with respect to the hyperbolic metric on $\hat{M}_{c}^{n}$ obtained by lifting the metric of $M_{c}^{n}$. Since $\hat{f} \circ \gamma=\hat{f}$, at any point $x \in M_{c}^{n}$ the normal spaces $N_{\hat{f}} \hat{M}(x)$ and $N_{\hat{f}} \hat{M}(\gamma(x))$ coincide, and for any $\xi \in N_{\hat{f}} \hat{M}(x)$ we have

$$
A_{\xi}^{\hat{f}} \circ \gamma_{*}=\gamma_{*} \circ A_{\xi}^{\hat{f}},
$$

or equivalently,

$$
\begin{equation*}
\gamma^{*} \hat{\alpha}=\hat{\alpha} \tag{5.23}
\end{equation*}
$$

In particular, given $x \in \hat{M}_{c}^{n}$ and an orthonormal basis $\xi_{1}, \ldots, \xi_{p+1}$ of $N_{\hat{f}} \hat{M}(x)$,

$$
\begin{aligned}
\operatorname{III}\left(\gamma_{*} X, \gamma_{*} Y\right) & =\sum_{r=1}^{p+1}\left\langle A_{\xi_{r}}^{\hat{f}} \gamma_{*} X, A_{\xi_{r}}^{\hat{f}} \gamma_{*} Y\right\rangle \\
& =\sum_{r=1}^{p+1}\left\langle\gamma_{*} A_{\xi_{r}}^{\hat{f}} X, \gamma_{*} A_{\xi_{r}}^{\hat{f}} Y\right\rangle \\
& =\sum_{r=1}^{p+1}\left\langle A_{\xi_{r}}^{\hat{f}} X, A_{\xi_{r}}^{\hat{f}} Y\right\rangle \\
& =I I I(X, Y)
\end{aligned}
$$

for all $X, Y \in T_{x} M$. Thus $\gamma$ is also an isometry with respect to III. Moreover, by the last assertion in Proposition 5.6, there exists a global diffeomorphism $\psi: \mathbb{R}^{n} \rightarrow \hat{M}_{c}^{n}$, which is an isometry with respect to the standard metric on $\mathbb{R}^{n}$ and the metric III on $\hat{M}_{c}^{n}$.

Let $\hat{\gamma}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be defined by

$$
\psi \circ \hat{\gamma}=\gamma \circ \psi
$$

Then $\hat{\gamma}$ is an isometry with respect to both the standard metric on $\mathbb{R}^{n}$ and the hyperbolic metric induced by $\psi$ from the metric of $\hat{M}_{c}^{n}$. On the other hand, it follows from (5.23) that the bases

$$
\left\|\eta_{1}(x)\right\| \gamma_{*} \psi_{*} \partial / \partial u_{1}(x), \ldots,\left\|\eta_{n}(x)\right\| \gamma_{*} \psi_{*} \partial / \partial u_{n}(x)
$$

and

$$
\left\|\eta_{1}(\gamma(x))\right\| \psi_{*} \partial / \partial u_{1}(\gamma(x)), \ldots,\left\|\eta_{n}(\gamma(x))\right\| \psi_{*} \partial / \partial u_{n}(\gamma(x))
$$

of $T_{\gamma(x)} \hat{M}$ coincide up to signs and permutations, by the uniqueness, up to signs, of a basis of unit-length vectors of $T_{\gamma(x)} \hat{M}$ with respect to which the second fundamental form of $\hat{f}$ is given by 5.5 . Since $\gamma_{*} \psi_{*}=\psi_{*} \hat{\gamma}_{*}$, it follows that $\hat{\gamma}_{*}$ takes

$$
\partial / \partial u_{1}(x), \ldots, \partial / \partial u_{n}(x) \text { into } \pm \partial / \partial u_{1}(\gamma(x)), \ldots, \pm \partial / \partial u_{n}(\gamma(x)),
$$

possibly permuting its elements. But $\hat{\gamma}$ is an isometry of $\mathbb{R}^{n}$ with respect to the standard metric, hence $\gamma_{*}$ is given by an orthogonal matrix whose columns are obtained by permuting the elements of the canonical basis of $\mathbb{R}^{n}$ and possibly changing some of their signs. There are precisely $n!2^{n}$ such matrices, hence there exists a positive integer $m$ such that $\gamma_{*}^{m}$ is the identity endomorphism. Thus

$$
\gamma^{m}(x)=x+V
$$

for some $V \in \mathbb{R}^{n}$, and $V$ must be nonzero, for otherwise $\gamma$ would have a fixed point.
Therefore

$$
d_{0}(x, \gamma(x))=\|V\|
$$

for any $x \in \mathbb{R}^{n}$, where $d_{0}$ is the standard Euclidean distance. Hence

$$
d(x, \gamma(x)) \leq \frac{1}{\sqrt{\tilde{c}-c}}\|V\|
$$

for any $x \in \mathbb{R}^{n}$, where $d$ is the hyperbolic distance. But there is no nontrivial isometry of the hyperbolic space with this property. This is a contradiction which shows that no such nontrivial $\gamma$ can exist.

### 5.6 Notes

Surfaces with constant Gauss curvature in space forms have been extensively studied by differential geometers in the nineteenth century and the beginning of the last century; see Bianchi [36] and the references therein, as well as Gálvez [198] and Spivak [317] for surveys of results of a global nature on the subject.

The study of isometric immersions of space forms into space forms with higher dimension and codimension was initiated by Cartan [67]. In particular, he proved part (i) of Theorem 5.1 by means of his theory of exteriorly orthogonal quadratic forms. Part (ii) was first proved by O'Neill [278]. The proof given here was taken from Moore [255], where the theory of flat bilinear forms was introduced.

Theorem 5.2 was proved by Cartan [67] for symmetric flat bilinear forms with respect to positive definite inner products, and by Moore [255] in the case of symmetric flat bilinear forms with respect to Lorentzian inner products. As a consequence of this result, Moore proved Theorem 5.5 in the case $c>\tilde{c}$, extending the dual result for $c<\tilde{c}$ previously obtained by Cartan in [67]. In the same paper, Moore was also able to describe what else may happen if the extra assumption needed in the Lorentzian version of Theorem 5.2 is dropped.

The existence of principal and asymptotic coordinates for isometric immersions $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{2 n-1}, c<\tilde{c}$, was shown by Moore [253], who observed in [255] that principal coordinates also exist for isometric immersions $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{2 n-1}, c>\tilde{c}$, that are free of weak-umbilic points.

The correspondence between isometric immersions $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{2 n-1}, c<\tilde{c}$, and solutions of the associated systems of partial differential equations was independently established by Aminov [17], [18] and Tenenblat-Terng [325] for $0=c<\tilde{c}$, and then extended to the case $0=c<\tilde{c}$ by Tenenblat [324]. The remaining cases have been considered by Dajczer-Tojeiro [137], [142].

A simple example of an isometric immersion $f: \mathbb{R}^{n} \rightarrow \mathbb{S}^{2 n-1}$ is the $n$-dimensional Clifford torus (see Exercise 1.7). An explicit example by Schur [314] of an isometric immersion $f: M_{c}^{n} \rightarrow \mathbb{R}^{2 n-1}$ with $c<0$ is the higher dimensional version of the pseudosphere given in Exercise 5.2.

Useful tools to produce further examples of isometric immersions of space forms into space forms are the Ribaucour and Bäcklund transformations. The former was extended from surface theory by Dajczer-Tojeiro [142], [143], and the latter by TenenblatTerng [325] for Euclidean submanifolds and by Tenenblat [324] for submanifolds of the sphere and the hyperbolic space.

Two $n$-dimensional submanifolds in $\mathbb{R}^{m}$ are said to be related by a Ribaucour transformation if they envelop a common congruence of $n$-spheres in $\mathbb{R}^{m}$ in such a way that their shape operators with respect to corresponding normal directions commute. Each of the two envelopes is said to be a Ribaucour transform of the other. Ribaucour transforms of a given submanifold can be parametrized in terms of the latter and the solutions of a linear system of partial differential equations. Given a submanifold of a certain class, one can then look for its Ribaucour transforms that belong to the same class. This yields a process to generate a family of new submanifolds within a certain class by starting with a given element of that class. This was carried out in [142] for the class of submanifolds of constant sectional curvature. Using this approach, a large family of further local explicit isometric immersions of the hyperbolic space $\mathbb{H}^{n}$ into $\mathbb{R}^{2 n-1}$ was obtained in [142], as well as of local isometric immersions of $\mathbb{S}^{n}$ into $\mathbb{R}^{2 n-1}$ that are free of weak-umbilic points.

In Guimarães-Tojeiro [204], based on a vectorial version of the Ribaucour transformation developed by Dajczer-Florit-Tojeiro [106], it was shown that from $k$ initial Ribaucour transforms of a given submanifold of constant curvature that also have the same constant curvature, one can produce a whole $k$-dimensional cube, all of whose remaining $2^{k}-(k+1)$ vertices are submanifolds with the same constant curvature given by explicit algebraic formulas.

Methods of soliton theory and integrable systems have been successfully applied to the study of isometric immersions of space forms into space forms. For instance, see [44], [177], [326] and [327].

Concerning global results, an important open problem remains is whether there exists a global isometric immersion of $\mathbb{H}^{n}$ into $\mathbb{R}^{2 n-1}$. This would extend the classical result of Hilbert on the nonimmersibility of the hyperbolic plane $\mathbb{H}^{2}$ in $\mathbb{R}^{3}$. Examples of isometric immersions of $\mathbb{H}^{n}$ into $\mathbb{R}^{4 n-3}$ have been constructed by Henke [216]. See Henke-Nettekoven [217] for isometric embeddings of $\mathbb{H}^{n}$ into $\mathbb{R}^{6 n-6}$ whose image is the graph of a smooth map $g: \mathbb{R}^{n} \rightarrow \mathbb{R}^{5 n-6}$. Blanusa [37] produced examples of isometric immersions of $\mathbb{H}^{n}$ into $\mathbb{S}^{6 n-4}$. For other related results see Mirandola-Vitório [248]. Theorem 5.14 is due to Nikolayevsky [266] and generalizes previous results by Pedit [289], Xavier [350] and Dajczer-Tojeiro [137].

The example in Exercise 5.2 of a $n$-dimensional submanifold of constant negative sectional curvature in Euclidean space $\mathbb{R}^{2 n+1}$ belongs to a family of multi-rotational submanifolds described by Dajczer-Tojeiro [137], whereas the examples in Exercise 5.5 of local isometric immersions of $\mathbb{S}_{c}^{3}$ into $\mathbb{R}^{5}$, with $0<c<1$, that are free of weak-umbilic points, have been taken from Manfio-Tojeiro [241].

### 5.7 Exercises

Exercise 5.1. Let $\beta: V^{n} \times V^{n} \rightarrow W^{n}$ be a flat symmetric bilinear form with respect to a Lorentzian inner product on $W^{n}$. Assume that there exists a light-like vector $\zeta \in W^{n}$ such that

$$
\begin{equation*}
\langle\beta(X, Y), \zeta\rangle=-\langle X, Y\rangle \tag{5.24}
\end{equation*}
$$

for all $X, Y \in V^{n}$. Given a time-like vector $\xi \in W^{n}$ with $\langle\xi, \zeta\rangle<0$, show that the bilinear form $\phi: V^{n} \times V^{n} \rightarrow \mathbb{R}$ given by $\phi=\langle\beta, \xi\rangle$ has at least one negative eigenvalue.
Hint: First notice that $\mathcal{N}(\beta)=\{0\}$ by (5.24). If $\mathcal{S}(\beta)$ is a nondegenerate subspace of $W^{n}$, conclude from Lemmas 4.10 and 4.14 that $\mathcal{S}(\beta)=W^{n}$. Then use Theorem 5.2 and (5.24) to show that there exists an orthonormal basis $X_{1}, \ldots, X_{n}$ of $V^{n}$ and an orthogonal basis $\eta_{1}, \ldots, \eta_{n}$ of $W^{n}$ such that

$$
\beta\left(X_{i}, X_{j}\right)=\delta_{i j} \eta_{i}, \quad 1 \leq i, j \leq n
$$

Exactly one of the vectors $\eta_{i}$ is time-like, say $\eta_{1}$, and $\left\langle\eta_{1}, \zeta\right\rangle=-1$ by (5.24). Conclude that

$$
\phi\left(X_{1}, X_{1}\right)=\left\langle\eta_{1}, \xi\right\rangle<0 .
$$

Now assume that $\mathcal{S}(\beta)$ is degenerate. Show that there exists a light-like vector $\rho \in W^{n}$ with $\langle\rho, \zeta\rangle=-1$ such that $\operatorname{dim} \mathcal{N}(\beta-\langle,\rangle \rho) \geq 2$. Now notice that

$$
\phi(X, X)=\langle\rho, \xi\rangle<0
$$

for any unit vector $X \in \mathcal{N}(\beta-\langle,\rangle \rho)$.

Exercise 5.2. Choose nonzero real numbers $a_{i}, 1 \leq i \leq n-1$, such that $\sum a_{i}^{2}=1$, and define an immersion from

$$
D=\left\{\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{R}^{n}: x_{n}<0\right\}
$$

into $\mathbb{R}^{2 n-1}$ by

$$
\begin{array}{rlr}
y_{2 i-1} & =a_{i} e^{x_{n}} \cos \left(x_{i} / a_{i}\right) \\
y_{2 i} & =a_{i} e^{x_{n}} \sin \left(x_{i} / a_{i}\right) \quad 1 \leq i \leq n-1 \\
y_{2 n-1} & =\int_{0}^{x_{n}}\left(1-e^{2 u}\right)^{\frac{1}{2}} d u .
\end{array}
$$

Show that the induced metric on $D$ is of constant negative curvature but it is not complete.

Exercise 5.3. Show that there exists an isometric immersion $f: \mathbb{R}^{n} \rightarrow \mathbb{Q}_{c}^{2 n-1}, c \neq 0$, free of weak-umbilic points if $c<0$.
Hint: Use Exercise 1.7
Exercise 5.4. Let $\phi: \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$
\phi(s)=\int_{0}^{s} \frac{\sqrt{d+(1-d) e^{2 \tau}}}{e^{2 \tau}+d} d \tau
$$

where $0<d<1$. Define $g, h: \mathbb{R} \rightarrow \mathbb{R}$ by

$$
g(s)=\sqrt{e^{2 s}+d} \cosh \phi \text { and } h(s)=\sqrt{e^{2 s}+d} \sinh \phi .
$$

Let $\mathbb{L}^{2 n}$ be the Lorentz space endowed with the inner product of signature $(-1,1, \ldots, 1)$ and let $F: \mathbb{R}^{n} \rightarrow \mathbb{L}^{2 n}$ be defined by

$$
F\left(u_{1}, u_{2}, \ldots, u_{n}\right)=\left(g\left(u_{1}\right), h\left(u_{1}\right), a_{1} e^{u_{1}+i\left(u_{2} / a_{1}\right)}, \ldots, a_{n-1} e^{u_{1}+i\left(u_{n-1} / a_{n-1}\right)}\right),
$$

where $\sum_{i=1}^{n-1} a_{i}^{2}=1$.
(i) Show that the Riemannian manifold

$$
M_{-1}^{n}=\mathbb{R} \times_{a_{1} e^{s}} \mathbb{S}^{1} \times_{a_{2} e^{s}} \cdots \times_{a_{n-1} e^{s}} \mathbb{S}^{1}
$$

where $a_{i}>0$ for $1 \leq i \leq n$, is complete and has constant sectional curvature -1 .
(ii) Show that $F$ induces an isometric embedding $f: M_{-1}^{n} \rightarrow \mathbb{H}_{-1 / d}^{2 n-1}$ free of weakumbilic points.

Exercise 5.5. Let $g: M^{2} \rightarrow \mathbb{S}^{3} \subset \mathbb{R}^{4}$ be a flat surface and let $g_{s}: M^{2} \rightarrow \mathbb{S}^{3} \subset \mathbb{R}^{4}$ be the family of its parallel surfaces, that is,

$$
g_{s}(x)=\cos s g(x)+\sin s N(x)
$$

where $N$ is a unit normal vector field to $g$ in $\mathbb{S}^{3}$. Define $f: M^{2} \times I \rightarrow \mathbb{R}^{5}=\mathbb{R}^{4} \times \mathbb{R}$, where $0 \in I \subset \mathbb{R}$ is an open interval, by

$$
f(x, s)=g_{s}(x)+b s \partial / \partial t
$$

where $b \in \mathbb{R}$ and $\partial / \partial t$ is a unit vector spanning the $\mathbb{R}$-factor in the orthogonal decomposition $\mathbb{R}^{5}=\mathbb{R}^{4} \times \mathbb{R}$.
(i) Prove that the metric induced by $f$ on the subset $V \subset M^{2} \times \mathbb{R}$ of regular points has constant sectional curvature $1 / \sqrt{1+b^{2}}$.
(ii) Show that $f$ has no weak-umbilic points and that $f(V) \subset \mathbb{S}^{3} \times \mathbb{R} \subset \mathbb{R}^{5}$.
(iii) Conclude that the map $f:(0, \pi / 2) \times(0, \pi / 2) \times \mathbb{R} \rightarrow \mathbb{R}^{5}$ given by

$$
f\left(s, t_{1}, t_{2}\right)=\left(\cos s \cos t_{1}, \cos s \sin t_{1}, \cos s \cos t_{2}, \cos s \sin t_{2}, b s\right), \quad b \in \mathbb{R}
$$

induces an isometric immersion of $\mathbb{S}_{c}^{3} \backslash X$ into $\mathbb{R}^{5}$ without weak-umbilic points, where $c=1 / \sqrt{1+b^{2}}$ and $X$ is the union of two circles of unit radii centered at the origin in orthogonal planes.

Exercise 5.6. Let $f: M_{c}^{n} \rightarrow \mathbb{Q}_{c, s}^{n+p}, s \in\{0,1\}$, be an isometric immersion with flat normal bundle and vanishing index of relative nullity. If $s=1$, assume further that the first normal spaces of $f$ are nondegenerate subspaces of the corresponding normal spaces at any point. Let $\left(u_{1}, \ldots, u_{n}\right)$ be the principal coordinates given locally on $M_{c}^{n}$ by Proposition 5.6 or Remark 5.7, depending on whether $s=0$ or $s=1$, respectively. Let $(v, h, V)$ be the triple associated with $f$ with respect to ( $u_{1}, \ldots, u_{n}$ ) and to any parallel orthonormal normal frame $\xi_{1}, \ldots, \xi_{p}$, with $\left\langle\xi_{j}, \xi_{j}\right\rangle=\epsilon_{j}$ for all $1 \leq j \leq n$, where $\left(\epsilon_{1}, \ldots, \epsilon_{n}\right)=(1, \ldots, 1)$ if $s=0$ and $\left(\epsilon_{1}, \ldots, \epsilon_{n}\right)=(-1,1, \ldots, 1)$ if $s=1$ (see Exercise 1.38).
(i) Show that $V$ takes values in $\mathbb{O}_{s}(n \times p)$, where $\mathbb{O}_{s}(n \times p)$ denotes the subspace of $M_{p \times n}(\mathbb{R})$ of all matrices that satisfy $V^{t} J V=J$, where $J=\operatorname{diag}\left(\epsilon_{1}, \ldots, \epsilon_{n}\right)$. Show also that $(v, h, V)$ satisfies the system of partial differential equations

$$
\left\{\begin{array}{l}
\text { (i) } \frac{\partial v_{i}}{\partial u_{j}}=h_{j i} v_{j}  \tag{5.25}\\
\text { (ii) } \frac{\partial h_{i j}}{\partial u_{i}}+\frac{\partial h_{j i}}{\partial u_{j}}+\sum_{k} h_{k i} h_{k j}+c v_{i} v_{j}=0 \\
\text { (iii) } \frac{\partial h_{i k}}{\partial u_{j}}=h_{i j} h_{j k} \\
\text { (iv) } \frac{\partial V_{i r}}{\partial u_{j}}=h_{j i} V_{j r}, \quad i \neq j \neq k \neq i
\end{array}\right.
$$

(ii) Show that, conversely, if $(v, h, V)$ is a solution of (5.25) on an open simply connected subset $U \subset \mathbb{R}^{n}$ such that $V$ takes values in $\mathbb{O}_{s}(n \times p)$ and $v_{i} \neq 0$ at
any point, then there exist an immersion $f: U \rightarrow \mathbb{Q}_{c, s}^{n+p}$ with flat normal bundle and vanishing index of relative nullity and a parallel orthonormal normal frame $\xi_{1}, \ldots, \xi_{p}$, with $\left\langle\xi_{j}, \xi_{j}\right\rangle=\epsilon_{j}$ for all $1 \leq j \leq n$, with respect to which $f$ has $(v, h, V)$ as associated triple and whose induced metric has constant sectional curvature $c$.

Hint for $(i)$ : Let $\eta_{1}, \ldots, \eta_{n}$ be the principal normal vector fields of $f$. By the assumption for $s=1,\left\langle\eta_{i}, \eta_{i}\right\rangle \neq 0$ for all $1 \leq i \leq n$, and by Proposition 5.6 and Remark 5.7, the coordinates $\left(u_{1}, \ldots, u_{n}\right)$ are such that

$$
v_{i}=\frac{1}{\sqrt{\left|\left\langle\eta_{i}, \eta_{i}\right\rangle\right|}}
$$

for all $1 \leq i \leq n$. Moreover, $\left\langle\eta_{i}, \eta_{j}\right\rangle=0$ if $1 \leq i \neq j \leq n$ by the Gauss equation. Let $\xi_{1}, \ldots, \xi_{p}$ be a parallel orthonormal normal frame with $\left\langle\xi_{j}, \xi_{j}\right\rangle=\epsilon_{j}$ for all $1 \leq j \leq n$, where $\left(\epsilon_{1}, \ldots, \epsilon_{n}\right)=(1, \ldots, 1)$ if $s=0$ and $\left(\epsilon_{1}, \ldots, \epsilon_{n}\right)=(-1,1, \ldots, 1)$ if $s=1$. Show that

$$
\begin{equation*}
\eta_{i}=\sum_{r=1}^{p} \frac{V_{i r}}{v_{i}} \xi_{r}, \quad 1 \leq i \leq n \tag{5.26}
\end{equation*}
$$

and check that the equations

$$
\begin{equation*}
\left\langle\eta_{i}, \eta_{j}\right\rangle=\epsilon_{i} \delta_{i j} v_{i}^{2}, \quad 1 \leq i, j \leq n \tag{5.27}
\end{equation*}
$$

imply that $V$ takes values in $\mathbb{O}_{s}(n \times p)$. Now use Exercise 1.38 and verify that system (1.51) reduces to (5.25).

Hint for (ii): Use Exercise 1.38 to obtain an isometric immersion $f: U \rightarrow \mathbb{Q}_{s}^{n+p}(c)$ that has $(v, h, V)$ as associated triple with respect to a parallel orthonormal normal frame with $\left\langle\xi_{j}, \xi_{j}\right\rangle=\epsilon_{j}$ for all $1 \leq j \leq n$, where $\left(\epsilon_{1}, \ldots, \epsilon_{n}\right)=(1, \ldots, 1)$ if $s=0$ and $\left(\epsilon_{1}, \ldots, \epsilon_{n}\right)=(-1,1, \ldots, 1)$ if $s=1$. Show that the principal normal vector fields of $f$ are given by (5.26), and prove that they satisfy (5.27) by using that $V$ takes values in $\mathbb{O}_{s}(n \times p)$. Conclude that $f$ has vanishing index of relative nullity, that the first normal spaces of $f$ are nondegenerate subspaces of the corresponding normal spaces if $s=1$, and that the induced metric $d s^{2}=\sum_{i=1}^{n} v_{i}^{2} d u_{i}^{2}$ has constant sectional curvature $c$.

Exercise 5.7. Let $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{2 n-1}$ be an isometric immersion with $c \neq \tilde{c}$. If $c>\tilde{c}$, assume that $f$ is free of weak-umbilic points. Let $(v, h, V)$ be the triple associated with $f$ with respect to the local principal coordinates $\left(u_{1}, \ldots, u_{n}\right)$ given by Corollary 5.11 and to any parallel orthonormal normal frame $\xi_{1}, \ldots, \xi_{n-1}$. Define the augmented matrix $\widehat{V} \in M_{n}(\mathbb{R})$ by

$$
\widehat{V}_{i r}=V_{i r}, \quad 1 \leq r \leq p, \quad \text { and } \widehat{V}_{i n}=\sqrt{|c-\tilde{c}|} v_{i}
$$

(i) Show that the augmented matrix $\widehat{V}$ takes values in $\mathbb{O}_{s}(n)$, where $s=0$ or $s=1$ depending on whether $\tilde{c}>c$ or $\tilde{c}<c$, respectively, and that the triple $(v, h, V)$ satisfies (5.25).
(ii) Show that, conversely, if $(v, h, V)$ is a solution of (5.25) on an open simply connected subset $U \subset \mathbb{R}^{n}$ such that $v_{i} \neq 0$ everywhere and such that the augmented matrix $\widehat{V}$ takes values in $\mathbb{O}_{s}(n)$, then there exist an immersion $f: U \rightarrow \mathbb{Q}_{\tilde{c}}^{2 n-1}$, where $\tilde{c}>c$ or $\tilde{c}<c$ depending on whether $s=0$ or $s=1$, respectively, and a parallel orthonormal normal frame $\xi_{1}, \ldots, \xi_{n-1}$ with respect to which $f$ has $(v, h, V)$ as associated triple and whose induced metric has constant sectional curvature $c$.
Hint for $(i)$ : Let $i: \mathbb{Q}_{\tilde{c}}^{2 n-1} \rightarrow \mathbb{Q}_{c, s}^{2 n}$ be an umbilical inclusion and define $\tilde{f}=i \circ f$. If $c>\tilde{c}$, use the assumption that $f$ is free of weak-umbilic points and the fact that the second fundamental forms of $f$ and $\tilde{f}$ are related by

$$
\begin{equation*}
\tilde{\alpha}=\alpha+\sqrt{|\tilde{c}-c|}\langle,\rangle \xi, \tag{5.28}
\end{equation*}
$$

where $\xi$ is one of the vector fields normal to $i$ with $\langle\xi, \xi\rangle=(\tilde{c}-c) /|\tilde{c}-c|$, to show that the first normal spaces of $\tilde{f}$ are nondegenerate subspaces of the corresponding normal spaces at any point. If $(v, h, V)$ is the triple associated with $f$ with respect to the coordinates $\left(u_{1}, \ldots, u_{n}\right)$ and a parallel orthonormal normal frame $\xi_{1}, \ldots, \xi_{n-1}$, show that $(v, h, \widehat{V})$ is the triple associated with $\tilde{f}$ with respect to $\left(u_{1}, \ldots, u_{n}\right)$ and the frame $i_{*} \xi_{1}, \ldots, i_{*} \xi_{p}, \xi$. Conclude from the preceding exercise that $\widehat{V}$ takes values in $\mathbb{O}_{s}(n)$, where $s=0$ or $s=1$ depending on whether $\tilde{c}>c$ or $\tilde{c}<c$, respectively, and that the triple $(v, h, V)$ satisfies (5.25).
Hint for (ii): Use Exercise 5.6 to obtain an isometric immersion $\tilde{f}: U \rightarrow \mathbb{Q}_{c, s}^{2 n}$ and a parallel orthonormal normal frame $\xi_{1}, \ldots, \xi_{n}$ with $(v, h, \widehat{V})$ as associated triple and induced metric $d s^{2}=\sum_{i=1}^{n} v_{i}^{2} d u_{i}^{2}$ of constant sectional curvature $c$. Show that $\xi_{n}$ is a parallel normal vector field satisfying

$$
\left\langle\xi_{n}, \xi_{n}\right\rangle=\frac{\tilde{c}-c}{|\tilde{c}-c|} \text { and } A_{\xi_{n}}^{\tilde{f}}=\sqrt{|\tilde{c}-c|} I .
$$

Conclude from Exercise 2.9 that $\tilde{f}(U)$ is contained in an umbilical hypersurface $\mathbb{Q}_{\tilde{c}}^{2 n-1}$, hence $\tilde{f}=i \circ f$ for some isometric immersion $f: U \rightarrow \mathbb{Q}_{\tilde{c}}^{2 n-1}$. Show that there exists a parallel orthonormal normal frame $\zeta_{1}, \ldots, \zeta_{n-1}$ such that $\xi_{i}=i_{*} \zeta_{i}$ for all $1 \leq i \leq n-1$ and that $(v, h, V)$ is the triple associated with $f$ with respect to the same coordinates and the frame $\zeta_{1}, \ldots, \zeta_{n-1}$.

Exercise 5.8. Prove that there is no isometric immersion $f: M_{c}^{n} \rightarrow \mathbb{S}_{\tilde{c}}^{n+p}$ with flat normal bundle if $0<c<\tilde{c}$.
Hint: Let $\tilde{f}=i \circ f$ be the composition of $f$ with an umbilical inclusion of $\mathbb{S}_{\tilde{c}}^{n+p}$ into $\mathbb{S}_{c}^{n+p+1}$. Arguing as in the proof of Proposition 5.10, show that the principal normal vector fields $\tilde{\eta}_{1}, \ldots, \tilde{\eta}_{n}$ of $\tilde{f}$ satisfy

$$
\sum_{i=1}^{n}\left\|\tilde{\tilde{i}}_{i}\right\|^{-2} \leq \frac{1}{\tilde{c}-c}
$$

Hence $\left\|\tilde{\eta}_{i}\right\| \geq \sqrt{\tilde{c}-c}$ for all $1 \leq i \leq n$. Then apply assertion (a) in Proposition 5.6.

## Chapter 6

## Submanifolds with nonpositive extrinsic curvature

The results of this chapter show that isometric immersions $f: M^{n} \rightarrow \tilde{M}^{m}$ with low codimension and nonpositive extrinsic curvature at any point must satisfy strong geometric conditions. That $f$ has nonpositive extrinsic curvature at any point means that the sectional curvature $K_{M}(\sigma)$ of $M^{n}$ along any plane $\sigma$ does not exceed the corresponding sectional curvature $K_{\tilde{M}}\left(f_{*} \sigma\right)$ of $\tilde{M}^{m}$. The simplest result along this line is that a two-dimensional surface with nonpositive curvature in $\mathbb{R}^{3}$ cannot be compact. This is a consequence of the fact that at a point of maximum of a distance function on a compact surface in $\mathbb{R}^{3}$ the Gaussian curvature must be positive. It turns out that the simple idea in the proof of this elementary fact has far-reaching generalizations for non-necessarily compact submanifolds in fairly general ambient Riemannian manifolds.

One of the main tools to extend this idea to higher dimensions and codimensions is an algebraic lemma due to Otsuki, whereas a key ingredient to handle the noncompact case is a maximum principle due to Omori [275] and Yau [347], and generalized by Pigola-Rigoli-Setti [292]. Using these tools, one can derive estimates for the extrinsic curvatures of submanifolds of certain Riemannian manifolds, under some sort of boundedness assumption. We present a theorem of this type for submanifolds of a product manifold $P^{m} \times \mathbb{R}^{\ell}$, obtained by Alías-Bessa-Montenegro [10], which generalizes several previous results.

Going in a different direction, the Chern-Kuiper inequalities show that if the extrinsic curvatures of a submanifold vanish at a point, then the index of relative nullity at that point must be positive, as long as the codimension of the submanifold does not exceed its dimension. If the extrinsic curvatures are only assumed to be nonpositive, a theorem by Florit states that the same conclusion holds under the sharp assumption that the codimension of the submanifold is not greater than half of its dimension. This is discussed in the last section.

### 6.1 Otsuki's lemma

Throughout this section, $V^{n}$ and $W^{p}$ denote real vector spaces of dimensions $n$ and $p$, respectively, endowed with positive definite inner products. For a symmetric bilinear form $\alpha: V^{n} \times V^{n} \rightarrow W^{p}$, we denote

$$
K_{\alpha}(X, Y)=\langle\alpha(X, X), \alpha(Y, Y)\rangle-\|\alpha(X, Y)\|^{2}
$$

for any pair of vectors $X, Y \in V^{n}$. If $\sigma$ is a two-dimensional subspace of $V^{n}$, we define

$$
K_{\alpha}(\sigma)=\frac{K_{\alpha}(X, Y)}{\|X \wedge Y\|^{2}}
$$

where $X, Y$ is any basis of $\sigma$ and

$$
\|X \wedge Y\|^{2}=\|X\|^{2}\|Y\|^{2}-\langle X, Y\rangle^{2} .
$$

Given an isometric immersion $f: M^{n} \rightarrow \tilde{M}^{m}$ with second fundamental form $\alpha$, for any $x \in M^{n}$ and any plane $\sigma \in T_{x} M$ the Gauss equation yields

$$
K_{\alpha}(\sigma)=K_{f}(\sigma)=K_{M}(\sigma)-K_{\tilde{M}}\left(f_{*} \sigma\right),
$$

which is called the extrinsic curvature of $f$ at $x$ along $\sigma$.
A basic tool in this chapter is the following algebraic lemma.
Lemma 6.1. Let $\alpha: V^{n} \times V^{n} \rightarrow W^{p}$ be a symmetric bilinear form. Assume that there exists a real number $\lambda \geq 0$ such that the following conditions hold:
(i) $K_{\alpha}(\sigma) \leq \lambda$ for every plane $\sigma \subset V^{n}$,
(ii) $\|\alpha(X, X)\|>\sqrt{\lambda}$ for every unit vector $X \in V^{n}$.

Then $p \geq n$.
Proof: Suppose $p<n$ and set

$$
S=\left\{X \in V^{n}:\|X\|=1\right\} .
$$

Let $X_{0} \in S$ be a minimum for $f \in C^{\infty}(S)$ defined by

$$
f(X)=\|\alpha(X, X)\|^{2}
$$

For any unit vector $Y \in T_{X_{0}} S$, the curve given by

$$
\gamma(t)=\cos t X_{0}+\sin t Y
$$

satisfies $\gamma(0)=X_{0}$ and $\gamma^{\prime}(0)=Y$. Then

$$
\begin{align*}
0=Y(f)\left(X_{0}\right) & =2\left\langle\left.\frac{d}{d t} \alpha(\gamma(t), \gamma(t))\right|_{t=0}, \alpha\left(X_{0}, X_{0}\right)\right\rangle \\
& =4\left\langle\alpha\left(X_{0}, Y\right), \alpha\left(X_{0}, X_{0}\right)\right\rangle \tag{6.1}
\end{align*}
$$

Using that $\gamma^{\prime \prime}(0)=-X_{0}$ we obtain

$$
\begin{equation*}
0 \leq Y Y(f)\left(X_{0}\right)=8\left\|\alpha\left(X_{0}, Y\right)\right\|^{2}+4\left\langle\alpha(Y, Y), \alpha\left(X_{0}, X_{0}\right)\right\rangle-4\left\|\alpha\left(X_{0}, X_{0}\right)\right\|^{2} \tag{6.2}
\end{equation*}
$$

Consider the linear map $B_{X_{0}}: T_{X_{0}} S \rightarrow W$ given by

$$
B_{X_{0}}(Y)=\alpha\left(X_{0}, Y\right)
$$

Then (6.1) implies that

$$
\left\langle B_{X_{0}}(Y), \alpha\left(X_{0}, X_{0}\right)\right\rangle=0
$$

for any $Y \in T_{X_{0}} S$. Hence

$$
\operatorname{dim} B_{X_{0}}\left(T_{X_{0}} S\right) \leq p-1,
$$

since $\alpha\left(X_{0}, X_{0}\right) \neq 0$ by condition (ii). Thus ker $B_{X_{0}} \neq\{0\}$, that is, there exists a unit vector $Y_{0}$ orthogonal to $X_{0}$ such that

$$
\alpha\left(X_{0}, Y_{0}\right)=0
$$

From (6.2) and the hypothesis, it follows that

$$
\begin{aligned}
0 & \leq\left\langle\alpha\left(Y_{0}, Y_{0}\right), \alpha\left(X_{0}, X_{0}\right)\right\rangle-\left\|\alpha\left(X_{0}, X_{0}\right)\right\|^{2} \\
& <\lambda-(\sqrt{\lambda})^{2} \\
& =0
\end{aligned}
$$

which is a contradiction.
Given a symmetric bilinear form $\alpha: V^{n} \times V^{n} \rightarrow W^{p}$, a vector $X \in V^{n}$ is said to be an asymptotic vector of $\alpha$ if $X \neq 0$ and $\alpha(X, X)=0$.

In the next statement and the sequel, we write $K_{\alpha} \leq 0$ (respectively, $K_{\alpha}<0$ ) as a shorthand for $K_{\alpha}(\sigma) \leq 0$ (respectively, $K_{\alpha}(\sigma)<0$ ) for any plane $\sigma \subset V^{n}$.

Corollary 6.2. Let $\alpha: V^{n} \times V^{n} \rightarrow W^{p}$ be a symmetric bilinear form.
(i) If $K_{\alpha} \leq 0$, then any subspace $S \subset V^{n}$ with $\operatorname{dim} S>p$ contains an asymptotic vector of $\alpha$.
(ii) If $K_{\alpha}<0$, then $p \geq n-1$.

Proof: (i) This is just an equivalent way of stating Lemma 6.1 for $\lambda=0$.
(ii) If there are no asymptotic vectors of $\alpha$, then the assertion follows from Lemma 6.1. Suppose $p<n-1$ and assume that there exists an asymptotic vector $X_{0} \in V^{n}$ of $\alpha$. Let $U$ be the orthogonal complement to $X_{0}$ in $V^{n}$, and consider the linear map $B_{X_{0}}: U \rightarrow W^{p}$ defined by

$$
B_{X_{0}}(Y)=\alpha\left(X_{0}, Y\right) .
$$

Since $\operatorname{dim} U=n-1>p$, there exists a nonzero vector $Y_{0} \in U$ such that $B_{X_{0}}\left(Y_{0}\right)=0$. This fact, together with $\alpha\left(X_{0}, X_{0}\right)=0$, contradicts the assumption.

In the proof of Theorem 6.9 below, the following stronger version of Lemma 6.1 for the case $\lambda=0$ will be needed.

Lemma 6.3. Let $\alpha: V^{n} \times V^{n} \rightarrow W^{p}$ be a symmetric bilinear form. If there is a subspace $S \subset V^{n}$ with no asymptotic vectors of $\alpha$ such that $\operatorname{dim} S>p$, then there exists a pair of linearly independent vectors $X, Y \in V^{n}$ such that

$$
\begin{equation*}
\alpha(X, X)=\alpha(Y, Y) \text { and } \alpha(X, Y)=0 \tag{6.3}
\end{equation*}
$$

Proof: Let $S \subset V^{n}$ be a subspace of dimension $m>p$. Restrict $\alpha$ to $S \times S$ and then extend it to a complex symmetric bilinear form $\alpha: S^{\mathbb{C}} \times S^{\mathbb{C}} \rightarrow W^{\mathbb{C}}$, where $S^{\mathbb{C}}=S \otimes \mathbb{C}$ and $W^{\mathbb{C}}=W \otimes \mathbb{C}$. Then, for $Z \in S^{\mathbb{C}}$, the equation

$$
\alpha(Z, Z)=0
$$

is equivalent to $p$ quadratic equations

$$
\alpha_{1}(Z, Z)=\cdots=\alpha_{p}(Z, Z)=0
$$

in $m$ variables. It is a well-known fact that $m>p$ implies the existence of a nontrivial solution $Z$ (see [207], p. 48), which cannot be real by assumption. If $Z=X+i Y$, then

$$
0=\alpha(Z, Z)=\alpha(X, X)-\alpha(Y, Y)+2 i \alpha(X, Y)
$$

Thus the vectors $X$ and $Y$ satisfy (6.3), which, together with the fact that there are no asymptotic vectors of $\alpha$ in $S$, implies that they must be linearly independent.

The following result is a direct consequence of part (ii) of Corollary 6.2.
Theorem 6.4. Let $f: M^{n} \rightarrow \tilde{M}^{n+p}$ be an isometric immersion. Assume that there exist a point $x_{0} \in M^{n}$ and a subspace $V_{x_{0}}$ of $T_{x_{0}} M$ of dimension $m \geq 2$ such that $K_{f}(\sigma)<0$ along every plane $\sigma \subset V_{x_{0}}$. Then $p \geq m-1$.

The preceding inequality is sharp, as it is shown by any isometric immersion $f: M_{c}^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{2 n-1}$ with $c<\tilde{c}$, e.g., the $n$-dimensional Clifford torus in $\mathbb{S}^{2 n-1}$ given in Exercise 3.1 or the higher dimensional version of the pseudosphere in Exercise 5.2.

For a compact Riemannian manifold one has the following generalization of Corollary 4.12. The noncompact case is treated in the next section.

Theorem 6.5. Let $M^{n}$ be a compact Riemannian manifold such that at any point $x \in M^{n}$ there exists a subspace $V_{x}$ of $T_{x} M$ with dimension $m \geq 2$ such that $K(\sigma) \leq 0$ for every plane $\sigma \subset V_{x}$. If $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ is an isometric immersion, then $p \geq m$.

Proof: Since $M^{n}$ is compact, by Corollary 1.6 there exists a point $x_{0} \in M^{n}$ such that $\alpha(X, X) \neq 0$ for every nonzero vector $X \in T_{x_{0}} M$. Furthermore, $K_{\alpha}(\sigma) \leq 0$ for every plane $\sigma \subset V_{x_{0}}$ by the Gauss equation. The statement then follows from part (i) of Corollary 6.2.

### 6.2 Cylindrically bounded submanifolds

In order to generalize Theorem 6.5 to noncompact submanifolds of more general ambient spaces, the idea is to show that, if compactness is replaced by some sort of boundedness, one can still find points on the submanifold satisfying the assumption in Otsuki's lemma (at least on suitable tangent subspaces), provided that the submanifold satisfies the maximum principle stated next.

The Omori-Yau maximum principle for the Hessian is said to hold on a given Riemannian manifold $M^{n}$ if for any function $g \in C^{2}(M)$ with $g^{*}=\sup _{M} g<+\infty$ there exists a sequence of points $\left\{x_{k}\right\}_{k \in \mathbb{N}}$ in $M^{n}$ satisfying:
(i) $g\left(x_{k}\right)>g^{*}-1 / k$,
(ii) $\left\|\operatorname{grad} g\left(x_{k}\right)\right\|<1 / k$,
(iii) Hess $g\left(x_{k}\right)(X, X) \leq(1 / k)\langle X, X\rangle$ for all $X \in T_{x_{k}} M$.

The next result provides conditions on a complete Riemannian manifold for the Omori-Yau maximum principle for the Hessian to hold. The function $r(x)$ denotes the distance to a reference point $p \in M^{n}$. It is a standard fact that $r(x)$ is smooth within the cut locus $\operatorname{cut}(p)$ of $p$.

Theorem 6.6. Let $M^{n}$ be a complete noncompact Riemannian manifold. Assume that the sectional curvature satisfies $K_{M}(x) \geq-G^{2}(r(x))$, where $G \in C^{1}([0,+\infty))$ satisfies

$$
\text { (i) } G(0)>0, \quad \text { (ii) } G^{\prime}(t) \geq 0 \quad \text { and } \frac{1}{G(t)} \notin L^{1}(+\infty) \text {. }
$$

Then the Omori-Yau maximum principle for the Hessian holds on $M^{n}$.
The following version of the Hessian comparison theorem will be used. Here $K_{M}^{\mathrm{rad}}$ denotes the radial sectional curvatures of $M^{n}$ with respect to $p$, that is, the sectional curvatures of tangent planes to $M^{n}$ containing the vector grad $r$.

Theorem 6.7. Let $M^{n}$ be a complete Riemannian manifold and let $D=M^{n} \backslash \operatorname{cut}(o)$ denote the domain of the normal geodesic coordinates centered at $o \in M^{n}$. Given $F \in C^{0}([0,+\infty))$, let $g$ be the solution of the problem

$$
\left\{\begin{array}{l}
g^{\prime \prime}(t)-F(t) g(t) \geq 0 \\
g(0)=0, g^{\prime}(0)=1
\end{array}\right.
$$

Assume that $K_{M}^{\mathrm{rad}}(x) \leq-F(r(x))$ on the ball $B_{r_{0}}(o)$, where $\left(0, r_{0}\right) \subset(0,+\infty)$ is the maximal interval where the function $g$ is positive. Then

$$
\operatorname{Hess} r(x) \geq \frac{g^{\prime}(r(x))}{g(r(x))}(\langle,\rangle-d r \otimes d r)
$$

on $D \cap B_{r_{0}}(o) \backslash\{o\}$.

Remark 6.8. If $K_{M}^{\mathrm{rad}} \leq b$ for some constant $b \in \mathbb{R}$ and $r(x)<\pi / \sqrt{b}$ if $b>0$, then

$$
\operatorname{Hess} r(x)(X, X) \geq C_{b}(r(x))\|X\|^{2}
$$

for any $X \in T_{x} M$ orthogonal to $\operatorname{grad} r(x)$, where

$$
C_{b}(t)= \begin{cases}\sqrt{b} \cot (\sqrt{b} t) & \text { if } b>0 \text { and } 0<t<\pi / 2 \sqrt{b} \\ 1 / t & \text { if } b=0 \text { and } t>0 \\ \sqrt{-b} \operatorname{coth}(\sqrt{-b} t) & \text { if } b<0 \text { and } t>0\end{cases}
$$

This is the usual statement of the Hessian comparison theorem, and follows by taking $g(t)=(1 / \sqrt{b}) \sin \sqrt{b} t, g(t)=t$ or $g(t)=(1 / \sqrt{-b}) \sinh \sqrt{-b} t$, according as $b>0$, $b=0$ or $b<0$, respectively.
Theorem 6.9. Let $f: M^{n} \rightarrow P^{m} \times \mathbb{R}^{\ell}, 2 \leq m \leq 2(n-\ell)-1$, be an isometric immersion between complete Riemannian manifolds such that $f(M) \subset B_{R}(o) \times \mathbb{R}^{\ell}$ with $0<R<\min \left\{\operatorname{inj}_{P}(o), \pi / 2 \sqrt{b}\right\}$, where $\operatorname{inj}_{P}(o)$ is the injectivity radius at o and $\pi / 2 \sqrt{b}$ is replaced by $+\infty$ if $b \leq 0$. Assume that the scalar curvature of $M^{n}$ satisfies

$$
s(x) \geq-A \rho^{2}(x)\left(\Pi_{j=1}^{N} \log ^{(j)}(\rho(x))\right)^{2}, \quad \rho(x) \gg 1
$$

for a constant $A>0$ and some integer $N \geq 1$, where $\rho$ is the distance function in $M^{n}$ to a point and $\log ^{(j)}$ stands for the $j$-iterated logarithm. If $K_{P} \leq b$ in $B_{R}(o)$, then

$$
\sup \left\{K_{f}(\sigma): x \in M^{n} \text { and } \sigma \subset T_{x} M\right\} \geq C_{b}^{2}(R)
$$

In particular,

$$
\sup _{M} K_{M} \geq C_{b}^{2}(R)+\inf _{B_{R}(o)} K_{P}
$$

Proof: Denote $N^{m+\ell}=P^{m} \times \mathbb{R}^{\ell}$, and let $\pi_{P}: N^{m+\ell} \rightarrow P^{m}$ be the projection onto $P^{m}$. Then the function

$$
\psi(t)= \begin{cases}1-\cos (\sqrt{b} t) & \text { if } b>0 \\ t^{2} & \text { if } b=0 \\ \cosh (\sqrt{-b} t) & \text { if } b<0\end{cases}
$$

where $t>0$ if $b \leq 0$ and $t<\pi / 2 \sqrt{b}$ if $b>0$, satisfies $\psi^{\prime \prime}=\psi^{\prime} C_{b}$. Define $h \in C^{\infty}(N)$ by $h=\psi \circ r \circ \pi_{P}$. Since $f(M) \subset B_{R}(o) \times \mathbb{R}^{\ell}$, then the function $g=h \circ f$ satisfies

$$
g^{*}=\sup _{M} g \leq \psi(R)<+\infty
$$

We may assume that $\sup K_{M}<+\infty$, for otherwise the estimate in the theorem is trivially satisfied. Then, since the scalar curvature is an average of sectional curvatures, it follows from the assumption on $s$ that

$$
K_{M}(x) \geq-C \rho^{2}(x)\left(\Pi_{j=1}^{N} \log ^{(j)}(\rho(x))\right)^{2}, \quad \rho(x) \gg 1
$$

for a constant $C>0$. According to Theorem 6.6, this curvature decay is sufficient to conclude that the Omori-Yau maximum principle for the Hessian holds on $M^{n}$. Thus there exists a sequence of points $\left\{x_{k}\right\}_{k \in \mathbb{N}}$ in $M^{n}$ such that
(i) $g\left(x_{k}\right)>g^{*}-1 / k$,
(ii) Hess ${ }^{M} g\left(x_{k}\right) \leq(1 / k)\langle$,$\rangle .$

The idea of the argument is to use part (ii) and (1.6) to estimate $\|\alpha(X, X)\|$ for $X$ in a suitable subspace $V_{x_{k}} \subset T_{x_{k}} M$, and then to apply Lemma 6.3 to $\left.\alpha\right|_{V_{x_{k}} \times V_{x_{k}}}$. This will imply the estimate in the statement.

By (1.5) we have

$$
\operatorname{grad}^{N} h(f(x))=f_{*} \operatorname{grad}^{M} g(x)+\left(\operatorname{grad}^{N} h(f(x))\right)^{\perp} .
$$

Note that

$$
\begin{equation*}
\operatorname{grad}^{N} h(z, y)=\psi^{\prime}(r(z)) \operatorname{grad}^{P} r(z) . \tag{6.4}
\end{equation*}
$$

Since $h$ only depends on $P^{m}$, from (1.6) and (6.4) we obtain
Hess $g(x)(X, X)=\operatorname{Hess}^{P} \psi \circ r(z(x))\left(X_{P}, X_{P}\right)+\psi^{\prime}(r(z(x)))\left\langle\operatorname{grad}^{P} r(z(x)), \alpha(X, X)\right\rangle$,
where $z(x)=\pi_{P}(f(x))$ and $X_{P}=\pi_{P *} f_{*} X$. Observe that

$$
\operatorname{Hess}^{P} \psi \circ r(z)\left(X_{P}, X_{P}\right)=\psi^{\prime \prime}(r(z))\left\langle\operatorname{grad}^{P} r, X_{P}\right\rangle^{2}+\psi^{\prime}(r(z)) \operatorname{Hess}^{P} r(z)\left(X_{P}, X_{P}\right) .
$$

Since $\psi^{\prime \prime}=\psi^{\prime} C_{b}$, the last two equations yield

$$
\begin{align*}
\text { Hess } g(x)(X, X)= & \psi^{\prime}(r(z(x)))\left[C_{b}(r(z(x)))\left\langle\operatorname{grad}^{P} r(z(x)), X_{P}\right\rangle^{2}\right. \\
& \left.+\left\langle\operatorname{grad}^{P} r(z(x)), \alpha(X, X)\right\rangle+\operatorname{Hess}^{P} r(z(x))\left(X_{P}, X_{P}\right)\right] . \tag{6.5}
\end{align*}
$$

Taking into account Remark 6.8, Theorem 6.7 gives

$$
\begin{align*}
\operatorname{Hess}^{P} r(z)(Y, Y) & =\operatorname{Hess}^{P} r(z)\left(Y^{\perp}, Y^{\perp}\right) \\
& \geq C_{b}(r(z))\left(\|Y\|^{2}-\left\langle\operatorname{grad}^{P} r, Y\right\rangle^{2}\right) \tag{6.6}
\end{align*}
$$

where $Y \in T_{z} P$ and $Y^{\perp}$ is defined by the orthogonal decomposition

$$
Y=\left\langle\operatorname{grad}^{P} r, Y\right\rangle \operatorname{grad}^{P} r+Y^{\perp}
$$

Consider the subspace $V_{x} \subset T_{x} M$ defined by

$$
f_{*} V_{x}=f_{*} T_{x} M \cap T_{z(x)} P \subset T_{f(x)} N
$$

whose dimension is at least $n-\ell \geq 2$. Since $X_{P}=f_{*} X$ for any $X \in V_{x}$, from 6.5) and (6.6) we obtain

$$
\begin{aligned}
\text { Hess } g(x)(X, X) & \geq \psi^{\prime}(r(z(x)))\left(C_{b}(r(z(x)))\|X\|^{2}+\left\langle\operatorname{grad}^{P} r(z(x)), \alpha(X, X)\right\rangle\right) \\
& \geq \psi^{\prime}(r(z(x)))\left(C_{b}(r(z(x)))\|X\|^{2}-\|\alpha(X, X)\|\right) .
\end{aligned}
$$

Hence

$$
\frac{1}{k}\|X\|^{2} \geq \psi^{\prime}\left(r\left(z\left(x_{k}\right)\right)\right)\left(C_{b}\left(r\left(z\left(x_{k}\right)\right)\right)\|X\|^{2}-\|\alpha(X, X)\|\right)
$$

for any $x_{k}$ and $X \in V_{x_{k}}$. Therefore

$$
\|\alpha(X, X)\| \geq\left(C_{b}\left(r\left(z\left(x_{k}\right)\right)\right)-\frac{1}{k \psi^{\prime}\left(r\left(z\left(x_{k}\right)\right)\right)}\right)\|X\|^{2}
$$

Since

$$
C_{b}\left(r\left(z\left(x_{k}\right)\right)\right)>\frac{1}{k \psi^{\prime}\left(r\left(z\left(x_{k}\right)\right)\right)}
$$

for $k$ sufficiently large, and

$$
\operatorname{dim} V_{x_{k}} \geq n-\ell>m+\ell-n=\operatorname{dim} N_{f} M\left(x_{k}\right),
$$

we can apply Lemma 6.3 to

$$
\left.\alpha\right|_{V_{x_{k}} \times V_{x_{k}}}: V_{x_{k}} \times V_{x_{k}} \rightarrow N_{f} M\left(x_{k}\right) .
$$

We obtain linearly independent vectors $X_{k}, Y_{k} \in V_{x_{k}}$ such that

$$
\alpha\left(X_{k}, X_{k}\right)=\alpha\left(Y_{k}, Y_{k}\right) \text { and } \alpha\left(X_{k}, Y_{k}\right)=0 .
$$

We may assume $\left\|X_{k}\right\| \geq\left\|Y_{k}\right\|$. Setting $\sigma_{k}=\operatorname{span}\left\{X_{k}, Y_{k}\right\}$, the Gauss equation yields

$$
\begin{aligned}
K_{f}\left(\sigma_{k}\right) & =K_{\alpha}\left(\sigma_{k}\right) \\
& \geq\left(\frac{\left\|\alpha\left(X_{k}, X_{k}\right)\right\|}{\left\|X_{k}\right\|^{2}}\right)^{2} \\
& \geq\left(C_{b}\left(r\left(z\left(x_{k}\right)\right)\right)-\frac{1}{k \psi^{\prime}\left(r\left(z\left(x_{k}\right)\right)\right)}\right)^{2}
\end{aligned}
$$

and the conclusion follows by letting $k \rightarrow+\infty$.
Remarks 6.10. (i) If follows from the proof that Theorem 6.9 is still valid under the weaker assumption that the weak maximum principle for the Hessian holds on $M^{n}$. The latter only requires conditions (i) and (iii) in the definition of the Omori-Yau maximum principle for the Hessian.
(ii) The weak maximum principle for the Hessian holds on $M^{n}$ if the manifold is complete and there exist $\varphi \in C^{2}(M)$ and a constant $k>0$ such that $\varphi(x) \rightarrow+\infty$ as $x \rightarrow \infty$ and

$$
\operatorname{Hess} \varphi \leq k \varphi\langle,\rangle
$$

outside a compact set of $M^{n}$; see Theorem 2.10 in [14].
For $\ell=0$, the assumption on $f$ reduces to $f(M)$ being bounded, and this implies, in particular, the following result.

Corollary 6.11. Let $f: M^{n} \rightarrow N^{m}, m \leq 2 n-1$, be an isometric immersion of a complete Riemannian manifold into a Hadamard manifold. Assume that the scalar curvature of $M^{n}$ is bounded from below. If $f$ has nonpositive extrinsic curvature at any point $x \in M^{n}$ and for every plane $\sigma \subset T_{x} M$, then $f(M)$ is unbounded.

### 6.3 Florit's nullity estimate

The aim of this section is to prove a sharp estimate of the index of relative nullity of an isometric immersion with nonpositive extrinsic curvature.

We start with a key fact on asymptotic vectors of a symmetric bilinear form $\alpha: V^{n} \times V^{n} \rightarrow W^{p}$ satisfying $K_{\alpha} \leq 0$. We always assume that $W^{p}$ is endowed with a positive definite inner product and denote by $A(\alpha)$ the set of asymptotic vectors of $\alpha$. For a given $X_{0} \in V^{n}$, set

$$
L_{\alpha}\left(X_{0}\right)=\alpha\left(X_{0},\right): V^{n} \rightarrow W^{p}
$$

Lemma 6.12. Let $\alpha: V^{n} \times V^{n} \rightarrow W^{p}$ be a symmetric bilinear form with $K_{\alpha} \leq 0$. Then for any $X_{0} \in A(\alpha)$ we have

$$
\mathcal{S}\left(\left.\alpha\right|_{\hat{V} \times \hat{V}}\right) \subset \hat{W}
$$

where $\hat{V}=\operatorname{ker} L_{\alpha}\left(X_{0}\right)$ and $\hat{W}=\operatorname{Im} L_{\alpha}\left(X_{0}\right)^{\perp}$.
Proof: Take $Z \in \hat{V}$ and $Y \in V^{n}$. Then for all $t \in \mathbb{R}$ we have

$$
\begin{aligned}
0 \geq K_{\alpha}\left(X_{0}+t Y, Z\right) & =\left\langle 2 t \alpha\left(X_{0}, Y\right)+t^{2} \alpha(Y, Y), \alpha(Z, Z)\right\rangle-t^{2}\|\alpha(Y, Z)\|^{2} \\
& =2 t\left\langle\alpha\left(X_{0}, Y\right), \alpha(Z, Z)\right\rangle+t^{2} K_{\alpha}(Y, Z) .
\end{aligned}
$$

Thus

$$
\left\langle\alpha\left(X_{0}, Y\right), \alpha(Z, Z)\right\rangle=0,
$$

and the statement follows using the symmetry of $\alpha$.
Given a symmetric bilinear form $\alpha: V^{n} \times V^{n} \rightarrow W^{p}$ with $K_{\alpha} \leq 0$, a vector subspace $T \subset V^{n}$ is called an asymptotic subspace of $\alpha$ if $\alpha(X, Y)=0$ for all $X, Y \in T$.

The next result provides a stronger version of part (i) of Corollary 6.2.
Proposition 6.13. Let $\alpha: V^{n} \times V^{n} \rightarrow W^{p}$ be a symmetric bilinear map with $K_{\alpha} \leq 0$. Then there exists an asymptotic subspace $T \subset V^{n}$ of $\alpha$ such that $\operatorname{dim} T \geq n-p$.

Proof: Using Lemma 6.12, we inductively construct a sequence of pairs of subspaces

$$
(V, W)=\left(V_{0}, W_{0}\right) \supset\left(V_{1}, W_{1}\right) \supset \cdots \supset\left(V_{k}, W_{k}\right) \supset \cdots
$$

and symmetric bilinear maps $\alpha_{k}=\left.\alpha\right|_{V_{k} \times V_{k}}: V_{k} \times V_{k} \rightarrow W_{k}$ such that

$$
n_{k}=\operatorname{dim} V_{k}=n-\sum_{i=0}^{k-1} r_{i} \text { and } p_{k}=\operatorname{dim} W_{k}=p-\sum_{i=0}^{k-1} r_{i}
$$

where

$$
r_{i}=\max \left\{\operatorname{dim} \operatorname{Im} L_{\alpha_{i}}(X): X \in A\left(\alpha_{i}\right)\right\} .
$$

In fact, assuming that $V_{i}, W_{i}$ and $\alpha_{i}, 0 \leq i \leq k$, have been defined, choose $X_{k} \in A\left(\alpha_{k}\right)$ such that

$$
\begin{aligned}
r_{k} & =\operatorname{dim} \operatorname{Im} L_{\alpha_{k}}\left(X_{k}\right) \\
& =\max \left\{\operatorname{dim} \operatorname{Im} L_{\alpha_{k}}(X): X \in A\left(\alpha_{k}\right)\right\}
\end{aligned}
$$

and set

$$
V_{k+1}=\operatorname{ker} L_{\alpha_{k}}\left(X_{k}\right), \quad W_{k+1}=\operatorname{Im} L_{\alpha_{k}}\left(X_{k}\right)^{\perp} \subset W_{k} \text { and } \alpha_{k+1}=\left.\alpha\right|_{V_{k+1} \times V_{k+1}}
$$

Then

$$
\mathcal{S}\left(\alpha_{k+1}\right) \subset W_{k+1}
$$

by Lemma 6.12, and therefore the construction can be extended up to $k+1$.
There must exist a positive integer $m$ such that $r_{m}=0$. This means that

$$
T=A\left(\alpha_{m}\right)=\mathcal{N}\left(\alpha_{m}\right)
$$

It follows from part $(i)$ of Corollary 6.2 that

$$
S \cap T=S \cap A\left(\alpha_{m}\right) \neq\{0\}
$$

for any subspace $S \subset V_{m}$ with $\operatorname{dim} S>p_{m}$. Therefore

$$
\operatorname{dim} T \geq n_{m}-p_{m}=n-p
$$

Moreover, since $\alpha_{m}=\left.\alpha\right|_{V_{m} \times V_{m}}$, then $T$ is an asymptotic subspace of $\alpha$.
The following result estimates the dimension of the kernel of a symmetric bilinear map $\alpha$ with $K_{\alpha} \leq 0$ in terms of the dimension of an asymptotic subspace of $\alpha$.

Proposition 6.14. Let $\alpha: V^{n} \times V^{n} \rightarrow W^{p}$ be a symmetric bilinear map with $K_{\alpha} \leq 0$. If $T$ is an asymptotic subspace of $\alpha$, then $\operatorname{dim} \mathcal{N}(\alpha) \geq \operatorname{dim} T-p$.

Proof: Set

$$
\beta=\left.\alpha\right|_{T^{\prime} \times T}: T^{\prime} \times T \rightarrow W^{p} .
$$

where $V^{n}=T^{\prime} \oplus T$. Take $Y_{0} \in R E(\beta)$ and define

$$
B_{Y_{0}}=\beta\left(Y_{0},\right)
$$

Given $Z^{\prime} \in \operatorname{ker} B_{Y_{0}} \subset T, Z \in T$ and $Y \in T^{\prime}$, using that $T$ is asymptotic gives

$$
\begin{aligned}
K_{\alpha}\left(Y_{0}+t Z, Y+s Z^{\prime}\right)= & \left\langle\alpha\left(Y_{0}, Y_{0}\right)+2 t \alpha\left(Y_{0}, Z\right), \alpha(Y, Y)+2 s \alpha\left(Y, Z^{\prime}\right)\right\rangle \\
& -\left\|\alpha\left(Y_{0}+t Z, Y+s Z^{\prime}\right)\right\|^{2}
\end{aligned}
$$

for all $s, t \in \mathbb{R}$. Since $\alpha\left(Y_{0}, Z^{\prime}\right)=0$ we obtain

$$
\begin{aligned}
K_{\alpha}\left(Y_{0}+t Z, Y+s Z^{\prime}\right)= & K_{\alpha}\left(Y_{0}, Y\right)-t^{2}\|\alpha(Z, Y)\|^{2}+2 t\left\langle\alpha\left(Y_{0}, Z\right), \alpha(Y, Y)\right\rangle \\
& -2 t\left\langle\alpha\left(Y_{0}, Y\right), \alpha(Z, Y)\right\rangle+2 s\left\langle\alpha\left(Y_{0}, Y_{0}\right), \alpha\left(Y, Z^{\prime}\right)\right\rangle \\
& +2 \operatorname{st}\left\langle\alpha\left(Y_{0}, Z\right), \alpha\left(Y, Z^{\prime}\right)\right\rangle .
\end{aligned}
$$

In view of the hypothesis on $K_{\alpha}$, that the right-hand side is linear in $s$ implies that

$$
\left\langle\alpha\left(Y_{0}, Y_{0}\right), \alpha\left(Y, Z^{\prime}\right)\right\rangle+2 t\left\langle\alpha\left(Y_{0}, Z\right), \alpha\left(Y, Z^{\prime}\right)\right\rangle=0
$$

for all $t \in \mathbb{R}$. Hence

$$
\left\langle\alpha\left(Y_{0}, Z\right), \alpha\left(Y, Z^{\prime}\right)\right\rangle=0,
$$

and therefore

$$
\beta\left(Y, \operatorname{ker} B_{Y_{0}}\right) \subset\left(B_{Y_{0}}(T)\right)^{\perp}
$$

for all $Y \in T^{\prime}$. This and Proposition 4.6 imply that $\alpha(Y, X)=0$ for all $Y \in T^{\prime}$ and that $X \in \operatorname{ker} B_{Y_{0}}$. But since ker $B_{Y_{0}} \subset T$, we see that ker $B_{Y_{0}} \subset \mathcal{N}(\alpha)$. Then

$$
\begin{aligned}
\operatorname{dim} \mathcal{N}(\alpha) & \geq \operatorname{dim} \operatorname{ker} B_{Y_{0}} \\
& =\operatorname{dim} T-\operatorname{dim} B_{Y_{0}}(T) \\
& \geq \operatorname{dim} T-p,
\end{aligned}
$$

as we wished.
Theorem 6.15. Let $f: M^{n} \rightarrow \tilde{M}^{n+p}$ be an isometric immersion between Riemannian manifolds. If there exists a point $x_{0} \in M^{n}$ such that $K_{f}(\sigma) \leq 0$ for all $\sigma \in T_{x_{0}} M$, then the index of relative nullity satisfies $\nu\left(x_{0}\right) \geq n-2 p$.

Proof: It follows from the Gauss equation and Propositions 6.13 and 6.14 .
That the estimate in Theorem 6.15 is sharp is shown by the following example.
Example 6.16. Let $U^{2} \subset \mathbb{R}^{3}$ be a surface with negative Gaussian curvature at some point $x_{0} \in U^{2}$. Then the product immersion of $p$ factors

$$
U^{2} \times \cdots \times U^{2} \rightarrow \mathbb{R}^{3 p}
$$

satisfies $\nu\left(x_{0}, \ldots, x_{0}\right)=0=n-2 p$.
The case in which a submanifold $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ has constant index of relative nullity $\nu=n-2 p$ was considered by Florit-Zheng [187]. They obtained the result given next without proof.

Theorem 6.17. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ be an isometric immersion of a Riemannian manifold with sectional curvature $K_{M} \leq 0$. Assume that the index of relative nullity satisfies $\nu=n-2 p$ at any point. Then each point of an open dense subset $U \subset M^{n}$ has an open neighborhood $V \subset U$ that splits as a Riemannian product of manifolds $V=M_{1}^{n_{1}} \times \cdots \times M_{p}^{n_{p}}$ with $K_{M_{i}} \leq 0$ such that $\left.f\right|_{V}=f_{1} \times \cdots \times f_{p}$ is a product of hypersurfaces $f_{j}: M_{j}^{n_{j}} \rightarrow \mathbb{R}^{n_{j}+1}, 1 \leq j \leq p$.

The above result has the following application.
Corollary 6.18. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ be an isometric immersion of a Riemannian manifold with sectional curvature $K_{M} \leq c$ and Ricci curvature Ric $c_{M}<c$. If $2 p \leq n$, then $c=0, n=2 p$ and $f$ splits locally as a product of $p$ surfaces in $\mathbb{R}^{3}$ with negative Gauss curvature. Moreover, if $M^{n}$ is a Hadamard manifold then the splitting is global.

### 6.4 Notes

The idea of the proof that any compact surface in $\mathbb{R}^{3}$ must have a point of positive Gauss curvature was first taken up by Tompkins [335], who has shown that a compact flat $n$-dimensional Riemannian manifold cannot be isometrically immersed in $\mathbb{R}^{2 n-1}$. This result inspired the seminal paper of Chern-Kuiper [86], where Lemma 6.1 was proved for dimensions $n=2,3$ and conjectured to be true for any dimension. This conjecture was proved by Otsuki [284] for $\lambda=0$ who, consequently, obtained Theorem 6.4 for all dimensions.

The Chern-Kuiper result gave rise to a long series of works, among others, by O'Neill [276], Moore [254], Jorge-Koutroufiotis [226] (Corollary 6.11), Pigola-RigoliSetti [292] and, finally, by Alías-Bessa-Montenegro [10], who obtained Theorem 6.9 on cylindrically bounded submanifolds. See also the work of Canevari-Freitas-Manfio [51]. The result on the maximum principle used in the proof of Theorem 6.6 is due to Pigola-Rigoli-Setti [292] and can be found in [14]. For the version of the Hessian comparison theorem used here we refer to [14] or [293]. Several related results, obtained with techniques similar to those in the proof of Theorem 6.9, are given in Chapter 5 of [14]. See also [9], [11], [12] and [13].

Finally, the nullity estimate on submanifolds with nonpositive extrinsic curvature given by Theorem 6.15 is due to Florit [181]. Concerning Theorem 6.17 (respectively, Corollary 6.18), Florit-Zheng [188] considered the more difficult case in which the index of relative nullity satisfies $\nu=n-2 p+1$ (respectively, the codimension of $f$ satisfies $2 p \leq n+1$ ). Theorem 6.17 has been extended to submanifolds of nonflat space forms by Florit [183].

### 6.5 Exercises

Exercise 6.1. Show that Corollary 6.11 is false if the ambient space is not simply connected.
Exercise 6.2. Let $\tilde{M}^{m}$ be a Riemannian manifold. Assume that there exists a constant $b \leq 0$ (respectively, $b>0$ ) such that the radial sectional curvatures $K_{\tilde{M}}^{\text {rad }}$ of $\tilde{M}^{m}$ along geodesics issuing from $o \in \tilde{M}^{m}$ satisfy $K_{\tilde{M}}^{\text {rad }}(x) \leq b$ for all $x \in \tilde{M}^{m}$ (respectively, for all $\left.x \in B_{\pi / \sqrt{b}}(o)\right)$. If $f: M^{n} \rightarrow \tilde{M}^{m}$ is an isometric immersion of a compact Riemannian manifold such that $f(M) \subset B_{\pi / \sqrt{b}}(o)$ if $b>0$, show that there exists $x_{0} \in M^{n}$ and $\xi \in N_{f} M\left(x_{0}\right)$ such that $A_{\xi}^{f}$ is positive definite.
Hint: Consider the function $\gamma: \tilde{M}^{m} \rightarrow \mathbb{R}$ given by $\gamma(x)=(1 / 2) r^{2}(x)$, where $r(x)$ is the distance from $x$ to $o$. Let $x_{0}$ be a point of $M^{n}$ where $g=\gamma \circ f$ attains its maximum. Show that $\xi=-\operatorname{grad} \gamma\left(f\left(x_{0}\right)\right) \in N_{f} M\left(x_{0}\right)$. Use the Hessian comparison theorem to obtain

$$
\operatorname{Hess} \gamma\left(f_{*} X, f_{*} X\right) \geq r\left(x_{0}\right) C_{b}\left(r\left(x_{0}\right)\right)\|X\|^{2}
$$

for all $X \in T_{x_{0}} M$. Then use formula (1.6) as in the proof of Corollary 1.6 to conclude that $A_{\xi}^{f}$ is positive definite.

Exercise 6.3. Show that the estimate given by Theorem 6.9 is sharp.
Exercise 6.4. Let $M^{n}=N_{1}^{n_{1}} \times N_{2}^{n_{2}}$ be the Riemannian product of two Riemannian manifolds. Assume that there exists a point $\left(x_{1}, x_{2}\right) \in M^{n}$ such that

$$
K_{N_{1}}\left(x_{1}\right), K_{N_{2}}\left(x_{2}\right) \leq c
$$

for some constant $c>0$. Show that there exists no isometric immersion of $M^{n}$ into $\mathbb{S}_{c}^{n+p}$ if $2 p<n$.

## Chapter 7

## Submanifolds with relative nullity

Several of the results of Chapters 4 and 6 have provided relevant geometric conditions under which a submanifold of a space form must have positive index of relative nullity at any point. The aim of this chapter is to study submanifolds that have this property.

According to Proposition 1.18, on each open subset where the index of relative nullity is a positive constant, the submanifold is foliated by totally geodesic submanifolds of the ambient space. It turns out that this imposes severe restrictions on complete submanifolds of low codimension, due to the fact that the leaves of the minimum relative nullity foliation of a complete submanifold are also complete. Among the main applications are Hartman's splitting theorem for complete Euclidean submanifolds with nonnegative Ricci curvature, and Dajczer-Gromoll's generalization of the rigidity of the totally geodesic inclusion of a round sphere $\mathbb{S}^{n}$ into $\mathbb{S}^{n+p}, p \leq n-1$. These results will be proved after developing the necessary tools, especially the splitting tensor of a totally geodesic foliation.

A useful parametrization of any oriented hypersurface with constant index of relative nullity of a space form, called the Gauss parametrization, is subsequently discussed. Some applications are provided, including a strong rigidity property of complete minimal hypersurfaces of the Euclidean space and the classification of Euclidean hypersurfaces with constant scalar curvature and type number two at any point.

The chapter ends with a discussion of intrinsically homogeneous hypersurfaces of space forms that includes a proof of Cartan's fundamental formula for isoparametric hypersurfaces.

### 7.1 The splitting tensor

Let $M^{n}$ be a Riemannian manifold and let $D$ be a smooth distribution on $M^{n}$. Let $D^{\perp}$ denote the distribution on $M^{n}$ that assigns to each $x \in M^{n}$ the orthogonal complement of $D(x)$ in $T_{x} M$. According to the orthogonal splitting $T M=D \oplus D^{\perp}$, we write

$$
X=X^{v}+X^{h}
$$

for any $X \in \mathfrak{X}(M)$. We denote

$$
\nabla_{X}^{h} Y=\left(\nabla_{X} Y\right)^{h}
$$

for all $X, Y \in \mathfrak{X}(M)$.
The splitting tensor $C$ of $D$ is the map $C: \Gamma(D) \times \Gamma\left(D^{\perp}\right) \rightarrow \Gamma\left(D^{\perp}\right)$ defined by

$$
C(T, X)=-\nabla_{X}^{h} T .
$$

It is clear that $C$ is $C^{\infty}(M)$-linear with respect to the second variable. That it is also $C^{\infty}(M)$-linear with respect to the first variable follows from

$$
C(\varphi T, X)=-\nabla_{X}^{h} \varphi T=-\varphi \nabla_{X}^{h} T=\varphi C(T, X) .
$$

Therefore the value of $C(T, X)$ at $x \in M^{n}$ depends only on the values of $T$ and $X$ at that point. Hence, for all $x \in M^{n}$ and $T \in D(x)$, the map $C$ gives rise to an endomorphism

$$
C_{T}: D^{\perp}(x) \rightarrow D^{\perp}(x),
$$

which we call the splitting tensor of $D$ at $x$ with respect to $T$. Accordingly, we usually regard $C$ as a map

$$
C: \Gamma(D) \rightarrow \Gamma\left(\operatorname{End}\left(D^{\perp}\right)\right)
$$

Notice that the distribution $D^{\perp}$ is integrable if and only if $C_{T}$ is self-adjoint for all $T \in \Gamma(D)$, for

$$
\begin{aligned}
\left\langle C_{T} X, Y\right\rangle-\left\langle X, C_{T} Y\right\rangle & =-\left\langle\nabla_{X}^{h} T, Y\right\rangle+\left\langle X, \nabla_{Y}^{h} T\right\rangle \\
& =\left\langle\nabla_{X} Y-\nabla_{Y} X, T\right\rangle \\
& =\langle[X, Y], T\rangle
\end{aligned}
$$

for all $X, Y \in \Gamma\left(D^{\perp}\right)$ and $T \in \Gamma(D)$. In this case, $C_{T}$ is precisely the shape operator with respect to $T$ of the inclusion of the leaves of $D^{\perp}$ into $M^{n}$.

Notice also that $C$ vanishes identically if and only if $D^{\perp}$ is totally geodesic, for

$$
\begin{aligned}
\left\langle C_{T} X, Y\right\rangle & =-\left\langle\nabla_{X} T, Y\right\rangle \\
& =\left\langle\nabla_{X} Y, T\right\rangle
\end{aligned}
$$

for all $X, Y \in \Gamma\left(D^{\perp}\right)$ and $T \in \Gamma(D)$. More generally, the image of the splitting tensor $C$ is spanned by the identity endomorphism of $D^{\perp}$ if and only if the distribution $D^{\perp}$ is umbilical. In fact, there exists $S \in \Gamma(D)$ such that

$$
C_{T}=\langle T, S\rangle I
$$

for all $T \in \Gamma(D)$ if and only if

$$
\begin{aligned}
\left\langle\nabla_{X} Y, T\right\rangle & =-\left\langle\nabla_{X} T, Y\right\rangle \\
& =\left\langle C_{T} X, Y\right\rangle \\
& =\langle T, S\rangle\langle X, Y\rangle
\end{aligned}
$$

for all $X, Y \in \Gamma\left(D^{\perp}\right)$ and $T \in \Gamma(D)$. This is the condition for $D^{\perp}$ to be umbilical with mean curvature vector field $S$.

For later use, in particular in the next section, given a smooth distribution $D$ on a Riemannian manifold $M^{n}$ and $B \in \Gamma\left(\operatorname{End}\left(D^{\perp}\right)\right)$, we define $\nabla_{X}^{h} B \in \Gamma\left(\operatorname{End}\left(D^{\perp}\right)\right)$ by

$$
\begin{equation*}
\left(\nabla_{X}^{h} B\right) Y=\nabla_{X}^{h} B Y-B \nabla_{X}^{h} Y \tag{7.1}
\end{equation*}
$$

for all $X \in \mathfrak{X}(M)$ and $Y \in \Gamma\left(D^{\perp}\right)$.

### 7.1.1 The splitting tensor of the relative nullity distribution

In the sequel we derive some useful formulas that are satisfied by the splitting tensor associated with the relative nullity distribution $\Delta$ of a submanifold of a space form. In fact, we consider a slightly more general situation as seen next.

Given an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$, let $D$ be a smooth totally geodesic distribution such that $D(x) \subset \Delta(x)$ for all $x \in M^{n}$. Since $\nabla_{S} X \in \Gamma\left(D^{\perp}\right)$ for all $S \in \Gamma(D)$ and $X \in \Gamma\left(D^{\perp}\right)$ because $D$ is totally geodesic, we write simply $\nabla_{S} C_{T}$ for the covariant derivative $\nabla_{S}^{h} C_{T}$ defined in 7.1 of the splitting tensor $C_{T} \in \Gamma\left(\operatorname{End}\left(D^{\perp}\right)\right)$ with respect to $S \in \Gamma(D)$. Thus

$$
\left(\nabla_{S} C_{T}\right) X=\nabla_{S} C_{T} X-C_{T}\left(\nabla_{S} X\right)
$$

for all $S \in \Gamma(D)$ and $X \in \Gamma\left(D^{\perp}\right)$.
Proposition 7.1. The differential equation

$$
\begin{equation*}
\nabla_{T} C_{S}=C_{S} C_{T}+C_{\nabla_{T} S}+c\langle T, S\rangle I \tag{7.2}
\end{equation*}
$$

holds for all $S, T \in \Gamma(D)$. In particular, the operator $C_{\gamma^{\prime}}$ along a unit-speed geodesic $\gamma$ contained in a leaf of $D$ satisfies the differential equation

$$
\begin{equation*}
\frac{D}{d t} C_{\gamma^{\prime}}=C_{\gamma^{\prime}}^{2}+c I . \tag{7.3}
\end{equation*}
$$

Proof: By the definition of the splitting tensor we have

$$
\left(\nabla_{T} C_{S}\right) X=-\nabla_{T} \nabla_{X}^{h} S-C_{S} \nabla_{T} X
$$

for all $X \in \Gamma\left(D^{\perp}\right)$ and $S, T \in \Gamma(D)$. Since $D$ is totally geodesic we obtain

$$
\nabla_{T} \nabla_{X}^{h} S=\nabla_{T}^{h} \nabla_{X}^{h} S \text { and } \nabla_{T}^{h} \nabla_{X}^{v} S=0 .
$$

Hence

$$
\begin{equation*}
\left(\nabla_{T} C_{S}\right) X=-\nabla_{T}^{h} \nabla_{X} S-C_{S} \nabla_{T} X . \tag{7.4}
\end{equation*}
$$

The Gauss equation gives

$$
\nabla_{X} \nabla_{T} S-\nabla_{T} \nabla_{X} S-\nabla_{[X, T]} S=R(X, T) S=c\langle T, S\rangle X
$$

Taking the $D^{\perp}$-components and using that $\nabla_{[X, T]^{v}}^{h} S=0$ yield

$$
\begin{aligned}
-\nabla_{T}^{h} \nabla_{X} S & =C_{\nabla_{T} S} X+\nabla_{\nabla_{X}^{h} T^{2}}^{h} S-\nabla_{\nabla_{T}^{h} X}^{h} S+c\langle T, S\rangle X \\
& =C_{\nabla_{T} S} X+C_{S} C_{T} X+C_{S} \nabla_{T} X+c\langle T, S\rangle X,
\end{aligned}
$$

and the result follows by substituting the preceding expression in (7.4).
Proposition 7.2. The differential equation

$$
\begin{equation*}
\left(\nabla_{X}^{h} C_{T}\right) Y-\left(\nabla_{Y}^{h} C_{T}\right) X=C_{\nabla_{X}^{v} T} Y-C_{\nabla_{Y}^{v} T} X \tag{7.5}
\end{equation*}
$$

holds for all $X, Y \in \Gamma\left(D^{\perp}\right)$ and $T \in \Gamma(D)$.
Proof: We first compute

$$
\begin{aligned}
\left(\nabla_{X}^{h} C_{T}\right) Y & =\nabla_{X}^{h} C_{T} Y-C_{T} \nabla_{X}^{h} Y \\
& =-\nabla_{X}^{h} \nabla_{Y}^{h} T-C_{T} \nabla_{X}^{h} Y \\
& =-\nabla_{X}^{h} \nabla_{Y} T+\nabla_{X}^{h} \nabla_{Y}^{v} T+\nabla_{\nabla_{X}^{h} Y}^{h} T \\
& =-\nabla_{X}^{h} \nabla_{Y} T-C_{\nabla_{Y}^{v} T} X+\nabla_{\nabla_{X}^{h} Y}^{h} T .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\left(\nabla_{X}^{h} C_{T}\right) Y-\left(\nabla_{Y}^{h} C_{T}\right) X & =-R^{h}(X, Y) T-\nabla_{[X, Y] v}^{h} T+C_{\nabla_{X}^{v} T} Y-C_{\nabla_{Y}^{v} T} X \\
& =C_{\nabla_{X}^{v} T} Y-C_{\nabla_{Y}^{v} T} X
\end{aligned}
$$

since $\nabla_{[X, Y]^{v}}^{h} T=0$ and $R(X, Y) T=0$ by the Gauss equation.
The shape operators in the following statement are considered restricted to $D^{\perp}$.
Proposition 7.3. The differential equation

$$
\begin{equation*}
\nabla_{T} A_{\xi}=A_{\xi} C_{T}+A_{\nabla \frac{1}{T} \xi} \tag{7.6}
\end{equation*}
$$

holds for all $T \in \Gamma(D)$ and $\xi \in \Gamma\left(N_{f} M\right)$. In particular, the endomorphism $A_{\xi} C_{T}$ of $D^{\perp}$ is symmetric, that is,

$$
\begin{equation*}
A_{\xi} C_{T}=C_{T}^{t} A_{\xi} . \tag{7.7}
\end{equation*}
$$

Proof: The Codazzi equation

$$
\left(\nabla_{T} A_{\xi}\right) X-A_{\nabla_{\frac{1}{T}} \xi} X=\left(\nabla_{X} A_{\xi}\right) T-A_{\nabla_{\frac{1}{X}} \xi} T
$$

for all $X \in \Gamma\left(D^{\perp}\right)$ and $T \in \Gamma(D)$ gives

$$
\left(\nabla_{T} A_{\xi}\right) X=-A_{\xi} \nabla_{X} T+A_{\nabla_{\frac{1}{T}} \xi} X,
$$

and (7.6) follows.

### 7.1.2 Submanifolds with umbilical conullity

Given an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ with constant index of relative nullity, the simplest possible structures of the splitting tensor of its relative nullity distribution $\Delta$ occur when either $C$ vanishes identically or $C(\Gamma(\Delta))$ is spanned by the identity endomorphism of the conullity distribution $\Delta^{\perp}$. As already observed right before Section 7.1.1, they correspond to the cases in which $\Delta^{\perp}$ is either totally geodesic or umbilical, respectively. In this section we determine the isometric immersions with either of these properties.

Given an isometric immersion $g: M^{n-k} \rightarrow \mathbb{R}^{m-k}$, set $M^{n}=M^{n-k} \times \mathbb{R}^{k}$ and define $f: M^{n} \rightarrow \mathbb{R}^{m}$ by $f=g \times \mathrm{id}$, where id: $\mathbb{R}^{k} \rightarrow \mathbb{R}^{k}$ is the identity map. We call $f$ a $k$-cylinder over $g$, or simply a cylinder over $g$. When it is unimportant which is the isometric immersion $g$, we just say that $f$ is a $k$-cylinder.

Proposition 7.4. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be an isometric immersion and let $D$ be a smooth totally geodesic distribution of rank $0<k<n$ such that $D(x) \subset \Delta(x)$ for all $x \in M^{n}$. If the distribution $D^{\perp}$ is totally geodesic, then $c=0$ and $f$ is locally a $k$-cylinder.

Proof: Since the splitting tensor $C$ of $D$ vanishes identically, it follows from (7.2) that $c=0$. Because the distribution $D^{\perp}$ is totally geodesic,

$$
\tilde{\nabla}_{X} f_{*} T=f_{*} \nabla_{X} T \in \Gamma\left(f_{*} D\right)
$$

for all $X \in \Gamma\left(D^{\perp}\right)$ and $T \in \Gamma(D)$, where $\tilde{\nabla}$ is the induced connection on $f^{*} T \mathbb{R}^{m}$. Thus $f_{*} D$ is constant in $\mathbb{R}^{m}$ along any leaf $\Sigma$ of $D^{\perp}$. Defining $M^{n-k}=\Sigma$ and $g=f \circ i$, where $i: \Sigma \rightarrow M^{n}$ is the inclusion, this implies that the immersion $g$ reduces codimension to $m-k$ and that $f$ coincides locally with the cylinder over $g$.

Corollary 7.5. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be an isometric immersion of a Riemannian product $M^{n}=M^{n-k} \times M^{k}$ such that the tangent spaces to the second factor are contained in the relative nullity subspaces of $f$ at any point. Then $c=0$ and there exist an isometric immersion $g: M^{n-k} \rightarrow \mathbb{R}^{m-k}$ and a local isometry $i: M^{k} \rightarrow \mathbb{R}^{k}$ such that $f=g \times i$, that is, $f(M)$ is an open subset of (the image of) a $k$-cylinder over $g$.

Let $g: M^{n-k} \rightarrow \mathbb{Q}_{\tilde{c}}^{m-k}$ be an isometric immersion and let $i: \mathbb{Q}_{\tilde{c}}^{m-k} \rightarrow \mathbb{Q}_{c}^{m}, \tilde{c} \geq c$, be an umbilical inclusion. Then the normal bundle of $\tilde{g}=i \circ g$ splits as

$$
N_{\tilde{g}} M^{n-k}=i_{*} N_{g} M^{n-k} \oplus N_{i} \mathbb{Q}_{\tilde{c}}^{m-k} .
$$

Thus we may regard $L=N_{i} \mathbb{Q}_{\tilde{c}}^{m-k}$ as a subbundle of $N_{\tilde{g}} M^{n-k}$. Define $f: L \rightarrow \mathbb{Q}_{c}^{m}$ by

$$
f(x, v)=\exp _{\tilde{g}(x)} v,
$$

where exp is the exponential map of $\mathbb{Q}_{c}^{m}$. We call the restriction of $f$ to the open subset of its regular points the generalized cone over $g$.

Proposition 7.6. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be an isometric immersion with constant index of relative nullity $\nu>0$. If the conullity distribution is umbilical, then $f$ coincides locally with the generalized cone over an isometric immersion $g: M^{n-\nu} \rightarrow \mathbb{Q}_{\tilde{c}}^{m-\nu}$ into an umbilical submanifold $\mathbb{Q}_{\tilde{c}}^{m-\nu}$ of $\mathbb{Q}_{c}^{m}$.

Proof: Let $j: \Sigma \rightarrow M^{n}$ be the inclusion of a leaf $\Sigma$ of $\Delta^{\perp}$ into $M^{n}$, and let $\tilde{g}=f \circ j$. Then the normal bundle $N_{\tilde{g}} \Sigma$ of $\tilde{g}$ splits as

$$
N_{\tilde{g}} \Sigma=f_{*} N_{j} \Sigma \oplus N_{f} M=f_{*} \Delta \oplus N_{f} M
$$

By assumption, there exists $S \in \Gamma(\Delta)$ such that

$$
C_{T}=\langle T, S\rangle I
$$

for all $T \in \Gamma(\Delta)$. Thus

$$
\begin{aligned}
\tilde{\nabla}_{X} f_{*} T & =f_{*} \nabla_{X} T+\alpha^{f}(X, T) \\
& =-f_{*} C_{T} X+f_{*} \nabla_{X}^{v} T \\
& =-\langle T, S\rangle f_{*} X+f_{*} \nabla_{X}^{v} T
\end{aligned}
$$

for all $T \in \Gamma(\Delta)$, where $\tilde{\nabla}$ is the induced connection on $f^{*} T \mathbb{Q}_{c}^{m}$. It follows that the subbundle $L=f_{*} \Delta$ of $N_{\tilde{g}} \Sigma$ is parallel with respect to the normal connection, and that the shape operator of $\tilde{g}$ with respect to any section $\eta=f_{*} T$ of $L$, with $T \in \Gamma(\Delta)$, is given by

$$
A_{\eta}^{\tilde{g}}=\langle T, S\rangle I .
$$

By Exercise 2.14, $\tilde{g}(\Sigma)$ is contained in an umbilical submanifold $\mathbb{Q}_{\tilde{c}}^{m-\nu}$ of $\mathbb{Q}_{c}^{m}$, that is, there exist an umbilical inclusion $i: \mathbb{Q}_{\tilde{c}}^{m-\nu} \rightarrow \mathbb{Q}_{c}^{m}$ and an isometric immersion $g: M^{n-\nu}=\Sigma \rightarrow \mathbb{Q}_{\tilde{c}}^{m-\nu}$ such that $\tilde{g}=i \circ g$. Moreover, at any $x \in \Sigma$ the fiber $L(x)=f_{*} \Delta(x)$ coincides with the normal space of $i$ at $g(x)$. Therefore the generalized cone over $g$ coincides locally with $f$.

### 7.2 Completeness of the relative nullity foliation

Most of the results of this section rely on the fundamental fact that the leaves of the minimum relative nullity distribution of a complete submanifold of a space form are also complete. This is a consequence of the following result.

Theorem 7.7. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be an isometric immersion and let $U \subset M^{n}$ be an open subset where the index of relative nullity $\nu=s$ is positive. If $\gamma:[0, b] \rightarrow M^{n}$ is a unit-speed geodesic such that $\gamma([0, b))$ is contained in a leaf of $\Delta$ in $U$, then $\Delta(\gamma(b))=$ $\mathcal{P}_{0}^{b}\left(\Delta(\gamma(0))\right.$, where $\mathcal{P}_{0}^{t}$ denotes the parallel transport along $\gamma$ from $\gamma(0)$ to $\gamma(t)$. Hence $\nu(\gamma(b))=s$. Moreover, the splitting tensor $C_{\gamma^{\prime}}$ extends smoothly to $\gamma(b)$ and (7.6) holds on $[0, b]$.

Proof: First we show that there exists a unique solution

$$
t \in[0, b) \mapsto T(t): \Delta^{\perp}(\gamma(t)) \rightarrow \Delta^{\perp}(\gamma(t))
$$

of the differential equation

$$
\begin{equation*}
\frac{D}{d t} T+C_{\gamma^{\prime}} T=0 \tag{7.8}
\end{equation*}
$$

with initial condition $T(0)=I$. To see this, choose an orthonormal basis $Y_{1}, \ldots, Y_{n-s}$ of $\Delta^{\perp}(\gamma(0))$ and parallel transport $Y_{j}$ along $\gamma[0, b)$ for $1 \leq j \leq n-s$. Since $\Delta$ is totally geodesic, then $\Delta^{\perp}$ is parallel along $\gamma$, and hence $Y_{j}(t) \in \Delta^{\perp}(\gamma(t))$ for all $t \in[0, b)$, $1 \leq j \leq n-s$. If $T(t): \Delta^{\perp}(\gamma(t)) \rightarrow \Delta^{\perp}(\gamma(t))$ is given by

$$
\begin{equation*}
T(t) Y_{j}(t)=\sum_{i=1}^{n-s} a_{i j}(t) Y_{i}(t) \tag{7.9}
\end{equation*}
$$

then (7.8) is equivalent to the ordinary linear differential matrix equation of first order

$$
\begin{equation*}
A^{\prime}(t)+C(t) A(t)=0 \tag{7.10}
\end{equation*}
$$

where $A(t)=\left(a_{i j}(t)\right)$ and $C(t)$ is the matrix of $C_{\gamma^{\prime}(t)}$ with respect to $Y_{1}(t), \ldots, Y_{n-s}(t)$. Thus the unique solution of (7.8) with initial condition $T(0)=I$ is given by 7.9), where $A(t)=\left(a_{i j}(t)\right)$ is the unique solution of the ordinary linear differential equation of first order 7.10 whose initial condition $A(0)$ is the identity matrix.

Next we argue that $T$ is also a solution on $[0, b)$ of the second order differential equation with a constant coefficient

$$
\frac{D^{2}}{d t^{2}} T+c T=0
$$

In fact, using (7.3) and (7.8) we have

$$
\begin{aligned}
-\frac{D}{d t}\left(\frac{D}{d t} T\right) & =\frac{D}{d t}\left(C_{\gamma^{\prime}} T\right) \\
& =\left(\frac{D}{d t} C_{\gamma^{\prime}}\right) T+C_{\gamma^{\prime}} \frac{D T}{d t} \\
& =\left(C_{\gamma^{\prime}}^{2}+c I\right) T-C_{\gamma^{\prime}}\left(C_{\gamma^{\prime}} T\right) \\
& =c T .
\end{aligned}
$$

It follows that $T$ extends smoothly to $t=b$ as an endomorphism of $\mathcal{P}_{0}^{b}\left(\Delta^{\perp}(\gamma(0))\right.$.
Now let $Z$ and $Y$ be parallel vector fields along $\gamma$ such that $Z$ is arbitrary and $Y(t) \in \Delta^{\perp}(\gamma(t))$ in $[0, b)$. Denoting by $X$ an extension of $\gamma^{\prime}$ in $U$, along $\gamma$ we have

$$
\begin{aligned}
\nabla_{\gamma^{\prime}}^{\perp} \alpha(T Y, Z) & =\left(\nabla_{X}^{\perp} \alpha\right)(T Y, Z)+\alpha(D T Y / d t, Z) \\
& =\left(\nabla_{T Y}^{\perp} \alpha\right)(X, Z)+\alpha(D T Y / d t, Z) \\
& =-\alpha\left(\nabla_{T Y} X, Z\right)+\alpha(D T Y / d t, Z) \\
& =\alpha\left(C_{\gamma^{\prime}} T Y+D T Y / d t, Z\right) \\
& =0 .
\end{aligned}
$$

Thus $\alpha(T Y, Z)$ is parallel along $\gamma$, and hence $T$ is invertible in $[0, b]$. Moreover, since $\mathcal{P}_{0}^{b}\left(\Delta(\gamma(0)) \subset \Delta(\gamma(b))\right.$ by continuity, it follows that $\mathcal{P}_{0}^{b}\left(\Delta^{\perp}(\gamma(0))=\Delta^{\perp}(\gamma(b))\right.$. In addition, $C_{\gamma^{\prime}}$ extends smoothly to $[0, b]$ as $C_{\gamma^{\prime}}=-D T / d t \circ T^{-1}$.

Corollary 7.8. Let $M^{n}$ be a complete Riemannian manifold and let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be an isometric immersion with positive index of relative nullity $\nu$ at any point. Then the leaves of the relative nullity distribution are complete on the open subset where $\nu=\nu_{0}$ is minimal.

### 7.2.1 The case of pairs of immersions

We discuss next the case of pairs of isometric immersions. The results in this section will be of use in Chapter 13 .

Given isometric immersions $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+q}$ with second fundamental forms $\alpha$ and $\hat{\alpha}$, respectively, endow the vector bundle $N_{f} M \oplus N_{\hat{f}} M$ with the indefinite metric of signature $(p, q)$ given by

$$
\langle\langle(\xi, \hat{\xi}),(\eta, \hat{\eta})\rangle\rangle_{N_{f} M \oplus N_{\hat{f}} M}=\langle\xi, \eta\rangle_{N_{f} M}-\langle\hat{\xi}, \hat{\eta}\rangle_{N_{\hat{f}} M}
$$

and the compatible connection $\nabla^{*}=\left({ }^{f} \nabla^{\perp},{ }^{f} \nabla^{\perp}\right)$. Then the symmetric bilinear map

$$
\beta=\alpha \oplus \hat{\alpha}: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \Gamma\left(N_{f} M \oplus N_{\hat{f}} M\right),
$$

which can be regarded as a section of $\operatorname{Hom}^{2}\left(T M, T M ; N_{f} M \oplus N_{\hat{f}} M\right)$, satisfies the Codazzi-type equation

$$
\begin{equation*}
\left(\nabla_{X}^{*} \beta\right)(Y, Z)=\left(\nabla_{Y}^{*} \beta\right)(X, Z) \tag{7.11}
\end{equation*}
$$

for all $X, Y, Z \in \mathfrak{X}(M)$. At $x \in M^{n}$ denote

$$
\Delta^{*}(x)=\mathcal{N}(\beta)(x)=\Delta_{f}(x) \cap \Delta_{\hat{f}}(x)
$$

and $\nu^{*}(x)=\operatorname{dim} \Delta^{*}(x)$. Clearly, the distribution $\Delta^{*}$ is smooth and totally geodesic along any open subset of $M^{n}$ where $\nu^{*}$ is constant.

Taking into account that the proof of Theorem 7.7 relies only on the fact that the second fundamental form of an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ satisfies the Codazzi equation, its proof can be easily adapted to yield the following results for pairs of isometric immersions.

Theorem 7.9. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+q}$ be isometric immersions, and let $U \subset M^{n}$ be an open set where the dimension of $\Delta^{*}$ satisfies $\nu^{*}=s>0$. If $\gamma:[0, b] \rightarrow M^{n}$ is a geodesic such that $\gamma([0, b))$ is contained in a leaf of $\Delta^{*}$ in $U$, then also $\nu^{*}(\gamma(b))=s$. Moreover, the splitting tensor $C_{\gamma^{\prime}}$ of $\Delta^{*}$ extends smoothly to $\gamma(b)$ and equation (7.6) holds on $[0, b]$.

Corollary 7.10. Let $M^{n}$ be a complete Riemannian manifold, and let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ and $\hat{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+q}$ be isometric immersions such that the index $\nu^{*}$ is positive at any point. Then the leaves of $\Delta^{*}$ are complete on the open subset where $\nu^{*}=\nu_{0}^{*}$ is minimal.

### 7.2.2 The spherical case

The completeness of the leaves of relative nullity has strong consequences for complete submanifolds of the sphere with positive index of relative nullity at any point, as shown by the main result in this section.

Theorem 7.11. Let $f: M^{n} \rightarrow \mathbb{S}_{c}^{m}$ be an isometric immersion of a complete Riemannian manifold with positive index of relative nullity $\nu$ at any point. Then, at any point where $\nu=\nu_{0}$ is minimal, and for any normal direction at that point, the numbers of positive and negative principal curvatures are equal.

Proof: Let

$$
U=\left\{x \in M^{n}: \nu(x)=\nu_{0}\right\}
$$

and let $\gamma:[0, \infty) \rightarrow M^{n}$ be a geodesic contained in a leaf $L$ of the relative nullity foliation of $U$. Take $\xi \in N_{\gamma(0)} M$ and let $\xi_{t}$ be its parallel transport along $\gamma$ with respect to the normal connection. It follows from (7.6) and (7.7) that the equation

$$
\begin{aligned}
\nabla_{\gamma^{\prime}(t)} A_{\xi_{t}} & =A_{\xi_{t}} C_{\gamma^{\prime}(t)} \\
& =C_{\gamma^{\prime}(t)}^{t} A_{\xi_{t}}
\end{aligned}
$$

holds for $\left.A_{\xi_{t}}\right|_{\Delta^{\perp}}$. Let $Z_{1}(t), \ldots, Z_{n-\nu_{0}}(t)$ be a parallel orthonormal frame of $\Delta^{\perp}$ along $\gamma$. Then the matrix $A(t)$ of $A_{\xi_{t} \mid \Delta_{\perp}}$ with respect to $Z_{1}(t), \ldots, Z_{n-\nu_{0}}(t)$ satisfies the linear differential equation

$$
A^{\prime}(t)=C^{t}(t) A(t)
$$

where $C(t)$ is the matrix of $C_{\gamma^{\prime}(t)}$ with respect to $Z_{1}(t), \ldots, Z_{n-\nu_{0}}(t)$. In Exercise 7.3 the reader is asked to prove that the rank of $A(t)$ is constant on $[0, \infty)$. It follows that $A_{\xi_{t}}$ has constant rank along $\gamma$, and hence the numbers of positive and negative eigenvalues remain constant along $\gamma$.

On the other hand, according to Corollary 7.8, $f$ embeds the leaf $L$ onto a totally geodesic sphere $\mathbb{S}^{\nu_{0}}$. Hence the antipodal map $I=-\mathrm{id}: \mathbb{S}_{c}^{m} \rightarrow \mathbb{S}_{c}^{m}$ induces an involution $\tau$ on $U$ satisfying $f \circ \tau=I \circ f$. Therefore, at any $x \in U$, the second fundamental form of $f$ satisfies

$$
A_{I_{*} \xi} \tau_{*} X=\tau_{*} A_{\xi} X
$$

for all $X \in T_{x} M$ and $\xi \in N_{f} M(x)$.
Let $i: \mathbb{S}_{c}^{m} \rightarrow \mathbb{R}^{m+1}$ denote the standard inclusion and let $\tilde{\nabla}$ be the induced connection on $(i \circ f)^{*} T \mathbb{R}^{m+1}$. Then

$$
\begin{aligned}
\tilde{\nabla}_{\gamma^{\prime}(t)} i_{*} \xi_{t} & =i_{*}\left(-f_{*} A_{\xi_{t}} \gamma^{\prime}(t)+\nabla_{\gamma^{\prime}(t)}^{\perp} \xi_{t}\right) \\
& =0,
\end{aligned}
$$

hence $i_{*} \xi_{t}$ is constant in $\mathbb{R}^{m+1}$ along $\gamma$. In particular, $\xi(\tau(x))=-I_{*} \xi(x)$, and thus

$$
A_{\xi(\tau(x))} \tau_{*} X=-\tau_{*} A_{\xi(x)} X
$$

It follows that the number of positive eigenvalues of $A_{\xi(x)}$ is equal to the number of negative eigenvalues of $A_{\xi(\tau(x))}$. Since the numbers of positive and negative eigenvalues are constant along $\gamma$, the statement follows.

Corollary 7.12. Let $f: M^{n} \rightarrow \mathbb{S}_{c}^{m}$ be an isometric immersion of a complete Riemannian manifold with positive index of relative nullity $\nu$ at any point. Suppose that at some point $x \in M^{n}$ where $\nu$ is minimal the Ricci curvature satisfies Ric $_{M} \geq c$. Then $f$ is totally geodesic.

Proof: If $f$ was not totally geodesic at $x$, we would have $\mathcal{H} \neq 0$ from (3.8), and then $\left.A_{\mathcal{H}}\right|_{\Delta^{\perp}}$ would be positive definite at $x$, in contradiction with Theorem 7.11.

By the preceding corollary, if an isometric immersion $f: M^{n} \rightarrow \mathbb{S}_{c}^{m}$ of a complete Riemannian manifold is not totally geodesic and has constant positive index of relative nullity, then $\operatorname{Ric}_{M}<c$ at any point. Examples of this situation are the minimal (homogeneous) isoparametric hypersurfaces with three distinct principal curvatures (cf. [249] and [250]). Recall that a hypersurface is called isoparametric if all of its principal curvatures are constant.

An important consequence of Corollary 7.12 is that the totally geodesic inclusion $i: \mathbb{S}_{c}^{n} \rightarrow \mathbb{S}_{c}^{n+p}$ is rigid if $1 \leq p \leq n-1$.

Corollary 7.13. If $f: \mathbb{S}_{c}^{n} \rightarrow \mathbb{S}_{c}^{n+p}$ is an isometric immersion with $1 \leq p \leq n-1$, then $f$ is totally geodesic.

Proof: Since the index of nullity $\mu$ is constant and equal to $n$, then Theorem 4.9 implies that $\nu \geq n-p>0$ at any point. The statement now follows from Corollary 7.12,

Example 7.14. There exist nontotally geodesic isometric immersions $f: \mathbb{S}_{c}^{n} \rightarrow \mathbb{S}_{c}^{2 n+1}$. For instance, let $\phi: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{2 n+2}$ be defined by

$$
\phi\left(x_{1}, \ldots, x_{n+1}\right)=\frac{1}{\sqrt{n+1}}\left(e^{i \sqrt{n+1} x_{1}}, \ldots, e^{i \sqrt{n+1} x_{n+1}}\right)
$$

and note that $\phi\left(\mathbb{R}^{n+1}\right) \subset \mathbb{S}^{2 n+1} \subset \mathbb{R}^{2 n+2}$. If $i: \mathbb{S}^{n} \rightarrow \mathbb{R}^{n+1}$ is the standard inclusion, then the map $f: \mathbb{S}^{n} \rightarrow \mathbb{S}^{2 n+1} \subset \mathbb{R}^{2 n+2}$, given by $f=\phi \circ i$, is a nontotally geodesic isometric immersion.

### 7.2.3 The Euclidean case

We now consider isometric immersions $f: M^{n} \rightarrow \mathbb{R}^{m}$ whose index of relative nullity is positive at any point. The simplest examples are the $k$-cylinders. The main result of this section is due to Hartman and states that these are the only possible complete examples with nonnegative Ricci curvature.

Theorem 7.15. Let $M^{n}$ be a complete manifold with nonnegative Ricci curvature and let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be an isometric immersion with minimal index of relative nullity $\nu_{0}>0$. Then $f$ is a $\nu_{0}$-cylinder.

The main step in the proof of Theorem 7.15 is Lemma 7.16 below. Given an isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$, that $f(M)$ contains $s$ linearly independent lines means that there exist $s$ everywhere minimizing geodesics in $M^{n}$ that intersect at some point, have linearly independent tangent vectors at that point and are mapped by $f$ onto straight lines in $\mathbb{R}^{m}$ (therefore span an $s$-dimensional affine subspace of $\mathbb{R}^{m}$ ).

Lemma 7.16. Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ be an isometric immersion such that $f\left(\mathbb{R}^{n}\right)$ contains $s$ linearly independent lines. Then $f$ is an s-cylinder.

Proof: We may suppose that $n=2$ and $s=1$. Let $L$ be a straight line in $\mathbb{R}^{2}$ such that $\tilde{L}=f(L)$ is a straight line in $\mathbb{R}^{m}$. First we show that any straight line $r$ orthogonal to $L$ is mapped by $f$ into the hyperplane $H$ through $f(L \cap r)$ orthogonal to $\tilde{L}$.

Choose coordinates $(x, y)$ in $\mathbb{R}^{2}$ and $(u, v)=\left(u_{1}, \ldots, u_{m-1}, v\right)$ in $\mathbb{R}^{m}$ such that $L$, $r, \tilde{L}$ and $H$ have equations $x=0, y=0, u=0$ and $v=0$, respectively. Write

$$
f(x, y)=(u(x, y), v(x, y)) .
$$

Since $f$ maps $L$ isometrically onto $\tilde{L}$ and the point $O=r \cap L$ is mapped by $f$ into $H$, we see that $u(0, y)=0$ and $v(0, y)=y$ for all $y \in \mathbb{R}$, after changing $v$ by $-v$, if necessary. Then, we must prove that $v(x, 0)=0$ for all $x \in \mathbb{R}$.

We denote by $d$ and $\tilde{d}$ the distances in $\mathbb{R}^{2}$ and $\mathbb{R}^{m}$, respectively. Take $p=\left(x_{0}, 0\right)$ and suppose that $v\left(x_{0}, 0\right) \neq 0$. Then we may assume that

$$
\begin{equation*}
v\left(x_{0}, 0\right)=c>0 \tag{7.12}
\end{equation*}
$$

Choose $q=\left(0, y_{0}\right) \in L$ such that

$$
\begin{equation*}
y_{0}<0 \text { and } 0<d(p, q)-d(O, q) \leq c / 2 \tag{7.13}
\end{equation*}
$$

From (7.12), (7.13) and the fact that $f$ is an isometric immersion, we obtain

$$
\left|y_{0}\right|+c \leq \tilde{d}(f(p), f(q)) \leq d(p, q) \leq\left|y_{0}\right|+c / 2,
$$

and that is not possible since $c>0$. Hence $v\left(x_{0}, 0\right)=0$, as we wished to prove.
It follows that the function $v(x, y)$ depends only on $y$, and hence $v(x, y)=y$, because $v(0, y)=y$. Now, from

$$
1=\|\partial f / \partial y\|^{2}=\left\|u_{y}(x, y)\right\|^{2}+1
$$

we conclude that $u_{y}(x, y)=0$, and thus $u(x, y)=u(x)$.
Proof of Theorem 7.15. It follows from Corollary 7.8 that $M^{n}$ contains $\nu_{0}$ linearly independent lines through each point where the index of relative nullity is minimal. By the splitting theorem of Cheeger-Gromoll, the Riemannian manifold $M^{n}$ is isometric to a Riemannian product $N^{n-\nu_{0}} \times \mathbb{R}^{\nu_{0}}$, and we may consider $f: N^{n-\nu_{0}} \times \mathbb{R}^{\nu_{0}} \rightarrow \mathbb{R}^{m}$. Fix a point $x_{0} \in N^{n-\nu_{0}}$ and let $\gamma: \mathbb{R} \rightarrow N^{n-\nu_{0}}$ be any smooth unit-speed curve such that $\gamma(0)=x_{0}$. Consider the isometric immersion $f_{\gamma}: \mathbb{R} \times \mathbb{R}^{\nu_{0}} \rightarrow \mathbb{R}^{m}$ given by

$$
f_{\gamma}(t, y)=f(\gamma(t), y)
$$

By Lemma 7.16 we have a splitting $\mathbb{R}^{m}=\mathbb{R}^{m-\nu_{0}} \times \mathbb{R}^{\nu_{0}}$ such that

$$
f_{\gamma}(t, y)=\left(h_{\gamma}(t), y\right)
$$

for some smooth map $h_{\gamma}: \mathbb{R} \rightarrow \mathbb{R}^{m-\nu_{0}}$. Since the subspace $\mathbb{R}^{\nu_{0}}$ in the orthogonal decomposition $\mathbb{R}^{m}=\mathbb{R}^{m-\nu_{0}} \times \mathbb{R}^{\nu_{0}}$ is the image $\mathbb{R}^{\nu_{0}}=f_{\gamma_{*}}(t, y) \mathbb{R}^{\nu_{0}}=f_{*}(\gamma(t)) \mathbb{R}^{\nu_{0}}$ of the factor $\mathbb{R}^{\nu_{0}}$ in the orthogonal decomposition $M^{n}=N^{n-\nu_{0}} \times \mathbb{R}^{\nu_{0}}$ for every $t \in \mathbb{R}$, it follows that the splitting $\mathbb{R}^{m}=\mathbb{R}^{m-\nu_{0}} \times \mathbb{R}^{\nu_{0}}$ depends neither on $x_{0}$ nor on the curve $\gamma$. Moreover, the map $h: N^{n-\nu_{0}} \rightarrow \mathbb{R}^{m-\nu_{0}}$ defined by

$$
h(x)=h_{\gamma}(t),
$$

where $\gamma$ is any smooth unit-speed curve in $N^{n-\nu_{0}}$ such that $\gamma(0)=x_{0}$ and $\gamma(t)=x$, is well defined, for if $\tilde{\gamma}$ is another such curve with $\tilde{\gamma}(0)=x_{0}$ and $\tilde{\gamma}(\tilde{t})=x$, then

$$
\left(h_{\gamma}(t), y\right)=f_{\gamma}(t, y)=f(\gamma(t), y)=f(\tilde{\gamma}(\tilde{t}), y)=f_{\tilde{\gamma}}(\tilde{t}, y)=\left(h_{\tilde{\gamma}}(\tilde{t}), y\right)
$$

for any $y \in \mathbb{R}^{\nu_{0}}$, hence $h_{\gamma}(t)=h_{\tilde{\gamma}}(\tilde{t})$. Thus

$$
f(x, y)=f(\gamma(t), y)=f_{\gamma}(t, y)=\left(h_{\gamma}(t), y\right)=(h(x), y)
$$

for all $x \in N^{n-\nu_{0}}$ and $y \in \mathbb{R}^{\nu_{0}}$, which implies that $h$ is an isometric immersion and concludes the proof.

The next result follows immediately from Theorem 7.15 and the Chern-Kuiper inequality (4.6).

Corollary 7.17. If $f: M^{n} \rightarrow \mathbb{R}^{n+p}, n \geq 2$ and $1 \leq p \leq n-1$, is an isometric immersion of a complete flat Riemannian manifold, then $f$ is a $(n-p)$-cylinder.

### 7.3 The Gauss parametrization

The aim of this section is to describe a useful parametrization of any oriented hypersurface $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ with constant index of relative nullity $\nu=k$.

The case of hypersurfaces of the sphere and the hyperbolic space will be derived by considering their cones in Euclidean and Lorentzian spaces, respectively. Since cones in the Lorentzian space over hypersurfaces of the hyperbolic space are Lorentzian hypersurfaces, we start by developing the theory both for hypersurfaces of Euclidean space $\mathbb{R}^{n+1}$ and for Lorentzian hypersurfaces of Lorentzian space $\mathbb{L}^{n+1}$.

Let $\mathbb{R}_{\mu}^{n+1}$ stand for either $\mathbb{R}^{n+1}$ or $\mathbb{L}^{n+1}$, and let $\mathbb{S}_{1, \mu}^{n} \subset \mathbb{R}_{\mu}^{n+1}$ denote either the Euclidean unit sphere $\mathbb{S}^{n} \subset \mathbb{R}^{n+1}$ or the Lorentzian unit sphere (or de Sitter space) $\mathbb{S}_{1,1}^{n} \subset \mathbb{L}^{n+1}$, depending on whether $\mu=0$ or $\mu=1$, respectively. Hence

$$
\mathbb{S}_{1, \mu}^{n}=\left\{x \in \mathbb{R}_{\mu}^{n+1}:\langle x, x\rangle=1\right\},
$$

where $\langle$,$\rangle is the inner product in \mathbb{R}_{\mu}^{n+1}$. The starting point for the Gauss parametrization of an oriented hypersurface $f: M^{n} \rightarrow \mathbb{R}_{\mu}^{n+1}$ with constant index of relative nullity
$\nu=k$, with Lorentzian induced metric if $\mu=1$, is the fact that the Gauss map $\eta: M^{n} \rightarrow \mathbb{S}_{1, \mu}^{n}$ of such a hypersurface is constant in $\mathbb{R}_{\mu}^{n+1}$ along the leaves of the relative nullity distribution $\Delta$. Hence, if $U \subset M^{n}$ is an open saturated subset (meaning that $U$ is a union of leaves of $\Delta$ ), then $\eta$ induces an immersion $g: L^{n-k} \rightarrow \mathbb{S}_{1, \mu}^{n}$ on the quotient space $L^{n-k}$ of relative nullity leaves, given by

$$
g \circ \pi=\eta,
$$

where $\pi: U \rightarrow L^{n-k}$ is the projection. We also refer to $g$ in the sequel as the Gauss map of $f$. If $\mu=1$ we also assume that the relative nullity subspaces of $f$ are timelike subspaces in the induced metric, which is always the case if $f$ is the cone over a hypersurface of the hyperbolic space $\mathbb{H}^{n} \subset \mathbb{L}^{n+1}$. With this assumption, even in this case the quotient space $L^{n-k}$ becomes a Riemannian manifold.

The support function $\gamma \in C^{\infty}(M)$ of $f$, given by

$$
\gamma=\langle f, i \circ \eta\rangle,
$$

where $i: \mathbb{S}_{1, \mu}^{n} \rightarrow \mathbb{R}_{\mu}^{n+1}$ is the inclusion map, is also constant along the leaves of $\Delta$, hence it induces a function $\bar{\gamma} \in C^{\infty}(L)$ given by $\bar{\gamma} \circ \pi=\gamma$. The Gauss parametrization allows to recover $f$ by means of $g$ and $\bar{\gamma}$, at least locally.

Denote $h=i \circ g$. The basic observation is that the subspaces $h_{*} T_{\bar{x}} L$ and $f_{*} \Delta^{\perp}(x)$ coincide for any $\bar{x}=\pi(x) \in L^{n-k}$. This follows from the fact that

$$
\begin{equation*}
h_{*} \bar{X}=i_{*} \eta_{*} X=-f_{*} A X \tag{7.14}
\end{equation*}
$$

for any $\bar{X}=\pi_{*} X \in T_{\bar{x}} L$. Therefore, if $\Lambda=N_{g} L$ denotes the normal bundle of $g$, then $i_{*} \Lambda(\bar{x})$ can be identified with $f_{*} \Delta(x)$. Moreover, given any cross section $\xi: L^{n-k} \rightarrow U$ to the submersion $\pi: U \rightarrow L^{n-k}$, we can define a diffeomorphism $\varphi_{\xi}$ from $U$ onto an open neighborhood $V$ of the zero section of $\Lambda$, in such a way that $\xi(L)$ is mapped into the zero section of $\Lambda$ and the leaf of $\Delta$ through $x$ in $U$ into the fiber of $\Lambda$ at $\bar{x}=\pi(x)$. Explicitly,

$$
\varphi_{\xi}(x)=(\bar{x}, f(x)-f(\xi(\bar{x}))) .
$$

If $\theta_{\xi}: V \rightarrow U$ is the inverse of $\varphi_{\xi}$, then

$$
f\left(\theta_{\xi}(\bar{x}, v)\right)=f(\xi(\bar{x}))+i_{*} v
$$

which is well defined and smooth on the whole normal bundle $\Lambda$, but may be singular outside $V$.

In the sequel, we work with a natural cross section $\xi$ defined locally as follows: given $\bar{x}=\pi(x) \in L^{n-k}$, let $\xi(\bar{x})$ be the unique point on the leaf $\Sigma$ of $\Delta$ through $x$ such that $f(\xi(\bar{x}))$ is closest to the origin among the points on the affine subspace $f(\Sigma)$. This is equivalent to requiring the position vector $f(\xi(\bar{x}))$ to be orthogonal to $f_{*} \Delta(x)$, that is, $f(\xi(\bar{x}))$ must belong to the orthogonal complement of $f_{*} \Delta(x)$ in $\mathbb{R}_{\mu}^{n+1}$, which is

$$
f_{*} \Delta^{\perp}(x) \oplus \operatorname{span}\{\eta(x)\}=h_{*} T_{\bar{x}} L \oplus \operatorname{span}\{h(\bar{x})\} .
$$

We claim that

$$
f(\xi(\bar{x}))=\bar{\gamma}(\bar{x}) h(\bar{x})+h_{*} \operatorname{grad} \bar{\gamma}(\bar{x})
$$

where $\operatorname{grad} \bar{\gamma}$ denotes the gradient of $\bar{\gamma}$ in the induced metric of $L^{n-k}$. First, using that the vector $f(x)-f(\xi(\bar{x}))$ belongs to $f_{*} \Delta(x)$, we obtain

$$
\begin{aligned}
\langle f(\xi(\bar{x})), h(\bar{x})\rangle & =\langle f(\xi(\bar{x})), i(\eta(x))\rangle=\langle f(x), i(\eta(x))\rangle=\gamma(x) \\
& =\bar{\gamma}(\bar{x}) .
\end{aligned}
$$

On the other hand, given $\bar{Z} \in T_{\bar{x}} L$ and $Z \in \Delta^{\perp}(x)$ such that $\pi_{*}(x) Z=\bar{Z}$, we have

$$
\begin{aligned}
\left\langle f(\xi(\bar{x})), h_{*} \bar{Z}\right\rangle & =\left\langle f(\xi(\bar{x})), i_{*} \eta_{*} Z\right\rangle=\left\langle f(x), i_{*} \eta_{*} Z\right\rangle=Z(\gamma)=\bar{Z}(\bar{\gamma}) \\
& =\left\langle h_{*} \nabla \bar{\gamma}, h_{*} \bar{Z}\right\rangle,
\end{aligned}
$$

and the claim follows.
Observe that the natural cross section $\xi$ we have just constructed is global provided that the relative nullity leaves are complete. It follows that

$$
\begin{equation*}
f\left(\theta_{\xi}(\bar{x}, v)\right)=\bar{\gamma}(\bar{x}) h(\bar{x})+h_{*} \operatorname{grad} \bar{\gamma}(\bar{x})+i_{*} v, \tag{7.15}
\end{equation*}
$$

called the Gauss parametrization of $f$.
We have just proved the converse statement of the following theorem.
Theorem 7.18. Let $g: L^{n-k} \rightarrow \mathbb{S}_{1, \mu}^{n}$ be an isometric immersion of a Riemannian manifold and let $\gamma \in C^{\infty}(L)$. Denote $h=i \circ g$, where $i: \mathbb{S}_{1, \mu}^{n} \rightarrow \mathbb{R}_{\mu}^{n+1}$ is the inclusion map, and consider the map $\psi: \Lambda \rightarrow \mathbb{R}_{\mu}^{n+1}$ defined on $\Lambda=N_{g} L$ by

$$
\begin{equation*}
\psi(y, w)=\gamma(y) h(y)+h_{*} \operatorname{grad} \gamma(y)+i_{*} w . \tag{7.16}
\end{equation*}
$$

Then, on the open subset of regular points, $\psi$ is an immersed hypersurface with constant index of relative nullity $\nu=k$.

Conversely, any hypersurface of $\mathbb{R}_{\mu}^{n+1}$ having constant index of relative nullity $\nu=k$, and time-like relative nullity subspaces in the induced metric if $\mu=1$, can be parametrized in this way, at least locally. The parametrization is global if the leaves of the relative nullity distribution are complete.

The direct statement in the preceding theorem is a consequence of the assertion in part (ii) of the following result, in which we collect other relations between the geometric data of $g$ and $\psi$ for later use.

Proposition 7.19. The following assertions hold:
(i) The map $\psi$ is regular at $(y, w) \in \Lambda$ if and only if the self-adjoint operator

$$
P_{w}(y)=\gamma(y) I+\operatorname{Hess} \gamma(y)-A_{w}
$$

on $T_{y} L$ is nonsingular, where $A_{w}$ is the shape operator of $g$ with respect to $w$.
(ii) On the open subset $V$ of regular points, $\psi$ is an immersed hypersurface having the map $G: \Lambda \rightarrow \mathbb{S}_{1, \mu}^{n}$, given by

$$
\begin{equation*}
G(y, w)=g(y) \tag{7.17}
\end{equation*}
$$

as a Gauss map of rank $n-k$.
(iii) For any $(y, w) \in V$ there exists a map $j=j(y, w): T_{y} L \rightarrow T_{(y, w)} \Lambda$, which is an isometry onto the orthogonal complement $\Delta^{\perp}(y, w)$ of the relative nullity subspace $\Delta(y, w)=N_{g} L(y)$ of $\psi$ at $(y, w)$, such that

$$
\begin{equation*}
\nabla_{\xi} j X=0 \tag{7.18}
\end{equation*}
$$

for all $\xi \in \Delta(y, w)$ and $X \in \mathfrak{X}(L)$ and

$$
\begin{equation*}
A j=-j P_{w}^{-1} \tag{7.19}
\end{equation*}
$$

where $A$ is the shape operator of $\psi$ at $(y, w)$ with respect to $G$.
(iv) For any $(y, w) \in V$ the splitting tensor $C_{\xi}: \Delta^{\perp}(y, w) \rightarrow \Delta^{\perp}(y, w)$ of $\Delta$ with respect to $\xi \in \Delta(y, w)=N_{g} L(y)$ is related to the shape operator $A_{\xi}$ of $g$ at $y$ by

$$
\begin{equation*}
C_{\xi} j=j A_{\xi} P_{w}^{-1} \tag{7.20}
\end{equation*}
$$

(v) The Levi-Civita connections of the metrics $\langle,\rangle^{\prime}$ and $\langle$,$\rangle on L^{n-k}$ and $V$ induced by $g$ and $\psi$, respectively, are related by

$$
\begin{equation*}
\left\langle\nabla_{P_{w}^{-1} X}^{\prime} Y, Z\right\rangle^{\prime}=\left\langle\nabla_{j X} j Y, j Z\right\rangle \tag{7.21}
\end{equation*}
$$

for all $X, Y, Z \in \mathfrak{X}(L)$.
(vi) The normal connection of $g$ and the Levi-Civita connection of $\langle$,$\rangle are related by$

$$
\begin{equation*}
\left\langle\nabla_{j X} \xi, \eta\right\rangle=\left\langle\nabla_{P_{w}^{-1} X}^{\perp} \xi, \eta\right\rangle \tag{7.22}
\end{equation*}
$$

for all $X \in \mathfrak{X}(L)$ and $\xi, \eta \in \Gamma\left(N_{g} L\right)$. Here $\xi \in \Gamma\left(N_{g} L\right)$ is also regarded as an element of $\Gamma(\Delta)$ by defining $\xi(y, w)=\xi(y)$ for all $(y, w) \in V$.

Proof: Given $y \in L^{n-k}$ and $w \in \Lambda(y)$, we denote by $\mathcal{V}(y, w)$ the vertical subspace of $T_{(y, w)} \Lambda$, the tangent space at $(y, w)$ to the fiber $\Lambda(y)$, which can be identified with $\Lambda(y)$ itself. Any vector in $T_{(y, w)} \Lambda$ that does not belong to $\mathcal{V}(y, w)$ is given by $\zeta_{*} X$, where $X \in T_{y} L$ and $\zeta \in \Gamma\left(N_{g} U\right)$ is a local section of $N_{g} L$ with $\zeta(y)=w$ on an open neighborhood $U$ of $y$. We choose such a local section satisfying

$$
\zeta(y)=w \text { and } \nabla \frac{\perp}{X} \zeta+\alpha(X, \operatorname{grad} \gamma)=0
$$

for all $X \in T_{y} L$ (see Exercise 7.6). Then, differentiating (7.16) we obtain

$$
\begin{equation*}
\psi_{*}(y, w) \zeta_{*}(y) X=h_{*}(y) P_{w}(y) X \tag{7.23}
\end{equation*}
$$

for all $X \in T_{y} L$. Since $\psi_{*}(y, w)$ is the identity map on $\mathcal{V}(y, w)$, the assertion in part (i) follows from 7.23).

On the open subset of regular points of $\psi$, one can also write (7.23) as

$$
\begin{equation*}
\psi_{*} j=h_{*}, \tag{7.24}
\end{equation*}
$$

where $j=j(y, w): T_{y} L \rightarrow T_{(y, w)} \Lambda$ is given by

$$
\begin{equation*}
j(y, w)=\zeta_{*}(y) P_{w}^{-1}(y) \tag{7.25}
\end{equation*}
$$

In other words, if $X \in T_{y} L$ and $\beta: I \rightarrow L^{n-k}$ is a smooth curve defined on an open interval $0 \in I$ such that $\beta(0)=y$ and $\beta^{\prime}(0)=P_{w}^{-1} X$, then

$$
\begin{equation*}
j(y, w) X=(\zeta \circ \beta)^{\prime}(0) . \tag{7.26}
\end{equation*}
$$

In particular, if $\hat{\pi}: \Lambda \rightarrow L^{n-k}$ is the projection, then

$$
\begin{equation*}
\hat{\pi}_{*} j=P_{w}^{-1} \tag{7.27}
\end{equation*}
$$

It follows from (7.24) that $j\left(T_{y} L\right)$ is the orthogonal complement of $\mathcal{V}(y, w)$ in $T_{(y, w)} \Lambda$ with respect to the metric induced by $\psi$, and that the map $G: \Lambda \rightarrow \mathbb{S}_{1, \mu}^{n}$, given by (7.17), is a Gauss map for $\psi$. Moreover,

$$
\begin{equation*}
-\psi_{*} A \zeta_{*}=i_{*} G_{*} \zeta_{*}=(i \circ G \circ \zeta)_{*}=h_{*} \tag{7.28}
\end{equation*}
$$

by the Weingarten equation of $\psi$. Using (7.24) and (7.28) we obtain

$$
\begin{aligned}
-\psi_{*} A j & =-\psi_{*} A \zeta_{*} P_{w}^{-1} \\
& =h_{*} P_{w}^{-1} \\
& =\psi_{*} j P_{w}^{-1},
\end{aligned}
$$

and (7.19) follows. Thus $A$ is nonsingular on $j\left(T_{y} L\right)$. Since $\mathcal{V}(y, w)$ clearly belongs to $\Delta(y, w)$ and $j\left(T_{y} L\right)=\mathcal{V}^{\perp}(y, w)$, it follows that $\mathcal{V}(y, w)=\Delta(y, w)$, and hence

$$
j\left(T_{y} L\right)=\Delta^{\perp}(y, w)
$$

Thus $G$ has rank $n-k$. That $j: T_{y} L \rightarrow \Delta^{\perp}(y, w)$ is an isometry follows from 7.24).
Now, using (7.24) for the second equality, we have

$$
\begin{aligned}
\psi_{*} \nabla_{\xi} j X & =\tilde{\nabla}_{\xi} \psi_{*} j X \\
& =\tilde{\nabla}_{\xi} h_{*} X \\
& =0
\end{aligned}
$$

for all $\xi \in \Delta(y, w)=N_{g} L(y)$ and $X \in \mathfrak{X}(L)$. This proves (7.18).

It remains to prove the assertions in parts (iv) to (vi). Using (7.24) and the Gauss formulas for $\psi$ and $g$, we obtain

$$
\begin{aligned}
\left\langle\nabla_{j X} j Y, j Z\right\rangle & =\left\langle\psi_{*} \nabla_{j X} j Y, \psi_{*} j Z\right\rangle \\
& =\left\langle\tilde{\nabla}_{j X} \psi_{*} j Y, \psi_{*} j Z\right\rangle \\
& =\left\langle\tilde{\nabla}_{P_{w}^{-1} X} h_{*} Y, h_{*} Z\right\rangle \\
& =\left\langle h_{*} \nabla_{P_{w}^{-1} X}^{\prime} Y, h_{*} Z\right\rangle \\
& =\left\langle\nabla_{P_{w}^{-1} X}^{\prime-} Y, Z\right\rangle^{\prime}
\end{aligned}
$$

for all $X, Y, Z \in \mathfrak{X}(L)$, which proves the assertion in part $(v)$. In the second and third terms in the preceding computation, as well as in the next computation, $\tilde{\nabla}$ stands both for the connection on $\psi^{*} T \mathbb{R}_{\mu}^{n+1}$ and for that on $h^{*} T \mathbb{R}_{\mu}^{n+1}$, and in the third equality we have used (7.26).

On the other hand, it follows from the Gauss formula for $\psi$ and the Weingarten formula for $h$ that

$$
\begin{aligned}
\psi_{*} \nabla_{j X} \xi & =\tilde{\nabla}_{j X} \psi_{*} \xi \\
& =\tilde{\nabla}_{P_{w}^{-1} X^{*}} \xi \\
& =-h_{*} A_{\xi} P_{w}^{-1} X+i_{*} \nabla_{P_{w}^{-1} X}^{\perp} \xi
\end{aligned}
$$

for all $X \in \mathfrak{X}(L)$ and $\xi \in \Gamma(\Delta)=\Gamma\left(N_{g} L\right)$. This implies that

$$
\begin{aligned}
\left\langle C_{\xi} j X, j Y\right\rangle & =-\left\langle\nabla_{j X} \xi, j Y\right\rangle \\
& =-\left\langle\psi_{*} \nabla_{j X} \xi, \psi_{*} j Y\right\rangle \\
& =\left\langle h_{*} A_{\xi} P_{w}^{-1} X, h_{*} Y\right\rangle \\
& =\left\langle A_{\xi} P_{w}^{-1} X, Y\right\rangle^{\prime} \\
& =\left\langle j A_{\xi} P_{w}^{-1} X, j Y\right\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
\left\langle\nabla_{j X} \xi, \eta\right\rangle & =\left\langle\psi_{*} \nabla_{j X} \xi, \psi_{*} \eta\right\rangle \\
& =\left\langle\nabla_{P_{w}^{-1} X}^{\perp} \xi, \eta\right\rangle
\end{aligned}
$$

for all $X, Y \in \mathfrak{X}(L)$ and $\xi, \eta \in \Gamma\left(N_{g} L\right)$. Thus the assertions in parts (iv) and (vi) hold.

Corollary 7.20. Let $c: I \rightarrow \mathbb{S}_{1, \mu}^{n}$ be a smooth curve parametrized by arc-length, let $e_{1}, \ldots, e_{n-1}$ be any frame of $N_{c} I$, and choose an arbitrary $\gamma \in C^{\infty}(I)$. Then the map $F: I \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}_{\mu}^{n+1}$, defined by

$$
F\left(s, t_{1}, \ldots, t_{n-1}\right)=\gamma(s) c(s)+\gamma^{\prime}(s) c^{\prime}(s)+\sum_{i=1}^{n-1} t_{i} e_{i}(s)
$$

parametrizes, at regular points, a flat hypersurface without totally geodesic points.
Conversely, any flat hypersurface of $\mathbb{R}_{\mu}^{n+1}$ without totally geodesic points, and time-like relative nullity subspaces if $\mu=1$, can be parameterized in this way, at least locally.

As an application of the formulas derived in Proposition 7.19, we characterize in the next result the hypersurfaces of $\mathbb{R}_{\mu}^{n+1}$ that carry a relative nullity distribution $\Delta$ of rank $\nu=k$, with time-like fibers if $\mu=1$, such that $\Delta^{\perp}$ is integrable.

Corollary 7.21. Under the assumptions of Theorem 7.18, the following assertions are equivalent:
(i) The distribution $\Delta^{\perp}$ is integrable.
(ii) The surface $g: L^{2} \rightarrow \mathbb{S}_{1, \mu}^{n}$ has flat normal bundle and $\left[\right.$ Hess $\left.\gamma, A_{w}\right]=0$ for all $w \in \Gamma\left(N_{g} L\right)$.

Proof: The distribution $\Delta^{\perp}$ is integrable if and only if the splitting tensor $C_{w^{\prime}}$ is symmetric for all $(y, w) \in V$ and $w^{\prime} \in \Delta(y, w)=N_{g} L(y)$. By (7.20) this is equivalent to

$$
A_{w^{\prime}} P_{w}^{-1}=P_{w}^{-1} A_{w^{\prime}}
$$

for all $y \in L^{2}$ and $w, w^{\prime} \in N_{g} L(y)$, which is the same as

$$
\begin{equation*}
\left[P_{w}, A_{w^{\prime}}\right]=0 \tag{7.29}
\end{equation*}
$$

for all $y \in L^{2}$ and $w, w^{\prime} \in N_{g} L(y)$. This condition is trivially satisfied if (ii) holds. Conversely, applying (7.29) to $w=0$ and $w^{\prime}$ yields

$$
\left[\operatorname{Hess} \gamma, A_{w}\right]=0
$$

for all $w \in \Gamma\left(N_{g} L\right)$, and hence

$$
\left[A_{w}, A_{w^{\prime}}\right]=0
$$

for all $y \in L^{2}$ and $w, w^{\prime} \in N_{g} L(y)$, that is, $g$ has flat normal bundle.
According to Theorem 7.18, the Gauss map of a hypersurface $f: M^{n} \rightarrow \mathbb{R}_{\mu}^{n+1}$ with constant index of relative nullity $k$, and time-like relative nullity subspaces if $\mu=1$, can be arbitrarily prescribed, and all hypersurfaces with a given Gauss map $g$ are parametrized by an arbitrary ("support") function. This also applies for $k=0$, in which case the map $\psi$ in 7.16 becomes

$$
\psi=\gamma h+h_{*} \operatorname{grad} \gamma,
$$

and is just the inverse of the Gauss map. This justifies the terminology "Gauss parametrization."

We now turn to the case of hypersurfaces of $\mathbb{Q}_{\epsilon}^{n+1}, \epsilon \in\{-1,1\}$.

Corollary 7.22. Let $g: L^{n-k} \rightarrow \mathbb{S}_{1, \mu}^{n+1}$ be an isometric immersion of a Riemannian manifold. Define

$$
\Lambda_{\epsilon}=\left\{(y, w) \in \Lambda=N_{g} L:\langle w, w\rangle=\epsilon\right\}, \epsilon=1-2 \mu
$$

and consider the map $\psi: \Lambda_{\epsilon} \rightarrow \mathbb{Q}_{\epsilon}^{n+1}$ defined by

$$
\begin{equation*}
\psi(y, w)=w \tag{7.30}
\end{equation*}
$$

Then the following assertions hold:
(i) On the open subset $V$ of regular points, $\psi$ is an immersed hypersurface with constant index of relative nullity $\nu=k$, having the map $G: \Lambda_{\epsilon} \rightarrow \mathbb{S}_{1, \mu}^{n+1}$, given by

$$
G(y, w)=g(y)
$$

as a Gauss map.
(ii) Conversely, any hypersurface of $\mathbb{Q}_{\epsilon}^{n+1}$ with constant index of relative nullity $\nu=k$ can be parametrized in this way, at least locally.
(iii) The map $\psi$ is regular at $(y, w) \in \Lambda_{\epsilon}$ if and only if the shape operator $A_{w}$ of $g$ at $y \in L^{n-k}$ is nonsingular.
(iv) At any $(y, w) \in V$, there exists an isometry $j=j(y, w): T_{y} L \rightarrow \Delta^{\perp}(y, w)$ onto the orthogonal complement of the relative nullity subspace $\Delta(y, w)$ of $\psi$ at $(y, w)$, such that the shape operator $A$ of $\psi$ at $(y, w)$ with respect to $G$ satisfies

$$
A j=j A_{w}^{-1}
$$

Proof: Extend $\psi$ to all of $\Lambda$ by 7.30 , thus parametrizing the cone over $\psi$ in $\mathbb{R}_{\mu}^{n+2}$ through the origin. By Proposition 7.19, on the open subset of regular points, the map $\psi: \Lambda \rightarrow \mathbb{R}_{\mu}^{n+2}$ is an immersed hypersurface of constant index of relative nullity $\nu=k+1$. Assertions (i), (iii) and (iv) then follow from the relation between the second fundamental forms of a hypersurface of $\mathbb{Q}_{\epsilon}^{n+1}$ and the cone over it in $\mathbb{R}_{\mu}^{n+2}$.

It remains to prove part (ii). Consider the cone over the hypersurface, which has zero support function and constant index of relative nullity $\nu=k+1$, with the same Gauss image $g: L^{n-k} \rightarrow \mathbb{S}_{1, \mu}^{n+1}$. Thus it can be parametrized in terms of the Gauss parametrization by $\psi: \Lambda=N_{g} L \rightarrow \mathbb{R}_{\mu}^{n+2}$ given by 7.30 . The restriction of $\psi$ to $\Lambda_{\epsilon}$ is therefore a parametrization of the original hypersurface.

The hypersurface $\psi: \Lambda_{\epsilon} \rightarrow \mathbb{R}_{\mu}^{n+2}$ defined by $(7.30)$ is called the polar map of $g: L^{n-k} \rightarrow \mathbb{S}_{1, \mu}^{n+1} \subset \mathbb{R}_{\mu}^{n+2}$. The polar maps of the standard embeddings of the real projective planes given by (3.3) provide examples of compact minimal hypersurfaces of the sphere with constant index of relative nullity $\nu=1$ (see Exercise 7.14). They belong to the family of Cartan's isoparametric hypersurfaces of $\mathbb{S}^{4}$ with three distinct principal curvatures.

Corollary 7.23. Let $c: I \rightarrow \mathbb{S}_{1, \mu}^{n+1}$ be a smooth curve parametrized by arc-length, let $e_{1}, \ldots, e_{n}$ be an orthonormal frame of $N_{c} I$ and let $F: I \times \mathbb{Q}_{\epsilon}^{n-1} \rightarrow \mathbb{Q}_{\epsilon}^{n+1} \subset \mathbb{R}_{\mu}^{n+2}$ be defined by

$$
F(s, t)=\sum_{j=1}^{n} i_{j}(t) i_{*} e_{j}(s),
$$

where $i=\left(i_{1}, \ldots, i_{n}\right): \mathbb{Q}_{\epsilon}^{n-1} \rightarrow \mathbb{R}_{\mu}^{n}$ is an inclusion. Then $F$ parametrizes, at regular points, a hypersurface of $\mathbb{Q}_{\epsilon}^{n+1}$ with constant sectional curvature $\epsilon$ that is free of totally geodesic points. Conversely, any hypersurface of $\mathbb{Q}_{\epsilon}^{n+1}$ with constant sectional curvature $\epsilon$ that is free of totally geodesic points can be parameterized in this way, at least locally.

### 7.3.1 Some applications

The Gauss parametrization has proved to be a powerful tool in the study of hypersurfaces of space forms with constant index of relative nullity. Some applications are given below. They include a strong rigidity property of complete minimal hypersurfaces of the Euclidean space, and the classification of Euclidean hypersurfaces with constant scalar curvature and index of relative nullity $\nu=n-2$ at any point.

First we prove the following simple but useful fact.
Lemma 7.24. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be an isometric immersion with index of relative nullity $\nu=k$ at any point. Assume that the leaves of relative nullity are complete and that the image of the induced Gauss map $g: L^{n-k} \rightarrow \mathbb{S}^{n}$ is contained in a totally geodesic submanifold $\mathbb{S}^{s}$ of $\mathbb{S}^{n}$. Then $f$ is a cylinder over an isometric immersion $f_{0}: M^{s} \rightarrow \mathbb{R}^{s+1}$ with index of relative nullity $\nu_{f_{0}}=k-n+s$.

Proof: Since the leaves of relative nullity are complete, there exists a global diffeomorphism $\theta: \Lambda \rightarrow M^{n}$ of the normal bundle $\Lambda$ of $g$ onto $M^{n}$ such that $f \circ \theta$ is given by (7.15). By the assumption that $g\left(L^{n-k}\right)$ is contained in a totally geodesic $\mathbb{S}^{s} \subset \mathbb{S}^{n}$, the normal bundle $\Lambda$ splits as $\Lambda=\Lambda_{1} \oplus \Lambda_{2}$, where $\Lambda_{1}$ is the normal bundle of $g$ in $\mathbb{S}^{s}$ and $\Lambda_{2}$ is the normal bundle of $\mathbb{S}^{s}$ in $\mathbb{S}^{n}$ along $g$. The latter being parallel along $g$ in $\mathbb{R}^{n+1}$, we can identify its fibers with a fixed subspace $\mathbb{R}^{n-s}$ of $\mathbb{R}^{n+1}$. Identifying $M^{n}$ with $\Lambda$, endowed with the metric induced by $\theta$, and denoting by $M^{s}$ the manifold $\Lambda_{1}$ endowed with the metric induced by $\left.\theta\right|_{\Lambda_{1}}$, it follows from (7.15) that the map $\phi: \Lambda_{1} \times \mathbb{R}^{n-s} \rightarrow \Lambda$ given by

$$
\phi\left(\left(y, w_{1}\right), w_{2}\right)=\left(y, w_{1}+w_{2}\right)
$$

is an isometry, and that $f \circ \phi$ splits accordingly as

$$
f \circ \phi=f_{0} \times \mathrm{id}
$$

where $f_{0}=\left.f \circ \theta\right|_{\Lambda_{1}}: M^{s} \rightarrow \mathbb{R}^{s+1}$.
Theorem 7.25. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be an isometric immersion with constant index of relative nullity $\nu=n-2$. Suppose that the leaves of relative nullity are complete
and that the mean curvature $H$ does not change sign along leaves. Then the induced Gauss map $g: L^{2} \rightarrow \mathbb{S}^{n}$ is a minimal surface and $f$ is a cylinder over an isometric immersion $f_{0}: M^{2+j} \rightarrow \mathbb{R}^{3+j}, 0 \leq j \leq 1$, with index of relative nullity $\nu_{f_{0}}=j$.

Proof: Let $\theta: \Lambda \rightarrow M^{n}$ be a global diffeomorphism of the normal bundle $\Lambda$ of $g$ onto $M^{n}$ such that $f \circ \theta$ is given by (7.15). By part (i) of Proposition 7.19, for any $(y, w) \in \Lambda$ the endomorphism

$$
P_{w}(y)=\gamma(y) I+\operatorname{Hess} \gamma(y)-A_{w}
$$

of $T_{y} L$ is nonsingular, where $A_{w}$ is the shape operator of $g$ with respect to $w$. Then part (iii) of the same proposition implies that

$$
\begin{aligned}
H(y, w) & =-\operatorname{tr} P_{w}(y)^{-1} \\
& =-\operatorname{tr} P_{w}(y) \operatorname{det} P_{w}^{-1}(y) .
\end{aligned}
$$

Since

$$
\operatorname{tr} P_{t w}(y)=\Delta \gamma(y)+2 \gamma(y)-\left(\operatorname{tr} A_{w}(y)\right) t
$$

and $H$ does not change sign along the leaf through $y$, then $\operatorname{tr} A_{w}(y)=0$.
Now suppose that there exist $y \in L^{2}$ and a two-dimensional linear subspace $S \subset N_{g} L(y)$ such that $A_{w} \neq 0$ for any $0 \neq w \in S$. Then the map $w \mapsto A_{w}$ is a linear isomorphism from $S$ onto the subspace of self-adjoint endomorphisms of $T_{y} L$ with vanishing trace. Therefore the image of the map $w \mapsto P_{w}(y)$ is the affine plane of all selfadjoint endomorphisms of $T_{y} L$ with the same trace as the operator $\gamma(y) I+\operatorname{Hess} \gamma(y)$. We have reached a contradiction by the fact that such affine plane always contains singular elements. Therefore, for any $y \in L^{2}$, the kernel of the linear map $w \rightarrow A_{w}$ must have codimension at most one in $T_{y} L$, that is, the first normal space of $g$ must have dimension at most one. Thus, by the real analyticity of minimal immersions, either $g$ is totally geodesic or its first normal spaces have dimension one at any point. In the latter case, Exercise 2.2 implies that $g(L)$ lies in a totally geodesic $\mathbb{S}^{3} \subset \mathbb{S}^{n}$, because a nontotally geodesic minimal surface cannot have index of relative nullity one. The conclusion follows from Lemma 7.24 ,

Theorem 7.26. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$, $n \geq 4$, be a minimal isometric immersion of a complete Riemannian manifold. Then any other minimal isometric immersion $\tilde{f}: M^{n} \rightarrow \mathbb{R}^{n+p}, p \geq 1$, is congruent to $i \circ f$, where $i: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+p}$ is a standard inclusion, unless $f$ is a cylinder over an isometric immersion $f_{0}: M^{2+j} \rightarrow \mathbb{R}^{3+j}$, $0 \leq j \leq 1$, with index of relative nullity $\nu_{f_{0}}=j$.

Proof: The statement is trivial if $f$ is totally geodesic. If there exists a point where the index of relative nullity $\nu$ is less than $n-2$, then the statement is true even locally by Theorem 3.11. Hence it remains to consider the case in which $\nu=n-2$ on an open and dense subset $U$ of $M^{n}$, which automatically contains a complete leaf of relative nullity through each of its points.

By Theorem 7.25, each connected component of $U$ splits as a Riemannian product $U=M^{2+j} \times \mathbb{R}^{n-2-j}$, with $0 \leq j \leq 1$, and $f$ splits accordingly as a product of isometric
immersions $f=f_{0} \times \operatorname{id}$, where $f_{0}: M^{2+j} \rightarrow \mathbb{R}^{3+j}$ has index of relative nullity $\nu_{f_{0}}=j$. By real analyticity, the splitting is global.

We conclude this chapter with a classification of the Euclidean hypersurfaces $M^{n}$ with nonzero constant scalar curvature $s$ and constant index of relative nullity $\nu=n-2$.

In the next statetement the scalar curvature $s$ is not normalized.
Theorem 7.27. Let $M^{n}$ be a Riemannian manifold with constant scalar curvature $s \neq 0$, and let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be an isometric immersion with index of relative nullity $\nu=n-2$ at any point. Then $f$ is locally a cylinder over an isometric immersion $f_{0}: L^{2} \rightarrow \mathbb{R}^{3}$. Moreover, if $M^{n}$ is complete then $s>0$ and $f$ is globally a cylinder over the standard inclusion of $\mathbb{S}_{s / 2}$ into $\mathbb{R}^{3}$.

Proof: In view of Lemma 7.24 , for the first assertion it suffices to show that the induced Gauss map $g: L^{2} \rightarrow \mathbb{S}^{n}$ of $f$ is totally geodesic. The last assertion then follows from Hilbert's theorem on the nonexistence of an isometric immersion of the hyperbolic plane $\mathbb{H}^{2}$ into $\mathbb{R}^{3}$ and Hilbert-Liebmann's theorem on the isometric rigidity of $\mathbb{S}_{s / 2}$ in $\mathbb{R}^{3}$.

Suppose that there exists $(y, w) \in \Lambda$ with $\|w\|=1$ such that $A_{w} \neq 0$, where $\Lambda$ is the normal bundle of $g$. Let $\theta: U \subset \Lambda \rightarrow V \subset M^{n}$ be a diffeomorphism of an open neighborhood of $(y, w)$ such that $f \circ \theta$ is given by (7.15). Set

$$
\begin{equation*}
P_{t}=\gamma I+\operatorname{Hess} \gamma-t A_{w} . \tag{7.31}
\end{equation*}
$$

Using an orthonormal basis of principal directions, let $\lambda_{1} \neq 0$ and $\lambda_{2}$ be the principal curvatures and $h_{i j}$ the components of Hess $\gamma$. Then the (not normalized) scalar curvature $s$ at $(y, t w)$ satisfies

$$
s^{-1}=\operatorname{det} P_{t}=\left(\gamma+h_{11}-t \lambda_{1}\right)\left(\gamma+h_{22}-t \lambda_{2}\right)-h_{12}^{2} .
$$

Since $s$ is constant, we see that
(i) $\operatorname{det} A_{w}=0$,
(ii) $\left(\gamma+h_{22}\right) \lambda_{1}+\left(\gamma+h_{11}\right) \lambda_{2}=0$,
(iii) $\operatorname{det} P_{0}=s^{-1}$.

It follows easily from $(i)$ that $g$ has index of relative nullity $\nu_{g}=1$ in a neighborhood of $y$, which we still denote by $V$. In particular, $L^{2}$ has constant Gauss curvature 1 on $V$. Let $X, Y$ be an orthonormal tangent frame such that $Y$ spans the relative nullity subspace of $g$ at each point of $V$. Then $\lambda_{2}=0$ on $V$, and thus

$$
\gamma+Y Y(\gamma)=0
$$

by part (ii). Since

$$
\nabla_{Y} Y=0=\nabla_{Y} X
$$

then $[X, Y]=\nabla_{X} Y$ is collinear with $X$, and we obtain

$$
\begin{aligned}
-X(\gamma) & =X\left\langle\nabla_{Y} \operatorname{grad} \gamma, Y\right\rangle \\
& =\left\langle\nabla_{X} \nabla_{Y} \operatorname{grad} \gamma, Y\right\rangle+\left\langle\nabla_{Y} \operatorname{grad} \gamma, \nabla_{X} Y\right\rangle \\
& =\langle R(X, Y) \operatorname{grad} \gamma, Y\rangle+\left\langle\nabla_{Y} \nabla_{X} \operatorname{grad} \gamma, Y\right\rangle+2\left\langle\nabla_{[X, Y]} \operatorname{grad} \gamma, Y\right\rangle \\
& =-X(\gamma)+2\left\langle\nabla_{[X, Y]} \operatorname{grad} \gamma, Y\right\rangle,
\end{aligned}
$$

because

$$
\left\langle\nabla_{X} \operatorname{grad} \gamma, Y\right\rangle=h_{12}
$$

is constant by part (iii). It follows that

$$
\left\langle\nabla_{[X, Y]} \operatorname{grad} \gamma, Y\right\rangle=0,
$$

and since $h_{12}^{2}=-s^{-1} \neq 0$ and $[X, Y]$ is collinear with $X$, this gives

$$
[X, Y]=\nabla_{X} Y=0
$$

Together with $\nabla_{Y} Y=0$, this implies that $V$ is flat, a contradiction.

### 7.4 Intrinsically homogeneous hypersurfaces

As an application of some of the results of this and previous chapters, in this section we study isometric immersions $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ of homogeneous Riemannian manifolds into space forms.

A Riemannian manifold $M^{n}$ is said to be homogeneous if its group of isometries acts transitively on $M^{n}$, that is, if for all $x, y \in M^{n}$ there exists an isometry $g$ of $M^{n}$ such that $g(x)=y$. A homogeneous Riemannian manifold is always complete.

Proposition 7.28. If $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ is an isometric immersion of a homogeneous Riemannian manifold, then the type number $\tau$ of $f$ either satisfies $\tau(x) \leq 1$ for all $x \in M^{n}$ or is constant on $M^{n}$.

Proof: Given $x \in M^{n}$, consider the $c$-nullity subspace

$$
\Gamma_{c}(x)=\left\{X \in T_{x} M: R(X, Y)=c(X \wedge Y) \text { for all } Y \in T_{x} M\right\}
$$

of $M^{n}$ at $x$. Since $M^{n}$ is homogeneous, for any $y \in M^{n}$ there is an isometry $g$ of $M^{n}$ such that $g(x)=y$. The differential $g_{*}$ at $x$ is a linear isomorphism of $T_{x} M$ onto $T_{y} M$ that preserves inner products. Thus

$$
\begin{aligned}
g_{*} \Gamma_{c}(x) & =\left\{g_{*} X \in T_{y} M: R(X, Y)=c(X \wedge Y) \text { for all } Y \in T_{x} M\right\} \\
& =\left\{g_{*} X \in T_{y} M: R\left(g_{*} X, g_{*} Y\right)=c\left(g_{*} X \wedge g_{*} Y\right) \text { for all } Y \in T_{x} M\right\} \\
& =\Gamma_{c}(y) .
\end{aligned}
$$

In particular, the dimension of $\Gamma_{c}$ is constant on $M^{n}$.
By the Gauss equation, we have $\tau(x) \leq 1$ at $x \in M^{n}$ if and only if $\Gamma_{c}(x)$ has dimension $n$. By the above argument, if $\tau(x) \leq 1$ for some $x \in M^{n}$, then $\tau \leq 1$ at any point. On the other hand, by Exercise 4.1 the relative nullity subspace $\Delta(x)$ of $f$ at $x$ coincides with $\Gamma_{c}(x)$ if $\tau(x) \geq 2$. Therefore, if $\tau(x) \geq 2$ for some point $x \in M^{n}$, then $\tau$ is constant on $M^{n}$.

Proposition 7.29. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ be an isometric immersion of a homogeneous Riemannian manifold. If $\tau(x) \geq 3$ for some $x \in M^{n}$, then the principal curvatures of $f$ are constant on $M^{n}$.

Proof: By Proposition 7.28, $\tau(x) \geq 3$ for all $x \in M^{n}$. Choose $x, y \in M^{n}$ and an isometry $g$ of $M^{n}$ such that $g(x)=y$. By Exercise 4.2, there exists an isometry $T$ of $\mathbb{Q}_{c}^{n+1}$ such that $T \circ f=f \circ g$. Therefore the shape operators of $f$ at $x$ and $y$ with respect to $\xi \in N_{f} M(x)$ and $T_{*} \xi \in N_{f} M(y)$, respectively, are related by

$$
A(y) g_{*}(x)=g_{*}(x) A(x) .
$$

It follows that $A(x)$ and $A(y)$ have the same eigenvalues.
In the study of isoparametric hypersurfaces in space forms, the following result is known as Cartan's fundamental formula.

Theorem 7.30. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ be an isoparametric hypersurface. If $\lambda_{1}, \ldots, \lambda_{g}$ are its pairwise distinct principal curvatures and $m_{1}, \ldots, m_{g}$ are the corresponding multiplicities, then

$$
\begin{equation*}
\sum_{j=1, j \neq i}^{g} m_{j} \frac{c+\lambda_{i} \lambda_{j}}{\lambda_{i}-\lambda_{j}}=0 \tag{7.32}
\end{equation*}
$$

for any $1 \leq i \leq g$.
Proof: Let $X_{1}, \ldots, X_{n}$ be a local orthonormal frame such that

$$
A X_{i}=\lambda_{i} X_{i}, \quad 1 \leq i \leq n
$$

where the principal curvatures are counted as many times as their multiplicities. The Gauss equation (1.18) gives

$$
\begin{equation*}
c+\lambda_{i} \lambda_{j}=\left\langle\nabla_{X_{i}} \nabla_{X_{j}} X_{j}, X_{i}\right\rangle-\left\langle\nabla_{X_{j}} \nabla_{X_{i}} X_{j}, X_{i}\right\rangle-\left\langle\nabla_{\left[X_{i}, X_{j}\right]} X_{j}, X_{i}\right\rangle \text { if } i \neq j . \tag{7.33}
\end{equation*}
$$

Denote $E_{i}=E_{\lambda_{i}}=\operatorname{ker}\left(A-\lambda_{i} I\right)$. Since the principal curvatures are constant, it follows from (1.19) that each $E_{i}$ is a totally geodesic distribution. In particular,

$$
\nabla_{X_{i}} X_{i} \in \Gamma\left(E_{i}\right), \quad 1 \leq i \leq n .
$$

Therefore

$$
\begin{align*}
\left\langle\nabla_{X_{i}} \nabla_{X_{j}} X_{j}, X_{i}\right\rangle & =X_{i}\left\langle\nabla_{X_{j}} X_{j}, X_{i}\right\rangle-\left\langle\nabla_{X_{j}} X_{j}, \nabla_{X_{i}} X_{i}\right\rangle \\
& =0 \tag{7.34}
\end{align*}
$$

and

$$
\begin{align*}
\left\langle\nabla_{X_{j}} \nabla_{X_{i}} X_{j}, X_{i}\right\rangle & =X_{j}\left\langle\nabla_{X_{i}} X_{j}, X_{i}\right\rangle-\left\langle\nabla_{X_{i}} X_{j}, \nabla_{X_{j}} X_{i}\right\rangle \\
& =-\left\langle\nabla_{X_{i}} X_{j}, \nabla_{X_{j}} X_{i}\right\rangle \tag{7.35}
\end{align*}
$$

if $\lambda_{i} \neq \lambda_{j}$. On the other hand,

$$
\begin{align*}
\left\langle\nabla_{\left[X_{i}, X_{j}\right]} X_{j}, X_{i}\right\rangle & =\left\langle\nabla_{\nabla_{X_{i}} X_{j}} X_{j}, X_{i}\right\rangle-\left\langle\nabla_{\nabla_{X_{j}} X_{i}} X_{j}, X_{i}\right\rangle \\
& =\sum_{k=1}^{n}\left(\left\langle\nabla_{X_{i}} X_{j}, X_{k}\right\rangle\left\langle\nabla_{X_{k}} X_{j}, X_{i}\right\rangle+\left\langle\nabla_{X_{j}} X_{i}, X_{k}\right\rangle\left\langle\nabla_{X_{k}} X_{i}, X_{j}\right\rangle\right) . \tag{7.36}
\end{align*}
$$

If $g=2$, then 7.32 just says that the two distinct principal curvatures $\lambda_{1}, \lambda_{2}$ satisfy

$$
c+\lambda_{1} \lambda_{2}=0
$$

and this follows from (7.33), (7.34), (7.35) and (7.36). From now on assume that $g \geq 3$. By (1.20), if $\lambda_{i} \neq \lambda_{j} \neq \lambda_{k} \neq \lambda_{i}$ then

$$
\begin{equation*}
\left(\lambda_{i}-\lambda_{j}\right)\left\langle\nabla_{X_{k}} X_{j}, X_{i}\right\rangle=\left(\lambda_{k}-\lambda_{i}\right)\left\langle\nabla_{X_{j}} X_{i}, X_{k}\right\rangle \tag{7.37}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\lambda_{i}-\lambda_{j}\right)\left\langle\nabla_{X_{k}} X_{i}, X_{j}\right\rangle=\left(\lambda_{j}-\lambda_{k}\right)\left\langle\nabla_{X_{i}} X_{j}, X_{k}\right\rangle . \tag{7.38}
\end{equation*}
$$

Substituting in (7.36) yields

$$
\begin{aligned}
\left(\lambda_{i}-\lambda_{j}\right)\left\langle\nabla_{\left[X_{i}, X_{j}\right]} X_{j}, X_{i}\right\rangle= & \sum_{\substack{k=1 \\
\lambda_{i} \neq \lambda_{j} \neq \lambda_{k} \neq \lambda_{i}}}^{n}\left(\left(\lambda_{k}-\lambda_{i}\right)\left\langle\nabla_{X_{i}} X_{j}, X_{k}\right\rangle\left\langle\nabla_{X_{j}} X_{i}, X_{k}\right\rangle\right. \\
& \left.+\left(\lambda_{j}-\lambda_{k}\right)\left\langle\nabla_{X_{i}} X_{j}, X_{k}\right\rangle\left\langle\nabla_{X_{j}} X_{i}, X_{k}\right\rangle\right) \\
= & \left(\lambda_{j}-\lambda_{i}\right) \sum_{\substack{k=1 \\
\lambda_{i} \neq \lambda_{j} \neq \lambda_{k} \neq \lambda_{i}}}^{n}\left\langle\nabla_{X_{i}} X_{j}, X_{k}\right\rangle\left\langle\nabla_{X_{j}} X_{i}, X_{k}\right\rangle \\
= & \left(\lambda_{j}-\lambda_{i}\right)\left\langle\nabla_{X_{i}} X_{j}, \nabla_{X_{j}} X_{i}\right\rangle .
\end{aligned}
$$

Hence

$$
\begin{equation*}
\left\langle\nabla_{\left[X_{i}, X_{j}\right]} X_{j}, X_{i}\right\rangle=-\left\langle\nabla_{X_{i}} X_{j}, \nabla_{X_{j}} X_{i}\right\rangle . \tag{7.39}
\end{equation*}
$$

It follows from (7.33), (7.34), (7.35), (7.37), (7.38) and (7.39) that

$$
\begin{aligned}
c+\lambda_{i} \lambda_{j} & =2\left\langle\nabla_{X_{i}} X_{j}, \nabla_{X_{j}} X_{i}\right\rangle \\
& =2 \sum_{k=1}^{n}\left\langle\nabla_{X_{i}} X_{j}, X_{k}\right\rangle\left\langle\nabla_{X_{j}} X_{i}, X_{k}\right\rangle \\
& =2 \sum_{\substack{k=1 \\
\lambda_{i} \neq \lambda_{j} \neq \lambda_{k} \neq \lambda_{i}}}^{n} \frac{\left(\lambda_{i}-\lambda_{j}\right)^{2}}{\left(\lambda_{i}-\lambda_{k}\right)\left(\lambda_{j}-\lambda_{k}\right)}\left\langle\nabla_{X_{k}} X_{i}, X_{j}\right\rangle^{2}
\end{aligned}
$$

if $\lambda_{i} \neq \lambda_{j}$. Using (1.20) again, we obtain

$$
\begin{aligned}
\sum_{\substack{i=1 \\
\lambda_{i} \neq \lambda_{j}}}^{n} \frac{c+\lambda_{i} \lambda_{j}}{\lambda_{i}-\lambda_{j}} & =2 \sum_{\substack{j, k=1 \\
\lambda_{i} \neq \lambda_{j} \neq \lambda_{k} \neq \lambda_{i}}}^{n} \frac{\lambda_{i}-\lambda_{j}}{\left(\lambda_{i}-\lambda_{k}\right)\left(\lambda_{j}-\lambda_{k}\right)}\left\langle\nabla_{X_{k}} X_{i}, X_{j}\right\rangle^{2} \\
& =-2 \sum_{\substack{j, k=1 \\
\lambda_{i} \neq \lambda_{j} \neq \lambda_{k} \neq \lambda_{i}}}^{n} \frac{\lambda_{i}-\lambda_{k}}{\left(\lambda_{i}-\lambda_{j}\right)\left(\lambda_{k}-\lambda_{j}\right)}\left\langle\nabla_{X_{j}} X_{i}, X_{k}\right\rangle^{2} \\
& =-\sum_{\substack{k=1 \\
\lambda_{k} \neq \lambda_{i}}}^{n} \frac{c+\lambda_{i} \lambda_{k}}{\lambda_{i}-\lambda_{k}}
\end{aligned}
$$

and formula (7.32) follows.
Theorem 7.31. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be an isoparametric hypersurface. Then $f(M)$ is an open subset of the image of the standard isometric immersion of $\mathbb{S}_{c}^{k} \times \mathbb{R}^{n-k}$ into $\mathbb{R}^{n+1}$ for some $c>0$ and $0 \leq k \leq n$.

Proof: After changing the sign of the unit normal vector field, if necessary, we can assume that there exist positive principal curvatures. Take $\lambda_{i}$ as the smallest positive principal curvature. Then all terms in the sum on the left-hand side of $(7.32)$ are nonpositive, and hence must be zero. Thus there are at most two distinct principal curvatures, and if there are two, then one must be zero.

If there is only one principal curvature, the result follows from Proposition 1.20 . If $\lambda$ is the only nonzero principal curvature with multiplicity $k$ and $\Delta$ is the relative nullity distribution of $f$, since $E_{\lambda}=\Delta^{\perp}$ is totally geodesic then Proposition 7.4 implies that $f$ is locally a cylinder over the restriction $g=\left.f\right|_{\Sigma}$ of $f$ to a leaf $\Sigma$ of $E_{\lambda}$. Since $g$ is umbilical with principal curvature $\lambda$, it is the standard embedding of $\mathbb{S}_{c}^{k}$ into $\mathbb{R}^{k+1}$, where $c=\lambda^{2}$. The global statement follows by applying Exercise 1.20 to the family of standard isometric immersions of $\mathbb{S}_{c}^{k} \times \mathbb{R}^{n-k}$ into $\mathbb{R}^{n+1}$.

Theorem 7.32. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be an isometric immersion of a homogeneous Riemannian manifold. Then $f$ is either a cylinder over a (complete) plane curve or the standard isometric embedding of $\mathbb{S}_{c}^{k} \times \mathbb{R}^{n-k}$ into $\mathbb{R}^{n+1}$ for some $c>0$ and $2 \leq k \leq n$.

Proof: By Proposition 7.28 , either $\tau(x) \leq 1$ for all $x \in M^{n}$ or $\tau$ is constant on $M^{n}$. If the first possibility holds, then $M^{n}$ is flat and $f$ is a cylinder over a (complete) plane curve by Corollary 7.17. If $\tau \geq 3$, then $f$ is isoparametric by Proposition 7.29, and the conclusion follows from Theorem 7.31. Finally, if $\tau=2$ one can apply Theorem 7.27, because a homogeneous Riemannian manifold has constant scalar curvature.

### 7.5 Notes

The proof of the completeness of the relative nullity foliation given in this book was basically taken from Ferus [171]. Other proofs were given by Abe [1], Maltz [240]
and Olmos-Vittone [274]. Completeness of umbilical foliations, that is, foliations by umbilical submanifolds, was studied by Reckziegel [297].

Hartman [205] proved the cylinder Theorem 7.15 under the stronger assumption that $M^{n}$ has nonnegative sectional curvatures, making use of Toponogov's theorem (see [77]), since the splitting theorem of Cheeger-Gromoll [78] was not yet available at that time. The result still holds if the assumption on the index of relative nullity is replaced by the hypothesis that the image of the immersion contains $\nu_{0}$ linearly independent lines. The hypersurface case was previously considered by Hartman-Nirenberg [206]. A generalization of Hartman's Theorem 7.15 was obtained by Freitas-Guimarães [195]. The structure of complete submanifolds in Euclidean space of rank two and arbitrary codimension will be discussed in Chapter 13 .

A description of the isometric immersions $f: \mathbb{H}_{c}^{n} \rightarrow \mathbb{H}_{c}^{n+1}$ that are free of umbilic points was given by Ferus [172], after the work of Nomizu [269] for $n=2$, and goes as follows: fix a unit-speed curve $\gamma: \mathbb{R} \rightarrow \mathbb{H}^{n}$ that has curvature $\kappa \leq 1$. The image by the exponential map of the normal bundle of $\gamma$ then yields a totally geodesic foliation $\mathcal{F}(\gamma)$ by complete totally geodesic leaves. Conversely, any complete totally geodesic foliation of $\mathbb{H}^{n}$ arises in this way by considering $\gamma$ as a trajectory of a unit vector field normal to the foliation. Now, given any smooth function $\lambda: \mathbb{R} \rightarrow \mathbb{R} \backslash\{0\}$, there exists an isometric immersion $f: \mathbb{H}_{c}^{n} \rightarrow \mathbb{H}_{c}^{n+1}$, without umbilical points, whose relative nullity foliation is $\mathcal{F}(\gamma)$, and whose second fundamental form satisfies $A \gamma^{\prime}=\lambda \gamma^{\prime}$ along $\gamma$. In particular, every foliation of $\mathbb{H}^{n}$ by totally geodesic hypersurfaces arises as the relative nullity foliation of a suitable isometric immersion $f: \mathbb{H}^{n} \rightarrow \mathbb{H}^{n+1}$ without umbilical points.

There is a description, due to Alexander-Portnoy [8], of the umbilical free isometric immersion $f: \mathbb{H}_{c}^{n} \rightarrow \mathbb{H}_{c}^{n+1}$, in the spirit of the Euclidean cylinder theorem. They proved that any such immersion takes the form of a hyperbolic $(n-1)$-cylinder over a uniquely determined parallelizing curve. If umbilical points are allowed, the situation becomes quite more complicated, but this has also been considered; see Abe-Haas [2] and Abe-Mori-Takahashi [3].

The existence of a complete irreducible minimal hypersurface $f: M^{3} \rightarrow \mathbb{R}^{4}$ with index of relative nullity $\nu=1$ at any point is an important open problem. It was shown by Hasanis-Savas Halilaj-Vlachos [209] that the problem has a negative answer if the scalar curvature of $M^{3}$ is bounded from below. In view of Exercise 3.9, the universal covering of such a hypersurface would provide a solution to the question of whether there exists a complete irreducible nonruled hypersurface $f: M^{3} \rightarrow \mathbb{R}^{4}$ that is not isometrically rigid. This is a main unsolved problem in the subject. We point out that the examples in [263] do not solve the problem, because they are 2-cylinders. The similar problem for hypersurfaces of the sphere and the hyperbolic space was investigated by Hasanis-Savas Halilaj-Vlachos [210, [211].

Theorem 7.11 on spherical submanifolds with positive index of relative nullity is due to Dajczer-Gromoll [108], and generalizes a sequence of results by O'Neill-Stiel [279], Ferus [174], Abe [1] and Rodríguez [303]. The existence of nontotally geodesic isometric immersions $f: \mathbb{S}_{c}^{n} \rightarrow \mathbb{S}_{c}^{2 n}$ remains an open problem for $n>2$. A positive answer for dimension $n=2$ was obtained by Ferus-Pinkall [178]. It is also unknown
whether there exist isometric immersions $f: \mathbb{S}_{c}^{n} \rightarrow \mathbb{S}_{\tilde{c}}^{2 n}$ with $c<\tilde{c}$, or if there exist nonumbilical isometric immersions $f: \mathbb{S}_{c}^{n} \rightarrow \mathbb{S}_{\tilde{c}}^{2 n}$ with $c>\tilde{c}$.

For additional information on isometric immersions of space forms into space forms we refer to the survey [40].

The Gauss parametrization for hypersurfaces in space forms with constant positive index of relative nullity was introduced by Dajczer-Gromoll [109], but was used long before by Sbrana [311, [312] in the case of Euclidean hypersurfaces with type number two. A complex Gauss parametrization for holomorphic hypersurfaces was discussed by Dajczer-Florit [102]. The result in Exercise 7.13 was obtained by Dajczer-Tojeiro [136].

Cartan's fundamental formula for isoparametric hypersurfaces in space forms appears in 69]. Cartan made use of it to determine all isoparametric hypersurfaces of Euclidean and hyperbolic spaces (see Theorem 7.31 and Exercise 8.6 in the next chapter). He also initiated the much richer theory of isoparametric hypersurfaces of the sphere, whose full classification has not yet been achieved. We refer to [74] for a discussion on this subject. See also Berndt-Console-Olmos 34 and the references therein for the generalization of this theory to higher codimension.

Intrinsically homogeneous hypersurfaces of Euclidean and hyperbolic spaces were classified by Takahashi [321, [322. Intrinsically homogeneous Euclidean submanifolds of codimension two, that is, isometric immersions $f: M^{n} \rightarrow \mathbb{R}^{n+2}$ of homogeneous Riemannian manifolds, were considered by Castro-Noronha [158]. The cases of the sphere and the hyperbolic space were also addressed by Castro-Noronha [159]. Extrinsically homogeneous submanifolds of space forms, that is, submanifolds that are orbits of closed subgroups of the isometry group of the ambient space, are discussed in [34].

Going in a different direction, Theorem 7.32 was generalized by Mercuri-Podesta-Seixas-Tojeiro [246] for cohomogeneity one complete Euclidean hypersurfaces, that is, isometric immersions $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ of complete Riemannian manifolds whose isometry groups act on $M^{n}$ with principal orbits of codimension one. Isometric actions of cohomogeneity one are special cases of polar actions, for which the orthogonal spaces to the orbits give rise to a totally geodesic distribution on the regular part of the manifold. Compact Euclidean hypersurfaces acted on polarly by a closed connected subgroup of its isometry group were described by Moutinho-Tojeiro [264].

### 7.6 Exercises

Exercise 7.1. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be an isometric immersion with constant positive index of relative nullity. Prove that the first normal spaces of $f$ are parallel along the leaves of the relative nullity distribution with respect to the induced connection on $f^{*} T \mathbb{Q}_{c}^{m}$.

## Exercise 7.2.

(i) Let $\mathcal{U}$ be a umbilical distribution on a Riemannian manifold $M^{n}$ with mean curvature vector field $\delta$. Show that its splitting tensor satisfies the differential
equations

$$
\begin{equation*}
\left(\nabla_{T}^{h} C_{S}\right) X=C_{S} C_{T} X+C_{\nabla_{T}^{v} S} X-R^{h}(T, X) S+\langle T, S\rangle\left(\langle X, \delta\rangle \delta-\nabla_{X}^{h} \delta\right) \tag{7.40}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\nabla_{X}^{h} C_{T}\right) Y-\left(\nabla_{Y}^{h} C_{T}\right) X=C_{\nabla_{X}^{v} T} Y-C_{\nabla_{Y}^{v} T} X-R^{h}(X, Y) T-\langle[X, Y], T\rangle \delta . \tag{7.41}
\end{equation*}
$$

(ii) If $\mathcal{U}=E_{\eta}$ is the distribution associated with a principal curvature normal vector field $\eta$ of an isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$, show that (7.40) and (7.41) take, respectively, the form

$$
\left(\nabla_{T}^{h} C_{S}\right) X=C_{S} C_{T} X+C_{\nabla_{T}^{v} S} X+\langle T, S\rangle\left(A_{\eta} X+\langle\delta, X\rangle \delta-\nabla_{X}^{h} \delta\right)
$$

and

$$
\left(\nabla_{X}^{h} C_{T}\right) Y-\left(\nabla_{Y}^{h} C_{T}\right) X=C_{\nabla_{X}^{v} T} Y-C_{\nabla_{Y}^{v} T} X-\langle[X, Y], T\rangle \delta .
$$

Exercise 7.3. Let $U: I \rightarrow M_{n}(\mathbb{R})$ be a solution on an interval $I \subset \mathbb{R}$ of the linear ordinary differential equation

$$
U^{\prime}(t)=B(t) U(t)
$$

where $B: I \rightarrow M_{n}(\mathbb{R})$ is continuous. Then $\operatorname{rank} U(t)$ is constant on $I$.
Hint: Observe that each column $U_{i}, 1 \leq i \leq n$, of $U$ is a solution of the linear ordinary differential equation

$$
X^{\prime}(t)=B(t) X(t)
$$

for $X: I \rightarrow \mathbb{R}^{n} \approx M_{n \times 1}(\mathbb{R})$. Let $\mathcal{S}$ be the space of solutions of this equation. By the existence and uniqueness theorem for solutions of such equation, for each $s \in I$ the map $\psi_{s}: \mathcal{S} \rightarrow \mathbb{R}^{n}$, given by

$$
\psi_{s}(X)=X(s)
$$

is an isomorphism. Therefore

$$
\begin{aligned}
\operatorname{rank} U\left(s_{1}\right) & =\operatorname{dim} \operatorname{span}\left\{U_{1}\left(s_{1}\right), \ldots, U_{n}\left(s_{1}\right)\right\} \\
& =\operatorname{dim} \operatorname{span}\left\{\psi_{s_{1}}\left(U_{1}\right), \ldots, \psi_{s_{1}}\left(U_{n}\right)\right\} \\
& =\operatorname{dim} \operatorname{span}\left\{U_{1}, \ldots, U_{n}\right\} \\
& =\operatorname{dim} \operatorname{span}\left\{\psi_{s_{2}}\left(U_{1}\right), \ldots, \psi_{s_{2}}\left(U_{n}\right)\right\} \\
& =\operatorname{dim} \operatorname{span}\left\{U_{1}\left(s_{2}\right), \ldots, U_{n}\left(s_{2}\right)\right\} \\
& =\operatorname{rank} U\left(s_{2}\right)
\end{aligned}
$$

for all $s_{1}, s_{2} \in I$.
Exercise 7.4. Show that the conclusion of Theorem 7.11 remains true if the completeness assumption is replaced by the weaker requirement that each point in the open subset

$$
U=\left\{x \in M^{n}: \nu(x)=\nu_{0}\right\}
$$

belongs to a geodesic arc that is contained in a leaf of the relative nullity distribution and contains a pair of points that are mapped into antipodal points of $\mathbb{S}_{c}^{m}$.

Exercise 7.5. Show that the conclusion of Theorem 7.15 is false without the assumption on the Ricci curvature.
Hint: See part (ii) of Exercise 1.23 .
Exercise 7.6. Let $\pi: E \rightarrow M$ be a vector bundle endowed with a linear connection $\nabla$. Given $x_{0} \in M$, let $\xi_{0} \in E_{x_{0}}=\pi^{-1}\left(x_{0}\right)$ and let $\phi: T_{x_{0}} M \rightarrow E_{x_{0}}$ be any linear map. Then there exist an open neighborhood $U \subset M$ of $x_{0}$ and $\xi \in \Gamma(U)$ such that $\xi\left(x_{0}\right)=\xi_{0}$ and $\nabla_{X} \xi=\phi(X)$ for any $X \in T_{x_{0}} M$.

Exercise 7.7. Let $f: M^{n} \rightarrow \mathbb{S}^{n+1}$ be an isometric immersion with index of relative nullity $\nu=k>0$ at any point. Show that $f$ is minimal if and only if the Gauss map $g: L^{n-k} \rightarrow \mathbb{S}^{n+1}$ of $f$ satisfies

$$
\operatorname{tr} A_{w}^{-1}(y)=0
$$

for all $y \in L^{n-k}$ and all $w \in N_{g} L(y)$. In particular, if $\nu=n-2$, conclude that $f$ is minimal if and only if $g$ is minimal.

Exercise 7.8. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be an isometric immersion with constant index of relative nullity $\nu=n-2$. Show that $f$ is minimal if and only if its Gauss map $g: L^{2} \rightarrow \mathbb{S}^{n}$ is a minimal surface and the support function $\gamma \in C^{\infty}(L)$ satisfies the differential equation

$$
\Delta \gamma+2 \gamma=0
$$

Exercise 7.9. Show that any isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ with constant mean curvature $H \neq 0$ is rigid, unless $f(M)$ is an open subset of a cylinder over an isometric immersion $f_{0}: L^{2} \rightarrow \mathbb{R}^{3}$ with constant mean curvature $H$ or $M^{n}$ is (isometric to) an open subset of $\mathbb{S}^{1} \times \mathbb{R}^{n-1}$ and $f$ is the restriction to $M^{n}$ of the cylinder over the standard inclusion of $\mathbb{S}^{1}$ into $\mathbb{R}^{2}$.

Hint: If the minimal index of relative nullity $\nu_{0}$ is greater than $n-2$, then the assertion follows from the Beez-Killing theorem. Suppose $\nu_{0}=n-2$, and let $\theta: V \subset \Lambda \rightarrow U \subset$ $M^{n}$ be a diffeomorphism of an open neighborhood of the zero section of the normal bundle $\Lambda$ of the induced Gauss map $g: L^{2} \rightarrow \mathbb{S}^{n}$ onto an open subset of $M^{n}$ where $\nu=n-2$ such that $f \circ \theta$ is given by (7.15). If there exists $(y, w) \in \Lambda$ such that $\|w\|=1$ and $A_{w}(y) \neq 0$, use that

$$
H=-\operatorname{tr} P_{t}^{-1}=-\operatorname{tr} P_{t} \operatorname{det} P_{t}^{-1}
$$

is constant in $t$, where $P_{t}$ is as in 7.31, to obtain
(i) $\operatorname{det} A_{w}=0$,
(ii) $h\left(\left(\gamma+h_{22}\right) \lambda_{1}+\left(\gamma+h_{11}\right) \lambda_{2}\right)=-\operatorname{tr} A_{w}$,
(iii) $\operatorname{tr} P_{0}=-H \operatorname{det} P_{0}$.

Conclude that $h_{12}^{2}=-H^{2}$, which is a contradiction. Then use Lemma 7.24 and the real analyticity of $f$. The case $\nu_{0}=n-1$ is easier and similar.

Exercise 7.10. Let $f: M^{n} \rightarrow \mathbb{S}^{n+1}, n \geq 4$, be an isometric immersion of a complete Riemannian manifold with constant index of relative nullity $\nu=n-2$ at any point. Prove that the mean curvature $H$ of $f$ must change sign along each leaf of the relative nullity distribution.

Hint: Assume otherwise, and let $\theta: \Lambda_{1} \rightarrow M^{n}$ be a global diffeomorphism of the unit normal bundle of the induced Gauss map $g: L^{2} \rightarrow M^{n}$ such that $f \circ \theta$ is given by (7.30). Since

$$
H(y, w)=\operatorname{tr} A_{w}^{-1}(y)=-\operatorname{tr} A_{-w}^{-1}(y)=-H(y,-w)
$$

for any $(y, w) \in \Lambda$, the assumption implies that $\operatorname{tr} A_{w}(y)=0$. Now use that the space of symmetric $2 \times 2$-matrices with trace zero has dimension two and that the codimension of $g$ is at least three; hence at any $y \in L^{2}$ there exists a unit vector $w \in N_{g} L(y)$ such that $A_{w}=0$.

Exercise 7.11. Let $M^{n}$ be an oriented complete Riemannian manifold whose notnormalized scalar curvature $s$ is bounded away from zero and let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be an isometric immersion with constant index of relative nullity $\nu=n-2$ at any point, and whose mean curvature $H$ is bounded from above (below) along leaves. Show that $f$ is a cylinder over an isometric immersion $f_{0}: M^{2} \rightarrow \mathbb{R}^{3}$ of a compact surface with positive curvature. In particular, $f$ is rigid.
Hint: In view of Lemma 7.24, for the first assertion it suffices to show that the induced Gauss map $g: L^{2} \rightarrow \mathbb{S}^{n}$ of $f$ is totally geodesic. Suppose that there exists $(y, w) \in$ $\Lambda=N_{g} L$ with $\|w\|=1$ such that $A_{w} \neq 0$, where $\Lambda$ is the normal bundle of $g$. Let $\theta: \Lambda \rightarrow M^{n}$ be a diffeomorphism such that $f \circ \theta$ is given by (7.15). Let $P_{t}$ be given by (7.31). If $h_{i j}$ are the components of the Hessian $H_{\gamma}$ with respect to an orthonormal frame of principal directions of $g$ and $\lambda_{1} \neq 0, \lambda_{2}$ are the principal curvatures, from the assumption that

$$
s^{-1}=\operatorname{det} P_{t}=\left(\gamma+h_{11}-t \lambda_{1}\right)\left(\gamma+h_{22}-t \lambda_{2}\right)-h_{12}^{2}
$$

is bounded in $t$, conclude that $\lambda_{2}=0$ and $\gamma+h_{22}=0$. Then use that

$$
\begin{aligned}
H & =-\operatorname{tr} P_{t}^{-1} \\
& =\operatorname{tr} P_{t} \operatorname{det} P_{t}^{-1} \\
& =\left(\delta \gamma+2 \gamma-t \lambda_{1}\right) h_{12}^{-2}
\end{aligned}
$$

to obtain a contradiction with the assumption on $H$. Finally, use the fact that $L^{2}$ is complete with Gaussian curvature $K$ bounded away from zero to conclude from Efimov's theorem that $K$ must be positive, and hence that $L^{2}$ is compact. For the rigidity of $f_{0}$, hence of $f$, use Minkowski's theorem.

Exercise 7.12. Let $i: \mathbb{S}^{3} \rightarrow \mathbb{R}^{4}$ be the standard inclusion and let $\gamma \in C^{\infty}(U)$ satisfy

$$
\Delta \gamma+3 \gamma=0
$$

on an open subset $U \subset \mathbb{S}^{3}$. Prove that the map $f: U \rightarrow \mathbb{R}^{4}$ given by

$$
f=\gamma i+i_{*} \operatorname{grad} \gamma
$$

defines, at regular points, a hypersurface with constant scalar curvature and vanishing Gauss-Kronecker curvature. Conversely, show that any hypersurface $f: M^{3} \rightarrow \mathbb{R}^{4}$ with constant scalar curvature and vanishing Gauss-Kronecker curvature can locally be parametrized in this way.

Exercise 7.13. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be an oriented complete hypersurface with constant support function $\gamma$. Show that

$$
f(M)=\mathbb{S}^{m} \times \mathbb{R}^{n-m} \subset \mathbb{R}^{n+1}, \quad 0 \leq m \leq n
$$

where the spherical factor is centered at the origin.
Exercise 7.14. Let $f: \mathbb{S}_{1 / 3}^{2} \rightarrow \mathbb{S}^{4}$ be the Veronese surface given by (3.3). Show that the polar map $\psi: N_{f}^{1} \mathbb{S}_{1 / 3}^{2} \rightarrow \mathbb{S}^{4}$ of $f$, given by

$$
\psi(y, w)=w
$$

is a minimal isoparametric hypersurface with three distinct principal curvatures and index of relative nullity $\nu=1$.

Exercise 7.15. Let $f: M^{n} \rightarrow \mathbb{L}^{n+1}$ be an isometric immersion with constant index of relative nullity $\nu=k$ of an oriented Riemannian manifold into Lorentzian space. Show that, similar to the case of hypersurfaces in Euclidean space, the Gauss map $\eta: M^{n} \rightarrow \mathbb{H}^{n}$ induces an immersion $g: L^{n-k} \rightarrow \mathbb{H}^{n}$, where $L^{n-k}$ is the quotient space of relative nullity leaves, and the support function

$$
\gamma=-\langle f, i \circ \eta\rangle,
$$

where $i: \mathbb{H}^{n} \rightarrow \mathbb{L}^{n+1}$ is the inclusion map, also induces a function $\gamma \in C^{\infty}\left(L^{n-k}\right)$. Show that the Gauss parametrization defined in the normal bundle of $g$ has in this case the form

$$
\psi(y, w)=\gamma(y) h(y)-h_{*} \operatorname{grad} \gamma(y)+i_{*} w
$$

where $h=i \circ g$.

## Chapter 8

## Isometric immersions of Riemannian products

The simplest way of constructing an immersion of a product manifold into a space form is to take an extrinsic product of immersions of the factors, a concept that will be discussed in this chapter. The metric induced on a product manifold by an extrinsic product of immersions is the Riemannian product of the metrics induced by the immersions of the factors, and its second fundamental form is adapted to the product structure of the manifold in the sense that the tangent spaces to each factor are preserved by all shape operators.

It is a basic fact that the latter condition characterizes extrinsic products of immersions among isometric immersions of Riemannian products into space forms. After giving a proof of this fact, in the rest of the chapter we present several results which assure that this condition is satisfied by a given isometric immersion of a Riemannian product under assumptions of both local and global nature.

### 8.1 Product manifolds

In this section we first introduce some terminology related to product manifolds and then state basic results on Riemannian product metrics.

A net $\mathcal{E}=\left(E_{i}\right)_{i=1, \ldots, r}$ on a differentiable manifold $M$ is a splitting $T M=\oplus_{i=1}^{r} E_{i}$ of its tangent bundle by a family of integrable subbundles. A net $\mathcal{E}=\left(E_{i}\right)_{i=1, \ldots, r}$ on a Riemannian manifold $M$ is called an orthogonal net if the subbundles of $\mathcal{E}$ are mutually orthogonal.

For a product manifold $M=\Pi_{i=1}^{r} M_{i}$ let $\pi_{i}: M \rightarrow M_{i}$ denote the canonical projection of $M$ onto $M_{i}$. The map $\tau_{i}^{\bar{x}}: M_{i} \rightarrow M$, for $\bar{x}=\left(\bar{x}_{1}, \ldots, \bar{x}_{r}\right) \in M$, stands for the inclusion of $M_{i}$ into $M$ given by

$$
\tau_{i}^{\bar{x}}\left(x_{i}\right)=\left(\bar{x}_{1}, \ldots, x_{i}, \ldots, \bar{x}_{r}\right), \quad 1 \leq i \leq r .
$$

The product net of a product manifold $M=\prod_{i=1}^{r} M_{i}$ is the net $\mathcal{E}=\left(E_{i}\right)_{i=1, \ldots, r}$ on $M$
defined by

$$
E_{i}(x)=\tau_{i *}^{x} T_{x_{i}} M_{i}, \quad 1 \leq i \leq r,
$$

for any $x=\left(x_{1}, \ldots, x_{r}\right) \in M$.
A metric $g$ on a product manifold $M=\prod_{i=1}^{r} M_{i}$ is the Riemannian product of metrics $g_{i}$ on $M_{i}, 1 \leq i \leq r$, if

$$
g=\sum_{i=1}^{r} \pi_{i}^{*} g_{i} .
$$

To describe the Levi-Civita connection of a Riemannian product it suffices to consider Riemannian products with only two factors. If $M=M_{1} \times M_{2}$ is a product manifold and $\mathcal{E}=\left(E_{1}, E_{2}\right)$ is its product net, then elements of $\Gamma\left(E_{1}\right)$ will always be denoted by the letters $X, Y, Z$, whereas those in $\Gamma\left(E_{2}\right)$ by $U, V, W$. The same applies to individual tangent vectors. A vector field $X \in \Gamma\left(E_{1}\right)$ (respectively, $V \in \Gamma\left(E_{2}\right)$ ) is said to be the lift of a vector field $\tilde{X} \in \mathfrak{X}\left(M_{1}\right)$ (respectively, $\left.\tilde{V} \in \mathfrak{X}\left(M_{2}\right)\right)$ if $\pi_{1 *} X=\tilde{X} \circ \pi_{1}$ (respectively, $\pi_{2 *} V=\tilde{V} \circ \pi_{2}$ ).

We denote the set of all lifts of vector fields in $M_{1}$ (respectively, $M_{2}$ ) by $\mathcal{L}\left(M_{1}\right)$ (respectively, $\mathcal{L}\left(M_{2}\right)$ ), and always denote vector fields in $M_{1}$ and $M_{2}$ with a tilde and use the same letters without the tilde to represent their lifts to $M$. Then the Levi-Civita connections $\nabla^{1}, \nabla^{2}$ and $\nabla$ of of $M_{1}, M_{2}$ and $M=M_{1} \times M_{2}$ are related by

$$
\begin{align*}
& \nabla_{X} Y \text { is the lift of } \nabla_{\tilde{X}}^{1} \tilde{Y},  \tag{8.1}\\
& \quad \nabla_{X} V=\nabla_{V} X=0,  \tag{8.2}\\
& \nabla_{V} W \text { is the lift of } \nabla_{\tilde{V}}^{2} \tilde{W} . \tag{8.3}
\end{align*}
$$

Observe that the formula $\nabla_{X} V=0$ (respectively, $\nabla_{V} X=0$ ) is tensorial in $X$ (respectively, $V$ ), hence it also holds for elements of $\Gamma\left(E_{1}\right)$ (respectively, $\Gamma\left(E_{2}\right)$ ) that are not necessarily lifts. On the other hand, they characterize elements of $\Gamma\left(E_{2}\right)$ (respectively, $\Gamma\left(E_{1}\right)$ ) that are lifts.

It follows from 8.1, 8.2 and 8.3 that $E_{1}$ and $E_{2}$ are totally geodesic distributions. In general, for a Riemannian product $M=\Pi_{i=1}^{r} M_{i}$, the product net $\mathcal{E}=\left(E_{i}\right)_{i=1, \ldots, r}$ is an orthogonal net such that $E_{i}$ and $E_{i}^{\perp}$ are totally geodesic for $1 \leq i \leq r$. Clearly, these conditions are equivalent to $E_{i}$ and $E_{i}^{\perp}$ being parallel distributions on $M$. The next proposition shows that Riemannian product metrics on a product manifold are characterized by this property of its product net.

Proposition 8.1. A Riemannian metric $g$ on a product manifold $M=\Pi_{i=1}^{r} M_{i}$ is a Riemannian product metric if and only if the product net $\mathcal{E}=\left(E_{i}\right)_{i=1, \ldots, r}$ of $M$ is an orthogonal net such that $E_{i}$ and $E_{i}^{\perp}$ are totally geodesic for $1 \leq i \leq r$.

Let $N$ be a smooth manifold endowed with a net $\mathcal{F}=\left(F_{i}\right)_{i=1, \ldots, r}$. A product representation of $\mathcal{F}$ is a smooth diffeomorphism $\psi: M \rightarrow N$ of a product manifold $M=\Pi_{i=1}^{r} M_{i}$ onto $N$ such that $\psi_{*} E_{i}(x)=F_{i}(\psi(x))$ for all $x \in M, 1 \leq i \leq r$, where $\mathcal{E}=\left(E_{i}\right)_{i=1, \ldots, r}$ is the product net of $M$. Therefore the restriction of the product
representation $\psi$ to the leaf of $E_{i}$ through any $x \in M$ is a smooth diffeomorphism onto the leaf of $F_{i}$ through $\psi(x)$.

In applying the results of this chapter on isometric immersions of Riemannian products, a first step is to recognize that a given submanifold of a space form is intrinsically a Riemannian product. The basic tool in that direction is the well-known theorem due to de Rham, which in the above terminology can be stated as follows.

Theorem 8.2. Let $M$ be a Riemannian manifold that carries an orthogonal net $\mathcal{E}=\left(E_{i}\right)_{i=1, \ldots, r}$ such that $E_{i}$ and $E_{i}^{\perp}$ are totally geodesic for $1 \leq i \leq r$. Then there exists locally (globally, if $M$ is simply connected and complete) a product representation $\psi: \Pi_{i=1}^{r} M_{i} \rightarrow M$ of $\mathcal{E}$ which is an isometry with respect to a Riemannian product metric on $\Pi_{i=1}^{r} M_{i}$.

For a Riemannian product $M=M_{1} \times M_{2}$ we list, for later use, the relations between the curvature tensors of $M, M_{1}$ and $M_{2}$. We denote by $R^{i}$ both the curvature tensor of $M_{i}, 1 \leq i \leq 2$, and its lift to $M$, which is the tensor whose value at $T_{1}, T_{2}, T_{3} \in$ $T_{z} M$ is the unique vector in $E_{i}(z)$ that projects to $R^{i}\left(\pi_{i *} T_{1}, \pi_{i *} T_{2}\right) \pi_{i *} T_{3}$ in $T_{\pi_{i}(z)} M_{i}$. The formulas

$$
\left\{\begin{array}{l}
R(X, Y) Z=R^{1}(X, Y) Z  \tag{8.4}\\
R(X, Y) V=R(V, W) X=R(X, U) V=0 \\
R(V, W) U=R^{2}(V, W) U
\end{array}\right.
$$

hold for all $X, Y, Z \in \Gamma\left(E_{1}\right)$ and $U, V, W \in \Gamma\left(E_{2}\right)$.

### 8.2 Extrinsic products of immersions

In this section we introduce the concept of an extrinsic product of immersions into Euclidean space and present some of its basic properties. After that we discuss the case of nonflat ambient spaces.

A map $f: M \rightarrow \mathbb{R}^{m}$ from a product manifold $M=\prod_{i=1}^{r} M_{i}$ is said to be the extrinsic product of the immersions $f_{i}: M_{i} \rightarrow \mathbb{R}^{m_{i}}, 1 \leq i \leq r$, if there exist an orthogonal decomposition $\mathbb{R}^{m}=\Pi_{j=0}^{r} \mathbb{R}^{m_{j}}$, with $\mathbb{R}^{m_{0}}$ possibly trivial, and $v \in \mathbb{R}^{m_{0}}$ (in case $\mathbb{R}^{m_{0}}$ is nontrivial) such that

$$
f(x)=\left(v, f_{1}\left(x_{1}\right), \ldots, f_{r}\left(x_{r}\right)\right)
$$

for all $x=\left(x_{1}, \ldots, x_{r}\right) \in M$.
A few elementary properties of an extrinsic product of immersions are collected in the next result whose proof is left to the reader.

Proposition 8.3. Let the map $f: M \rightarrow \mathbb{R}^{m}$ from a product manifold $M=\Pi_{i=1}^{r} M_{i}$ be an extrinsic product of immersions $f_{i}: M_{i} \rightarrow \mathbb{R}^{m_{i}}, 1 \leq i \leq r$. Then the following assertions hold:
(i) The differential of $f$ at any $x=\left(x_{1}, \ldots, x_{r}\right) \in M$ is given by

$$
f_{*} \tau_{i}^{x} X_{i}=f_{i *} X_{i} \quad 1 \leq i \leq r,
$$

for all $X_{i} \in T_{x_{i}} M_{i}$.
(ii) The map $f$ is an immersion whose induced metric is the Riemannian product of the Riemannian metrics on $M_{i}$ induced by $f_{i}, 1 \leq i \leq r$.
(iii) The normal space of $f$ at $x \in M$ is

$$
N_{f} M(x)=\mathbb{R}^{m_{0}} \oplus_{i=1}^{r} N_{f_{i}} M_{i}\left(x_{i}\right) .
$$

(iv) The second fundamental form of $f$ at $x \in M$ is given by

$$
\alpha^{f}\left(\tau_{i *}^{x} X_{i}, \tau_{i *}^{x} Y_{i}\right)=\alpha^{f_{i}}\left(X_{i}, Y_{i}\right)
$$

for all $X_{i}, Y_{i} \in T_{x_{i}} M_{i}$, and

$$
\alpha^{f}\left(\tau_{i *}^{x} X_{i}, \tau_{j *}^{x} X_{j}\right)=0, \quad i \neq j,
$$

for all $X_{i} \in T_{x_{i}} M_{i}$ and $X_{j} \in T_{x_{j}} M_{j}$.
The notion of an extrinsic product of immersions can be extended to the cases in which the target manifold is the sphere or the hyperbolic space. It will be convenient to use the notations

$$
\mathbb{S}^{n}(r)=\mathbb{S}_{1 / r^{2}}^{n}=\left\{X \in \mathbb{R}^{n+1}:\langle X, X\rangle=r^{2}\right\}
$$

and

$$
\mathbb{H}^{n}(r)=\mathbb{H}_{-1 / r^{2}}^{n}=\left\{X=\left(x_{0}, \ldots, x_{n}\right) \in \mathbb{L}^{n+1}:\langle X, X\rangle=-r^{2}, x_{0}>0\right\}
$$

A map $f: M \rightarrow \mathbb{S}^{m}$ from a product manifold $M=\prod_{i=1}^{r} M_{i}$ into the sphere $\mathbb{S}^{m} \subset$ $\mathbb{R}^{m+1}$ is said to be the extrinsic product of the immersions $f_{i}: M_{i} \rightarrow \mathbb{S}^{m_{i}-1}\left(r_{i}\right) \subset \mathbb{R}^{m_{i}}$, $1 \leq i \leq r$, if there exist an orthogonal decomposition $\mathbb{R}^{m+1}=\prod_{j=0}^{r} \mathbb{R}^{m_{j}}$, with $\mathbb{R}^{m_{0}}$ possibly trivial, and $v \in \mathbb{R}^{m_{0}}$ (in case $\mathbb{R}^{m_{0}}$ is nontrivial), with

$$
\|v\|^{2}+\sum_{i=1}^{r} r_{i}^{2}=1
$$

such that

$$
f(x)=\left(v, f_{1}\left(x_{1}\right), \ldots, f_{r}\left(x_{r}\right)\right)
$$

for all $x=\left(x_{1}, \ldots, x_{r}\right) \in M$.
Extrinsic products of immersions into hyperbolic space are of three different types. First, given an orthogonal decomposition

$$
\begin{equation*}
\mathbb{L}^{m+1}=\mathbb{L}^{m_{1}} \times \Pi_{i=2}^{r+1} \mathbb{R}^{m_{i}} \tag{8.5}
\end{equation*}
$$

with $\mathbb{R}^{m_{r+1}}$ possibly trivial, let $v \in \mathbb{R}^{m_{r+1}}$ (in case $\mathbb{R}^{m_{r+1}}$ is nontrivial) and let

$$
f_{1}: M_{1} \rightarrow \mathbb{H}^{m_{1}-1}\left(r_{1}\right) \subset \mathbb{L}^{m_{1}} \text { and } f_{i}: M_{i} \rightarrow \mathbb{S}^{m_{i}-1}\left(r_{i}\right) \subset \mathbb{R}^{m_{i}}, \quad 2 \leq i \leq r
$$

be immersions with

$$
-r_{1}^{2}+\sum_{i=2}^{r} r_{i}^{2}+\|v\|^{2}=-1
$$

The map $f: M=\Pi_{i=1}^{r} M_{i} \rightarrow \mathbb{H}^{m} \subset \mathbb{L}^{m+1}$, given by

$$
\begin{equation*}
f(x)=\left(f_{1}\left(x_{1}\right), \ldots, f_{r}\left(x_{r}\right), v\right) \tag{8.6}
\end{equation*}
$$

for all $x=\left(x_{1}, \ldots, x_{r}\right) \in M$, is said to be the extrinsic product of $f_{1}, \ldots, f_{r}$.
The other two types of extrinsic products of immersions into $\mathbb{H}^{m}$ arise by composing extrinsic products of immersions into either Euclidean space $\mathbb{R}^{n}$ or the sphere $\mathbb{S}^{n}$ of dimension $n<m$ with an umbilical inclusion of either of these spaces into $\mathbb{H}^{m}$.

### 8.3 The basic decomposition theorems

In this section it is discussed, in terms of the second fundamental form, when an isometric immersion of a Riemannian product manifold into a space form must be an extrinsic product of isometric immersions.

The second fundamental form of an isometric immersion $f: M \rightarrow \mathbb{Q}_{c}^{m}$ of a Riemannian manifold $M$ endowed with a net $\mathcal{E}=\left(E_{i}\right)_{i=1, \ldots, r}$ is said to be adapted to the net if

$$
\alpha^{f}\left(E_{i}, E_{j}\right)=0 \text { for } 1 \leq i \neq j \leq r .
$$

If $M=\Pi_{i=1}^{r} M_{i}$ is a product manifold and $f: M \rightarrow \mathbb{R}^{m}$ is the extrinsic product of immersions $f_{i}: M_{i} \rightarrow \mathbb{R}^{m_{i}}, 1 \leq i \leq r$, it follows from part (iv) of Proposition 8.3 that the second fundamental form of $f$ is adapted to the product net of $M$. The next result shows that extrinsic products of isometric immersions are characterized by this property among isometric immersions of Riemannian products into Euclidean space.

Theorem 8.4. Let $f: M \rightarrow \mathbb{R}^{m}$ be an isometric immersion of a Riemannian product manifold $M=\Pi_{i=1}^{r} M_{i}$ whose second fundamental form is adapted to the product net of $M$. Then $f$ is an extrinsic product of isometric immersions.

Proof: There is no loss in generality in assuming that $r=2$. Let $\mathcal{E}=\left(E_{1}, E_{2}\right)$ be the product net of $M$. For all $x, y \in M, X_{1} \in E_{1}(x)$ and $X_{2} \in E_{2}(y)$ we have

$$
\begin{equation*}
f_{*} X_{1} \perp f_{*} X_{2} \tag{8.7}
\end{equation*}
$$

To see this, write $x=\left(x_{1}, x_{2}\right)$ and denote $L=M_{1} \times\left\{x_{2}\right\}=\tau_{1}^{x}\left(M_{1}\right)$. For $\hat{X} \in E_{2}(x)$, let $\bar{X}=\pi_{2 *}(x) \hat{X} \in T_{x_{2}} M_{2}$. Set $\hat{X}(z)=\tau_{2^{*}}^{z} \bar{X}$ for any $z=\left(z_{1}, x_{2}\right) \in L$. Then $\hat{X}$ is a
parallel vector field along $L$ with respect to (the pulled-back to $L$ of) the Levi-Civita connection of $M$. Set

$$
\xi=f_{*} \hat{X}: L \rightarrow \mathbb{R}^{m}
$$

The assumption on the second fundamental form of $f$ gives

$$
\xi_{*}(X)=\alpha^{f}(X, \hat{X})=0
$$

for all $X \in \mathfrak{X}(L)$. Therefore $\xi=f_{*} \hat{X}$ is constant, and 8.7) follows.
Define linear subspaces $\mathbb{R}^{m_{i}}$ of $\mathbb{R}^{m}, 1 \leq i \leq 2$, by

$$
\mathbb{R}^{m_{i}}=\operatorname{span}\left\{f_{*}(x) X_{i}: x \in M \text { and } X_{i} \in E_{i}(x)\right\}
$$

By (8.7) the subspaces $\mathbb{R}^{m_{1}}$ and $\mathbb{R}^{m_{2}}$ are orthogonal. Consider the orthogonal decomposition

$$
\mathbb{R}^{m}=\mathbb{R}^{m_{0}} \oplus \mathbb{R}^{m_{1}} \oplus \mathbb{R}^{m_{2}}
$$

and let $P_{i}: \mathbb{R}^{m} \rightarrow \mathbb{R}^{m_{i}}, 0 \leq i \leq 2$, denote the orthogonal projections. Then

$$
\left(P_{i} \circ f\right)_{*} X_{j}=0, \quad 1 \leq i \neq j \leq 2
$$

whereas

$$
\left(P_{0} \circ f\right)_{*} X_{j}=0, \quad 1 \leq j \leq 2,
$$

for any $X_{j} \in E_{j}$. Therefore the map $P_{i} \circ f, 1 \leq i \leq 2$, is constant along the fibers of the projection $\pi_{i}: M \rightarrow M_{i}$, whereas $P_{0} \circ f$ has a constant value $v \in \mathbb{R}^{m_{0}}$. Fixed $\bar{x}=\left(\bar{x}_{1}, \bar{x}_{2}\right) \in M$, define $f_{i}: M_{i} \rightarrow \mathbb{R}^{m_{i}}, 1 \leq i \leq 2$, by

$$
f_{i}=P_{i} \circ f \circ \tau_{i}^{\bar{x}} .
$$

Then $f_{i}$ is an isometric immersion such that

$$
P_{i} \circ f=f_{i} \circ \pi_{i}, \quad 1 \leq i \leq 2 .
$$

Hence

$$
f=\sum_{i=0}^{2} P_{i} \circ f=v+f_{1} \circ \pi_{1}+f_{2} \circ \pi_{2}
$$

and thus $f$ is the extrinsic product of $f_{1}$ and $f_{2}$.
Remark 8.5. It follows from the proof of Theorem 8.4 that the decomposition of an isometric immersion $f: M \rightarrow \mathbb{R}^{m}$ of a Riemannian product manifold $M=\Pi_{i=1}^{r} M_{i}$ as an extrinsic product of isometric immersions $f_{i}: M_{i} \rightarrow \mathbb{R}^{m_{i}}, 1 \leq i \leq r$, is unique as soon as $f_{i}$ is assumed to be substantial for all $1 \leq i \leq r$.

Corollary 8.6. Let $f: M \rightarrow \mathbb{S}^{m}$ be an isometric immersion of a Riemannian product manifold $M=\Pi_{i=1}^{r} M_{i}$ whose second fundamental form is adapted to the product net of $M$. Then $f$ is an extrinsic product of isometric immersions.

Proof: Applying Theorem 8.4 to $\tilde{f}=i \circ f$, where $i: \mathbb{S}^{m} \rightarrow \mathbb{R}^{m+1}$ is the umbilical inclusion, implies that there exist an orthogonal decomposition

$$
\mathbb{R}^{m+1}=\Pi_{j=0}^{r} \mathbb{R}^{m_{j}}
$$

with $\mathbb{R}^{m_{0}}$ possibly trivial, isometric immersions $\tilde{f}_{j}: M_{j} \rightarrow \mathbb{R}^{m_{j}}, 1 \leq j \leq r$, and $v \in \mathbb{R}^{m_{0}}$ such that

$$
\tilde{f}(x)=\left(v, \tilde{f}_{1}\left(x_{1}\right), \ldots, \tilde{f}_{r}\left(x_{r}\right)\right)
$$

for all $x \in M$. Since $\tilde{f}(M) \subset \mathbb{S}^{m} \subset \mathbb{R}^{m+1}$, there exist $r_{j}>0,1 \leq j \leq r$, such that $\tilde{f}_{j}\left(M_{j}\right) \subset \mathbb{S}^{m_{j}-1}\left(r_{j}\right) \subset \mathbb{R}^{m_{j}}$ and

$$
\|v\|^{2}+\sum_{j=1}^{r} r_{j}^{2}=1
$$

Defining $f_{j}: M_{j} \rightarrow \mathbb{S}^{m_{j}-1}\left(r_{j}\right)$ by $\tilde{f}_{j}=i_{j} \circ f_{j}, 1 \leq j \leq r$, where $i_{j}: \mathbb{S}^{m_{j}-1}\left(r_{j}\right) \rightarrow \mathbb{R}^{m_{j}}$ is the umbilical inclusion, we conclude that $f$ is the extrinsic product of $f_{1}, \ldots, f_{r}$.

To state and prove the version of Theorem 8.4 for isometric immersions into hyperbolic space, we first extend Theorem 8.4 to isometric immersions of Riemannian products into Lorentzian space.

Theorem 8.7. Let $f: M \rightarrow \mathbb{L}^{m}$ be an isometric immersion of a Riemannian product manifold $M=\Pi_{i=1}^{r} M_{i}$ whose second fundamental form is adapted to the product net of $M$. Then one of the following possibilities holds:
(i) There exist an orthogonal decomposition

$$
\begin{equation*}
\mathbb{L}^{m}=\mathbb{L}^{m_{1}} \times \Pi_{i=2}^{r+1} \mathbb{R}^{m_{i}} \tag{8.8}
\end{equation*}
$$

with $\mathbb{R}^{m_{r+1}}$ possibly trivial, a vector $v \in \mathbb{R}^{m_{r+1}}$ (in case $\mathbb{R}^{m_{r+1}}$ is nontrivial) and substantial isometric immersions $f_{1}: M_{1} \rightarrow \mathbb{L}^{m_{1}}, f_{i}: M_{i} \rightarrow \mathbb{R}^{m_{i}}, 2 \leq i \leq r$, such that

$$
\begin{equation*}
f\left(x_{1}, \ldots, x_{r}\right)=\left(f_{1}\left(x_{1}\right), \ldots, f_{r}\left(x_{r}\right), v\right) . \tag{8.9}
\end{equation*}
$$

(ii) There exist an orthogonal decomposition

$$
\mathbb{L}^{m}=\Pi_{i=1}^{r} \mathbb{R}^{m_{i}} \times \mathbb{L}^{m_{r+1}}
$$

substantial isometric immersions $f_{i}: M_{i} \rightarrow \mathbb{R}^{m_{i}}, 1 \leq i \leq r$, and $v \in \mathbb{L}^{m_{r+1}}$ such that

$$
\begin{equation*}
f\left(x_{1}, \ldots, x_{r}\right)=\left(f_{1}\left(x_{1}\right), \ldots, f_{r}\left(x_{r}\right), v\right) . \tag{8.10}
\end{equation*}
$$

(iii) There exist $1 \leq s \leq r$, orthogonal decompositions

$$
\begin{equation*}
\mathbb{L}^{m_{1}}=\Pi_{i=1}^{s} \mathbb{R}^{m_{i}} \times \mathbb{L}^{2} \text { and } \mathbb{L}^{m}=\mathbb{L}^{m_{1}} \times \Pi_{i=s+1}^{r+1} \mathbb{R}^{m_{i}} \tag{8.11}
\end{equation*}
$$

with $\mathbb{R}^{m_{r+1}}$ possibly trivial, a vector $v \in \mathbb{R}^{m_{r+1}}$ (in case $\mathbb{R}^{m_{r+1}}$ is nontrivial), a function $\varphi \in C^{\infty}\left(M_{1} \times \cdots \times M_{s}\right)$, substantial isometric immersions $f_{i}: M_{i} \rightarrow \mathbb{R}^{m_{i}}$, $1 \leq i \leq r$, and a pseudo-orthonormal basis $w, \bar{w}$ of $\mathbb{L}^{2}$ with $\langle w, w\rangle=0=\langle\bar{w}, \bar{w}\rangle$ and $\langle w, \bar{w}\rangle=1$, and $\delta \in\{0,1\}$ such that

$$
\begin{equation*}
f\left(x_{1}, \ldots, x_{r}\right)=\left(g\left(x_{1}, \ldots, x_{s}\right), f_{s+1}\left(x_{s+1}\right), \ldots, f_{r}\left(x_{r}\right), v\right) \tag{8.12}
\end{equation*}
$$

where

$$
g\left(x_{1}, \ldots, x_{s}\right)=\left(f_{1}\left(x_{1}\right), \ldots, f_{s}\left(x_{s}\right), \varphi\left(x_{1}, \ldots, x_{s}\right) w+\delta \bar{w}\right) .
$$

Proof: Let $\mathcal{E}=\left(E_{1}, \ldots, E_{r}\right)$ be the product net of $M$. Define linear subspaces $W_{i}$ of $\mathbb{L}^{m}, 1 \leq i \leq r$, by

$$
W_{i}=\operatorname{span}\left\{f_{*}(x) X_{i}: x \in M \text { and } X_{i} \in E_{i}(x)\right\} .
$$

Arguing as in the proof of Theorem 8.4, we see that the subspaces $W_{i}$ are mutually orthogonal. We distinguish two cases.

First suppose that all the $W_{i}$ are nondegenerate subspaces of $\mathbb{L}^{m}$. Let $W_{r+1}$ be a (possibly trivial) subspace of $\mathbb{L}^{m}$ defined by requiring that

$$
\mathbb{L}^{m}=W_{1} \oplus \cdots \oplus W_{r} \oplus W_{r+1}
$$

be an orthogonal sum decomposition. Let $P_{i}: \mathbb{L}^{m} \rightarrow W_{i}$ denote the orthogonal projections for $1 \leq i \leq r+1$. Since $W_{i}$ is time-like for exactly one $i \in\{1, \ldots, r+1\}$, we may assume that either $i=1$ or $i=r+1$. Since $W_{1}, \ldots, W_{r+1}$ are mutually orthogonal,

$$
\left(P_{i} \circ f\right)_{*} X_{j}=0, \quad 1 \leq i \neq j \leq r,
$$

whereas

$$
\left(P_{r+1} \circ f\right)_{*} X_{j}=0, \quad 1 \leq j \leq r,
$$

for all $X_{j} \in \Gamma\left(E_{j}\right)$. Therefore the map $P_{i} \circ f$ is constant along the fibers of the projection $\pi_{i}: M \rightarrow M_{i}$ for $1 \leq i \leq r$, while $P_{r+1} \circ f$ has a constant value $v$ on $M$. Fixed $\bar{x}=\left(\bar{x}_{1}, \ldots, \bar{x}_{r}\right) \in M$, define $f_{i}: M_{i} \rightarrow W_{i}$ for $1 \leq i \leq r$ by

$$
f_{i}=P_{i} \circ f \circ \tau_{i}^{\bar{x}}
$$

Then $f_{i}$ is an isometric immersion such that

$$
P_{i} \circ f=f_{i} \circ \pi_{i}, \quad 1 \leq i \leq r .
$$

Hence

$$
f=\sum_{i=1}^{r+1} P_{i} \circ f=\sum_{i=1}^{r} f_{i} \circ \pi_{i}+v
$$

If $W_{1}$ is time-like, writing $W_{1}=\mathbb{L}^{m_{1}}$ and $W_{i}=\mathbb{R}^{m_{i}}$ for $2 \leq i \leq r+1$, we obtain an orthogonal decomposition of $\mathbb{L}^{m+1}$ as in (8.8) with respect to which $f$ is given by (8.9).

If $W_{r+1}$ is time like, we can write $W_{i}=\mathbb{R}^{m_{i}}$ for $1 \leq i \leq r$ and $W_{r+1}=\mathbb{L}^{m_{r+1}}$. We obtain an orthogonal decomposition of $\mathbb{L}^{m+1}$ as in (8.11) with respect to which $f$ is given by 8.10).

The second case to consider is when some of the $W_{i}$ are degenerate subspaces of $\mathbb{L}^{m}$. In this case, without loss of generality, we may assume that there exists $1 \leq s \leq r$ such that $W_{1}, \ldots, W_{s}$ are degenerate, while $W_{s+1}, \ldots, W_{r}$ are nondegenerate, and thus necessarily space-like if $s<r$. Then there exists a one-dimensional light-like subspace $L_{0}$ such that $W_{i} \cap W_{i}^{\perp}=L_{0}$ for all $1 \leq i \leq s$. Choose a distinct light-like line $L_{1}$ orthogonal to $W_{s+1}, \ldots, W_{r}$. Set $\hat{W}_{i}=\bar{W}_{i} \cap \bar{L}_{1}^{\perp}, 1 \leq i \leq r$, so that

$$
W_{i}=\hat{W}_{i} \oplus L_{0}, \quad 1 \leq i \leq s, \quad \text { and } W_{i}=\hat{W}_{i}, \quad s+1 \leq i \leq r .
$$

Defining

$$
\hat{W}_{r+1}=\left(L_{0} \oplus \hat{W}_{1} \oplus \cdots \oplus \hat{W}_{r} \oplus L_{1}\right)^{\perp},
$$

it follows that $\mathbb{L}^{m}$ decomposes as

$$
\mathbb{L}^{m}=L_{0} \oplus \hat{W}_{1} \oplus \cdots \oplus \hat{W}_{r} \oplus \hat{W}_{r+1} \oplus L_{1} .
$$

Let $\hat{P}_{i}: \mathbb{L}^{m} \rightarrow \hat{W}_{i}, 1 \leq i \leq r+1$, be the orthogonal projection. Arguing as in the preceding case, we see that the map $\hat{P}_{i} \circ f$ is constant along the fibers of the projection $\pi_{i}: M \rightarrow M_{i}$ for $1 \leq i \leq r$, the components of $f$ in $L_{1}$ and $\hat{W}^{r+1}$ are constant and the component of $f$ in $L_{0}$ is constant along the fibers of the projection of $M$ onto $M_{1} \times \ldots \times M_{s}$. Thus there exist isometric immersions $f_{i}: M_{i} \rightarrow \hat{W}_{i}, 1 \leq i \leq r$, $\varphi \in C^{\infty}\left(M_{1} \times \cdots \times M_{s}\right)$, and vectors $w \in L_{0}, \bar{w} \in L_{1}$ and $v \in \hat{W}_{r+1}$ (in case $\hat{W}_{r+1}$ is nontrivial) such that

$$
f=\varphi w+\sum_{i=1}^{r} f_{i} \circ \pi_{i}+\bar{w}+v .
$$

If $\bar{w} \neq 0$, we can assume that $w$ has been chosen so that $\langle w, \bar{w}\rangle=1$.
Now write $\mathbb{R}^{m_{i}}=\hat{W}_{i}$ for $1 \leq i \leq r+1$ and $\mathbb{L}^{2}=L_{0} \oplus L_{1}$. We obtain an orthogonal decomposition as in (8.11) with respect to which $f$ is given by (8.12) for

$$
g=\sum_{i=1}^{s} f_{i} \circ \pi_{i}+\varphi w+\delta \bar{w} .
$$

Corollary 8.8. Let $f: M \rightarrow \mathbb{H}^{m}$ be an isometric immersion of a Riemannian product manifold $M=\prod_{i=1}^{r} M_{i}$ whose second fundamental form is adapted to the product net of $M$. Then either $f$ is an extrinsic product given by (8.6) of substantial isometric immersions

$$
f_{1}: M_{1} \rightarrow \mathbb{H}^{m_{1}-1}\left(r_{1}\right) \subset \mathbb{L}^{m_{1}} \text { and } f_{i}: M_{i} \rightarrow \mathbb{S}^{m_{i}-1}\left(r_{i}\right) \subset \mathbb{R}^{m_{i}}, \quad 2 \leq i \leq r
$$

with respect to an orthogonal decomposition of $\mathbb{L}^{m+1}$ as in (8.5), or $f$ is the composition of an extrinsic product of immersions into either $\mathbb{R}^{n}$ or $\mathbb{S}^{n}, n<m$, with an umbilical inclusion of either of these spaces into $\mathbb{H}^{m}$.

Proof: Apply Theorem 8.7 to $\tilde{f}=i \circ f$, where $i: \mathbb{H}^{m} \rightarrow \mathbb{L}^{m+1}$ is the umbilical inclusion. Assume first that the assertion in case $(i)$ of Theorem 8.7 holds for $\tilde{f}$. Namely, there exist an orthogonal decomposition

$$
\mathbb{L}^{m+1}=\mathbb{L}^{m_{1}} \times \Pi_{i=2}^{r+1} \mathbb{R}^{m_{i}}
$$

with $\mathbb{R}^{m_{r+1}}$ possibly trivial, substantial isometric immersions $\tilde{f}_{1}: M_{1} \rightarrow \mathbb{L}^{m_{1}}, \tilde{f}_{i}: M_{i} \rightarrow$ $\mathbb{R}^{m_{i}}, 2 \leq i \leq r$, and $v \in \mathbb{R}^{m_{r+1}}$ (in case $\mathbb{R}^{m_{r+1}}$ is nontrivial) such that

$$
\tilde{f}\left(x_{1}, \ldots, x_{r}\right)=\left(\tilde{f}_{1}\left(x_{1}\right), \ldots, \tilde{f}_{r}\left(x_{r}\right), v\right)
$$

$\operatorname{From}\langle\tilde{f}, \tilde{f}\rangle=-1$ we obtain

$$
\sum_{i=1}^{r}\left\langle\tilde{f}_{i} \circ \pi_{i}, \tilde{f}_{i} \circ \pi_{i}\right\rangle+\langle v, v\rangle=-1
$$

Thus each $\left\langle\tilde{f}_{i}, \tilde{f}_{i}\right\rangle, 1 \leq i \leq r$, is constant, and hence there exist isometric immersions $f_{1}: M_{1} \rightarrow \mathbb{H}^{m_{1}-1}\left(r_{1}\right)$ and $f_{j}: M_{j} \rightarrow \mathbb{S}^{m_{j}-1}\left(r_{j}\right), 2 \leq j \leq r$, such that

$$
-r_{1}^{2}+\sum_{j=2}^{r} r_{j}^{2}+\langle v, v\rangle=-1
$$

and $\tilde{f}_{j}=i_{j} \circ f_{j}, 1 \leq j \leq r$, where $i_{1}: \mathbb{H}^{m_{1}-1}\left(r_{1}\right) \rightarrow \mathbb{L}^{m_{1}}$ and $i_{j}: \mathbb{S}^{m_{j}-1}\left(r_{j}\right) \rightarrow \mathbb{R}^{m_{j}}$, $2 \leq j \leq r$, are umbilical inclusions. Therefore $f$ is the extrinsic product of $f_{1}, \ldots, f_{r}$.

Next suppose that $\tilde{f}$ satisfies the conclusion in part (ii) of Theorem 8.7. Thus there exist an orthogonal decomposition

$$
\mathbb{L}^{m+1}=\Pi_{i=1}^{r} \mathbb{R}^{m_{i}} \times \mathbb{L}^{m_{r+1}}
$$

substantial isometric immersions $\tilde{f}_{i}: M_{i} \rightarrow \mathbb{R}^{m_{i}}, 1 \leq i \leq r$, and $v \in \mathbb{L}^{m_{r+1}}$ such that

$$
\begin{equation*}
\tilde{f}\left(x_{1}, \ldots, x_{r}\right)=\left(\tilde{f}_{1}\left(x_{1}\right), \ldots, \tilde{f}_{r}\left(x_{r}\right), v\right) \tag{8.13}
\end{equation*}
$$

Using that $\langle\tilde{f}, \tilde{f}\rangle=-1$, it follows from (8.13) that $\left\langle\tilde{f}_{i}, \tilde{f}_{i}\right\rangle=r_{i}^{2}$ is constant, with

$$
\sum_{i=1}^{r} r_{i}^{2}+\langle v, v\rangle=-1
$$

Thus $\tilde{f}_{j}\left(M_{j}\right) \subset \mathbb{S}^{m_{j}-1}\left(r_{j}\right) \subset \mathbb{R}^{m_{j}}, 1 \leq j \leq r$, so we can write $\tilde{f}_{j}=i_{j} \circ f_{j}$, where $f_{j}: M_{j} \rightarrow \mathbb{S}^{m_{j}-1}\left(r_{j}\right), 1 \leq j \leq r$, are isometric immersions and $i_{j}: \mathbb{S}^{m_{j}-1}\left(r_{j}\right) \rightarrow \mathbb{R}^{m_{j}}$ are umbilical inclusions.

Denote $\mathbb{R}^{n+1}=\Pi_{i=1}^{r} \mathbb{R}^{m_{i}}$ and let $\bar{f}: M \rightarrow \mathbb{S}^{n} \subset \mathbb{R}^{n+1}$ be given by

$$
\bar{f}\left(x_{1}, \ldots, x_{r}\right)=\left(\tilde{f}_{1}\left(x_{1}\right), \ldots, \tilde{f}_{r}\left(x_{r}\right)\right)
$$

Then $\bar{f}$ is the extrinsic product of $f_{1}, \ldots, f_{r}$ into $\mathbb{S}^{n}$ and $f=i \circ \bar{f}$, where

$$
i: \mathbb{S}^{n} \rightarrow \mathbb{H}^{m} \subset \mathbb{L}^{m+1}=\mathbb{R}^{n+1} \times \mathbb{R}^{m_{r+1}}
$$

is the umbilical inclusion given by

$$
i(x)=(x, v) .
$$

Finally, suppose that $\tilde{f}$ satisfies the conclusion in part (iii) of Theorem 8.7. Thus there exist $1 \leq s \leq r$, orthogonal decompositions

$$
\mathbb{L}^{m_{1}}=\Pi_{i=1}^{s} \mathbb{R}^{m_{i}} \times \mathbb{L}^{2} \text { and } \mathbb{L}^{m+1}=\mathbb{L}^{m_{1}} \times \Pi_{i=s+1}^{r+1} \mathbb{R}^{m_{i}}
$$

with $\mathbb{R}^{m_{r+1}}$ possibly trivial, a vector $v \in \mathbb{R}^{m_{r+1}}$ (in case $\mathbb{R}^{m_{r+1}}$ is nontrivial), $\varphi \in$ $C^{\infty}\left(M_{1} \times \cdots \times M_{s}\right)$, substantial isometric immersions $f_{i}: M_{i} \rightarrow \mathbb{R}^{m_{i}}, 1 \leq i \leq r$, a pseudo-orthonormal basis $w, \bar{w}$ of $\mathbb{L}^{2}$ with $\langle w, w\rangle=0=\langle\bar{w}, \bar{w}\rangle$ and $\langle w, \bar{w}\rangle=1$, and $\delta \in\{0,1\}$ such that

$$
\tilde{f}\left(x_{1}, \ldots, x_{r}\right)=\left(g\left(x_{1}, \ldots, x_{s}\right), f_{s+1}\left(x_{s+1}\right), \ldots, f_{r}\left(x_{r}\right), v\right)
$$

where

$$
g\left(x_{1}, \ldots, x_{s}\right)=\left(f_{1}\left(x_{1}\right), \ldots, f_{s}\left(x_{s}\right), \varphi\left(x_{1}, \ldots, x_{s}\right) w+\delta \bar{w}\right) .
$$

Since $\langle\tilde{f}, \tilde{f}\rangle=-1$, the case $\delta=0$ is ruled out and we obtain

$$
2 \varphi\left(x_{1}, \ldots, x_{s}\right)+\sum_{i=1}^{r}\left\langle f_{i}\left(x_{i}\right), f_{i}\left(x_{i}\right)\right\rangle+\langle v, v\rangle=-1
$$

for all $x=\left(x_{1}, \ldots, x_{r}\right) \in M$. In particular, $\left\langle f_{i}, f_{i}\right\rangle=r_{i}^{2}$ is constant, that is, $f_{i}$ takes values in $\mathbb{S}^{m_{i}-1}\left(r_{i}\right)$, for $s+1 \leq i \leq r$. In summary,

$$
\begin{equation*}
\tilde{f}=\sum_{i=1}^{r} f_{i} \circ \pi_{i}+\bar{w}-\frac{1}{2}\left(\sum_{i=1}^{r}\left\langle f_{i} \circ \pi_{i}, f_{i} \circ \pi_{i}\right\rangle+r^{2}\right) w+v \tag{8.14}
\end{equation*}
$$

where

$$
r^{2}=1+\langle v, v\rangle .
$$

Write $\mathbb{R}^{n}=\Pi_{i=1}^{r} \mathbb{R}^{m_{i}}$ and let $\bar{f}: M \rightarrow \mathbb{R}^{n}$ be the extrinsic product of $f_{1}, \ldots, f_{r}$ into $\mathbb{R}^{n}$ given by

$$
\bar{f}\left(x_{1}, \ldots, x_{r}\right)=\left(f_{1}\left(x_{1}\right), \ldots, f_{r}\left(x_{r}\right)\right) .
$$

Then $f=j \circ \bar{f}$, where $j: \mathbb{R}^{n} \rightarrow \mathbb{H}^{m} \subset \mathbb{L}^{m+1}=\mathbb{R}^{n} \times \mathbb{L}^{2} \times \mathbb{R}^{m_{r+1}}$ is the umbilical inclusion of $\mathbb{R}^{n}$ into $\mathbb{H}^{m}$ given by

$$
j(x)=\left(x, \bar{w}-\frac{1}{2}\left(\|x\|^{2}+r^{2}\right) w, v\right) .
$$

### 8.4 Local conditions for decomposability

In this section we describe sufficient conditions of local nature for the second fundamental form of an isometric immersion of a Riemannian product manifold to be adapted to its product net.

### 8.4.1 Curvature conditions

The main result of this section implies that an isometric immersion of a Riemannian product manifold $M^{n}=\Pi_{i=0}^{k} M_{i}^{n_{i}}$ into $\mathbb{R}^{n+k}$, with $n_{i} \geq 2$ for all $1 \leq i \leq k$, is an extrinsic product of hypersurface immersions if no $M_{i}^{n_{i}}, 1 \leq i \leq k$, contains an open subset of flat points. The proof relies on the next result.

Proposition 8.9. Let $f: M^{n} \rightarrow \mathbb{R}^{n+k}$ be an isometric immersion of a Riemannian product $M^{n}=\Pi_{i=0}^{k} M_{i}^{n_{i}}$, with $n_{i} \geq 2$ for all $1 \leq i \leq k$. If $M_{i}^{n_{i}}$ is not flat at $x_{i}$ for all $1 \leq i \leq k$, then the second fundamental form is adapted to the product net $\mathcal{E}=\left(E_{i}\right)_{i=0, \ldots, k}$ of $M^{n}$ at $x=\left(x_{0}, x_{1}, \ldots, x_{k}\right)$. Moreover, if $n_{0}>0$ then $M_{0}^{n_{0}}$ is flat at $x_{0}$ and $E_{0}(x)$ belongs to the relative nullity subspace of $f$ at $x$.

Proof: We first prove the result when $n_{0}=0$ and $n_{i}=2$ for $1 \leq i \leq n$. At $x \in M^{n}$, take an orthonormal basis $e_{1}, \ldots, e_{2 k}$ of $T_{x} M$ such that $e_{1}, e_{2} \in E_{1}(x), \ldots, e_{2 k-1}, e_{2 k} \in E_{k}(x)$. Let $\omega^{1}, \ldots, \omega^{2 k}$ denote the dual basis and $K_{i}$ the sectional curvature of $M_{i}^{n_{i}}$ at $x_{i}$, $1 \leq i \leq k$. Define the symmetric bilinear forms $B_{i}, 1 \leq i \leq k$, by

$$
B_{i}= \begin{cases}\sqrt{-K_{i}}\left(\omega^{2 i-1} \otimes \omega^{2 i-1}+\omega^{2 i} \otimes \omega^{2 i}\right) & \text { if } K_{i}<0 \\ \sqrt{K_{i}}\left(\omega^{2 i-1} \otimes \omega^{2 i-1}-\omega^{2 i} \otimes \omega^{2 i}\right) & \text { if } K_{i}>0\end{cases}
$$

Set $B=B_{1} \oplus \cdots \oplus B_{k}: T_{x} M \times T_{x} M \rightarrow \mathbb{R}^{k}$ and consider the symmetric bilinear form

$$
\beta=\alpha \oplus B: T_{x} M \times T_{x} M \rightarrow N_{f} M(x) \oplus \mathbb{R}^{k} .
$$

It follows from the Gauss equations and the curvature relations (8.4) that $\beta$ is flat with respect to the positive inner product on $N_{f} M(x) \oplus \mathbb{R}^{k}$ that makes the decomposition orthogonal and whose restriction to each factor is the standard inner product on it.

Since $K_{i} \neq 0$ for all $1 \leq i \leq k$, the nullity subspace of the bilinear form $\beta$ is trivial. Then, by Theorem 5.2, there exists a basis $X_{1}, \ldots, X_{2 k}$ of $T_{x} M$ that diagonalizes $\beta$. In particular, the nullity subspace of each $B_{i}, 1 \leq i \leq k$, is spanned by a subset of this basis. Since

$$
E_{i}(x)=\cap_{j \neq i} \mathcal{N}\left(B_{j}\right),
$$

also $E_{i}(x)$ is spanned by a subset of this basis for $1 \leq i \leq k$. Hence we can arrange that $X_{1}, X_{2} \in E_{1}(x), \ldots, X_{2 k-1}, X_{2 k} \in E_{k}(x)$, and the result follows in this case.

To treat the general case, choose an orthonormal set of vectors $e_{1}, \ldots, e_{2 k} \in T_{x} M$ such that $e_{2 j-1}, e_{2 j}$ span a plane in $E_{j}(x)$ with nonzero sectional curvature for all $1 \leq j \leq k$. We assume without loss of generality that $\alpha\left(e_{2 j}, e_{2 j}\right) \neq 0,1 \leq j \leq k$. If we restrict $\alpha$ to the subspace spanned by $e_{1}, \ldots, e_{2 k}$ and apply the argument of the preceding paragraph, we find that $\alpha\left(e_{2 i}, e_{2 j}\right)=0$ for $1 \leq i \neq j \leq k$. Since the sectional curvature of the two-plane spanned by $e_{2 i}$ and $e_{2 j}$ is zero when $i \neq j$, it follows from the Gauss equation that

$$
\left\langle\alpha\left(e_{2 i}, e_{2 i}\right), \alpha\left(e_{2 j}, e_{2 j}\right)\right\rangle=0, \quad 1 \leq i \neq j \leq k .
$$

Therefore the vectors $\alpha\left(e_{2 j}, e_{2 j}\right), 1 \leq j \leq k$, form an orthogonal basis of $N_{f} M(x)$. Similarly we can conclude that

$$
\left\langle\alpha\left(e_{2 i-1}, e_{2 i}\right), \alpha\left(e_{2 j}, e_{2 j}\right)\right\rangle=0=\left\langle\alpha\left(e_{2 i-1}, e_{2 i-1}\right), \alpha\left(e_{2 j}, e_{2 j}\right)\right\rangle
$$

for $1 \leq i \neq j \leq k$, from which it follows that the normal vectors $\alpha\left(e_{2 i-1}, e_{2 i}\right)$ and $\alpha\left(e_{2 i-1}, e_{2 i-1}\right)$ are scalar multiples of $\alpha\left(e_{2 i}, e_{2 i}\right)$. After a rotation of $e_{2 i-1}$ and $e_{2 i}$, we can arrange that $\alpha\left(e_{2 i-1}, e_{2 i}\right)=0$, so that the normal vectors $\alpha\left(e_{2 i-1}, e_{2 i-1}\right), 1 \leq i \leq k$, also form a basis of the normal space.

If $X \in E_{i}(x), 0 \leq i \leq k$, and $j \neq i$, then the Gauss equation gives

$$
\left\langle\alpha\left(X, e_{2 j}\right), \alpha\left(e_{2 r-1}, e_{2 r-1}\right)\right\rangle=0, \quad 1 \leq r \leq k .
$$

Thus $\alpha\left(X, e_{2 j}\right)=0$, and hence

$$
\left\langle\alpha(X, X), \alpha\left(e_{2 j}, e_{2 j}\right)\right\rangle=\left\langle\alpha\left(X, e_{2 j}\right), \alpha\left(X, e_{2 j}\right)\right\rangle=0 \text { if } i \neq j .
$$

Therefore $\alpha(X, X)=0$ if $i=0$, and $\alpha(X, X)$ is a scalar multiple of $\alpha\left(e_{2 i}, e_{2 i}\right)$ if $1 \leq i \leq k$. If $Y \in E_{j}(x)$, with $1 \leq j \leq k$ and $j \neq i$, this and the Gauss equation give

$$
\|\alpha(X, Y)\|^{2}=\langle\alpha(X, X), \alpha(Y, Y)\rangle=0
$$

Moreover, the fact that $\alpha(X, X)=0$ for all $X \in E_{0}(x)$ implies that $E_{0}(x)$ belongs to the relative nullity subspace of $f$ at $x$ and that $M_{0}^{n_{0}}$ is flat at $x_{0}$.

Theorem 8.10. Let $f: M^{n} \rightarrow \mathbb{R}^{n+k}$ be an isometric immersion of a Riemannian product manifold $M^{n}=\prod_{i=0}^{k} M_{i}^{n_{i}}$. If the subset of points of $M_{i}^{n_{i}}$ at which all sectional curvatures vanish has empty interior for all $1 \leq i \leq k$, then $M_{0}^{n_{0}}$ is flat and $f(M)$ is an open subset of a $n_{0}$-cylinder over an extrinsic product of hypersurface immersions.

Proof: The subset $S \subset M^{n}$ given by

$$
S=\left\{x=\left(x_{0}, x_{1}, \ldots, x_{k}\right): M_{i}^{n_{i}} \text { is not flat at } x_{i} \text { for all } 1 \leq i \leq k\right\}
$$

is dense on $M^{n}$ and, by Proposition 8.9, at any point $x=\left(x_{0}, x_{1}, \ldots, x_{k}\right) \in S$ the second fundamental form of $f$ is adapted to the product net $\mathcal{E}=\left(E_{i}\right)_{i=0, \ldots, k}$ of $M^{n}$, $M_{0}^{n_{0}}$ is flat at $x_{0}$ and $E_{0}(x)$ belongs to the relative nullity subspace of $f$ at $x$. Therefore these conditions hold at any point of $M^{n}$. It follows from Corollary 7.5 that $f$ is locally a $n_{0}$-cylinder over

$$
g=f \circ \mu_{\bar{x}_{0}}: \bar{M}^{n-n_{0}}=\Pi_{i=1}^{k} M_{i}^{n_{i}} \rightarrow \mathbb{R}^{n-n_{0}+k}
$$

where $\bar{x}_{0} \in M_{0}^{n_{0}}$ is arbitrary and $\mu_{\bar{x}_{0}}: \Pi_{i=1}^{k} M_{i}^{n_{i}} \rightarrow M^{n}$ is the inclusion of $\Pi_{i=1}^{k} M_{i}^{n_{i}}$ into $M^{n}$ given by

$$
\begin{equation*}
\mu_{\bar{x}_{0}}\left(x_{1}, \ldots, x_{k}\right)=\left(\bar{x}_{0}, x_{1}, \ldots, x_{k}\right) . \tag{8.15}
\end{equation*}
$$

Since $\mu_{\bar{x}_{0}}$ is totally geodesic, by Exercise 1.6 the second fundamental forms of $g$ and $f$ are related by

$$
\alpha^{g}(X, Y)=\alpha^{f}\left(\mu_{\bar{x}_{0 *}} X, \mu_{\bar{x}_{0} *} Y\right)
$$

for all $\bar{x}=\left(x_{1}, \ldots, x_{k}\right) \in \bar{M}^{n-n_{0}}$ and $X, Y \in T_{\bar{x}} \bar{M}$. Moreover, if $\bar{\varepsilon}=\left(\bar{E}_{i}\right)_{i=1, \ldots, k}$ is the product net of $\bar{M}$, then it is clear that $\mu_{\bar{x}_{0} *} \bar{E}_{i}(\bar{x})=E_{i}\left(\mu_{\bar{x}_{0}}(\bar{x})\right)$ for all $1 \leq i \leq k$. Therefore $\alpha^{g}$ is adapted to the product structure of $\bar{M}^{n-n_{0}}$, and hence $g$ is an extrinsic product of hypersurface immersions by Theorem 8.4.

In the case of Riemannian products with only two factors, a complete description of their isometric immersions in space forms with codimension less than or equal to two is given in the sequel without proof. Notice that, as a particular case of Theorem 8.10, it follows that if $f: L^{p} \times M^{n} \rightarrow \mathbb{R}^{p+n+1}$ is an isometric immersion of a Riemannian product such that $p+n \geq 3$ and the subset of points of, say, $M^{n}$, at which all sectional curvatures vanish has empty interior, then $f\left(L^{p} \times M^{n}\right)$ is an open subset of a $p$-cylinder over a hypersurface $h: M^{n} \rightarrow \mathbb{R}^{n+1}$. In case the ambient space is the sphere or hyperbolic space one has the following result.

Theorem 8.11. Let $f: L^{p} \times M^{n} \rightarrow \mathbb{Q}_{c}^{p+n+1}, c \neq 0$, be an isometric immersion of a Riemannian product manifold. If $p+n \geq 3$ then $f$ is an extrinsic product of local isometries $i_{1}: L^{p} \rightarrow \mathbb{Q}_{c_{1}}^{p}$ and $i_{2}: M^{n} \rightarrow \mathbb{Q}_{c_{2}}^{n}, 1 / c_{1}+1 / c_{2}=1 / c$.

The case of codimension two is quite more involved, and can be regarded as a first step towards generalizations of Theorem 8.10 to higher codimension.

Theorem 8.12. Let $f: L^{p} \times M^{n} \rightarrow \mathbb{R}^{p+n+2}$ be an isometric immersion of a Riemannian product. Assume that either $L^{p}$ or $M^{n}$ has dimension at least two and has no open subsets where all sectional curvatures vanish. Then there exists an open dense subset of $L^{p} \times M^{n}$ each of whose points lies in an open product neighborhood $U=L_{0}^{p} \times M_{0}^{n} \subset L^{p} \times M^{n}$ such that one of the following possibilities holds:
(i) The immersion $\left.f\right|_{U}$ is an extrinsic product of isometric immersions $h_{1}: L_{0}^{p} \rightarrow \mathbb{R}^{p+k_{1}}$ and $h_{2}: M_{0}^{n} \rightarrow \mathbb{R}^{n+k_{2}}, k_{1}+k_{2}=2$.

$$
\begin{aligned}
& \mathbb{R}^{p+k_{1}} \times \mathbb{R}^{n+k_{2}}=\mathbb{R}^{p+n+2} \\
& \left.\left.\right|_{h_{1}}\right|_{h_{2}} ^{p} /\left.f\right|_{U}=h_{1} \times h_{2} \\
& L_{0}^{p}
\end{aligned}
$$

(ii) The immersion $\left.f\right|_{U}$ is a composition $\left.f\right|_{U}=H \circ g$, where $g$ is an extrinsic product of isometric immersions $h_{1}: L_{0}^{p} \rightarrow \mathbb{R}^{p+k_{1}}$ and $h_{2}: M_{0}^{n} \rightarrow \mathbb{R}^{n+k_{2}}, k_{1}+k_{2}=1$, and $H: W \rightarrow \mathbb{R}^{p+n+2}$ is an isometric immersion of an open subset $W \subset \mathbb{R}^{p+n+1}$ that contains $g(U)$.

$$
\begin{aligned}
& \mathbb{R}^{p+k_{1}} \times \mathbb{R}^{n+k_{2}} \supset W \\
& \overbrace{L_{1}^{p}}^{h_{1}} \stackrel{h}{2}^{h_{2}} M_{0}^{n} \xrightarrow[\left.f\right|_{U}=H \circ\left(h_{1} \times h_{2}\right)]{ } \mathbb{R}^{p+n+2}
\end{aligned}
$$

In the case of a nonflat ambient space one has the following result.
Theorem 8.13. Let $f: L^{p} \times M^{n} \rightarrow \mathbb{Q}_{c}^{p+n+2}, c \neq 0$, be an isometric immersion of a Riemannian product manifold such that either $n \geq 3$ or $p \geq 3$. Then there exists an open dense subset of $L^{p} \times M^{n}$ each of whose points lies in an open product neighborhood $U=L_{0}^{p} \times M_{0}^{n} \subset L^{p} \times M^{n}$ such that one of the following possibilities holds:
(i) The immersion $\left.f\right|_{U}$ is the extrinsic product of isometric immersions $h_{1}: L_{0}^{p} \rightarrow \mathbb{Q}_{c_{1}}^{p+k_{1}}$ and $h_{2}: M_{0}^{n} \rightarrow \mathbb{Q}_{c_{2}}^{n+k_{2}}, k_{1}+k_{2}=1$.

(ii) The immersion $\left.f\right|_{U}$ is a composition $\left.f\right|_{U}=H \circ g$, where $g$ is an extrinsic product of local isometries $i_{1}: L_{0}^{p} \rightarrow \mathbb{Q}_{c_{1}}^{p}$ and $i_{2}: M_{0}^{n} \rightarrow \mathbb{Q}_{c_{2}}^{n}$, and $H: W \rightarrow \mathbb{Q}_{c}^{p+n+2}$ is an isometric immersion of an open subset $W \subset \mathbb{Q}_{c}^{p+n+1}$ that contains $g(U)$.


As an example showing that for $c \neq 0$ the assumption that either $n \geq 3$ or $p \geq 3$ is indeed necessary, one may take any local isometric immersion of $\mathbb{R}^{3}$ into $\mathbb{S}^{5}$ that is not an extrinsic product of a unit speed parametrization $\alpha: I \rightarrow \mathbb{S}^{1}\left(r_{1}\right) \subset \mathbb{R}^{2}$ of an open subset of a circle of radius $r_{1}$ and any $\alpha \times g: I \times U \rightarrow \mathbb{S}^{1}\left(r_{1}\right) \times \mathbb{S}^{3}\left(r_{2}\right), r_{1}^{2}+r_{2}^{2}=1$, where $\alpha: I \rightarrow \mathbb{R}^{2}$ is a of an open subset of a circle of radius $r_{1}$ and any isometric immersion $g: U \rightarrow \mathbb{S}^{3}\left(r_{2}\right)$ of an open subset $U \subset \mathbb{R}^{2}$. Such immersions have been discussed in Chapter 5.

### 8.4.2 Conditions on the s-nullities

Another decomposition theorem of local nature for isometric immersions of Riemannian products is presented next, now under assumptions on the $s$-nullities of the second fundamental form.

Theorem 8.14. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}, 2 p<n$, be an isometric immersion of a Riemannian product manifold $M^{n}=\Pi_{i=1}^{k} M_{i}^{n_{i}}$. If at any point of $M^{n}$ the s-nullities of $f$ satisfy $\nu_{s}<n-2 s, 1 \leq s \leq p$, then $f$ is an extrinsic product of isometric immersions.

Proof: The proof relies on Lemma 8.15 below, which implies that the second fundamental form of $f$ at any point of $M^{n}$ is adapted to the product structure of $M^{n}$. The result then follows from Theorem 8.4, Theorem 8.8 and Corollary 8.6.

Let $\beta: V \times V \rightarrow W$ be a symmetric bilinear form, where $V$ and $W$ are real vector spaces of dimension $n$ and $p$, respectively, equipped with inner products. The multilinear map $R: V \times V \times V \times V \rightarrow \mathbb{R}$ defined by

$$
R(x, y, z, w)=\langle\beta(x, w), \beta(y, z)\rangle-\langle\beta(x, z), \beta(y, w)\rangle
$$

has then the algebraic properties of the curvature tensor.
Lemma 8.15. Let $V=V_{1} \oplus V_{2}$ be an orthogonal splitting such that

$$
R(x, y, z, u)=R(x, y, u, v)=R(x, u, v, w)=0
$$

for all $x, y, z \in V_{1}$ and $u, v, w \in V_{2}$. If $n>2 p$ and the $s$-nullities of $\beta$ satisfy $\nu_{s}<n-2 s$ for $1 \leq s \leq p$, then $\beta(x, y)=0$ for all $x \in V_{1}$ and $y \in V_{2}$.

Proof: Let $S \subset W$ denote the subspace

$$
S=\operatorname{span}\left\{\beta(x, y): x \in V_{1} \text { and } y \in V_{2}\right\} .
$$

For each $x \in V_{1}$, define a linear map $B_{x}: V_{2} \rightarrow S$ by

$$
B_{x}(y)=\beta(x, y) .
$$

Fix $x \in V_{1}$ such that $B_{x}$ has maximal rank, i.e.,

$$
\operatorname{rank} B_{x} \geq \operatorname{rank} B_{y}
$$

for all $y \in V_{1}$. Thus $D=\operatorname{ker} B_{x} \subset V_{2}$ satisfies $\operatorname{dim} D \leq \operatorname{dim} \operatorname{ker} B_{y}$ for all $y \in V_{1}$.
We first argue that

$$
\begin{equation*}
D \subset \operatorname{ker} B_{y} \text { for all } y \in V_{1} \tag{8.16}
\end{equation*}
$$

Consider an orthogonal splitting $V_{2}=D \oplus E$, and let $e_{1}, \ldots, e_{\ell}$ be an orthonormal basis of $E$. From $R\left(x, y, v, e_{j}\right)=0$ we obtain

$$
\begin{equation*}
\left\langle B_{x} e_{j}, B_{y} v\right\rangle=\left\langle B_{x} v, B_{y} e_{j}\right\rangle=0, \quad 1 \leq j \leq \ell \tag{8.17}
\end{equation*}
$$

for all $v \in D$ and $y \in V_{1}$.
The rank of $B_{x+t y}$ is at most $\ell$ for all $t \in \mathbb{R}$. Thus the vectors $B_{x+t y} v=t B_{y} v$, $B_{x+t y} e_{j}, 1 \leq j \leq \ell$, are linearly dependent. Hence the Gramm determinant of these vectors is an identically zero polynomial in $t$. By (8.17) the term of lowest order is $t^{2}\left\|B_{y} v\right\|^{2} G$, where $G$ is the Gramm determinant of the linearly independent vectors $B_{x} e_{j}, 1 \leq j \leq \ell$. It follows that $B_{y} v=0$ for all $y \in V_{1}$ and $v \in D$, and this is 8.16.

Next, we prove that

$$
\begin{equation*}
\beta(u, v)-\sum_{i, j=1}^{\ell}\left\langle u, e_{i}\right\rangle\left\langle v, e_{j}\right\rangle \beta\left(e_{i}, e_{j}\right) \in S^{\perp} \tag{8.18}
\end{equation*}
$$

for all $u, v \in V_{2}$. From (8.16) we obtain

$$
\beta(y, v)=\sum_{j=1}^{\ell}\left\langle v, e_{j}\right\rangle \beta\left(y, e_{j}\right)
$$

for all $y \in V_{1}$. Then $R(y, u, w, v)=0$ yields

$$
\langle\beta(u, v), \beta(y, w)\rangle=\sum_{j=1}^{\ell}\left\langle v, e_{j}\right\rangle\left\langle\beta\left(y, e_{j}\right), \beta(u, w)\right\rangle
$$

for all $y \in V_{1}$ and $u, v, w \in V_{2}$. In particular,

$$
\left\langle\beta(w, u), \beta\left(y, e_{j}\right)\right\rangle=\sum_{i=1}^{\ell}\left\langle u, e_{i}\right\rangle\left\langle\beta\left(y, e_{i}\right), \beta\left(e_{j}, w\right)\right\rangle .
$$

Hence

$$
\langle\beta(u, v), \beta(y, w)\rangle=\sum_{i, j=1}^{\ell}\left\langle u, e_{i}\right\rangle\left\langle v, e_{j}\right\rangle\left\langle\beta\left(y, e_{i}\right), \beta\left(e_{j}, w\right)\right\rangle .
$$

Using

$$
\left\langle\beta\left(y, e_{i}\right), \beta\left(e_{j}, w\right)\right\rangle=\left\langle\beta(y, w), \beta\left(e_{i}, e_{j}\right)\right\rangle
$$

we obtain

$$
\left\langle\beta(u, v)-\sum_{i, j=1}^{\ell}\left\langle u, e_{i}\right\rangle\left\langle v, e_{j}\right\rangle \beta\left(e_{i}, e_{j}\right), \beta(y, w)\right\rangle=0
$$

for all $y \in V_{1}$ and $u, v, w \in V_{2}$, and this is (8.18).
By (8.16) we have $\beta(u, y)=0$ if $u \in D$ and $y \in V_{1}$, and (8.18) implies that $\beta_{S}(u, v)=0$ if $u \in D$ and $v \in V_{2}$. Therefore

$$
\beta_{S}(u, e)=0 \text { if } u \in D \text { and } e \in V .
$$

Suppose that $s=\operatorname{dim} S \neq 0$. Choose vectors $x_{j} \in V_{j}, j=1,2$, such that $B_{x_{j}}: V_{k} \rightarrow S$, $j \neq k$, has maximal rank. It follows from the above that

$$
\beta_{S}(b, e)=0
$$

for all $b \in \operatorname{ker} B_{x_{1}} \oplus \operatorname{ker} B_{x_{2}}$ and $e \in V$. Hence

$$
\nu_{s} \geq \operatorname{dim} \operatorname{ker} B_{x_{1}}+\operatorname{dim} \operatorname{ker} B_{x_{2}} \geq n-2 s,
$$

and this contradicts our assumption on the $s$-nullities.

### 8.5 Global conditions for decomposability

This section describes conditions of a global nature under which an isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{n+k}$ of a Riemannian product $M^{n}=\Pi_{i=1}^{k} M_{i}^{n_{i}}$ must necessarily be an extrinsic product of hypersurface immersions. Namely, it is shown that this is the case whenever each $M_{i}^{n_{i}}, 1 \leq i \leq k$, is complete, nonflat and does not have a "flat strip", that is, does not contain any open subset isometric to $I \times \mathbb{R}^{n_{i}-1}$ where $I \subset \mathbb{R}$ is an interval. In particular, this implies that any isometric immersion $f: M^{n}=\prod_{i=1}^{k} M_{i}^{n_{i}} \rightarrow \mathbb{R}^{n+k}$ of a compact Riemannian product is an extrinsic product of hypersurface immersions.

The proof will require several preliminary steps. The first one is the following generalization of the Chern-Kuiper inequality.

Theorem 8.16. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ be an isometric immersion and let $\Gamma(x)$ be its nullity subspace at $x \in M^{n}$. If $\Gamma^{\perp}(x)$ splits as an orthogonal sum of nontrivial subspaces

$$
\Gamma^{\perp}(x)=T_{1} \oplus \cdots \oplus T_{\ell}
$$

that are invariant by all curvature endomorphisms $R(X, Y), X, Y \in T_{x} M$, then

$$
\nu(x) \leq \mu(x) \leq \nu(x)+p-\ell .
$$

Proof: Choose unit vectors $Y_{j} \in T_{j}, 1 \leq j \leq \ell$, and define

$$
S=\left(\Gamma(x) \cap \Delta^{\perp}(x)\right) \oplus \operatorname{span}\left\{Y_{1}, \ldots, Y_{\ell}\right\}
$$

Observe that $R\left(Y_{i}, Y_{j}\right)=0$ if $i \neq j$, because

$$
\left\langle R\left(Y_{i}, Y_{j}\right) X, Z\right\rangle=\left\langle R(X, Z) Y_{i}, Y_{j}\right\rangle=0
$$

for all $X, Z \in T_{x} M$. Thus $R(X, Y)=0$ for all $X, Y \in S$, and

$$
\operatorname{dim} S=\mu(x)-\nu(x)+\ell
$$

To conclude the proof, it suffices to show that if $S \subset T_{x} M$ is a subspace such that $S \cap \Delta(x)=0$ and $R(X, Y)=0$ for all $X, Y \in S$ then $\operatorname{dim} S \leq p$. To see this, it suffices to prove that there exists $Z \in T_{x} M$ such that the linear map $B_{Z}: S \rightarrow N_{f} M(x)$ defined by

$$
B_{Z}=\left.\alpha(Z, \cdot)\right|_{S}
$$

is injective. Assume otherwise. Let $Z \in T_{x} M$ be such that the subspace $B_{Z}(S)$ has maximal dimension, and let $0 \neq Y \in S$ be such that $B_{Z} Y=0$. Since $Y \notin \Delta(x)$, there exists $W \in T_{x} M$ such that $\alpha(Y, W) \neq 0$. From $R(X, Y)=0$ we obtain

$$
\left\langle B_{Z} X, \alpha(Y, W)\right\rangle=\left\langle B_{Z} Y, \alpha(X, W)\right\rangle=0
$$

for all $X \in S$. Thus $0 \neq \alpha(Y, W) \perp B_{Z}(S)$. In particular, the vector

$$
\alpha(Z+t W, Y)=t \alpha(W, Y) \in B_{Z+t W}(S)
$$

is perpendicular to $B_{Z}(S)$. Hence, for small $t>0$, the dimension of $B_{Z+t W}(S)$ exceeds that of $B_{Z}(S)$, which is a contradiction.

The above result provides the following estimate of the index of relative nullity of an isometric immersion of a Riemannian product into Euclidean space.

Corollary 8.17. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ be an isometric immersion of a Riemannian product $M^{n}=\Pi_{i=1}^{k} M_{i}^{n_{i}}$. Let $\ell$ be the number of factors $M_{i}^{n_{i}}$ at $x=\left(x_{1}, \ldots, x_{k}\right) \in M^{n}$ that have a nonzero sectional curvature at $x_{i}$. Then

$$
\nu(x) \leq \mu(x) \leq \nu(x)+p-\ell .
$$

The following consequence of Corollary 8.17 will play a key role in the proof of the main result in this section.

Corollary 8.18. Let $f: M^{n} \rightarrow \mathbb{R}^{n+k}$ be an isometric immersion of a Riemannian product $M^{n}=\Pi_{i=1}^{k} M_{i}^{n_{i}}$. If the second fundamental form of $f$ is not adapted to the product net of $M^{n}$ at $x=\left(x_{1}, \ldots, x_{k}\right) \in M^{n}$, then

$$
\begin{equation*}
\mu(x) \geq \nu(x) \geq \mu(x)-r(x) \geq r(x)>0 \tag{8.19}
\end{equation*}
$$

where $r(x)$ denotes the number of factors $M_{i}^{n_{i}}$ that are flat at $x_{i}$.
Proof: If the second fundamental form of $f$ is not adapted to the product net of $M^{n}$ at $x=\left(x_{1}, \ldots, x_{k}\right) \in M^{n}$, then at least one factor $M_{i}^{n_{i}}$ is flat at $x_{i}$ by Proposition 8.9. Thus $r(x)>0$. Moreover, $\mu(x) \geq 2 r(x)$, hence $\mu(x)-r(x) \geq r(x)$. Finally, the first two inequalities in 8.19) follow from Corollary 8.17.

The next tool in the proof of Theorem 8.23 below is the following intrinsic version of Theorem 7.7 .

Proposition 8.19. Let $U$ be an open subset of a complete Riemannian manifold $M^{n}$ where the index of nullity $\mu=s$ is constant. Then the distribution of nullity $\Gamma$ in $U$ is integrable and its leaves are totally geodesic submanifolds of $M^{n}$. Moreover, if $\sigma:[0, a] \rightarrow M^{n}$ is a geodesic such that $\sigma([0, a))$ is contained in a leaf of $\Gamma$ in $U$ then also $\mu(\sigma(a))=s$.

Proof: The second Bianchi identity

$$
\begin{equation*}
\left(\nabla_{X} R\right)_{Y, Z} V+\left(\nabla_{Z} R\right)_{X, Y} V+\left(\nabla_{Y} R\right)_{Z, X} V=0 \tag{8.20}
\end{equation*}
$$

implies that

$$
R(Y, Z) \nabla_{X} V=0
$$

for all $X, V \in \Gamma(\Gamma)$ and $Y, Z \in \mathfrak{X}(M)$. Thus $\Gamma$ is totally geodesic.
For the proof of the second part, take an orthonormal frame $Y_{1}, \ldots, Y_{n-s}$ spanning $\Gamma^{\perp}$ on a neighborhood of $\sigma([0, a))$ in $U$ where the vector fields are parallel along $\sigma$, and
let $X \in \Gamma(\Gamma)$ be a unit vector field such that $X=\sigma^{\prime}$ and $\nabla_{Y_{j}} X=0,1 \leq j \leq n-s$, along $\sigma$. To see that $\mu(\sigma(a))>s$ is not possible, take a parallel vector $Z$ along $\sigma$ such that $Z(\sigma(a)) \in \Gamma(\sigma(a))$. Applying 8.20 to $X, Y=Y_{i}, Z=Y_{j}, 1 \leq i \neq j \leq n-s$, and $V=Z$ implies that along $\sigma$ we have

$$
\nabla_{\sigma^{\prime}} R\left(Y_{i}, Y_{j}\right) Z=0
$$

hence $Z(\sigma(0)) \in \Gamma(\sigma(0))$.
Consider an isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ of a Riemannian product manifold $M^{n}=\Pi_{i=1}^{k} M_{i}$ whose product net $\mathcal{E}=\left(E_{i}\right)_{i=1, \ldots, k}$. At $x=\left(x_{1}, \ldots, x_{k}\right) \in M^{n}$ denote $\Gamma_{i}(x)=\tau_{i *}^{x} \Gamma_{i}\left(x_{i}\right)$ where $\Gamma_{i}\left(x_{i}\right)$ is the nullity space of $M_{i}$ at $x_{i}$. Since the curvature operator $R(X, Y)$ vanishes when $X \in E_{i}(x)$ and $Y \in E_{j}(x), 1 \leq i \neq j \leq k$, then

$$
\Gamma_{i}(x)=\Gamma(x) \cap E_{i}(x) \text { and } \oplus_{i=1}^{k} \Gamma_{i}(x)=\oplus_{i=1}^{k} \pi_{i} \Gamma(x)=\Gamma(x)
$$

where $\pi_{i}: T_{x} M \rightarrow E_{i}(x)$ denotes the orthogonal projection.
Let $\Delta_{i}(x)=\tau_{i *}^{x} \Delta_{i}\left(x_{i}\right)$, where $\Delta_{i}\left(x_{i}\right)$ is the relative nullity subspace of $f \circ \tau_{i}^{x}$ at $x_{i} \in M_{i}$. If $X \in E_{i}(x)$ and $Y \in E_{j}(x), 1 \leq i \neq j \leq k$, it follows from the Gauss equation that $\alpha(X, Y)=0$ whenever $\alpha(X, X)=0$. This implies that

$$
\Delta_{i}(x)=\Delta(x) \cap E_{i}(x)
$$

where $\Delta(x)$ is the relative nullity subspace of $f$ at $x$. But in this case we can only assert that

$$
\begin{equation*}
\oplus_{i=1}^{k} \Delta_{i}(x) \subset \Delta(x) \subset \oplus_{i=1}^{k} \pi_{i} \Delta(x) \subset \Gamma(x) \tag{8.21}
\end{equation*}
$$

with equality holding at the first inclusion if and only if it holds at the second one. The third inclusion follows from $\Delta(x) \subset \Gamma(x)$ and $\pi_{i} \Gamma(x) \subset \Gamma(x)$.

The relative nullity subspace $\Delta(x)$ at a point $x \in M^{n}$ is said to conform to the product structure of $M^{n}$ if equality holds in the first (hence the second) inclusion of (8.21).

Clearly, if the second fundamental form $\alpha$ is adapted to the product net at $x$ then $\Delta(x)$ conforms to the product structure of $M^{n}$ at $x$.

Proposition 8.20. Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be an isometric immersion of a Riemannian product manifold $M^{n}=\Pi_{i=1}^{k} M_{i}^{n_{i}}$. Let $U$ be an open subset of $M_{1}^{n_{1}}(x)=\tau_{1}^{x}\left(M_{1}\right)$ at $x \in M^{n}$ such that the subspaces $\Delta_{1}(y)$ have the same dimension for all $y \in U$. Let $\sigma$ be a unit speed geodesic with $\sigma(0)=x$ such that $\sigma(s) \in U$ and $\sigma^{\prime}(s) \in \Delta_{1}(\sigma(s))$ for $0 \leq s<a$. If the second fundamental form of $f$ satisfies

$$
\alpha\left(\pi_{1} X, \pi_{j} X\right)(\sigma(a))=0, \quad 2 \leq j \leq k
$$

for any $X \in T_{\sigma(a)} M$ then the same holds at $x$.

Proof: Choose an orthonormal tangent frame $e_{1}, \ldots, e_{n}$ on a neighborhood of $\sigma([0, a])$ in $M^{n}$ such that $e_{i}$ is parallel along $\sigma$ and $e_{i}(\sigma(s)) \in E_{1}\left(\sigma(s)\right.$ for all $1 \leq i \leq n_{1}$ and any $s \in[0, a]$. Moreover, assume that $e_{1}=\sigma^{\prime}$ along $\sigma$ and let $\xi$ be any unit normal vector field parallel along $\sigma$ in the normal connection. The Codazzi equation along $\sigma$ yields

$$
\nabla_{e_{1}} A_{\xi} e_{j}+A_{\xi} \nabla_{e_{j}} e_{1}=0, \quad 2 \leq j \leq n_{1} .
$$

For any $r \geq n_{1}+1$, along $\sigma$ we obtain the system of ordinary linear differential equations

$$
e_{1}\left\langle A_{\xi} e_{j}, e_{r}\right\rangle+\sum_{i=1}^{n_{1}}\left\langle\nabla_{e_{j}} e_{1}, e_{i}\right\rangle\left\langle A_{\xi} e_{i}, e_{r}\right\rangle=0, \quad 2 \leq j \leq n_{1},
$$

and the claim follows from the uniqueness of solutions of such a system with a given initial condition.

A similar argument gives the following result.
Proposition 8.21. Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be an isometric immersion of a Riemannian product manifold $M^{n}=\Pi_{i=1}^{k} M_{i}$. Let $x \in M^{n}$ and let $U$ be an open subset of $M_{1}(x)$ that contains $x$ and has a neighborhood $W$ in $M^{n}$ such that

$$
\alpha\left(\pi_{1} X, \pi_{j} X\right)=0, \quad 2 \leq j \leq k,
$$

for all $y \in W$ and $X \in T_{y} M$. Let $\sigma$ be a unit speed geodesic with $\sigma(0)=x$ such that $\sigma(s) \in U$ and $\sigma^{\prime}(s) \in \Delta_{1}(\sigma(s))$ for $0 \leq s<a$. If the second fundamental form is adapted to the product net of $M^{n}$ at $\sigma(a)$, then it is also adapted to the product net of $M^{n}$ at $x$.

The following is the last result needed for the proof of Theorem 8.23 .
Proposition 8.22. Let $f: M^{n} \rightarrow \mathbb{R}^{n+k}$ be an isometric immersion of a complete Riemannian product manifold $M^{n}=\Pi_{i=1}^{k} M_{i}^{n_{i}}$. Then the relative nullity spaces conform to the product structure of $M^{n}$, unless one of the factors $M_{i}^{n_{i}}$ is everywhere flat.

Proof: Assume that there are points $x \in M^{n}$ at which the relative nullity subspaces do not conform to the product structure of $M^{n}$, that is, at which $\Delta(x)$ is a proper subset of $\oplus_{i=1}^{k} \pi_{i} \Delta(x)$. Let $V \subset M^{n}$ be the open subset consisting of all such points and let $W \subset M^{n}$ be the open subset of points where the second fundamental form of $f$ is not adapted to the product net of $M^{n}$. Since $V \subset W$ then (8.19) holds on $V$. Let $V_{1} \subset V$ be the subset where $\nu$ reaches its minimum. Then the subset $V_{0} \subset V_{1}$ where $\mu$ is minimum is an open set on which both $\nu$ and $\mu$ are constant and positive.

Take $x \in V_{0}$ and a geodesic $\sigma$ such that $\sigma(0)=x, \sigma^{\prime}(0) \in \Delta(x)$ and $\sigma[0, b) \subset V_{0}$. Thus $\sigma[0, b)$ is contained in a leaf of relative nullity in $V_{0}$. Since both $\Delta$ and the distributions $E_{i}, 1 \leq i \leq k$, of the product net $\mathcal{E}=\left(E_{i}\right)_{i=1, \ldots, k}$ of $M^{n}$ are parallel along $\left.\sigma\right|_{[0, b]}$, if $\Delta(x)$ does not conform to the product structure at $x$ then the same holds for $\Delta(\sigma(b))$. Thus $\sigma(b) \in V$. Moreover, by Theorem 7.7 and Proposition 8.19, $\nu$ and $\mu$ do
not change at $\sigma(b)$, hence $\sigma(b) \in V_{0}$. It follows that $\sigma$ does not leave $V_{0}$, and therefore the leaf of $\Delta$ through $x$ is complete. It remains to show that this can only happen if one of the factors is everywhere flat.

Let $r=r(x)$ be the number of factors $M_{i}^{n_{i}}$ that are flat at $x_{i}$. By Corollary 8.18,

$$
\nu(x) \geq \mu(x)-r(x) \geq r(x)>0 .
$$

Reorder the factors so that the first $r$ are flat at $x$. By (8.21) we have $\oplus_{i=1}^{r} \pi_{i} \Delta(x) \subset$ $\Gamma(x)$. Moreover, since $\oplus_{i=1}^{r} \pi_{i} \Delta(x)$ is strictly larger than $\Delta(x)$, by the preceding inequality its dimension is at least $\mu(x)-r+1$. Thus its codimension in $\Gamma(x)$ is at most $r-1$. Since $\Gamma(x)=\oplus_{i=1}^{k} \pi_{i} \Gamma(x)$, the codimension of $\oplus_{i=1}^{r} \pi_{i} \Delta(x)$ in $\oplus_{i=1}^{r} \pi_{i} \Gamma(x)$ is at most $r-1$. Because we have ordered the factors in such a way that the latter is all of $\oplus_{i=1}^{r} \pi_{i} T_{x} M$, we conclude that there must exist $1 \leq i \leq r$ such that $\pi_{i} \Delta(x)=\pi_{i} T_{x} M$, and we may assume that $i=1$.

Take any $X \in \pi_{1} T_{x} M$. Then there exists $Y \perp \pi_{1} T_{x} M$ such that $Z=X+Y \in$ $\Delta(x)$. Let $\sigma=\sigma_{1} \times \cdots \times \sigma_{k}$ be the complete geodesic in $M^{n}$ such that $\sigma(0)=x$ and $\sigma^{\prime}(0)=Z$. Since the leaf of $\Delta$ through $x$ is complete then $\sigma$ lies entirely in $V_{0}$. The distribution $\Gamma(\sigma(s))$ is parallel along $\sigma$, thus $\pi_{1} T_{\sigma(s)} M \subset \Gamma(\sigma(s))$ holds for any value of $s$ since it holds at $x$. But then $M_{1}$ is flat at $\sigma_{1}(s)$ for any $s$. Since $X$ is arbitrary and $\sigma_{1}$ is a complete geodesic with $\sigma_{1}(0)=x_{1}$ and $\sigma_{1}^{\prime}(0)=X$, it follows that $M_{1}$ is everywhere flat.

We are now in a position to state and prove the main result of this section.
Theorem 8.23. Let $M_{1}^{n_{1}}, \ldots, M_{k}^{n_{k}}$ be complete nonflat Riemannian manifolds such that no $M_{i}^{n_{i}}$ contains an open subset isometric to $I \times \mathbb{R}^{n_{i}-1}$, where $I \subset \mathbb{R}$ is an open interval. Then any isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{n+k}$ of the Riemannian product manifold $M^{n}=\Pi_{i=1}^{k} M_{i}^{n_{i}}$ is an extrinsic product of hypersurface immersions.

Proof: Let $f: M^{n} \rightarrow \mathbb{R}^{n+k}$ be an isometric immersion of the Riemannian product $M^{n}=\Pi_{i=1}^{k} M_{i}^{n_{i}}$, where $M_{i}^{n_{i}}$ is complete and nonflat for all $1 \leq i \leq k$. Assuming that $f$ is not an extrinsic product of hypersurfaces, we will show that some $M_{i}^{n_{i}}$ must contain an open subset isometric to $I \times \mathbb{R}^{n_{i}-1}$.

By Theorem 8.4, the open subset $V \subset M^{n}$ of points where the second fundamental form of $f$ is not adapted to the product net of $M^{n}$ is nonempty. If $x=\left(x_{1}, \ldots, x_{k}\right) \in V$ and $r(x)$ denotes the number of factors $M_{i}^{n_{i}}$ that are flat at $x_{i}$, then

$$
\mu(x) \geq \nu(x) \geq r(x)>0
$$

by Corollary 8.18 .
Let $U$ be a connected component of $V$ where $\nu$ reaches its minimum. The relative nullity spaces have constant dimension on $U$ and conform to the product structure of $M^{n}$ by Proposition 8.22 . Hence the subspaces $\Delta_{i}$ have constant dimension along $U$ for any $1 \leq i \leq k$, because the dimension of each $\Delta_{i}$ does not increase in a neighborhood of a point, and since $\oplus_{i=1}^{r} \Delta_{i}(x)=\Delta(x)$, if one dimension decreases another has to increase.

Now, for any point $x=\left(x_{1}, \ldots, x_{k}\right) \in U$ the codimension of $\Delta(x)$ in $\Gamma(x)$ is at most $r=r(x)$ by (8.19). Ordering the factors so that the first $r$ factors $M_{i}^{n_{i}}$ are flat at $x_{i}$, it follows that the sum of the codimensions of $\pi_{i} \Delta_{i}(x)$ in $E_{i}(x)=\pi_{i} T_{x} M, 1 \leq i \leq r$, is at most $r$. Therefore for some $1 \leq i \leq r$, say $i=1$, the dimension of $\Delta_{1}(x)=\pi_{1} \Delta(x)$ is at least $n_{1}-1$. We conclude that $U$ carries a distribution $\Delta_{1}$ of dimension either $n_{1}-1$ or $n_{1}$ and that each $\Delta_{1}(x)=\tau_{1 *}^{x} \Delta_{1}\left(x_{1}\right)$, where $\Delta_{1}\left(x_{1}\right)$ is the relative nullity subspace of $f \circ \tau_{1}^{x}$ at $x_{1} \in M_{1}^{n_{1}}$. It follows from Proposition 1.18 and the fact that $\tau_{1}^{x}$ is totally geodesic for all $x \in M^{n}$ that $\Delta_{1}$ is integrable and that its leaves are totally geodesic in $U$.

Suppose first that the second fundamental form of $f$ satisfies

$$
\begin{equation*}
\alpha\left(\pi_{1} X, \pi_{j} X\right)=0, \quad 2 \leq j \leq k, \tag{8.22}
\end{equation*}
$$

for all $x \in U$ and $X \in T_{x} M$. For a given $y \in U$, set $S=M_{1}(y) \cap U$. For any $x \in S$ and any geodesic tangent to $\Delta_{1}$ with $\sigma(0)=x$ such that $\sigma([0, b))$ lies in $S$, we have $\sigma(b) \in V$ by Proposition 8.21. But since $\Delta_{1} \subset \Delta$, it follows from Theorem 7.7 that $\nu$ does not change at $\sigma(b)$, so $\sigma(b) \in S$ by the definition of $U$. Thus the leaves of the distribution $\Delta_{1}$ on $S$ are complete.

Now suppose that the second fundamental form of $f$ does not satisfy condition (8.22) at some $y \in U$. Define $S$ to be subset of points of $M_{1}(y) \cap U$ that condition is not satisfied. By Proposition 8.20, for any geodesic tangent to $\Delta_{1}$ with $\sigma(0)=x$ such that $\sigma([0, b))$ lies in $S$, we have $\sigma(b) \in V$. As before, $\sigma(b) \in U$, and hence $\sigma(b) \in S$ by the definition of $S$. Again the leaves of the distribution $\Delta_{1}$ on $S$ are complete.

In any case, it has been shown that $M_{1}(x)$ contains a nonempty open subset where the leaves of the relative nullity spaces of $\left.f\right|_{M_{1}(x)}$ have dimension $n_{1}-1$ and are complete. It remains to show that for any connected component of $S$ the leaves are are carried into parallel $\left(n_{1}-1\right)$ - dimensional parallel affine subspaces. But for that it is sufficient to check that the proof of Lemma 7.16 works in this case.

Corollary 8.24. Let $M_{1}^{n_{1}}, \ldots, M_{k}^{n_{k}}$ be compact nonflat Riemannian manifolds. Then any isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{n+k}$ of the Riemannian product manifold $M^{n}=$ $\Pi_{i=1}^{k} M_{i}^{n_{i}}$ is an extrinsic product of hypersurface immersions.

It was proved in [7], by means of an example for the simplest case of a product of only two factors, that Theorem 8.23 is no longer true if some factor contains a flat strip. A complete understanding of the geometric situation in this case is achieved by the following result stated without proof.

Theorem 8.25. Let $f: M^{n} \rightarrow \mathbb{R}^{n+2}$ be an isometric immersion of a Riemannian product manifold $M^{n}=M_{1}^{n_{1}} \times M_{2}^{n_{2}}$ of complete nonflat Riemannian manifolds. Then there is an open dense subset each of whose points lies in an open product neighborhood $U=U_{1} \times U_{2}$, with $U_{j} \subset M_{j}^{n_{j}}, 1 \leq j \leq 2$, such that one of the following possibilities holds:
(i) $\left.f\right|_{U}$ is an extrinsic product of immersions.
(ii) Each $U_{j}$ is isometric to $I_{j} \times \mathbb{R}^{n_{j}-1}$ and $f$ is a cylinder over an isometric immersion $g: I_{1} \times I_{2} \subset \mathbb{R}^{2} \rightarrow \mathbb{R}^{4}$.
(iii) One $U_{j}$ is isometric to $I_{j} \times \mathbb{R}^{n_{j}-1}$ for some $1 \leq j \leq 2$ and $\left.f\right|_{U}$ is a cylinder over an isometric immersion $\tilde{f}: U_{i} \times I_{j} \rightarrow \mathbb{R}^{n_{i}+3}, 1 \leq i \neq j \leq 2$, which is a composition $\tilde{f}=h \circ g$ of a cylinder $g: U_{i} \times I_{j} \rightarrow \mathbb{R}^{n_{i}+2}$ over an isometric immersion $k: U_{i} \rightarrow \mathbb{R}^{n_{i}+1}$ with an isometric immersion $h: V \rightarrow \mathbb{R}^{n_{i}+3}$ of an open subset $V \subset \mathbb{R}^{n_{i}+2}$ containing $g\left(U_{i} \times I_{j}\right)$.

### 8.6 Notes

The basic decomposition Theorem 8.4 for isometric immersions of Riemannian products into Euclidean space was proved by Moore [252]. Its versions for isometric immersions of Riemannian products into the sphere and the hyperbolic space were obtained by Molzan [251]. The extension of Theorem 8.4 for isometric immersions of Riemannian products into Lorentzian space, namely, Theorem 8.7, on which relies our proof of Corollary 8.8, does not seem to have already appeared in the literature. It will also be used in Chapter 9 in the proof of the generalization of Theorem 8.4 for conformal immersions of Riemannian products into Euclidean space due to Tojeiro [330].

The local Theorem 8.10 for isometric immersions into Euclidean space, as well as Corollary 8.24 for compact Riemannian products, were proved by Moore [252]. The results on isometric immersions with codimension at most two, namely Theorems 8.11, 8.12 and 8.13 , have been taken from Dajczer-Tojeiro [144]. The local Theorem 8.14 is due to Dajczer-Vlachos [151].

Most of the results in Section 8.5, including Theorem 8.23 for complete Riemannian products, are due to Alexander-Maltz [7]. Theorem 8.25 was obtained by Barbosa-Dajczer-Tojeiro [28].

The classification of Einstein hypersurfaces of space forms in Exercise 8.4 is due to Fialkow [179], whereas that of hypersurfaces with parallel Ricci tensor in Exercise 8.5 was obtained by Reckziegel [298]. Exercise 8.7 was taken from Ejiri [166].

### 8.7 Exercises

Exercise 8.1. Let $M^{n}$ be a Riemannian manifold and let $\Phi \in \Gamma(\operatorname{End}(T M))$ be a parallel symmetric tensor on $M^{n}$, that is,

$$
\left(\nabla_{X} \Phi\right) Y=\nabla_{X} \Phi Y-\Phi \nabla_{X} Y=0
$$

for all $X, Y \in \mathfrak{X}(M)$.
(i) Show that the eigenvalues $\lambda_{1}, \ldots, \lambda_{k}$ of $\Phi$ are constant on $M^{n}$ and that the corresponding eigenbundles $E_{1}, \ldots, E_{k}$ are parallel.
(ii) Conclude that there exists locally (globally, if $M^{n}$ is simply connected and complete) a product representation $\psi: \Pi_{i=1}^{k} M_{i} \rightarrow M^{n}$ of $\mathcal{E}=\left(E_{i}\right)_{i=1, \ldots, k}$ which is an isometry with respect to a Riemannian product metric on $\prod_{i=1}^{k} M_{i}$.

Hint for $(i)$ : Let $\lambda_{1}, \ldots, \lambda_{k}$ be the eigenvalues of $\Phi$ at $x \in M^{n}$. Given any $y \in M^{n}$, let $\gamma$ be a smooth curve on $M^{n}$ joining $x$ to $y$. For each $1 \leq i \leq k$, choose $e_{i} \in T_{x} M$ such that $\Phi e_{i}=\lambda_{i} e_{i}$ and denote by $E_{i}$ the parallel transport of $e_{i}$ along $\gamma$ and by $X$ the tangent vector to $\gamma$. Then

$$
\nabla_{X}\left(\Phi E_{i}\right)=\left(\nabla_{X} \Phi\right) E_{i}+\Phi\left(\nabla_{X} E_{i}\right)=0
$$

and

$$
\nabla_{X}\left(\lambda_{i} E_{i}\right)=\lambda_{i} \nabla_{X} E_{i}=0
$$

By the uniqueness of parallel transport along $\gamma$ it follows that $\Phi E_{i}=\lambda_{i} E_{i}$ at $y$.
Exercise 8.2. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ be a hypersurface whose shape operator $A$ satisfies

$$
R(X, Y) A=\nabla_{X} \nabla_{Y} A-\nabla_{Y} \nabla_{X} A-\nabla_{[X, Y]} A=0
$$

for all $X, Y \in \mathfrak{X}(M)$ (in particular if $\nabla A=0$ ). Show that the principal curvatures $\lambda_{1}, \ldots, \lambda_{n}$ of $f$ satisfy

$$
\left(\lambda_{i} \lambda_{j}+c\right)\left(\lambda_{i}-\lambda_{j}\right)=0
$$

for all $1 \leq i, j \leq n$, and conclude that $f$ has at most two distinct principal curvatures at any point.
Hint: Let $x \in M^{n}$ be arbitrary and let $e_{1}, \ldots, e_{n}$ be an orthonormal basis of $T_{x} M$ such that $A e_{i}=\lambda_{i} e_{i}$ for all $1 \leq i \leq n$. For each $\lambda$, denote $E_{\lambda}=\operatorname{ker}(A-\lambda I)$. Since $R A=0$, the endomorphisms $R(X, Y) A$ and $A$ commute for all $X, Y \in T_{x} M$. In particular,

$$
\begin{aligned}
\lambda_{j} R\left(e_{i}, e_{j}\right) e_{j} & =R\left(e_{i}, e_{j}\right)\left(A e_{j}\right) \\
& =A R\left(e_{i}, e_{j}\right) e_{j} .
\end{aligned}
$$

Thus $R\left(e_{i}, e_{j}\right) e_{j} \in E_{\lambda_{j}}$, and hence

$$
\left\langle R\left(e_{i}, e_{j}\right) e_{j}, e_{i}\right\rangle=0
$$

whenever $\lambda_{i} \neq \lambda_{j}$. On the other hand, the Gauss equation

$$
R\left(e_{i}, e_{j}\right)=\left(\lambda_{i} \lambda_{j}+c\right)\left(e_{i} \wedge e_{j}\right)
$$

gives

$$
\left\langle R\left(e_{i}, e_{j}\right) e_{j}, e_{i}\right\rangle=\lambda_{i} \lambda_{j}+c .
$$

Exercise 8.3. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ be a nonumbilical hypersurface with parallel shape operator. Show that either $c=0$ and $f(M)$ is an open subset of a cylinder over an
umbilical isometric immersion $g: \mathbb{Q}_{\tilde{c}}^{k} \rightarrow \mathbb{R}^{k+1}$ with $\tilde{c}>0$ and $1 \leq k \leq n-1$, or $c \neq 0$ and $f(M)$ is an open subset of the image of an extrinsic product of identity maps

$$
\mathrm{id}_{1}: \mathbb{Q}_{c_{1}}^{n_{1}} \rightarrow \mathbb{Q}_{c_{1}}^{n_{1}} \text { and } \mathrm{id}_{2}: \mathbb{Q}_{c_{2}}^{n_{2}} \rightarrow \mathbb{Q}_{c_{2}}^{n_{2}},
$$

with $n_{1}+n_{2}=n$ and $1 / c_{1}+1 / c_{2}=1 / c$.
Hint: Use Exercises 8.1 and 8.2 together with Theorem 8.4 for $c=0$, and Corollaries 8.6 and 8.8 for $c \neq 0$.

Exercise 8.4. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}, n \geq 3$, be an isometric immersion of an Einstein manifold with Ric $=\rho\langle$,$\rangle . Show that$
(i) if $\rho>(n-1) c$, then $f$ is umbilical, and hence $M^{n}$ has constant sectional curvature $\tilde{c}=\rho /(n-1)$.
(ii) if $\rho=(n-1) c$, then $f$ has index of relative nullity $\nu \geq n-1$, and hence $M^{n}$ has constant sectional curvature $c$.
(iii) if $\rho<(n-1) c$, then $c>0, n \geq 4, \rho=(n-2) c$ and $f(M)$ is an open subset of the image of an extrinsic product of identity maps

$$
\mathrm{id}_{1}: \mathbb{S}_{c_{1}}^{k} \rightarrow \mathbb{S}_{c_{1}}^{k} \text { and id } \mathrm{id}_{2}: \mathbb{S}_{c_{2}}^{n-k} \rightarrow \mathbb{S}_{c_{2}}^{n-k}, \quad 2 \leq k \leq n-2
$$

with

$$
\begin{equation*}
c_{1}=\frac{n-2}{k-1} c \text { and } c_{2}=\frac{n-2}{n-k-1} c . \tag{8.23}
\end{equation*}
$$

Hint: Argue in a way similar to that suggested by the hint of Exercise 3.11. First use (3.7) to show that the principal curvatures $\lambda_{1}, \ldots, \lambda_{n}$ of $f$ satisfy

$$
\lambda_{j}^{2}-r \lambda_{j}+\rho-(n-1) c=0, \quad 1 \leq j \leq n,
$$

where $r=n H$. If $\rho=(n-1) c$, conclude that at most one principal curvature does not vanish. If $\rho>(n-1) c$, show that assuming $f$ to have two distinct principal curvatures at some point leads to a contradiction. If $\rho<(n-1) c$, write

$$
\lambda_{1}=\cdots=\lambda_{p}=\nu \text { and } \lambda_{p+1}=\cdots=\lambda_{n}=\mu
$$

for some $1 \leq p \leq n$, with $\mu \neq \nu$ everywhere, and show that

$$
(p-1) \nu^{2}+(n-p-1)(\rho-(n-1) c)=0,
$$

which implies that $p>1, p<n-1$ and

$$
\nu^{2}=-\frac{n-p-1}{p-1}(\rho-(n-1) c) .
$$

Conclude that $n \geq 4$, that $\nu, \mu$ and $p$ are all constant on $M^{n}$, and that both $E_{\nu}$ and $E_{\mu}$ are parallel distributions on $M^{n}$. At any $x \in M^{n}$, take $X \in E_{\nu}(x)$ and $Y \in E_{\mu}(x)$
and show that the sectional curvature $K(X, Y)$ of $M^{n}$ along the plane spanned by $X$ and $Y$ satisfies

$$
0=K(X, Y)=\nu \mu+c=\rho-(n-2) c .
$$

Now apply Theorem 8.2 and Corollary 8.6 to conclude that $M^{n}$ is locally isometric to $\mathbb{S}_{c_{1}}^{k} \times \mathbb{S}_{c_{2}}^{n-k}$, with $2 \leq k \leq n-2$ and $c_{1}, c_{2}$ as in (8.23), and that $f$ is locally the extrinsic product of the identity maps $\mathrm{id}_{1}: \mathbb{S}_{c_{1}}^{k} \rightarrow \mathbb{S}_{c_{1}}^{k}$ and id ${ }_{2}: \mathbb{S}_{c_{2}}^{n-k} \rightarrow \mathbb{S}_{c_{2}}^{n-k}$. Finally, make use of Exercise 1.20 for the global conclusion.

Exercise 8.5. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ be a hypersurface with parallel Ricci tensor that is neither Einstein nor a hypersurface with parallel second fundamental form. Show that $c=0$ and that $f(M)$ is an open subset of a cylinder over a surface $g: M_{\tilde{c}}^{2} \rightarrow \mathbb{R}^{3}$ with constant Gaussian curvature.

Hint: First use Exercise 8.1 to show that the eigenvalues $\rho_{1}, \ldots, \rho_{k}$ of the endomorphism $T \in \Gamma(\operatorname{End}(T M))$ of $M^{n}$ associated to the Ricci tensor Ric of $M^{n}$ are constant and that the eigenbundle $E_{i}=\operatorname{ker}\left(T-\rho_{i} I\right)$ is parallel for $1 \leq i \leq k$. Observe also that $k \geq 2$ by the assumption that $M^{n}$ is not an Einstein manifold. Now use (3.7) to show that, if $\Lambda(x)$ denotes the set of principal curvatures of $f$ at $x$, then for each $\lambda \in \Lambda(x)$ the number

$$
\sigma_{\lambda}=(n-1) c-(\lambda-n H(x)) \lambda
$$

is an eigenvalue of $T$. Thus $\Lambda(x)=\cup_{i=1}^{k} \Lambda_{i}(x)$, with

$$
\Lambda_{i}(x)=\left\{\lambda \in \Lambda(x): \sigma_{\lambda}=\rho_{i}\right\},
$$

and

$$
E_{i}(x)=\oplus_{\lambda \in \Lambda_{i}(x)} E_{\lambda}
$$

where $E_{\lambda}=\operatorname{ker}(A-\lambda I)$, and hence the second fundamental form of $f$ is adapted to the net $\mathcal{E}=\left(E_{i}\right)_{i=1, \ldots, k}$. Now use the local de Rham Theorem 8.2 together with Theorem 8.4 for $c=0$ and Corollaries 8.6 and 8.8 for $c \neq 0$ to conclude that there exists locally a product representation $\psi: M_{1} \times \cdots \times M_{k} \rightarrow M$ of $\mathcal{E}=\left(E_{i}\right)_{i=1, \ldots, k}$, which is an isometry with respect to a Riemannian product metric on $M_{1} \times \cdots \times M_{k}$, such that $f \circ \psi$ is an extrinsic product of isometric immersions $f_{1}, \ldots, f_{k}$ of the factors $M_{1}, \ldots, M_{k}$.

If $c \neq 0$, observe that, by dimension reasons, each $f_{i}$ is actually a local isometry, and show that this implies $f$ to have parallel second fundamental form, in contradiction with the assumption. If $c=0$, note that exactly one of $f_{1}, \ldots, f_{k}$, say, $f_{1}$, has codimension one, while the others are local isometries. Argue that this implies that $k=2, \lambda=0$ is a principal curvature of $f$ with multiplicity $n-2$, and that $f_{1}$ is a nonumbilical isometric immersion $f_{1}: M_{\tilde{c}}^{2} \rightarrow \mathbb{R}^{3}$ of a two-dimensional Riemannian manifold of constant Gaussian curvature $\tilde{c}=\rho_{1}$.

Exercise 8.6. Let $f: M^{n} \rightarrow \mathbb{H}_{c}^{n+1}$ be an isoparametric hypersurface. Show that $f(M)$ is an open subset of the image of the standard isometric immersion of $\mathbb{S}_{c_{1}}^{k} \times \mathbb{H}_{c_{2}}^{n-k}$ into $\mathbb{H}_{c}^{n+1}$ for some $0 \leq k \leq n$, where $1 / c_{1}+1 / c_{2}=1 / c$.

Exercise 8.7. Let $f: M_{1}^{n_{1}} \times M_{2}^{n_{2}} \rightarrow \mathbb{Q}_{c}^{m}$, with $c \leq 0$ and $n_{1}, n_{2} \geq 2$, be a minimal isometric immersion. Show that $c=0$ and that $f$ is an extrinsic product of minimal isometric immersions.

Hint: At $x=\left(x_{1}, x_{2}\right) \in M=M_{1}^{n_{1}} \times M_{2}^{n_{2}}$, consider orthonormal bases $u_{1}, \ldots, u_{n_{1}}$ of $T_{x_{1}} M_{1}$ and $v_{1}, \ldots, v_{n_{2}}$ of $T_{x_{2}} M_{2}$. Denoting also $u_{i}=\tau_{1 *}^{x} u_{i} \in T_{x} M, 1 \leq i \leq n_{1}$, and $v_{j}=\tau_{2 *}^{x} v_{j} \in T_{x} M, 1 \leq j \leq n_{2}$, the Gauss equation for $f$ gives

$$
0=\left\langle R\left(u_{i}, v_{j}\right) v_{j}, u_{i}\right\rangle=c+\left\langle\alpha\left(u_{i}, u_{i}\right), \alpha\left(v_{j}, v_{j}\right)\right\rangle-\left\|\alpha\left(u_{i}, v_{j}\right)\right\|^{2} .
$$

Sum in both indices and use the minimality condition to obtain

$$
\begin{aligned}
n_{1} n_{2} c & =\left\|\sum_{i} \alpha\left(u_{i}, u_{i}\right)\right\|^{2}+\sum_{i, j}\left\|\alpha\left(u_{i}, v_{j}\right)\right\|^{2} \\
& =\left\|\sum_{j} \alpha\left(v_{j}, v_{j}\right)\right\|^{2}+\sum_{i, j}\left\|\alpha\left(u_{i}, v_{j}\right)\right\|^{2},
\end{aligned}
$$

and then conclude the proof using Theorem 8.4.

## Chapter 9

## Conformal immersions

In this chapter we initiate the study of conformal immersions. Our approach is based on the fact that, to any conformal immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ of a Riemannian manifold $M^{n}$ into Euclidean space, one can naturally associate an isometric immersion $F: M^{n} \rightarrow \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$ into the light-cone $\mathbb{V}^{m+1}$ of Lorentzian space $\mathbb{L}^{m+2}$, called its isometric light-cone representative.

The starting point is the fact that $\mathbb{R}^{m}$ can be isometrically embedded into (the upper half $\mathbb{V}_{+}^{m+1}$ of) $\mathbb{V}^{m+1}$, giving rise to a model of Euclidean space as a hypersurface of $\mathbb{V}^{m+1}$. In particular, this is a very convenient setting for dealing with Moebius geometric notions, one main reason being that Moebius transformations of Euclidean space are linearized in this model. Namely, they are given by linear orthogonal transformations of $\mathbb{L}^{m+2}$ that preserve $\mathbb{V}_{+}^{m+1}$. Another reason is that spheres and affine subspaces of codimension $k$ in $\mathbb{R}^{m}$ have a neat description in terms of $k$-dimensional space-like subspaces in $\mathbb{L}^{m+2}$. In particular, the space of oriented hyperspheres of $\mathbb{R}^{m}$ can be naturally identified with the de Sitter space $\mathbb{S}_{1,1}^{m+1} \subset \mathbb{L}^{m+2}$. Moreover, orthogonality between spheres and affine subspaces is easily handled in this model, for it corresponds to orthogonality between the corresponding space-like subspaces.

We provide an elementary and self-contained account of the above facts. They are first applied to study envelopes of congruences of spheres in Euclidean space and their relation to Dupin principal normal vector fields. A conformal version of the Gauss parametrization is then discussed, which allows to parametrize a Euclidean hypersurface that envelops a $k$-parameter congruence of hyperspheres in terms of the locus of their centers and their radii.

We proceed by showing that the classical theorem by Liouville on the classification of conformal maps between open subsets of Euclidean space of dimension $m \geq 3$ can be derived as a consequence of the isometric rigidity of the light-cone hypersurface model of $\mathbb{R}^{m}$. We provide a short and direct proof of the latter fact, thus yielding a simple proof of Liouville's theorem.

We then derive a Fundamental theorem of submanifolds within the context of Moebius geometry. It implies that a Euclidean submanifold is completely determined, up to Moebius transformations of the ambient space, by the so-called Moebius metric, Moebius second fundamental form, Blaschke tensor, Moebius form and the normal
connection of the submanifold.
Our main interest is on the study of conformal deformations of Euclidean submanifolds, which will be pursued in subsequent chapters. Here, after defining the conformal versions of the $s$-nullities introduced in Section 2.2.2, we present a conformal counterpart of Theorem 4.23 on the isometric rigidity of Euclidean submanifolds of codimension at most five. It provides sufficient conditions in terms of the conformal $s$-nullities for a Euclidean submanifold with codimension at most four to be conformally rigid, and builds on the fact that conformal congruence between Euclidean submanifolds turns out to be equivalent to isometric congruence between their isometric light-cone representatives.

Finally, we present a conformal version of Moore's decomposition Theorem 8.4, which gives a complete description of all conformal immersions of a Riemannian product of dimension $n \geq 3$ into Euclidean space whose second fundamental forms are adapted to the product net of the manifold. This yields, in particular, a classification of Euclidean submanifolds that carry a Dupin principal normal with umbilical conullity, a special case of which will be needed in our proof of Cartan's theorem on the classification of conformally deformable Euclidean hypersurfaces in Chapter 17.

### 9.1 The paraboloid model

Let $\mathbb{L}^{m+2}$ be the $(m+2)$-dimensional Lorentzian space with the metric induced by the inner product

$$
\langle v, w\rangle=-v_{0} w_{0}+v_{1} w_{1}+\cdots+v_{m+1} w_{m+1}
$$

for $v=\left(v_{0}, \ldots, v_{m+1}\right)$ and $w=\left(w_{0}, \ldots, w_{m+1}\right)$. The set of light-like vectors

$$
\mathbb{V}^{m+1}=\left\{v \in \mathbb{L}^{m+2}:\langle v, v\rangle=0, v \neq 0\right\}
$$

is called the light cone of $\mathbb{L}^{m+2}$. Notice that the submanifold $\mathbb{V}^{m+1}$ has two connected components and inherits a degenerate metric from $\mathbb{L}^{m+2}$.

For any given $w \in \mathbb{V}^{m+1}$, the intersection

$$
\mathbb{E}^{m}=\mathbb{E}_{w}^{m}=\left\{v \in \mathbb{V}^{m+1}:\langle v, w\rangle=1\right\}
$$

of $\mathbb{V}^{m+1}$ with the affine hyperplane

$$
\left\{u \in \mathbb{L}^{m+2}:\langle u, w\rangle=1\right\}
$$

is a model of the $m$-dimensional Euclidean space. To see this, fix $v \in \mathbb{E}^{m}$ and a linear isometry

$$
C: \mathbb{R}^{m} \rightarrow(\operatorname{span}\{v, w\})^{\perp} \subset \mathbb{L}^{m+2}
$$

Then the map $\Psi=\Psi_{v, w, C}: \mathbb{R}^{m} \rightarrow \mathbb{L}^{m+2}$, given by

$$
\begin{equation*}
\Psi(x)=v+C x-\frac{1}{2}\|x\|^{2} w \tag{9.1}
\end{equation*}
$$

is an isometric embedding such that $\Psi\left(\mathbb{R}^{m}\right)=\mathbb{E}^{m}$. That $\Psi$ is an isometric immersion follows from

$$
\begin{equation*}
\Psi_{*}(x) X=C X-\langle X, x\rangle w \tag{9.2}
\end{equation*}
$$

for all $x \in \mathbb{R}^{m}$ and $X \in T_{x} \mathbb{R}^{m}=\mathbb{R}^{m}$, where $\langle$,$\rangle is also the inner product of \mathbb{R}^{m}$.
We call $(v, w, C)$ as above an admissible triple. Note that if $(v, w, C)$ and $(\bar{v}, \bar{w}, \bar{C})$ are admissible triples, then the linear map in $\mathbb{L}^{m+2}$ given by

$$
T v=\bar{v}, T w=\bar{w} \text { and } T \circ C=\bar{C}
$$

belongs to $\mathbb{O}_{1}(m+2)$, that is, it is orthogonal with respect to the inner product in $\mathbb{L}^{m+2}$, and satisfies

$$
T \circ \Psi_{v, w, C}=\Psi_{\bar{v}, \bar{w}, \bar{C}} .
$$

The isometric immersion $\Psi: \mathbb{R}^{m} \rightarrow \mathbb{V}^{m+1}$ shares with any isometric immersion of a Riemannian manifold into the light cone the following property.

Proposition 9.1. Let $f: M^{n} \rightarrow \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$ be an isometric immersion of a Riemannian manifold. Then the position vector field $f$ is a light-like parallel normal vector field such that

$$
\begin{equation*}
\left\langle\alpha^{f}(X, Y), f\right\rangle=-\langle X, Y\rangle \tag{9.3}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}(M)$.
Proof: Differentiating $\langle f, f\rangle=0$ we see that $f$ is a normal vector field. The remaining assertions follow from

$$
\tilde{\nabla}_{X} f=f_{*} X
$$

where $\tilde{\nabla}$ denotes the connection in $\mathbb{L}^{m+2}$.
The normal bundle of the isometric embedding $\Psi: \mathbb{R}^{m} \rightarrow \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$ given by (9.1) is spanned by the position vector field $\Psi$ and the constant vector field $w$, that is,

$$
N_{\Psi} \mathbb{R}^{m}=\operatorname{span}\{\Psi, w\} .
$$

Differentiating $\langle\Psi, w\rangle=1$ twice gives

$$
\begin{equation*}
\left\langle\alpha^{\Psi}(X, Y), w\right\rangle=0 \tag{9.4}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}\left(\mathbb{R}^{m}\right)$. Thus, by (9.3) and (9.4) we have

$$
\begin{align*}
\alpha^{\Psi}(X, Y) & =\left\langle\alpha^{\Psi}(X, Y), \Psi\right\rangle w+\left\langle\alpha^{\Psi}(X, Y), w\right\rangle \Psi \\
& =-\langle X, Y\rangle w \tag{9.5}
\end{align*}
$$

for all $X, Y \in \mathfrak{X}\left(\mathbb{R}^{m}\right)$.
Any nonflat space form $\mathbb{Q}_{c}^{m}$ also has a hypersurface of $\mathbb{V}^{m+1}$ as a model. In fact, fix $z \in \mathbb{L}^{m+2}$ with $\langle z, z\rangle=-1 / c$ and a linear isometry

$$
B: \mathbb{R}_{\mu}^{m+1} \rightarrow\{z\}^{\perp} \subset \mathbb{L}^{m+2}
$$

where $\mu \in\{0,1\}$ and $\mathbb{R}_{\mu}^{m+1}$ is either $\mathbb{R}^{m+1}$ or $\mathbb{L}^{m+1}$ according to whether $c>0$ or $c<0$, respectively. The isometric embedding $T_{B, z}: \mathbb{Q}_{c}^{m} \rightarrow \mathbb{L}^{m+2}$ defined by

$$
\begin{equation*}
x \in \mathbb{Q}_{c}^{m} \subset \mathbb{R}_{\mu}^{m+1} \mapsto z+B x \tag{9.6}
\end{equation*}
$$

has as image the intersection of $\mathbb{V}^{m+1}$ with the affine hyperplane

$$
\left\{u \in \mathbb{L}^{m+2}:\langle u, z\rangle=-1 / c\right\}
$$

if $c>0$, and one of the two branches of this intersection if $c<0$.

### 9.2 The space of Euclidean hyperspheres

Hyperspheres in Euclidean space $\mathbb{R}^{m}$ have a neat description in its model $\mathbb{E}^{m}$. Let $S \subset \mathbb{R}^{m}$ be a hypersphere with (constant) mean curvature $h$ with respect to a unit normal vector field $N$. Differentiating the map $\rho: S \rightarrow \mathbb{L}^{m+2}$ given by

$$
\rho(x)=\Psi_{*}(x) N(x)+h \Psi(x)
$$

and using (9.5), we obtain

$$
\begin{aligned}
\rho_{*} X & =\Psi_{*} \bar{\nabla}_{X} N+h \Psi_{*} X \\
& =\Psi_{*}(-h X)+h \Psi_{*} X \\
& =0
\end{aligned}
$$

where $\bar{\nabla}$ is the derivative of $\mathbb{R}^{m}$. Hence $\rho(S)=\{z\}$ for some unit space-like vector $z \in \mathbb{L}^{m+2}$ satisfying $\langle\Psi(x), z\rangle=0$ for all $x \in S$. It follows that

$$
\Psi(S)=\mathbb{E}^{m} \cap\{z\}^{\perp},
$$

and from now on we write $S=\mathbb{E}^{m} \cap\{z\}^{\perp}$ for short. Observe that $S$ is an affine hyperplane if and only if

$$
0=h=\langle z, w\rangle .
$$

Note also that changing the unit normal vector field $N$ by $-N$, and hence the corresponding mean curvature $h$, makes the unit space-like vector $z$ change its sign.

In summary, unit space-like vectors in $\mathbb{L}^{m+2}$ are in one-to-one correspondence with oriented hyperspheres or affine hyperplanes of $\mathbb{R}^{m}$. Hence the space of oriented hyperspheres and affine hyperplanes of $\mathbb{R}^{m}$ is naturally identified with the de Sitter space

$$
\mathbb{S}_{1,1}^{m+1}=\left\{u \in \mathbb{L}^{m+2}:\langle u, u\rangle=1\right\}
$$

Let $S=\mathbb{E}^{m} \cap\{z\}^{\perp}$ be a hypersphere with (Euclidean) center $x_{0}$ and radius $r$, oriented by its inward pointing unit normal vector field

$$
x \in S \mapsto N(x)=\frac{1}{r}\left(x_{0}-x\right)
$$

with corresponding mean curvature $h=1 / r$. Using (9.2), it follows easily that the associated unit space-like vector

$$
\begin{equation*}
z=\Psi_{*}(x) N(x)+h \Psi(x) \tag{9.7}
\end{equation*}
$$

for any $x \in S$, is given by

$$
\begin{equation*}
z=\frac{1}{r} \Psi\left(x_{0}\right)+\frac{r}{2} w . \tag{9.8}
\end{equation*}
$$

On the other hand, if $S=\mathbb{E}^{m} \cap\{z\}^{\perp}$ is an affine hyperplane oriented by a unit normal vector $N$, then $z=\Psi_{*}(x) N, x \in S$, is given by

$$
z=C N-c w
$$

where $c \in \mathbb{R}$ is the oriented distance from $S$ to the origin in $\mathbb{R}^{m}$. Hence the space of (oriented) affine hyperplanes in $\mathbb{R}^{m}$ is identified with the image $\mathbb{S}_{1,1}^{m+1} \cap\{w\}^{\perp}$ of the embedding $\phi: \mathbb{S}^{m-1} \times \mathbb{R} \rightarrow \mathbb{S}_{1,1}^{m+1}$ defined by

$$
\phi(x, t)=C x-t w,
$$

with $\phi(x, t)$ representing the affine hyperplane with unit normal vector $x$ and oriented distance $t \in \mathbb{R}$ to the origin in $\mathbb{R}^{m}$.

The relative position of two hyperspheres has a simple description in this model. Given hyperspheres or affine hyperplanes

$$
S_{i}=\mathbb{E}^{m} \cap\left\{z_{i}\right\}^{\perp}, \quad 1 \leq i \leq 2
$$

then they intersect along an $(m-2)$-dimensional sphere or affine subspace, have a unique common point (or are two parallel affine hyperplanes) or do not intersect if and only if the subspace $V$ spanned by $z_{1}$ and $z_{2}$ is space-like, degenerate or time-like, respectively. In the first case, if $N_{x}^{1}$ and $N_{x}^{2}$ are the unit normal vectors of $S_{1}$ and $S_{2}$, respectively, at $x \in S_{1} \cap S_{2}$, then

$$
\left\langle N_{x}^{1}, N_{x}^{2}\right\rangle=\left\langle z_{1}, z_{2}\right\rangle
$$

In particular, $S_{1}$ and $S_{2}$ intersect orthogonally if and only if $\left\langle z_{1}, z_{2}\right\rangle=0$.
We also see that any ( $m-2$ )-dimensional sphere or affine subspace is given by $\mathbb{E}^{m} \cap V^{\perp}$ for some two-dimensional space-like subspace $V \subset \mathbb{L}^{m+2}$, affine subspaces being characterized by the fact that $w \in V^{\perp}$. More generally, the space of spheres and affine subspaces of codimension $k$ is naturally identified in this way with the Grassmannian of $k$-dimensional space-like subspaces of $\mathbb{L}^{m+2}$.

### 9.3 Envelopes of congruences of hyperspheres

Given a smooth map $h: M^{n} \rightarrow \mathbb{R}^{m}$ and a positive function $r \in C^{\infty}(M)$, the family of hyperspheres

$$
x \in M^{n} \mapsto S(h(x), r(x))
$$

centered at $h(x) \in \mathbb{R}^{m}$ with radius $r(x)$ is said to be a congruence of hyperspheres.
By the discussion in the preceding section, if $\Psi=\Psi_{v, w, C}: \mathbb{R}^{m} \rightarrow \mathbb{L}^{m+2}$ is the isometric embedding onto

$$
\mathbb{E}^{m}=\mathbb{E}_{w}^{m}=\left\{u \in \mathbb{V}^{m+1}:\langle u, w\rangle=1\right\} \subset \mathbb{L}^{m+2}
$$

given by (9.1 in terms of $w \in \mathbb{V}^{m+1}, v \in \mathbb{E}^{m}$ and a linear isometry $C: \mathbb{R}^{m} \rightarrow\{v, w\}^{\perp}$, then the congruence of hyperspheres $S(h(x), r(x)), x \in M^{n}$, can be identified with the map $S: M^{n} \rightarrow \mathbb{S}_{1,1}^{m+1} \subset \mathbb{L}^{m+2}$ defined by

$$
\begin{equation*}
S(x)=\frac{1}{r(x)} \Psi(h(x))+\frac{r(x)}{2} w . \tag{9.9}
\end{equation*}
$$

The congruence of hyperspheres $S(h(x), r(x)), x \in M^{n}$, is said to be a $k$-parameter congruence of hyperspheres if the map $S$ has rank $k$ everywhere. Since

$$
\begin{equation*}
S_{*} X=\frac{1}{r(x)} \Psi_{*} h_{*} X-\frac{X(r)}{r^{2}(x)} \Psi(h(x))+\frac{X(r)}{2} w \tag{9.10}
\end{equation*}
$$

for all $x \in M^{n}$ and $X \in T_{x} M$, this is equivalent to requiring that $\operatorname{ker} h_{*}(x) \cap \operatorname{ker} r_{*}(x)$ have dimension $n-k$ for all $x \in M^{n}$. From now on, a map $S: M^{n} \rightarrow \mathbb{S}_{1,1}^{m+1}$ of rank $k$ will be called itself a $k$-parameter congruence of hyperspheres in $\mathbb{R}^{m}$.

In a similar way, a map $S: M^{n} \rightarrow \mathbb{S}_{1,1}^{m+1} \cap\{w\}^{\perp}$ of rank $k$ is called a $k$-parameter congruence of affine hyperplanes. One can always write such a map as

$$
\begin{equation*}
S(x)=C(i \circ g)(x)-\gamma(x) w \tag{9.11}
\end{equation*}
$$

where $g: M^{n} \rightarrow \mathbb{S}^{m-1}$ is a smooth map, $i: \mathbb{S}^{m-1} \rightarrow \mathbb{R}^{m}$ is the inclusion and $\gamma \in C^{\infty}(M)$. Hence $S$ having rank $k$ is equivalent to $\operatorname{ker} g_{*} \cap \operatorname{ker} \gamma_{*}$ having dimension $n-k$ everywhere. Notice that $S(x)$ represents the affine hyperplane in $\mathbb{R}^{m}$ having $(i \circ g)(x)$ as a unit normal vector and $\gamma(x)$ as its oriented distance to the origin in $\mathbb{R}^{m}$.

An immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ is said to envelop the congruence of hyperspheres determined by $h: M^{n} \rightarrow \mathbb{R}^{m}$ and $r \in C^{\infty}(M)$ if

$$
f(x) \in S(h(x), r(x)) \text { and } f_{*} T_{x} M \subset T_{f(x)} S(h(x), r(x))
$$

that is, if

$$
\begin{equation*}
\|f(x)-h(x)\|^{2}=r^{2}(x) \text { and }\left\langle f_{*} X, f(x)-h(x)\right\rangle=0 \tag{9.12}
\end{equation*}
$$

for all $x \in M^{n}$ and $X \in T_{x} M$.
Notice that (9.12) implies that

$$
\left\langle h_{*} X, f(x)-h(x)\right\rangle=-r X(r)
$$

for all $x \in M^{n}$ and $X \in T_{x} M$. Thus

$$
\operatorname{ker} h_{*}(x) \subset \operatorname{ker} r_{*}(x),
$$

and hence

$$
\operatorname{ker} h_{*}(x) \cap \operatorname{ker} r_{*}(x)=\operatorname{ker} h_{*}(x)
$$

for all $x \in M^{n}$.
Accordingly, an immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ is said to envelop the congruence of affine hyperplanes determined by $g: M^{n} \rightarrow \mathbb{S}^{m-1}$ and $\gamma \in C^{\infty}(M)$ if

$$
\begin{equation*}
\langle f(x), i(g(x))\rangle=\gamma(x) \text { and }\left\langle f_{*} X, i(g(x))\right\rangle=0 \tag{9.13}
\end{equation*}
$$

for all $x \in M^{n}$ and $X \in T_{x} M$.
Similarly, it follows from (9.13) that

$$
\operatorname{ker} g_{*}(x) \cap \operatorname{ker} \gamma_{*}(x)=\operatorname{ker} g_{*}(x)
$$

for all $x \in M^{n}$.
Proposition 9.2. An immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ envelops a congruence of hyperspheres $S: M^{n} \rightarrow \mathbb{S}_{1,1}^{m+1}$ (respectively, affine hyperplanes $S: M^{n} \rightarrow \mathbb{S}_{1,1}^{m+1} \cap\{w\}^{\perp}$ ) if and only if

$$
\begin{equation*}
\langle(\Psi \circ f)(x), S(x)\rangle=0 \text { and }\left\langle(\Psi \circ f)_{*} X, S(x)\right\rangle=0 \tag{9.14}
\end{equation*}
$$

for all $x \in M^{n}$ and $X \in T_{x} M$.
Proof: Using (9.1) and (9.2), it is easily checked that if $f: M^{n} \rightarrow \mathbb{R}^{m}$ is an immersion and $S: M^{n} \rightarrow \mathbb{S}_{1,1}^{m+1}$ (respectively, $S: M^{n} \rightarrow \mathbb{S}_{1,1}^{m+1} \cap\{w\}^{\perp}$ ) is a congruence of hyperspheres (respectively, affine hyperplanes), then the equations in (9.14) are equivalent to those in 9.12 (respectively, (9.13)).

According to Proposition 9.2, if an immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ envelops a congruence of hyperspheres $S: M^{n} \rightarrow \mathbb{S}_{1,1}^{m+1}$, then $S$ can be regarded as a unit space-like normal vector field along $\Psi \circ f$ orthogonal to the light-like position vector field $\Psi \circ f$.

Note also that (9.14) implies that

$$
\left\langle S_{*} X,(\Psi \circ f)(x)\right\rangle=0
$$

for all $x \in M^{n}$ and $X \in T_{x} M$. Therefore, if $S$ is an immersion, then $\Psi \circ f$ can be seen as a light-like normal vector field along $S$. In particular, an ( $m-1$ )-parameter congruence of hyperspheres $S: M^{m-1} \rightarrow \mathbb{S}_{1,1}^{m+1}$ in $\mathbb{R}^{m}$ can have at most two envelopes. Moreover, it has no envelopes if its induced metric is time-like, and it has two distinct envelopes if and only if its induced metric is Riemannian.

Let $e_{0}, e_{1}, \ldots, e_{m+1}$ be a pseudo-orthonormal basis of $\mathbb{L}^{m+2}$ with $e_{0}=v$ and $e_{m+1}=-(1 / 2) w$. Observe that $\left\langle e_{0}, e_{m+1}\right\rangle=-1 / 2$. Then we may write the isometric immersion $\Psi: M^{n} \rightarrow \mathbb{L}^{m+2}$ with respect to this basis as

$$
\Psi(x)=\left(1, x,\|x\|^{2}\right)
$$

for all $x \in \mathbb{R}^{m}$. Thus the congruence of hyperspheres $S: M^{n} \rightarrow \mathbb{S}_{1,1}^{m+1}$, given by (9.9) in terms of the map $h: M^{n} \rightarrow \mathbb{R}^{m}$ and $r \in C^{\infty}(M)$, can be written as

$$
\begin{equation*}
S=\frac{1}{r}\left(1, h,\|h\|^{2}-r^{2}\right) . \tag{9.15}
\end{equation*}
$$

Notice that $h$ and $r$ can be recovered from $S=\left(S_{0}, \ldots, S_{m+1}\right)$ by

$$
\begin{equation*}
h=r\left(S_{1}, \ldots, S_{m}\right) \text { and } r=1 / S_{0} . \tag{9.16}
\end{equation*}
$$

Lemma 9.3. The map $S: M^{n} \rightarrow \mathbb{S}_{1,1}^{m+1} \subset \mathbb{L}^{m+2}$ given by 9.15) is an immersion with Riemannian induced metric if and only if $h$ is an immersion and the gradient of $r$ with respect to the metric induced by $h$ satisfies $\|$ gradr $\|<1$.

Proof: From 9.10 we obtain

$$
\left\langle S_{*} X, S_{*} X\right\rangle=\frac{1}{r^{2}}\left(\left\langle h_{*} X, h_{*} X\right\rangle-(X(r))^{2}\right)
$$

for all $X \in \mathfrak{X}(M)$. It follows that $S$ is an immersion with Riemannian induced metric if and only if $h$ is an immersion and

$$
\left\langle h_{*} X, h_{*} X\right\rangle>(X(r))^{2}
$$

for all $X \in \mathfrak{X}(M)$. If $h$ is an immersion, the preceding inequality is trivially satisfied if $\langle X, \operatorname{grad} r\rangle=0$, and for $X=\operatorname{grad} r$ it reduces to $\|\operatorname{grad} r\|<1$ with respect to the metric induced by $h$.

Proposition 9.4. If a hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ envelops a $k$-parameter congruence of hyperspheres $S: M^{n} \rightarrow \mathbb{S}_{1,1}^{n+2}, 1 \leq k \leq n-1$, then $f$ has a principal curvature $\lambda$ such that $\operatorname{ker} S_{*}(x) \subset E_{\lambda}(x)$ for all $x \in M^{n}$, with $\operatorname{ker} S_{*}(x)=E_{\lambda}(x)$ for all $x$ in an open dense subset of $M^{n}$, on which $\lambda$ is constant along $E_{\lambda}$.

Conversely, any hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ that carries a Dupin principal curvature of multiplicity $n-k$ envelops a $k$-parameter congruence of hyperspheres.

Proof: Let the smooth map $h: M^{n} \rightarrow \mathbb{R}^{n+1}$ and $r \in C^{\infty}(M), r>0$, be such that (9.12) holds and $\operatorname{ker} h_{*}(x)=\operatorname{ker} h_{*}(x) \cap \operatorname{ker} r_{*}(x)$ has rank $n-k$ for all $x \in M^{n}$. It follows from (9.12) that $\eta \in \Gamma\left(f^{*} T \mathbb{R}^{n+1}\right)$, defined by

$$
\eta(x)=\frac{1}{r(x)}(h(x)-f(x))
$$

for all $x \in M^{n}$, is a unit normal vector field to $f$. If $T \in \operatorname{ker} h_{*}(x)$, then

$$
\begin{align*}
f_{*} A_{\eta}^{f} T & =-\tilde{\nabla}_{T} \eta \\
& =\frac{T(r)}{r^{2}(x)}(h(x)-f(x))-\frac{1}{r(x)}\left(h_{*} T-f_{*} T\right) \\
& =\frac{1}{r(x)} f_{*} T . \tag{9.17}
\end{align*}
$$

Therefore $\lambda=1 / r$ is a principal curvature of $f$ such that $\operatorname{ker} h_{*}(x) \subset E_{\lambda}(x)$ for all $x \in M^{n}$. To complete the proof of the direct statement, it suffices to show that $\lambda$ cannot have constant multiplicity greater than $n-k$ on any open subset $U \subset M^{n}$. Assume otherwise, and let $T \in E_{\lambda}(x)$ for $x \in U$. Since $\lambda$ is constant on $U$ along $E_{\lambda}$ by Proposition 1.22 , it follows from (9.17) that $T \in \operatorname{ker} h_{*}(x)$, a contradiction.

Conversely, let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be a hypersurface that carries a Dupin principal curvature $\lambda$ of multiplicity $n-k$ with respect to a unit normal vector field $N$. By Proposition 1.22, the map $h: M^{n} \rightarrow \mathbb{R}^{n+1}$ given by

$$
h(x)=f(x)+\frac{1}{\lambda(x)} N(x)
$$

and the function $r=1 / \lambda$ determine a congruence of hyperspheres $S: M^{n} \rightarrow \mathbb{S}_{1,1}^{n+2}$ that is enveloped by $f$. Moreover,

$$
E_{\lambda}(x) \subset \operatorname{ker} h_{*}(x) \cap \operatorname{ker} r_{*}(x)=\operatorname{ker} S_{*}(x)
$$

for all $x \in M^{n}$. On the other hand, given $x \in M^{n}$ and $T \in \operatorname{ker} h_{*}(x) \cap \operatorname{ker} r_{*}(x)$, the proof of the direct statement has shown that $T \in E_{\lambda}(x)$. Therefore $S$ has rank $k$.

Proposition 9.5. A hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ envelops a $k$-parameter congruence of affine hyperplanes $S: M^{n} \rightarrow \mathbb{S}_{1,1}^{n+2} \cap\{w\}^{\perp}, 1 \leq k \leq n-1$, if and only if it carries a relative nullity distribution $\Delta$ of rank $n-k$. Moreover, the subbundle $\operatorname{ker} S_{*}$ coincides with $\Delta$.

Proof: Let the smooth map $g: M^{n} \rightarrow \mathbb{S}^{n}$ and $\gamma \in C^{\infty}(M)$ be such that

$$
\operatorname{ker} g_{*}(x)=\operatorname{ker} g_{*}(x) \cap \operatorname{ker} \gamma_{*}(x)=\operatorname{ker} S_{*}(x)
$$

has rank $k$ for all $x \in M^{n}$ and (9.13) holds. Then $\eta=i \circ g$ is a unit normal vector field to $f$, and if $T \in \operatorname{ker} g_{*}(x)$, then $A_{\eta}^{f} T=0$. On the other hand, if $A_{\eta}^{f} T=0$, then $T \in \operatorname{ker} g_{*}(x)=\operatorname{ker} S_{*}(x)$, as follows by differentiating (9.13). Thus $\operatorname{ker} S_{*}$ is the relative nullity distribution of $f$.

Conversely, if $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ carries a relative nullity distribution $\Delta$ of rank $n-k$, then the Gauss map $g: M^{n} \rightarrow \mathbb{S}^{n}$ of $f$ and its support function $\gamma=\langle f,(i \circ g)\rangle$, where $i: \mathbb{S}^{n} \rightarrow \mathbb{R}^{n+1}$ is the inclusion map, determine a $k$-parameter congruence of affine hyperplanes that is enveloped by $f$.

### 9.3.1 The conformal Gauss parametrization

Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be an orientable hypersurface carrying a nowhere vanishing Dupin principal curvature $\lambda$ with constant multiplicity $n-k$ with respect to a unit normal vector field $N$. By Proposition 9.4, the hypersurface $f$ envelops a $k$-parameter congruence of hyperspheres. Namely, the principal curvature $\lambda$ and the map

$$
h=f+\frac{1}{\lambda} N
$$

are constant along the leaves of the eigenbundle $E_{\lambda}$, and $f$ envelops the $k$-parameter congruence of hyperspheres determined by $h$ and $1 / \lambda$.

Let $\pi: M^{n} \rightarrow L^{k}$ be the projection onto the quotient space of leaves of $E_{\lambda}$. Let $g: L^{k} \rightarrow \mathbb{R}^{n+1}$ and $r \in C^{\infty}(L)$ be defined by

$$
h=g \circ \pi \text { and } \lambda^{-1}=r \circ \pi .
$$

Next we show how to recover $f$ in terms of $g$ and $r$.
Using that

$$
\begin{align*}
f(x) & =h(x)-\frac{1}{\lambda(x)} N(x) \\
& =g(y)-r(y) N(x) \tag{9.18}
\end{align*}
$$

for all $x \in M^{n}$ and $y=\pi(x)$, we obtain

$$
\begin{aligned}
0 & =\left\langle f_{*} Y, N\right\rangle \\
& =\left\langle h_{*} Y-Y\left(\lambda^{-1}\right) N-\lambda^{-1} N_{*} Y, N\right\rangle \\
& =\left\langle g_{*} \pi_{*} Y, N\right\rangle-\pi_{*}(Y)(r) \\
& =\left\langle N-g_{*} \operatorname{grad} r, g_{*} \pi_{*} Y\right\rangle
\end{aligned}
$$

for all $Y \in T_{x} M$. Thus the tangent component $N^{T}(x)$ to $g$ of $N(x)$ at $y$ is

$$
\begin{equation*}
N^{T}(x)=g_{*} \operatorname{grad} r(y) \tag{9.19}
\end{equation*}
$$

Let $\varphi: M^{n} \rightarrow N_{g}^{1} L$ be the map into the unit normal bundle of $g$ given by

$$
\varphi(x)=(y, u), \quad y=\pi(x)
$$

where

$$
u=\frac{1}{{\sqrt{1-\|\operatorname{grad} r(y)\|^{2}}}^{\perp}} N^{\perp}(x)
$$

It is easily checked that $\varphi_{*}$ is everywhere injective, hence $\varphi$ is a local diffeomorphism onto an open subset $U \subset N_{g}^{1} L$. In fact, the map $\varphi$ is injective, and thus a global diffeomorphism, if $n-k \geq 2$ and the leaves of $E_{\lambda}$ are assumed to be complete, and hence compact. For if $\varphi(x)=\varphi(z)$, then $\pi(x)=\pi(z)$ and $N^{\perp}(x)=N^{\perp}(z)$, which imply that $N(x)=N(z)$, and hence $f(x)=f(z)$ by 9.18). Since the restriction $\left.f\right|_{\sigma}$ of $f$ to any leaf $\sigma$ of $E_{\lambda}$ is umbilical, compactness of $\sigma$ implies that $\left.f\right|_{\sigma}$ is a covering map of $\sigma$ onto a round sphere $\mathbb{S}^{n-k}$, and hence a diffeomorphism because $\mathbb{S}^{n-k}$ is simply connected if $n-k \geq 2$. It follows that $x=z$.

Denoting by $\theta$ a local (global, if $n-k \geq 2$ and the leaves of $E_{\lambda}$ are complete) inverse of $\varphi$, we conclude from (9.18) and (9.19) that

$$
\begin{equation*}
f \circ \theta(y, u)=g(y)-r(y) g_{*} \operatorname{grad} r(y)-r(y) \sqrt{1-\|\operatorname{grad} r\|^{2}} u \tag{9.20}
\end{equation*}
$$

for all $(y, u) \in U$. This map is called the conformal Gauss parametrization of $f$.
We have thus proved the converse statement of the following theorem.

Theorem 9.6. Let $g: V^{k} \rightarrow \mathbb{R}^{n+1}, 1 \leq k \leq n-1$, be an isometric immersion and let $r \in C^{\infty}(V)$ be a positive function such that $\|$ grad $r \|<1$. Then the map $\phi: N_{g}^{1} V \rightarrow \mathbb{R}^{n+1}$, defined by

$$
\begin{equation*}
\phi(y, u)=g(y)-r(y) g_{*} \operatorname{grad} r(y)-r(y) \sqrt{1-\|\operatorname{gradr}\|^{2}} u, \tag{9.21}
\end{equation*}
$$

parametrizes, on the open subset of regular points, a hypersurface that carries a Dupin principal curvature of multiplicity $n-k$.

Conversely, if $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ is an orientable hypersurface with a nowhere vanishing Dupin principal curvature $\lambda$ of multiplicity $n-k$, then there exist locally (globally, if $n-k \geq 2$ and the leaves of $E_{\lambda}$ are complete) an isometric immersion $g: V^{k} \rightarrow \mathbb{R}^{n+1}$, a positive function $r \in C^{\infty}(V)$ with $\|$ grad $r \|<1$, and a diffeomorphism $\theta: U \rightarrow M^{n}$ of an open subset $U \subset N_{g}^{1} V$ such that $f \circ \theta$ is given by 9.20.

Proof: Define a unit vector field $N \in \Gamma\left(\phi^{*} T \mathbb{R}^{n+1}\right)$ by

$$
N(y, u)=g_{*} \operatorname{grad} r(y)+\sqrt{1-\|\operatorname{grad} r\|^{2}} u
$$

Then

$$
\phi(y, u)=g(y)-r(y) N(y, u)
$$

for all $(y, u) \in N_{g}^{1} V$. Differentiating $\phi$ with respect to a vertical vector $W$ at $(y, u)$ gives

$$
\begin{equation*}
\phi_{*} W=-r N_{*} W \tag{9.22}
\end{equation*}
$$

On the other hand, any non-vertical tangent vector to $N_{g}^{1} V$ at $(y, u)$ can be written as $\zeta_{*} X$ for some $X \in T_{y} V$ and some local section of $N_{g}^{1} V$ with $\zeta(y)=u$ on an open neighborhood of $y$. We have

$$
\begin{equation*}
\phi_{*} \zeta_{*} X=g_{*} X-X(r) N-r N_{*} \zeta_{*} X \tag{9.23}
\end{equation*}
$$

Using that $N$ is a unit vector field, it follows from (9.22) and (9.23) that

$$
\left\langle\phi_{*} W, N\right\rangle=0=\left\langle\phi_{*} \zeta_{*} X, N\right\rangle
$$

for any vertical vector $W$ at $(y, u)$ and any $X \in T_{y} V$. Thus $N$ is normal to $\phi$ along the open subset of its regular points. Moreover, it follows from (9.22) that $\lambda=1 / r$ is a principal curvature of $\phi$ and that the vertical subspaces of $N_{1} V$ belong to the eigenspaces correspondent to $\lambda$. Furthermore, from (9.23) we obtain

$$
r \phi_{*}\left(A_{N}^{\phi}-\lambda I\right) \zeta_{*} X=X(r) N-g_{*} X
$$

for any $X \in \mathfrak{X}(V)$. Hence the vertical subbundle of $N_{g}^{1} V$ is precisely the eigenbundle of $\phi$ correspondent to $\lambda$. Therefore $\lambda$ is a Dupin principal curvature of multiplicity $n-k$.

The last result of this section gives a necessary and sufficient condition for a $k$-parameter congruence of hyperspheres of $\mathbb{R}^{n+1}, k<n$, to be enveloped by an $n$-dimensional hypersurface.

Corollary 9.7. A $k$-parameter congruence of hyperspheres of $\mathbb{R}^{n+1}, 2 \leq k \leq n-1$, determined by an immersion $s: V^{k} \rightarrow \mathbb{S}_{1,1}^{n+2} \subset \mathbb{L}^{n+3}$, is enveloped by an $n$-dimensional hypersurface of $\mathbb{R}^{n+1}$ if and only if the metric induced by $s$ is Riemannian.

Proof: If a $k$-parameter congruence of hyperspheres of $\mathbb{R}^{n+1}$ is determined by an immersion $s: V^{k} \rightarrow \mathbb{S}_{1,1}^{n+2} \subset \mathbb{L}^{n+3}$ with Riemannian induced metric, by Lemma 9.3 the pair $(g, r)$ associated with it is such that $g$ is an immersion and the gradient of $r$ in the metric induced by $g$ satisfies $\left\|\nabla^{g} r\right\|<1$. Hence it is enveloped by the hypersurface parametrized as in 9.21 by means of $(g, r)$.

Conversely, if a $k$-parameter congruence of hyperspheres of $\mathbb{R}^{n+1}$ determined by an immersion $s: V^{k} \rightarrow \mathbb{S}_{1,1}^{n+2} \subset \mathbb{L}^{n+3}$ is enveloped by a hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}$, then $f$ carries a Dupin principal curvature of multiplicity $n-k$. Hence $f$ can be parametrized as in 9.21 in terms of the pair $(g, r)$ associated to $s$. By Theorem 9.6, the map $g$ is an immersion and $\left\|\nabla^{g} r\right\|<1$, hence the metric induced by $s$ is Riemannian by Lemma 9.3 .

### 9.3.2 Envelopes and Dupin principal normal vector fields

As pointed out at the end of Section 9.2 , the space of spheres and affine subspaces of codimension $s$ in $\mathbb{R}^{m}$ is naturally identified with the Grassmannian $G_{s, m+2}$ of $s$-dimensional space-like subspaces of $\mathbb{L}^{m+2}$. Therefore, a congruence of $(m-s)$ dimensional spheres in $\mathbb{R}^{m}$ can be defined as a smooth map $S: M^{n} \rightarrow G_{s, m+2}$. Alternatively, it may be given by an $s$-tuple of smooth maps $S_{i}: M^{n} \rightarrow \mathbb{S}_{1,1}^{m+1} \subset \mathbb{L}^{m+2}$, $1 \leq i \leq s$, with

$$
\left\langle S_{i}(x), S_{j}(x)\right\rangle=0
$$

for all $x \in M^{n}$ and $1 \leq i \neq j \leq s$, where $\langle$,$\rangle stands for the inner product of \mathbb{L}^{m+2}$. In this case, the $(m-s)$-dimensional sphere in $\mathbb{R}^{m}$ determined by $S(x)$ is the intersection of the hyperspheres determined by $S_{1}(x), \ldots, S_{s}(x)$. Such an $s$-tuple will be referred to in the sequel as an orthogonal $s$-tuple of smooth maps $S_{i}: M^{n} \rightarrow \mathbb{S}_{1,1}^{m+1} \subset \mathbb{L}^{m+2}$, $1 \leq i \leq s$.

The congruence of $(m-s)$-dimensional spheres in $\mathbb{R}^{m}$ determined by the orthogonal $s$-tuple of smooth maps $S_{i}: M^{n} \rightarrow \mathbb{S}_{1,1}^{m+1} \subset \mathbb{L}^{m+2}, 1 \leq i \leq s$, is said to be a $k$-parameter congruence of $(m-s)$-dimensional spheres in $\mathbb{R}^{m}$ if $\operatorname{ker}\left(S_{1}\right)_{*} \cap \cdots \cap \operatorname{ker}\left(S_{s}\right)_{*}$ has rank $n-k$ everywhere.

An immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ envelops the congruence of $(m-s)$-dimensional spheres in $\mathbb{R}^{m}$ determined by the orthogonal s-tuple of smooth maps $S_{i}: M^{n} \rightarrow \mathbb{S}_{1,1}^{m+1} \subset$ $\mathbb{L}^{m+2}, 1 \leq i \leq s$, if it envelops $S_{i}$ for all $1 \leq i \leq s$.

We have the following extension of Proposition 9.4.
Proposition 9.8. If an immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ envelops a $(n-q)$-parameter congruence of $n$-dimensional spheres determined by the orthogonal $(m-n)$-tuple of smooth maps

$$
S_{i}: M^{n} \rightarrow \mathbb{S}_{1,1}^{m+1} \subset \mathbb{L}^{m+2}, \quad 1 \leq i \leq m-n
$$

then it carries a principal normal vector field $\eta$ such that

$$
D(x)=\operatorname{ker}\left(S_{1}\right)_{*}(x) \cap \cdots \cap \operatorname{ker}\left(S_{m-n}\right)_{*}(x) \subset E_{\eta}(x)
$$

for all $x \in M^{n}$, with $D(x)=E_{\eta}(x)$ for all $x$ in an open dense subset of $M^{n}$, on which $\eta$ is a Dupin principal normal vector field of multiplicity $q$.

Conversely, any simply connected submanifold $f: M^{n} \rightarrow \mathbb{R}^{m}$ that carries a Dupin principal normal vector field of multiplicity $q$ envelops a $(n-q)$-parameter congruence of $n$-dimensional spheres.

Proof: First we prove the converse statement. Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be a simply connected submanifold that carries a Dupin principal normal vector field $\eta$ with multiplicity $q$. It follows from (1.29) that the subbundle $\{\eta\}^{\perp}$ of $N_{f} M$ is parallel with respect to the normal connection along $E_{\eta}$. Moreover, by the Ricci equation,

$$
\left\langle R^{\perp}(T, S) \xi, \mu\right\rangle=0
$$

for all $T, S \in \Gamma\left(E_{\eta}\right)$ and $\xi, \mu \in \Gamma\left(\{\eta\}^{\perp}\right)$. Since $M^{n}$ is simply connected, there exists an orthonormal frame $\xi_{1}, \ldots, \xi_{m-n-1}$ of $\{\eta\}^{\perp}$ such that

$$
\nabla_{T}^{\perp} \xi_{i}=0
$$

for all $T \in \Gamma\left(E_{\eta}\right)$ and $1 \leq i \leq m-n-1$. For each $1 \leq i \leq m-n-1$, write $\xi_{i}=i \circ g_{i}$, with $g_{i}: M^{n} \rightarrow \mathbb{S}^{m-1}$, and let $S_{i}: M^{n} \rightarrow \mathbb{S}_{1,1}^{m+1}$ be the congruence of affine hyperplanes in $\mathbb{R}^{m}$ given by (9.11) in terms of $g_{i}$ and $\gamma_{i}=\left\langle f, \xi_{i}\right\rangle \in C^{\infty}(M)$. From

$$
\left\langle f, \xi_{i}\right\rangle=\gamma_{i} \text { and }\left\langle f_{*} X, \xi_{i}\right\rangle=0
$$

for all $X \in T_{x} M$, it follows that $S_{i}$ is enveloped by $f$, and since $A_{\xi_{i}}^{f} T=0=\nabla_{T}^{\frac{1}{T}} \xi_{i}$ for all $T \in E_{\eta}$, then $E_{\eta} \subset \operatorname{ker}\left(\xi_{i}\right)_{*}=\operatorname{ker}\left(g_{i}\right)_{*}=\operatorname{ker}\left(S_{i}\right)_{*}$.

Now let $S_{m-n}: M^{n} \rightarrow \mathbb{S}_{1,1}^{m+1}$ be the congruence of hyperspheres in $\mathbb{R}^{m}$ given by (9.9) in terms of the smooth map $h: M^{n} \rightarrow \mathbb{R}^{m}$ defined by

$$
h=f+\frac{1}{\|\eta\|^{2}} \eta
$$

and $r=1 /\|\eta\| \in C^{\infty}(M)$. Then (9.12) holds for $h$ and $r$, hence $f$ envelops $S_{m-n}$. Moreover, $E_{\eta}(x) \subset \operatorname{ker} h_{*}(x)=\operatorname{ker}\left(\bar{S}_{m-n}\right)_{*}(x)$ for all $x \in M^{n}$ by Proposition 1.22.

Therefore we have an orthogonal $(m-n)$-tuple of congruences of hyperspheres $S_{1}, \ldots, S_{m-n}$, all of which are enveloped by $f$. Consequently, it defines a congruence of $n$-dimensional spheres that is enveloped by $f$, with $E_{\eta}(x) \subset D(x)$ for all $x \in M^{n}$. The opposite inclusion will be shown next in the proof of the direct statement.

So assume that $f$ envelops the $(n-q)$-parameter congruence of $n$-dimensional spheres determined by the orthogonal ( $m-n$ )-tuple of smooth maps $S_{i}: M^{n} \rightarrow \mathbb{S}_{1,1}^{m+1} \subset$ $\mathbb{L}^{m+2}, 1 \leq i \leq m-n$. Thus

$$
D(x)=\operatorname{ker}\left(S_{1}\right)_{*}(x) \cap \cdots \cap \operatorname{ker}\left(S_{m-n}\right)_{*}(x)
$$

has dimension $q$ for all $x \in M^{n}$. We may assume that $S_{i}$ is a congruence of affine hyperplanes determined by smooth maps $g_{i}: M^{n} \rightarrow \mathbb{S}^{m-1}$ and $\gamma_{i} \in C^{\infty}(M)$ for all $1 \leq i \leq m-n-1$, and that $S_{m-n}$ is a congruence of hyperspheres determined by a smooth map $h: M^{n} \rightarrow \mathbb{R}^{m}$ and $r \in C^{\infty}(M), r>0$.

Since $f$ envelops $S_{m-n}$, it follows that (9.12) holds for $h: M^{n} \rightarrow \mathbb{R}^{m}$ and $r \in$ $C^{\infty}(M)$. Therefore $\eta \in \Gamma\left(f^{*} T \mathbb{R}^{m}\right)$, defined by

$$
\eta(x)=\frac{1}{r^{2}(x)}(h(x)-f(x)),
$$

is a normal vector field along $f$. We prove next that $\eta$ is a principal normal vector field such that $D(x) \subset E_{\eta}(x)$ for all $x \in M^{n}$ and $\nabla_{T}^{\frac{1}{T}} \eta=0$ for all $T \in D(x)$.

Given $T \in D(x)$, from $T \in \operatorname{ker}\left(S_{m-n}\right)_{*}(x)=\operatorname{ker} h_{*}(x)=\operatorname{ker} h_{*}(x) \cap \operatorname{ker} r_{*}(x)$ we obtain

$$
\begin{aligned}
-f_{*} A_{\eta}^{f} T+\nabla_{T}^{\frac{1}{T}} \eta & =\tilde{\nabla}_{T} \eta \\
& =T\left(1 / r^{2}\right)(h(x)-f(x))+\frac{1}{r^{2}(x)}\left(h_{*} T-f_{*} T\right) \\
& =-\frac{1}{r^{2}(x)} f_{*} T \\
& =-\|\eta\|^{2} f_{*} T
\end{aligned}
$$

which implies that

$$
A_{\eta}^{f} T=\|\eta\|^{2} T \text { and } \nabla_{T}^{\perp} \eta=0 .
$$

On the other hand, since $f$ envelops $S_{i}, 1 \leq i \leq m-n-1$, by (9.13) the maps $\xi_{i}=i \circ g_{i}: M^{n} \rightarrow \mathbb{R}^{m}$ and $\gamma_{i} \in C^{\infty}(M)$, where $i: \mathbb{S}^{m-1} \rightarrow \mathbb{R}^{m}$ is the inclusion, satisfy

$$
\left\langle f(x), \xi_{i}(x)\right\rangle=\gamma_{i}(x) \text { and }\left\langle f_{*} X, \xi_{i}(x)\right\rangle=0
$$

for all $x \in M^{n}, X \in T_{x} M$ and $1 \leq i \leq m-n-1$. In particular, $\xi_{i}$ is a unit normal vector field along $f$. Moreover,

$$
\left\langle\xi_{i}(x), \xi_{j}(x)\right\rangle=0=\left\langle\xi_{i}(x), \eta(x)\right\rangle=0
$$

for all $x \in M^{n}$ and $1 \leq i \neq j \leq m-n-1$, because $S_{i}: M^{n} \rightarrow \mathbb{S}_{1,1}^{m+1} \subset \mathbb{L}^{m+2}, 1 \leq i \leq$ $m-n$, form an orthogonal $(m-n)$-tuple of smooth maps. Thus $\xi_{1}, \ldots, \xi_{m-n-1}$ is an orthonormal frame of $\{\eta\}^{\perp} \subset N_{f} M$. Since $T \in \operatorname{ker}\left(S_{i}\right)_{*}(x)=\operatorname{ker}\left(g_{i}\right)_{*}(x)=\operatorname{ker}\left(\xi_{i}\right)_{*}(x)$, $1 \leq i \leq m-n-1$, then $A_{\xi_{i}}^{f} T=0$. It follows that $D(x) \subset E_{\eta}(x)$ for all $x \in M^{n}$ and that $\nabla \frac{1}{T} \eta=0$ for all $T \in D(x)$.

To complete the proof of the direct statement, it suffices to show that $\eta$ cannot have constant multiplicity greater than $q$ on any open subset $U \subset M^{n}$. Indeed, if otherwise and $T \in E_{\eta}(x)$ for $x \in U$, since $\eta$ is parallel along $E_{\lambda}$ on $U$ by Proposition 1.22, then $T \in \operatorname{ker} h_{*}(x)$ by 9.17), a contradiction.

### 9.4 The light-cone representative

Two Riemannian metrics $\langle$,$\rangle and \langle,\rangle^{\prime}$ on a manifold $M^{n}$ are conformal if there exists a positive function $\varphi \in C^{\infty}(M)$ such that

$$
\langle,\rangle^{\prime}=\varphi^{2}\langle,\rangle
$$

The function $\varphi$ is called the conformal factor of $\langle,\rangle^{\prime}$ with respect to $\langle$,$\rangle . A conformal$ structure on $M^{n}$ is an equivalence class of conformal Riemannian metrics. Clearly, every Riemannian manifold has a canonical conformal structure determined by its metric.

Given an immersion $f: M^{n} \rightarrow \bar{M}^{m}$ between differentiable manifolds, since conformal metrics on $\bar{M}^{m}$ are pulled-back by $f$ to conformal metrics on $M^{n}$, a conformal structure on $\bar{M}^{m}$ induces a conformal structure on $M^{n}$, the conformal structure on $M^{n}$ induced by $f$. If $M^{n}$ is already endowed with a conformal structure, the map $f$ is said to be conformal if such conformal structure coincides with that induced by $f$. In particular, if $M^{n}$ and $\bar{M}^{m}$ are Riemannian manifolds, then $f$ is conformal if its induced metric $\langle,\rangle_{f}$ is conformal to the Riemannian metric $\langle$,$\rangle of M^{n}$, and we call the conformal factor of $\langle,\rangle_{f}$ with respect to $\langle$,$\rangle the conformal factor of f$.

Let $f: M^{n} \rightarrow \mathbb{V}^{m+1}$ be an immersion of a differentiable manifold $M^{n}$ into the light-cone. Given a positive function $\mu \in C^{\infty}(M)$, the map $h: M^{n} \rightarrow \mathbb{V}^{m+1}$ given by $h=\mu f$ is also an immersion, and the induced metrics $\langle,\rangle_{f}$ and $\langle,\rangle_{h}$ are related by

$$
\langle,\rangle_{h}=\mu^{2}\langle,\rangle_{f}
$$

This follows from

$$
h_{*}(X)=X(\mu) f+\mu f_{*}(X),
$$

bearing in mind that the position vector $f$ is a light-like normal vector field.
The next result summarizes some consequences of this simple but useful observation. We follow the notations of the first section, and assume that a triple $(v, w, C)$ has been fixed with $w_{0}<0$, so that $\mathbb{E}^{m} \subset \mathbb{V}_{+}^{m+1}$, the upper half of $\mathbb{V}^{m+1}$. Let $\Pi: \mathbb{V}_{+}^{m+1} \backslash \mathbb{R} w \rightarrow \mathbb{E}^{m}$ denote the projection onto $\mathbb{E}^{m}$ given by

$$
\Pi(u)=\frac{u}{\langle u, w\rangle},
$$

where $\mathbb{R} w=\{t w: t<0\}$.
Proposition 9.9. Let $M^{n}$ be a Riemannian manifold. Then the following assertions hold:
(i) Any conformal immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ with conformal factor $\varphi \in C^{\infty}(M)$ gives rise to an isometric immersion $\mathcal{J}(f): M^{n} \rightarrow \mathbb{V}_{+}^{m+1}$ given by

$$
\mathcal{J}(f)=\frac{1}{\varphi} \Psi \circ f
$$

(ii) Any isometric immersion $F: M^{n} \rightarrow \mathbb{V}_{+}^{m+1} \backslash \mathbb{R} w$ gives rise to a conformal immersion $\mathcal{C}(F): M^{n} \rightarrow \mathbb{R}^{m}$ given by

$$
\Psi \circ \mathcal{C}(F)=\Pi \circ F,
$$

whose conformal factor is $1 /\langle F, w\rangle$.
(iii) For any conformal immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ and for any isometric immersion $F: M^{n} \rightarrow \mathbb{V}_{+}^{m+1} \backslash \mathbb{R} w$ one has

$$
\mathfrak{C}(\mathcal{J}(f))=f \text { and } \mathcal{J}(\mathcal{C}(F))=F .
$$

We call $\mathcal{J}(f): M^{n} \rightarrow \mathbb{V}_{+}^{m+1}$ the isometric light-cone representative of the conformal immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$, and write $\mathcal{J}_{v, w, C}(f)$ when we need to emphasize the dependence on the triple $(v, w, C)$.

Examples 9.10. (i) The isometric embedding $T_{B, z}: \mathbb{Q}_{c}^{m} \rightarrow \mathbb{V}^{m+1}$ defined by (9.6) gives rise to a conformal diffeomorphism of $\mathbb{Q}_{c}^{m}$ (minus one point if $c>0$ ) onto $\mathbb{R}^{m}$.
(ii) Fix an orthogonal decomposition $\mathbb{L}^{m+2}=V \oplus W$, with $V$ time-like, and linear isometries $C: \mathbb{L}^{k+1} \rightarrow V$ and $D: \mathbb{R}^{m-k+1} \rightarrow W$. Define an isometric immersion

$$
L_{C, D}: \mathbb{H}_{-c}^{k} \times \mathbb{S}_{c}^{m-k} \rightarrow \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}
$$

by

$$
(X, Y) \mapsto C X+D Y
$$

The map $\Theta: \mathbb{H}_{-c}^{k} \times \mathbb{S}_{c}^{m-k} \rightarrow \mathbb{R}^{m}$ given by

$$
\begin{equation*}
\Theta=\mathcal{C}\left(L_{C, D}\right) \tag{9.24}
\end{equation*}
$$

defines a conformal diffeomorphism onto the complement of the ( $k-1$ )-dimensional sphere that is mapped onto $\mathbb{E}^{m} \cap V$ by the isometry $\Psi: \mathbb{R}^{m} \rightarrow \mathbb{E}^{m}$.

Next we compute $\mathcal{C}(T \circ \Psi)$ for some special elements $T$ in $\mathbb{O}_{1}(m+2)$. A similarity of $\mathbb{R}^{m}$ with ratio $\lambda \in(0,+\infty)$ is a map $L: \mathbb{R}^{m} \rightarrow \mathbb{R}^{m}$ such that

$$
\|L x-L y\|=\lambda\|x-y\|
$$

for all $x, y \in \mathbb{R}^{n}$.
Proposition 9.11. The following holds:
(i) If $R \in \mathbb{O}_{1}(m+2)$ is the reflection

$$
R(u)=u-2\langle u, z\rangle z
$$

with respect to the hyperplane in $\mathbb{L}^{m+2}$ orthogonal to a unit space-like vector $z$, with $\langle z, w\rangle \neq 0$, then

$$
\begin{equation*}
\mathcal{C}(R \circ \Psi)=I \tag{9.25}
\end{equation*}
$$

is the inversion with respect to the hypersphere $S=\mathbb{E}^{m} \cap\{z\}^{\perp}$.
(ii) If $G \in \mathbb{O}_{1}(m+2)$ satisfies $G(w)=\lambda w$ for some $\lambda \in(0,+\infty)$, then

$$
\begin{equation*}
\mathcal{C}(G \circ \Psi)=L \tag{9.26}
\end{equation*}
$$

is a similarity of ratio $\lambda$. Conversely, for any similarity $L$ of ratio $\lambda \in(0,+\infty)$ there exists $G \in \mathbb{O}_{1}(m+2)$ satisfying $G(w)=\lambda w$ such that 9.26) holds. In particular, isometries of $\mathbb{R}^{m}$ correspond in this way to the elements of $\mathbb{O}_{1}(m+2)$ that fix $w$.

Proof: (i) Writing $z$ as in (9.8) in terms of the center $x_{0}$ and radius $r$ of $S=\mathbb{E}^{m} \cap\{z\}^{\perp}$, a straightforward computation yields

$$
\begin{aligned}
R \circ \Psi(x) & =\Psi(x)-2\langle\Psi(x), z\rangle z \\
& =\frac{\left\|x-x_{0}\right\|^{2}}{r^{2}}\left(v+C(I(x))-(1 / 2)\|I(x)\|^{2} w\right) \\
& =\frac{\left\|x-x_{0}\right\|^{2}}{r^{2}} \Psi(I(x))
\end{aligned}
$$

where

$$
I(x)=x_{0}+\frac{r^{2}}{\left\|x-x_{0}\right\|^{2}}\left(x-x_{0}\right), \quad x \neq x_{0}
$$

is the inversion with respect to $S$. Thus $\Pi \circ R \circ \Psi=\Psi \circ I$, which gives 9.25).
(ii) If $G \in \mathbb{O}_{1}(m+2)$ satisfies $G(w)=\lambda w$ for some $\lambda \in(0,+\infty)$, then

$$
\begin{aligned}
\langle G \circ \Psi, w\rangle & =\left\langle G \circ \Psi, \frac{1}{\lambda} G w\right\rangle \\
& =\frac{1}{\lambda}\langle G \circ \Psi, G w\rangle \\
& =\frac{1}{\lambda}\langle\Psi, w\rangle \\
& =\frac{1}{\lambda} .
\end{aligned}
$$

Therefore $L=\mathcal{C}(G \circ \Psi)$ is given by

$$
\Psi \circ L=\Pi \circ G \circ \Psi=\lambda G \circ \Psi,
$$

and hence $L$ is a similarity of ratio $\lambda$.
For the converse, we use that any similarity $L$ of $\mathbb{R}^{m}$ of ratio $\lambda$ is given by

$$
L(x)=\lambda B(x)+x_{0}
$$

for some $x_{0} \in \mathbb{R}^{m}$ and $B \in \mathbb{O}(m)$. Define

$$
\bar{v}=\frac{1}{\lambda}\left(v+C x_{0}-(1 / 2)\left\|x_{0}\right\|^{2} w\right)
$$

and $\bar{w}=\lambda w$. Then $\bar{v}, \bar{w} \in \mathbb{V}^{n+1}$ and $\langle\bar{v}, \bar{w}\rangle=1$. Moreover, $\bar{C}: \mathbb{R}^{m} \rightarrow \mathbb{L}^{m+2}$ given by

$$
\bar{C}(x)=C B(x)-\left\langle B(x), x_{0}\right\rangle w
$$

is a linear isometry onto $\{\bar{v}, \bar{w}\}^{\perp}$; hence $(\bar{v}, \bar{w}, \bar{C})$ is an admissible triple. Now let $G \in \mathbb{O}_{1}(m+2)$ be defined by

$$
G(v)=\bar{v}, \quad G(w)=\bar{w} \text { and } G \circ C=\bar{C}
$$

It is easily checked that

$$
\begin{aligned}
\Psi(L(x)) & =v+C(L(x))-\frac{1}{2}\|L(x)\|^{2} w \\
& =\lambda G(\Psi(x))
\end{aligned}
$$

which is equivalent to (9.26).
Let $\mathbb{O}_{1}^{+}(m+2)$ denote the set of elements of $\mathbb{O}_{1}(m+2)$ that preserve $\mathbb{V}_{+}^{m+1}$.
Corollary 9.12. For any $T \in \mathbb{O}_{1}^{+}(m+2)$ there exists a composition $I \circ L$ of $a$ similarity $L$ and an inversion I with respect to a hypersphere of unit radius (possibly with I replaced by the identity map) such that

$$
\begin{equation*}
\mathcal{C}(T \circ \Psi)=I \circ L \tag{9.27}
\end{equation*}
$$

Proof: Define $(\bar{v}, \bar{w}, \bar{C})$ by

$$
\bar{v}=T(v), \quad \bar{w}=T(w) \text { and } \bar{C}=T \circ C .
$$

If $\bar{w}=\lambda w$ for some $\lambda \in(0,+\infty)$, the statement (with $I$ replaced by the identity map) follows from part (ii) of Proposition 9.11. Otherwise, consider the reflection

$$
R(u)=u-2\langle u, z\rangle z
$$

determined by the unit space-like vector

$$
z=\frac{1}{\langle\bar{w}, w\rangle} \bar{w}+\frac{1}{2} w
$$

and let $G \in \mathbb{O}_{1}(m+2)$ be given by

$$
G(w)=R(\bar{w})=-\frac{1}{2}\langle\bar{w}, w\rangle w, \quad G(v)=R(\bar{v}) \text { and } G \circ C=R \circ \bar{C} .
$$

Then $R \circ G$ takes $w$ to $\bar{w}, v$ to $\bar{v}$ and $R \circ G \circ C=\bar{C}$, hence $R \circ G=T$. By part (i) of Proposition 9.11, the map $\mathcal{C}(R \circ \Psi)=I$ is an inversion with respect to the hypersphere of unit radius $S=\mathbb{E}^{m} \cap\{z\}^{\perp}$, whereas $\mathcal{C}(G \circ \Psi)=L$ is a similarity of ratio $\lambda=-(1 / 2)\langle\bar{w}, w\rangle$ by part (ii) of Proposition 9.11. Then, from

$$
\begin{aligned}
\Pi \circ T \circ \Psi & =\Pi \circ R \circ G \circ \Psi \\
& =\Pi \circ R \circ \Psi \circ L \\
& =\Psi \circ I \circ L,
\end{aligned}
$$

we obtain (9.27).

### 9.5 Rigidity of the paraboloid model

In Theorem 16.1 we will prove a general rigidity theorem for hypersurfaces of the light-cone. Here we give a short proof of the rigidity of the (restriction to any open subset $U \subset \mathbb{R}^{m}$ of the) isometric embedding $\Psi: \mathbb{R}^{m} \rightarrow \mathbb{V}^{m+1}$.

Theorem 9.13. Let $F: U \rightarrow \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$ be an isometric immersion of a connected open subset $U \subset \mathbb{R}^{m}, m \geq 3$. Then $F=\left.\Psi_{\bar{v}, \bar{w}, \bar{C}}\right|_{U}$ for some admissible triple $(\bar{v}, \bar{w}, \bar{C})$.

Proof: We first prove that

$$
\alpha^{F}(X, Y)=0
$$

for all $x \in U$ and $X, Y \in T_{x} U=\mathbb{R}^{m}$ with $\langle X, Y\rangle=0$. Extend $X, Y$ to constant vector fields on $\mathbb{R}^{m}$. By Proposition 9.1,

$$
\tilde{\nabla}_{Y} F_{*} X=\alpha^{F}(X, Y)=\omega_{X}(Y) F
$$

for some one-form $\omega_{X}$ on the intersection $\mathcal{H} \cap U$ of $U$ with the affine hyperplane $\mathcal{H}$ through $x$ orthogonal to $X$. Let $\tilde{R}$ denotes the curvature tensor of $\mathbb{L}^{m+2}$. Then

$$
\begin{aligned}
0 & =\tilde{R}(Y, Z) F_{*} X \\
& =\tilde{\nabla}_{Y} \tilde{\nabla}_{Z} F_{*} X-\tilde{\nabla}_{Z} \tilde{\nabla}_{Y} F_{*} X-\tilde{\nabla}_{[Y, Z]} F_{*} X \\
& =d \omega_{X}(Y, Z) F+\omega_{X}(Z) F_{*} Y-\omega_{X}(Y) F_{*} Z
\end{aligned}
$$

for all $Y, Z \in\{X\}^{\perp}$. Since $m \geq 3$, we can take $Y, Z$ linearly independent. Then the vector fields $F, F_{*} Z$ and $F_{*} Y$ are also linearly independent because $F$ is an immersion and the position vector $F$ is a nonzero normal vector field. Thus the preceding equation implies that $\omega_{X}$ is identically zero.

It follows that $F$ is umbilic (see Exercise 1.21), that is, there exists $\bar{w} \in \Gamma\left(N^{F} U\right)$ such that

$$
\begin{equation*}
\alpha^{F}(X, Y)=-\langle X, Y\rangle \bar{w} \tag{9.28}
\end{equation*}
$$

for all $x \in U$ and $X, Y \in T_{x} U=\mathbb{R}^{m}$. From the Gauss equation of $F$, we see that $\bar{w}$ is a light-like vector field. Now (9.3) and (9.28) yield

$$
\begin{aligned}
\langle X, Y\rangle\langle\bar{w}, F\rangle & =-\left\langle\alpha^{F}(X, Y), F\right\rangle \\
& =\langle X, Y\rangle
\end{aligned}
$$

for all $X, Y \in \mathbb{R}^{n}$, thus $\langle\bar{w}, F\rangle=1$ everywhere. We show next that $\bar{w}$ is in fact a constant vector field. First, $\tilde{\nabla}_{X} \bar{w}$ has no tangent component, for $A_{\bar{w}}^{F}=0$. On the other hand, its normal component is

$$
\left\langle\tilde{\nabla}_{X} \bar{w}, F\right\rangle \bar{w}+\left\langle\tilde{\nabla}_{X} \bar{w}, \bar{w}\right\rangle F=0
$$

as follows by differentiating $\langle\bar{w}, \bar{w}\rangle=0$ and $\langle\bar{w}, F\rangle=1$. Then, writing $\Psi=\left.\Psi_{v, w, C}\right|_{U}$ for short, we have a vector bundle isometry $\tau: N_{\Psi} U \rightarrow N_{F} U$, given by $\tau(\Psi)=F$ and $\tau(w)=\bar{w}$, that preserves second fundamental forms, because of (9.5) and 9.28), and
normal connections, for $\{\Psi, w\}$ and $\{F, \bar{w}\}$ are parallel frames. By Theorem 1.25 , there exists $T \in \mathbb{O}_{1}(m+2)$ such that

$$
F=T \circ \Psi=\left.\Psi_{\bar{v}, \bar{w}, \bar{C}}\right|_{U},
$$

where $\bar{v}=T v, \bar{w}=T w$ and $\bar{C}=T \circ C$.
If an admissible triple $(v, w, C)$ has been fixed, by Theorem 9.13 any isometric immersion $F: U \rightarrow \mathbb{V}^{m+1}$ is given by $F=\left.T \circ \Psi_{v, w, C}\right|_{U}$ for some $T \in \mathbb{O}_{1}(m+2)$.

As a consequence of Theorem 9.13 , we now prove that conformal transformations of $\mathbb{R}^{m}, m \geq 3$, that is, conformal maps $f: U \rightarrow \mathbb{R}^{m}$ defined on connected open subsets $U \subset \mathbb{R}^{m}$, are given in its paraboloid model by the elements of $\mathbb{O}_{1}^{+}(m+2)$.

Corollary 9.14. For any conformal map $f: U \rightarrow \mathbb{R}^{m}, m \geq 3$, on a connected open subset $U \subset \mathbb{R}^{m}$ there exists $T \in \mathbb{O}_{1}^{+}(m+2)$ such that $f=\left.\mathcal{C}(T \circ \Psi)\right|_{U}$.

Proof: Let $F=\mathcal{J}(f): U \rightarrow \mathbb{V}_{+}^{m+1}$ be the isometric light-cone representative of $f$. From Theorem 9.13 we see that $F=\left.T \circ \Psi\right|_{U}$ for some $T \in \mathbb{O}_{1}^{+}(m+2)$. Then

$$
f=\mathcal{C}(F)=\left.\mathcal{C}(T \circ \Psi)\right|_{U},
$$

as we wished.
We can now derive from Corollary 9.14 the classical theorem of Liouville on conformal maps on connected open subsets of $\mathbb{R}^{m}, m \geq 3$.

Corollary 9.15. If $f: U \rightarrow \mathbb{R}^{m}, m \geq 3$, is a conformal map on a connected open subset $U \subset \mathbb{R}^{m}$, then there exist a similarity $L$ and an inversion $I$ with respect to a hypersphere of unit radius (possibly with I replaced by the identity map) such that $f=I \circ L$.

Proof: By Corollary 9.14 , there exists $T \in \mathbb{O}_{1}^{+}(m+2)$ such that $f=\left.\mathcal{C}(T \circ \Psi)\right|_{U}$. The statement now follows from Corollary 9.12 .

Remark 9.16. In Exercise 9.4 , the reader is asked to prove that, conversely, Corollary 9.15 implies Corollary 9.14, and that this, in turn, implies Theorem 9.13. In summary, Liouville's theorem is equivalent to the rigidity of the paraboloid model of $\mathbb{R}^{m}$ as a hypersurface of the light-cone $\mathbb{V}^{m+1}$.

### 9.6 The second fundamental form of the light-cone representative

The next result computes the relation between the second fundamental forms and normal connections of a conformal immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ and its isometric light-cone representative $\mathcal{J}(f): M^{n} \rightarrow \mathbb{V}_{+}^{m+1} \subset \mathbb{L}^{m+2}$.

Proposition 9.17. Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be a conformal immersion with conformal factor $\varphi \in C^{\infty}(M)$ and let $F=\mathcal{J}(f): M^{n} \rightarrow \mathbb{V}_{+}^{m+1} \subset \mathbb{L}^{m+2}$ be its isometric light-cone representative. Then the following assertions hold:
(i) The map $\phi: \Gamma\left(N_{f} M\right) \rightarrow \Gamma\left(N_{F} M\right)$ given by

$$
\begin{equation*}
\phi(\xi)=\Psi_{*} \xi+H_{\xi} \Psi \circ f \tag{9.29}
\end{equation*}
$$

where $H_{\xi}=\left\langle\mathcal{H}^{f}, \xi\right\rangle$, defines a vector bundle isometry of $N_{f} M$ onto a subbundle $V$ of $N_{F} M$, which is parallel with respect to the normal connection on $N_{f} M$ and the connection on $V$ induced from the normal connection on $N_{F} M$.
(ii) The Lorentzian plane bundle $V^{\perp}$ has the position vector $F$ and the vector field

$$
\begin{equation*}
\zeta=-\Psi_{*}\left(f_{*} \operatorname{grad} \varphi^{-1}+\varphi \mathcal{H}^{f}\right)-\frac{\varphi}{2}\left(\left\|\operatorname{grad} \varphi^{-1}\right\|^{2}+\left\|\mathcal{H}^{f}\right\|^{2}\right) \Psi \circ f+\varphi w \tag{9.30}
\end{equation*}
$$

as a pseudo-orthonormal frame, with

$$
\begin{equation*}
\langle F, F\rangle=0=\langle\zeta, \zeta\rangle \text { and }\langle F, \zeta\rangle=1 \tag{9.31}
\end{equation*}
$$

Here grad and || || are calculated with respect to the metric $\langle$,$\rangle of M^{n}$.
(iii) The second fundamental form of $F$ splits, according to the orthogonal decomposition $N_{F} M=V \oplus V^{\perp}$, as

$$
\begin{equation*}
\alpha^{F}(X, Y)=\phi\left(\beta^{f}(X, Y)\right)-\psi(X, Y) F-\langle X, Y\rangle \zeta \tag{9.32}
\end{equation*}
$$

where

$$
\begin{equation*}
\beta^{f}(X, Y)=\frac{1}{\varphi}\left(\alpha^{f}(X, Y)-\langle X, Y\rangle_{f} \mathcal{H}^{f}\right) \tag{9.33}
\end{equation*}
$$

and

$$
\begin{align*}
\psi(X, Y)= & \varphi\left\langle\beta^{f}(X, Y), \mathcal{H}^{f}\right\rangle+\frac{\varphi^{2}}{2}\left(\left\|\operatorname{grad} \varphi^{-1}\right\|^{2}+\left\|\mathcal{H}^{f}\right\|^{2}\right)\langle X, Y\rangle \\
& -\varphi \operatorname{Hess}^{-1}(X, Y) \tag{9.34}
\end{align*}
$$

for all $X, Y \in \mathfrak{X}(M)$. Here $\langle,\rangle_{f}=\varphi^{2}\langle$,$\rangle stands for the metric induced by f$ and Hess for the Hessian with respect to $\langle$,$\rangle .$
(iv) The vector field $\zeta$ satisfies

$$
\begin{equation*}
{ }^{F} \nabla_{X}^{\perp} \zeta=-\varphi \phi\left({ }^{f} \nabla \frac{\perp}{X} \mathcal{H}^{f}+\beta^{f}\left(X, \operatorname{grad} \varphi^{-1}\right)\right) . \tag{9.35}
\end{equation*}
$$

Proof: (i) Differentiating $F=\varphi^{-1} \Psi \circ f$ gives

$$
F_{*} X=X\left(\varphi^{-1}\right) \Psi \circ f+\varphi^{-1}(\Psi \circ f)_{*} X
$$

for all $X \in \mathfrak{X}(M)$. Therefore

$$
\left\langle\phi(\xi), F_{*} X\right\rangle=0
$$

for all $X \in \mathfrak{X}(M)$ and $\xi \in \Gamma\left(N_{f} M\right)$. Hence $\phi(\xi) \in \Gamma\left(N_{F} M\right)$ for all $\xi \in \Gamma\left(N_{f} M\right)$. Since $\Psi$ is an isometric immersion and the position vector $\Psi$ is a light-like normal vector field along $\Psi$, it follows that

$$
\langle\phi(\xi), \phi(\eta)\rangle=\langle\xi, \eta\rangle
$$

for all $\xi, \eta \in \Gamma\left(N_{f} M\right)$. Hence $\phi$ defines a vector bundle isometry of $N_{f} M$ onto a subbundle $V$ of $N_{F} M$. To prove that $\phi$ is parallel with respect to the normal connection on $N_{f} M$ and the connection on $V$ induced from the normal connection on $N_{F} M$, we have to show that

$$
\begin{equation*}
\left\langle{ }^{F} \nabla_{X}^{\perp} \phi(\xi), \phi(\eta)\right\rangle=\left\langle{ }^{f} \nabla_{X}^{\perp} \xi, \eta\right\rangle \tag{9.36}
\end{equation*}
$$

for all $\xi, \eta \in \Gamma\left(N_{f} M\right)$. Differentiating (9.29) with respect to the connection $\bar{\nabla}$ of $\mathbb{L}^{m+2}$, using the Gauss formula of $\Psi$ and 9.5 we obtain

$$
\begin{equation*}
\bar{\nabla}_{X} \phi(\xi)=\Psi_{*}\left(-f_{*} A_{\xi}^{f} X+H_{\xi} f_{*} X+{ }^{f} \nabla_{X}^{\perp} \xi\right)+X\left(H_{\xi}\right)(\Psi \circ f), \tag{9.37}
\end{equation*}
$$

and (9.36) follows.
(ii) It is easily checked that $F, \zeta \in \Gamma\left(V^{\perp}\right)$ and that conditions 9.31) hold.
(iii) We compute

$$
\alpha^{F}(X, Y)=\bar{\nabla}_{Y} F_{*} X-F_{*} \nabla_{Y} X,
$$

where $\nabla$ is the Levi-Civita connection of $M^{n}$ with respect to $\langle$,$\rangle . We have$

$$
\begin{aligned}
\bar{\nabla}_{Y} F_{*} X= & Y X\left(\varphi^{-1}\right) \Psi \circ f+X\left(\varphi^{-1}\right)(\Psi \circ f)_{*} Y+Y\left(\varphi^{-1}\right)(\Psi \circ f)_{*} X \\
& -\varphi\langle X, Y\rangle w+\varphi^{-1} \Psi_{*} f_{*} \tilde{\nabla}_{Y} X+\varphi^{-1} \Psi_{*} \alpha^{f}(X, Y) \\
= & Y X\left(\varphi^{-1}\right) \Psi \circ f-\varphi\langle X, Y\rangle w+\varphi^{-1} \Psi_{*} \alpha^{f}(X, Y) \\
& +\Psi_{*} f_{*}\left(X\left(\varphi^{-1}\right) Y+Y\left(\varphi^{-1}\right) X+\varphi^{-1} \tilde{\nabla}_{Y} X\right),
\end{aligned}
$$

where $\tilde{\nabla}$ is the Levi-Civita connection of the metric induced by $f$. On the other hand,

$$
F_{*} \nabla_{Y} X=\left(\nabla_{Y} X\right)\left(\varphi^{-1}\right) \Psi \circ f+\varphi^{-1} \Psi_{*} f_{*} \nabla_{Y} X .
$$

Using the formula (see Exercise 9.1)

$$
\nabla_{Y} X=\tilde{\nabla}_{Y} X+\varphi\left(X\left(\varphi^{-1}\right) Y+Y\left(\varphi^{-1}\right) X-\langle X, Y\rangle \operatorname{grad} \varphi^{-1}\right),
$$

we obtain
$\alpha^{F}(X, Y)=\Psi_{*}\left(\langle X, Y\rangle f_{*} \operatorname{grad} \varphi^{-1}+\varphi^{-1} \alpha^{f}(X, Y)\right)+\operatorname{Hess} \varphi^{-1}(X, Y) \Psi \circ f-\varphi\langle X, Y\rangle w$.
In particular,
$\left\langle\alpha^{F}(X, Y), \zeta\right\rangle=\frac{\varphi^{2}}{2}\left(\left\|\mathcal{H}^{f}\right\|^{2}-\left\|\operatorname{grad} \varphi^{-1}\right\|^{2}\right)\langle X, Y\rangle-\left\langle\alpha^{f}(X, Y), \mathcal{H}^{f}\right\rangle+\varphi \operatorname{Hess} \varphi^{-1}(X, Y)$,
which is the opposite of the expression on the right-hand side of (9.34).

A straightforward computation now gives

$$
\begin{aligned}
\left(\alpha^{F}(X, Y)\right)_{V} & =\alpha^{F}(X, Y)-\left\langle\alpha^{F}(X, Y), \zeta\right\rangle F-\left\langle\alpha^{F}(X, Y), F\right\rangle \zeta \\
& =\alpha^{F}(X, Y)-\varphi^{-1}\left\langle\alpha^{F}(X, Y), \zeta\right\rangle \Psi \circ f+\langle X, Y\rangle \zeta \\
& =\phi\left(\beta^{f}(X, Y)\right)
\end{aligned}
$$

where $\beta^{f}$ is given by (9.33), and 9.32 follows.
(iv) From 9.37) we obtain

$$
\begin{aligned}
\left\langle\bar{\nabla}_{X} \phi(\xi), \zeta\right\rangle & =\left\langle A_{\xi}^{f} X, \operatorname{grad} \varphi^{-1}\right\rangle_{f}+\varphi X\left(H_{\xi}\right)-\varphi\left\langle{ }^{f} \nabla_{X}^{\perp} \xi, \mathcal{H}^{f}\right\rangle-H_{\xi}\left\langle X, \operatorname{grad} \varphi^{-1}\right\rangle_{f} \\
& =\varphi\left\langle\left\langle^{f} \nabla_{X}^{\perp} \mathcal{H}^{f}+\beta^{f}\left(X, \operatorname{grad} \varphi^{-1}\right), \xi\right\rangle\right. \\
& =\varphi\left\langle\phi\left({ }^{( } \nabla_{X}^{\perp} \mathcal{H}^{f}+\beta^{f}\left(X, \operatorname{grad} \varphi^{-1}\right)\right), \phi(\xi)\right\rangle,
\end{aligned}
$$

and 9.35 follows.

### 9.7 Conformal congruence of submanifolds

Two immersions $f, g: M^{n} \rightarrow \mathbb{R}^{m}$ are said to be conformally congruent if $g=\tau \circ f$ for some conformal transformation $\tau$ of $\mathbb{R}^{m}$. A conformal immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ is called conformally rigid if any other conformal immersion $g: M^{n} \rightarrow \mathbb{R}^{m}$ is conformally congruent to $f$.

A basic tool for proving conformal rigidity of a Euclidean submanifold is the following result, which reduces the problem to proving the isometric rigidity of its isometric light-cone representative.

Proposition 9.18. Two conformal immersions $f, g: M^{n} \rightarrow \mathbb{R}^{m}$ are conformally congruent if and only if their isometric light-cone representatives $\mathcal{J}(f), \mathcal{J}(g): M^{n} \rightarrow$ $\mathbb{V}_{+}^{m+1} \subset \mathbb{L}^{m+2}$ are isometrically congruent.

Proof: Assume first that $\mathcal{J}(f)=T \circ \mathcal{J}(g)$ for some $T \in \mathbb{O}_{1}(m+2)$. Then $\mathcal{T}=\mathcal{C}(T \circ \Psi)$ is well defined and

$$
f=\mathfrak{C}(T \circ \mathcal{J}(g))=\mathfrak{C}(T \circ \Psi) \circ g=\mathcal{T} \circ g .
$$

Conversely, if $f=\mathfrak{T} \circ g$ for a conformal diffeomorphism of $\mathbb{R}^{m}$, then $\mathcal{J}(\mathcal{T})=T \circ \Psi$ for some $T \in \mathbb{O}_{1}^{+}(m+2)$ by Theorem 9.13 . Then

$$
\begin{aligned}
\mathcal{J}(f)=\mathcal{J}(\mathcal{T} \circ g) & =\varphi_{\mathcal{T} \circ g}^{-1} \Psi \circ \mathcal{T} \circ g \\
& =\left(\varphi_{\mathcal{J}} \circ g\right)^{-1} \varphi_{g}^{-1} \Psi \circ \mathcal{T} \circ g \\
& =\varphi_{g}^{-1} \mathcal{J}(\mathcal{T}) \circ g \\
& =\varphi_{g}^{-1} T \circ \Psi \circ g \\
& =T \circ \mathcal{J}(g),
\end{aligned}
$$

where $\varphi_{g}, \varphi_{\mathcal{J}}$ and $\varphi_{\mathcal{J} \circ g}$ are the conformal factors of $g, \mathcal{T}$ and $\mathcal{T} \circ g$, respectively.
Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be an isometric immersion. The traceless second fundamental form $\gamma^{f}: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \Gamma\left(N_{f} M\right)$ of $f$ is defined by

$$
\begin{equation*}
\gamma^{f}(X, Y)=\alpha^{f}(X, Y)-\langle X, Y\rangle \mathcal{H}^{f} \tag{9.38}
\end{equation*}
$$

Its norm, given at each $x \in M^{n}$ by

$$
\left\|\gamma^{f}(x)\right\|^{2}=\sum_{i, j=1}^{n}\left\|\gamma^{f}(x)\left(X_{i}, X_{j}\right)\right\|^{2},
$$

where $X_{1}, \ldots, X_{n}$ is an orthonormal basis of $T_{x} M$, vanishes precisely at the umbilical points of $f$. We leave to the reader to check, using Exercise 9.2, that the metric on the subset $M_{0} \subset M^{n}$ of nonumbilical points of $f$, defined by

$$
\langle X, Y\rangle^{*}=\rho^{2}\langle X, Y\rangle
$$

with

$$
\rho=\rho_{f}=\sqrt{n /(n-1)}\left\|\gamma^{f}\right\|,
$$

is invariant under conformal changes of the metric on the ambient space. Thus it is an invariant representative of the induced conformal structure on $M^{n}$.

The metric $\langle,\rangle^{*}$ is called the Moebius metric of $M^{n}$ determined by $f$. The map $F: M_{0} \rightarrow \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$ defined by

$$
F=\rho \Psi \circ f
$$

is called the Moebius representative of $f$.
Proposition 9.19. Two immersions $f, g: M^{n} \rightarrow \mathbb{R}^{m}$ free of umbilic points are conformally congruent if and only if their Moebius representatives $F, G: M^{n} \rightarrow \mathbb{V}_{+}^{m+1} \subset \mathbb{L}^{m+2}$ are isometrically congruent.

Proof: If $f, g: M^{n} \rightarrow \mathbb{R}^{m}$ are conformally congruent, then their Moebius metrics coincide by the above discussion. Let $\langle,\rangle^{*}$ denote their common Moebius metric, and endow $M^{n}$ with $\langle,\rangle^{*}$. Then $f$ and $g$ become conformal immersions with conformal factors $\rho_{f}^{-1}$ and $\rho_{g}^{-1}$, respectively. Hence the Moebius representatives $F$ and $G$ of $f$ and $g$, respectively, coincide with their isometric light-cone representatives. It follows from Proposition 9.18 that $F$ and $G$ are isometrically congruent.

Conversely, assume that $F$ and $G$ are isometrically congruent. Then, in particular, they induce the same metric on $M^{n}$, that is, the Moebius metrics of $f$ and $g$ coincide. As before, if $\langle,\rangle^{*}$ is this common metric and $M^{n}$ is endowed with $\langle,\rangle^{*}$, then $f$ and $g$ become conformal immersions with conformal factors $\rho_{f}^{-1}$ and $\rho_{g}^{-1}$, respectively, and their Moebius representatives $F$ and $G$, respectively, coincide with their isometric light-cone representatives. By the converse statement of Proposition 9.18, the immersions $f$ and $g$ are conformally congruent.

### 9.8 A Fundamental theorem in Moebius geometry

In this section it is shown that a submanifold $f: M^{n} \rightarrow \mathbb{R}^{m}$ that is free of umbilic points is completely determined, up to conformal transformations of $\mathbb{R}^{m}$, by the Moebius metric on $M^{n}$ determined by $f$, its normal connection and its Moebius second fundamental form defined below.

Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be an isometric immersion free of umbilic points and let $F: M^{n} \rightarrow \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$ be its Moebius representative. As shown previously, the $\operatorname{map} \phi: \Gamma\left(N_{f} M\right) \rightarrow \Gamma\left(N_{F} M\right)$ given by (9.29) defines a vector bundle isometry of $N_{f} M$ onto a subbundle $V$ of $N_{F} M$, which is parallel with respect to the normal connection on $N_{f} M$ and the connection on $V$ induced from the normal connection on $N_{F} M$. The subbundle $V$ of $N_{F} M$ is called the Moebius normal bundle of $f$.

Let $\zeta=\zeta_{f}$ be the vector field defined by 9.30 with $\varphi=\rho^{-1}$, that is,

$$
\zeta=-\Psi_{*}\left(f_{*} \operatorname{grad}^{*} \rho+\rho^{-1} \mathcal{H}^{f}\right)-\frac{1}{2 \rho}\left(\left\|\operatorname{grad}^{*} \rho\right\|_{*}^{2}+\left\|\mathcal{H}^{f}\right\|_{*}^{2}\right) \Psi \circ f+\frac{1}{\rho} w .
$$

Here and in the sequel, a subscript or superscript "*" refers to the Moebius metric $\langle,\rangle^{*}$. The vector fields $F$ and $\zeta$ form a pseudo-orthonormal frame of the Lorentzian normal plane bundle $\mathbb{L}^{2}=V^{\perp}$ such that

$$
\langle F, F\rangle=0=\langle\zeta, \zeta\rangle \text { and }\langle F, \zeta\rangle=1
$$

The Moebius second fundamental form of $f: M^{n} \rightarrow \mathbb{R}^{m}$ is the symmetric section $\beta=\beta^{f}$ of $\operatorname{Hom}^{2}\left(T M, T M ; N_{f} M\right)$ defined by

$$
\beta(X, Y)=\rho\left(\alpha^{f}(X, Y)-\langle X, Y\rangle \mathcal{H}^{f}\right)
$$

for all $X, Y \in \mathfrak{X}(M)$.
By Proposition 9.17, the $V$-component of the second fundamental form $\alpha^{F}$ of $F$ is given by

$$
\alpha_{V}^{F}=\phi \circ \beta .
$$

The Moebius third fundamental form $\operatorname{III}_{\beta} \in \Gamma\left(T^{*} M \otimes T^{*} M\right)$ associated with $\beta$ is defined by

$$
I I I_{\beta}(X, Y)=\sum_{i=1}^{n}\left\langle\beta\left(X, X_{i}\right), \beta\left(Y, X_{i}\right)\right\rangle
$$

where $X_{1}, \ldots, X_{n}$ is an orthonormal frame with respect to the Moebius metric.

Notice that the norm of $\beta$ with respect to the Moebius metric satisfies

$$
\begin{align*}
\|\beta\|_{*}^{2} & =\operatorname{tr} I I I_{\beta} \\
& =\sum_{i, j=1}^{n}\left\langle\beta\left(X_{i}, X_{j}\right), \beta\left(X_{i}, X_{j}\right)\right\rangle \\
& =\sum_{i, j=1}^{n} \frac{1}{\rho^{2}}\left\langle\gamma^{f}\left(\tilde{X}_{i}, \tilde{X}_{j}\right), \gamma^{f}\left(\tilde{X}_{i}, \tilde{X}_{j}\right)\right\rangle \\
& =\frac{n-1}{n} \tag{9.39}
\end{align*}
$$

where $\tilde{\gamma}^{f}$ is the traceless second fundamental form of $f$ and $\tilde{X}_{i}=\rho X_{i}, 1 \leq i \leq n$, hence $\tilde{X}_{1}, \ldots, \tilde{X}_{n}$ is an orthonormal frame with respect to the metric induced by $f$.

The Blaschke tensor $\psi=\psi_{f}$ of $f$ is the symmetric bilinear form given by

$$
\psi(X, Y)=\frac{1}{\rho}\left\langle\beta(X, Y), \mathcal{H}^{f}\right\rangle+\frac{1}{2 \rho^{2}}\left(\left\|\operatorname{grad}^{*} \rho\right\|_{*}^{2}+\left\|\mathcal{H}^{f}\right\|_{*}^{2}\right)\langle X, Y\rangle^{*}-\frac{1}{\rho} \operatorname{Hess}^{*} \rho(X, Y)
$$

for all $X, Y \in \mathfrak{X}(M)$. The Moebius form $\omega=\omega_{f} \in \Gamma\left(T^{*} M \otimes N_{f} M\right)$ of $f$ is the normal bundle valued one-form defined by

$$
\omega(X)=-\frac{1}{\rho}\left(\nabla \frac{\perp}{X} \mathcal{H}^{f}+\beta\left(X, \operatorname{grad}^{*} \rho\right)\right)
$$

where the gradient is computed with respect to the Moebius metric.
By Proposition 9.17,

$$
\psi(X, Y)=-\left\langle\alpha^{F}(X, Y), \zeta\right\rangle
$$

and

$$
\begin{equation*}
\phi(\omega(X))={ }^{F} \nabla_{X}^{\perp} \zeta \tag{9.40}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}(M)$.
Proposition 9.20. The Blaschke tensor is given in terms of the Moebius metric and the Moebius third fundamental form by

$$
\begin{equation*}
(n-2) \psi(X, Y)=\operatorname{Ric}^{*}(X, Y)+I I I_{\beta}(X, Y)-\frac{n^{2} s^{*}+1}{2 n}\langle X, Y\rangle^{*} \tag{9.41}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}(M)$. In particular,

$$
\begin{equation*}
\operatorname{tr} \psi=\frac{n^{2} s^{*}+1}{2 n}=\frac{n}{2}\left\langle\mathcal{H}^{F}, \mathcal{H}^{F}\right\rangle . \tag{9.42}
\end{equation*}
$$

Proof: By Proposition 9.17, the second fundamental form of $F$ is given by

$$
\begin{equation*}
\alpha^{F}(X, Y)=\phi(\beta(X, Y))-\psi(X, Y) F-\langle X, Y\rangle^{*} \zeta \tag{9.43}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}(M)$. Taking traces in the preceding equation gives

$$
\begin{equation*}
n \mathcal{H}^{F}=-\operatorname{tr} \psi F-n \zeta . \tag{9.44}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
n\left\langle\mathcal{H}^{F}, \mathcal{H}^{F}\right\rangle=2 \operatorname{tr} \psi . \tag{9.45}
\end{equation*}
$$

On the other hand, from (3.4) we obtain

$$
\begin{equation*}
\operatorname{Ric}^{*}(X, Y)=n\left\langle\alpha^{F}(X, Y), \mathcal{H}^{F}\right\rangle-\sum_{i=1}^{n}\left\langle\beta\left(X, X_{i}\right), \beta\left(Y, X_{i}\right)\right\rangle-2 \psi(X, Y) \tag{9.46}
\end{equation*}
$$

Thus

$$
\begin{aligned}
\operatorname{Ric}^{*}(X) & =\frac{1}{n-1} \operatorname{Ric}^{*}(X, X) \\
& =\frac{n}{n-1}\left\langle\alpha^{F}(X, X), \mathcal{H}^{F}\right\rangle-\frac{1}{n-1} \sum_{i=1}^{n}\left\langle\beta\left(X, X_{i}\right), \beta\left(X, X_{i}\right)\right\rangle-\frac{2}{n-1} \psi(X, X) .
\end{aligned}
$$

Using (9.39) we obtain

$$
\begin{align*}
s^{*} & =\frac{1}{n} \sum_{j=1}^{n} \operatorname{Ric}^{*}\left(X_{j}\right) \\
& =\frac{n}{n-1}\left\langle\mathcal{H}^{F}, \mathcal{H}^{F}\right\rangle-\frac{1}{n(n-1)} \sum_{i, j=1}^{n}\left\langle\beta\left(X_{i}, X_{j}\right), \beta\left(X_{i}, X_{j}\right)\right\rangle-\frac{2}{n(n-1)} \operatorname{tr} \psi \\
& =\frac{n}{n-1}\left\langle\mathcal{H}^{F}, \mathcal{H}^{F}\right\rangle-\frac{1}{n^{2}}-\frac{2}{n(n-1)} \operatorname{tr} \psi \tag{9.47}
\end{align*}
$$

Now (9.42) follows from (9.45) and (9.47). Substituting in (9.46) and using (9.43) and (9.44) yield (9.41).

Proposition 9.21. The following relations hold:
(i) The conformal Gauss equation

$$
\begin{align*}
\left\langle R^{*}(X, Y) Z, W\right\rangle^{*}= & \langle\beta(X, W), \beta(Y, Z)\rangle-\langle\beta(X, Z), \beta(Y, W)\rangle \\
& +\psi(X, W)\langle Y, Z\rangle^{*}+\psi(Y, Z)\langle X, W\rangle^{*} \\
& -\psi(X, Z)\langle Y, W\rangle^{*}-\psi(Y, W)\langle X, Z\rangle^{*} \tag{9.48}
\end{align*}
$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$.
(ii) The conformal Codazzi equations

$$
\begin{equation*}
\left({ }^{f} \nabla_{X}^{\perp} \beta\right)(Y, Z)-\left({ }^{f} \nabla_{Y}^{\perp} \beta\right)(X, Z)=\omega((X \wedge Y) Z) \tag{9.49}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\nabla_{X} \psi\right)(Y, Z)-\left(\nabla_{Y} \psi\right)(X, Z)=\langle\omega(X), \beta(Y, Z)\rangle-\langle\omega(Y), \beta(X, Z)\rangle \tag{9.50}
\end{equation*}
$$

for all $X, Y, Z \in \mathfrak{X}(M)$.
(ii) The conformal Ricci equations

$$
\begin{equation*}
d \omega(X, Y)=\beta(X, \hat{\psi} Y)-\beta(Y, \hat{\psi} X) \tag{9.51}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle R^{\perp}(X, Y) \xi, \eta\right\rangle=\left\langle\left[B_{\xi}, B_{\eta}\right] X, Y\right\rangle^{*} \tag{9.52}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in \Gamma\left(N_{f} M\right)$, with $\hat{\psi}, B_{\xi} \in \Gamma(E n d(T M))$ given by

$$
\langle\hat{\psi} X, Y\rangle^{*}=\psi(X, Y) \text { and }\left\langle B_{\xi} X, Y\right\rangle^{*}=\langle\beta(X, Y), \xi\rangle .
$$

Proof: The Gauss equation for $F$ and

$$
\alpha^{F}(X, Y)=\phi(\beta(X, Y))-\psi(X, Y) F-\langle X, Y\rangle^{*} \zeta
$$

give (9.48). Using (9.36) and (9.40 we obtain

$$
\begin{aligned}
{ }^{F} \nabla_{X}^{\perp} \alpha^{F}(Y, Z)= & \phi\left({ }^{f} \nabla_{X}^{\perp} \beta(Y, Z)\right)-\langle\omega(X), \beta(Y, Z)\rangle F-X \psi(Y, Z) F \\
& -X\langle Y, Z\rangle \zeta-\langle Y, Z\rangle^{F} \nabla_{X}^{\perp} \zeta
\end{aligned}
$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Therefore the $V$-component of the Codazzi equation

$$
\left({ }^{F} \nabla_{X}^{\perp} \alpha^{F}\right)(Y, Z)=\left({ }^{F} \nabla_{Y}^{\perp} \alpha^{F}\right)(X, Z)
$$

gives (9.49), whereas the $F$-component yields (9.50). Finally, the Ricci equation

$$
R^{\perp}(X, Y) \zeta=\alpha^{F}\left(X, A_{\zeta}^{F} Y\right)-\alpha^{F}\left(A_{\zeta}^{F} X, Y\right)
$$

of $F$ yields (9.51), and (9.52) is equivalent to the Ricci equation of $f$.
Theorem 9.22. (Fundamental theorem of submanifolds in Moebius geometry) Existence: Let $\left(M^{n},\langle\rangle,\right)$ be a simply connected Riemannian manifold, let $\mathcal{E}$ be a Riemannian vector bundle of rank $p$ over $M^{n}$ with compatible connection $\nabla^{\mathcal{E}}$ and curvature tensor $R^{\varepsilon}$, and let $\beta^{\varepsilon}$ be a symmetric section of $\operatorname{Hom}^{2}(T M, T M ; \mathcal{E})$ such that

$$
\operatorname{tr} \beta^{\varepsilon}=0 \text { and }\left\|\beta^{\varepsilon}\right\|=\sqrt{(n-1) / n}
$$

For each $\xi \in \Gamma(\mathcal{E})$, define $B_{\xi}^{\mathcal{\varepsilon}} \in \Gamma(\operatorname{End}(T M))$ by

$$
\left\langle B_{\xi}^{\varepsilon} X, Y\right\rangle=\left\langle\beta^{\varepsilon}(X, Y), \xi\right\rangle .
$$

Assume that there exist $\psi \in \Gamma\left(T^{*} M \otimes T^{*} M\right)$ and $\omega \in \Gamma\left(T^{*} M \otimes \mathcal{E}\right)$ such that Eqs. 9.48) to 9.52) are satisfied for all $\xi, \eta \in \Gamma(\mathcal{E})$. Then there exist an immersion $f: M^{n} \rightarrow$ $\mathbb{R}^{n+p}$. free of umbilic points and a vector bundle isometry $\phi^{\varepsilon}: \mathcal{E} \rightarrow N_{f} M$ such that $\langle$, is the Moebius metric on $M^{n}$ determined by $f$,

$$
\beta^{f}=\phi^{\varepsilon} \circ \beta^{\varepsilon}, \quad \psi_{f}=\psi, \quad \omega_{f}=\phi^{\varepsilon} \circ \omega \text { and }{ }^{f} \nabla^{\perp} \phi^{\varepsilon}=\phi^{\varepsilon} \nabla^{\varepsilon} \text {. }
$$

Uniqueness: Let $f, g: M^{n} \rightarrow \mathbb{R}^{m}$ be immersions free of umbilic points that determine the same Moebius metric on the manifold $M^{n}$. Suppose further that there exists a vector bundle isometry $\mathfrak{T}: N_{f} M \rightarrow N_{g} M$ such that

$$
\mathcal{T}^{f} \nabla^{\perp}={ }^{g} \nabla^{\perp} \mathcal{T} \text { and } \mathcal{T} \circ \beta^{f}=\beta^{g} .
$$

Then there exists a conformal transformation $\tau$ of $\mathbb{R}^{m}$ such that $\tau \circ f=g$.
Proof: First we prove existence. Let $M \times \mathbb{L}^{2}$ be the trivial Lorentzian plane bundle over $M^{n}$ endowed with its canonical connection and choose a parallel pseudo-orthonormal frame $e_{1}, e_{2}$ of $M \times \mathbb{L}^{2}$ with $\left\langle e_{1}, e_{2}\right\rangle=1$. Let $\tilde{\varepsilon}$ be the Whitney sum of $\mathcal{E}$ with $M \times \mathbb{L}^{2}$ endowed with the compatible connection $\nabla^{\tilde{\varepsilon}}$ given by

$$
\nabla_{X}^{\tilde{\varepsilon}} \xi=\nabla_{X}^{\varepsilon} \xi-\langle\omega(X), \xi\rangle e_{2}, \quad \nabla_{X}^{\tilde{\varepsilon}} e_{1}=0 \text { and } \nabla_{X}^{\tilde{\varepsilon}} e_{2}=\omega(X) .
$$

Define a symmetric section $\alpha^{\tilde{\varepsilon}}$ of $\operatorname{Hom}^{2}(T M, T M ; \tilde{\varepsilon})$ by

$$
\alpha^{\tilde{\varepsilon}}(X, Y)=\beta^{\varepsilon}(X, Y)-\psi(X, Y) e_{1}-\langle X, Y\rangle e_{2}
$$

for all $X, Y \in \mathfrak{X}(M)$. Define $A_{\xi}^{\tilde{\varepsilon}} \in \Gamma(\operatorname{End}(T M))$ for $\xi \in \Gamma(\mathcal{E})$ by

$$
\left\langle A_{\xi}^{\tilde{\varepsilon}} X, Y\right\rangle=\left\langle\alpha^{\tilde{\varepsilon}}(X, Y), \xi\right\rangle
$$

and set

$$
A_{e_{1}}^{\tilde{\varepsilon}}=-I \text { and } A_{e_{2}}^{\tilde{\varepsilon}}=-\hat{\psi} .
$$

It follows from (9.48) that the Gauss equation for an isometric immersion of $M^{n}$ into $\mathbb{L}^{n+p+2}$ is satisfied. The Codazzi equation follows from (9.49) and (9.50), whereas the Ricci equation is a consequence of (9.51) and (9.52).

By the Fundamental theorem of submanifolds, there exist an isometric immersion $F: M^{n} \rightarrow \mathbb{L}^{n+p+2}$ and a parallel vector bundle isometry $\Phi: \tilde{\varepsilon} \rightarrow N_{F} M$ such that

$$
\alpha^{F}=\Phi \circ \alpha^{\tilde{\varepsilon}} \text { and }{ }^{F} \nabla^{\perp} \Phi=\Phi \nabla^{\tilde{\varepsilon}} .
$$

Set $h=F-\Phi\left(e_{1}\right)$. From $\nabla^{\tilde{\varepsilon}} e_{1}=0$ and $A_{e_{1}}^{\tilde{\varepsilon}}=-I$ we obtain

$$
\begin{aligned}
h_{*} X & =F_{*} X+F_{*} A_{\Phi\left(e_{1}\right)}^{F} X \\
& =0
\end{aligned}
$$

for any $X \in \mathfrak{X}(M)$. Thus $h$ is a constant vector in $\mathbb{L}^{n+p+2}$, which we may assume to be 0 . Thus $F=\Phi\left(e_{1}\right)$ takes values in $\mathbb{V}^{n+p+1}$, and we may write

$$
F=\varphi^{-1} \Psi \circ f
$$

for some $\varphi \in C^{\infty}(M)$ and some map $f: M^{n} \rightarrow \mathbb{R}^{n+p}$, which is necessarily conformal with conformal factor $\varphi$.

On one hand,

$$
\begin{aligned}
\alpha^{F}(X, Y) & =\Phi\left(\alpha^{\tilde{\varepsilon}}(X, Y)\right) \\
& =\Phi\left(\beta^{\varepsilon}(X, Y)\right)-\psi(X, Y) F-\langle X, Y\rangle \Phi\left(e_{2}\right)
\end{aligned}
$$

Since $\operatorname{tr} \beta^{\varepsilon}=0$, it follows that

$$
n \mathcal{H}^{F}=\operatorname{tr} \psi F-n \Phi\left(e_{2}\right) .
$$

On the other hand,

$$
\alpha^{F}(X, Y)=\phi^{f}\left(\beta^{f}(X, Y)\right)-\psi_{f}(X, Y) F-\langle X, Y\rangle \zeta_{f} .
$$

Thus

$$
n \mathcal{H}^{F}=-\operatorname{tr} \psi_{f} F-n \zeta_{f} .
$$

In particular,

$$
\operatorname{tr} \psi=\frac{n}{2}\left\langle\mathcal{H}^{F}, \mathcal{H}^{F}\right\rangle=\operatorname{tr} \psi_{f} .
$$

Hence $\zeta_{f}=\Phi\left(e_{2}\right)$, and therefore

$$
\begin{equation*}
\phi^{f} \circ \beta^{f}=\Phi \circ \beta^{\varepsilon} \text { and } \psi=\psi_{f} . \tag{9.53}
\end{equation*}
$$

Define $\phi^{\varepsilon}: \mathcal{E} \rightarrow N_{f} M$ by

$$
\left.\Phi\right|_{\varepsilon}=\phi^{f} \circ \phi^{\varepsilon}
$$

Then

$$
\beta^{f}=\phi^{\varepsilon} \circ \beta^{\varepsilon}
$$

by the first equation in (9.53). Moreover,

$$
\begin{aligned}
\phi^{f}\left(\omega^{f}(X)\right) & ={ }^{F} \nabla^{\perp} \zeta_{f} \\
& ={ }^{F} \nabla^{\perp} \Phi\left(e_{2}\right) \\
& =\Phi \nabla^{\tilde{\varepsilon}} e_{2} \\
& =\Phi \omega(X) \\
& =\phi^{f}\left(\phi^{\varepsilon} \omega(X)\right)
\end{aligned}
$$

for all $X \in \mathfrak{X}(M)$. Hence $\omega^{f}=\phi^{\varepsilon} \circ \omega$.
It remains to show that $\langle$,$\rangle is the Moebius metric of f$, or equivalently, that

$$
\varphi^{-1}=\rho_{f}=\sqrt{n /(n-1)}\left\|\gamma^{f}\right\|
$$

where $\gamma^{f}$ is the traceless second fundamental form of $f$ given by 9.38.
Let $I I_{\beta^{f}} \in \operatorname{Hom}^{2}\left(T M, T M ; N_{f} M\right)$ be defined by

$$
I I I_{\beta^{f}}(X, Y)=\sum_{i=1}^{n}\left\langle\beta^{f}\left(X, X_{i}\right), \beta^{f}\left(Y, X_{i}\right)\right\rangle
$$

for all $X, Y \in \mathfrak{X}(M)$, where $X_{1}, \ldots, X_{n}$ is an orthonormal frame with respect to $\langle$,$\rangle .$ The norm of $\beta^{f}$ with respect to $\langle$,$\rangle satisfies$

$$
\begin{align*}
\left\|\beta^{f}\right\|^{2} & =\operatorname{tr} I I I_{\beta}^{f} \\
& =\sum_{i, j=1}^{n}\left\langle\beta^{f}\left(X_{i}, X_{j}\right), \beta^{f}\left(X_{i}, X_{j}\right)\right\rangle \\
& =\sum_{i, j=1}^{n} \varphi^{2}\left\langle\gamma^{f}\left(\tilde{X}_{i}, \tilde{X}_{j}\right), \gamma^{f}\left(\tilde{X}_{i}, \tilde{X}_{j}\right)\right\rangle \\
& =\frac{n-1}{n} \varphi^{2} \rho_{f}^{2} \tag{9.54}
\end{align*}
$$

where $\tilde{X}_{i}=\varphi^{-1} X_{i}, 1 \leq i \leq n$. Hence $\tilde{X}_{1}, \ldots, \tilde{X}_{n}$ is an orthonormal frame with respect to the metric induced by $f$. Since $\left\|\beta^{\varepsilon}\right\|^{2}=(n-1) / n$ by assumption, it follows from (9.54) that $\rho_{f}=\varphi^{-1}$, as wished.

Now we prove uniqueness. Let $F, G: M^{n} \rightarrow \mathbb{V}_{+}^{m+1} \subset \mathbb{L}^{m+2}$ be the Moebius representatives of $f$ and $g$. Let $\phi^{f}: N_{f} M \rightarrow V_{f}$ and $\phi^{g}: N_{g} M \rightarrow V_{g}$ be the vector bundle isometries of $N_{f}$ and $N_{g} M$ onto the Moebius normal bundles of $f$ and $g$, respectively. Then $\Phi: N_{F} M \rightarrow N_{G} M$, defined by

$$
\Phi \circ \phi^{f}=\phi^{g} \circ \mathcal{T}, \quad \Phi(F)=G \text { and } \Phi\left(\zeta_{f}\right)=\zeta_{g},
$$

is also a vector bundle isometry. By assumption,

$$
\begin{aligned}
{ }^{g} \nabla_{X}^{\perp} \beta^{g}(Y, Z) & ={ }^{g} \nabla_{X}^{\perp} \mathcal{T}\left(\beta^{f}(Y, Z)\right) \\
& =\mathcal{T}^{f} \nabla_{X}^{\perp} \beta^{f}(Y, Z)
\end{aligned}
$$

for all $X, Y, Z \in \mathfrak{X}(M)$. On the other hand, by (9.49) we have

$$
\begin{aligned}
\omega_{g}((X \wedge Y) Z) & =\left({ }^{g} \nabla_{X}^{\perp} \beta^{g}\right)(Y, Z)-\left({ }^{g} \nabla_{Y}^{\perp} \beta^{g}\right)(X, Z) \\
& =\mathcal{T}\left(\left({ }^{f} \nabla_{X}^{\perp} \beta^{f}\right)(Y, Z)-\left({ }^{f} \nabla_{Y}^{\perp} \beta^{f}\right)(X, Z)\right) \\
& =\mathcal{T}\left(\omega_{f}((X \wedge Y) Z)\right)
\end{aligned}
$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Hence $\omega_{g}=\mathcal{T} \circ \omega_{f}$. It follows that

$$
\begin{aligned}
{ }^{G} \nabla_{X}^{\perp} \Phi\left(\phi^{f}(\xi)\right) & ={ }^{G} \nabla_{X}^{\perp} \phi^{g}(\mathcal{T}(\xi)) \\
& =\phi^{g}\left({ }^{g} \nabla_{X}^{\perp} \mathcal{T}(\xi)\right)-\left\langle\omega_{g}(X), \mathcal{T}(\xi)\right\rangle G \\
& =\phi^{g}\left(\mathcal{T}\left({ }^{f} \nabla_{X}^{\perp} \xi\right)\right)-\left\langle\mathcal{T}\left(\omega_{f}(X)\right), \mathcal{T}(\xi)\right\rangle \Phi(F) \\
& =\Phi\left(\phi^{f}\left({ }^{f} \nabla_{X}^{\perp} \xi\right)-\left\langle\omega_{f}(X), \xi\right\rangle F\right) \\
& =\Phi^{F} \nabla_{X}^{\perp} \phi^{f}(\xi)
\end{aligned}
$$

and

$$
\begin{aligned}
{ }^{G} \nabla_{X}^{\perp} \Phi\left(\zeta_{f}\right) & ={ }^{G} \nabla_{X}^{\perp} \zeta_{g} \\
& =\phi^{g}\left(\omega_{g}(X)\right) \\
& =\phi^{g}\left(\mathcal{T}\left(\omega_{f}(X)\right)\right) \\
& =\Phi\left(\phi^{f}\left(\omega_{f}(X)\right)\right) \\
& =\Phi\left({ }^{F} \nabla_{X}^{\perp} \zeta_{f}\right) .
\end{aligned}
$$

Hence $\Phi$ is parallel with respect to the normal connections of $F$ and $G$. Moreover, the Moebius third fundamental forms $I I I_{\beta}^{f}$ and $I I I_{\beta}^{g}$ coincide by the assumption on the Moebius second fundamental forms of $f$ and $g$. Hence $\psi_{g}=\psi_{f}$ by (9.41), and therefore

$$
\begin{aligned}
\alpha^{G}(X, Y) & =\phi^{g}\left(\beta^{g}(X, Y)\right)-\psi_{g}(X, Y) G-\langle X, Y\rangle \zeta_{g} \\
& =\phi^{g}\left(\mathcal{T}\left(\beta^{f}(X, Y)\right)\right)-\psi_{f}(X, Y) \Phi(F)-\langle X, Y\rangle \Phi\left(\zeta_{f}\right) \\
& =\Phi\left(\phi^{f}\left(\beta^{f}(X, Y)\right)\right)-\psi_{f}(X, Y) \Phi(F)-\langle X, Y\rangle \Phi\left(\zeta_{f}\right) \\
& =\Phi\left(\alpha^{F}(X, Y)\right) .
\end{aligned}
$$

Hence $\Phi$ also preserves the second fundamental forms of $F$ and $G$. Thus $F$ and $G$ are congruent by the uniqueness part of Theorem 1.25, and hence $f$ and $g$ are conformally congruent by Proposition 9.19.

For a hypersurface $f, g: M^{n} \rightarrow \mathbb{R}^{n+1}$, the statement of Theorem 9.22 simplifies considerably in terms of the Moebius shape operator of $f$ with respect to a unit normal vector field $N$, defined by

$$
S^{f}=\rho_{f}^{-1}\left(A_{N}^{f}-H^{f} I\right)
$$

Corollary 9.23. (Fundamental theorem of hypersurfaces in Moebius geometry)
(i) Existence: Let $\left(M^{n},\langle\rangle,\right)$ be a simply connected Riemannian manifold and let $B \in$ $\Gamma(\operatorname{End}(T M))$ be a symmetric tensor such that

$$
\operatorname{tr} B=0 \text { and }\|B\|=\sqrt{(n-1) / n}
$$

Assume that there exist a symmetric tensor $\hat{\psi} \in \Gamma(E n d(T M))$ and $\omega \in \Gamma\left(T^{*} M\right)$ such that

$$
\begin{gathered}
R(X, Y)=B X \wedge B Y+\hat{\psi} X \wedge Y+X \wedge \hat{\psi} Y \\
\left(\nabla_{X} B\right) Y-\left(\nabla_{Y} B\right) X=\omega(X) Y-\omega(Y) X \\
\left(\nabla_{X} \hat{\psi}\right) Y-\left(\nabla_{Y} \hat{\psi}\right) X=\omega(X) B Y-\omega(Y) B X
\end{gathered}
$$

and

$$
d \omega(X, Y)=\langle[\hat{\psi}, B] X, Y\rangle
$$

for all $X, Y \in \mathfrak{X}(M)$. Then there exist a hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ free of umbilic points such that $\langle$,$\rangle is the Moebius metric on M^{n}$ determined by $f$ and $B$ is the Moebius shape operator with respect to one of the smooth unit normal vector fields along $f$.
(ii) Uniqueness: Let $f, g: M^{n} \rightarrow \mathbb{R}^{n+1}$ be immersions free of umbilic points that determine the same Moebius metric on $M^{n}$. Assume that, for each $x \in M^{n}$, the Moebius shape operators $S_{N}^{f}$ and $S_{\phi(N)}^{g}$ of $f$ and $g$ at $x$ with respect to $N$ and $\phi(N)$, where $N \in N_{f} M(x)$ is a unit vector and $\phi: N_{f} M \rightarrow N_{g} M$ is one of the two possible vector bundle isometries, coincide. Then there exists a conformal transformation $\tau$ of $\mathbb{R}^{n}$ such that $\tau \circ f=g$.

### 9.9 Conformal rigidity of Euclidean submanifolds

This section provides a conformal version of Theorem 4.23, giving sufficient conditions for a conformal immersion $f: M^{n} \rightarrow \mathbb{R}^{n+p}$, with $p \leq 4$ and $n>2 p+2$, to be conformally rigid, in terms of the conformal $s$-nullities defined next.

The conformal s-nullity $\nu_{s}^{c}(x), 1 \leq s \leq p$, of an immersion $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ at $x \in M^{n}$ is defined as

$$
\nu_{s}^{c}(x)=\max \left\{\operatorname{dim} \mathcal{N}\left(\alpha_{U^{s}}^{f}(x)-\langle,\rangle \zeta\right): U^{s} \subset N_{f} M(x) \text { and } \zeta \in U_{s}\right\}
$$

where $\langle$,$\rangle stands for the metric on M^{n}$ induced by $f$.
In other words, $\nu_{s}^{c}(x)$ is the maximal dimension of a subspace $W \subset T_{x} M$ for which there exist an $s$-dimensional subspace $U^{s} \subset N_{f} M(x)$ and a normal vector $\zeta \in U^{s}$ such that

$$
\left.A_{\xi}\right|_{W}=\langle\xi, \zeta\rangle I
$$

for all $\xi \in U^{s}$. In this way, the conformal $s$-nullity is a natural extension of the maximal multiplicity of the principal curvatures of a hypersurface. It follows easily from part (i) of Exercise 9.2 that $\nu_{s}^{c}$ is invariant under conformal changes of the metric of the ambient space.

Theorem 9.24. A conformal immersion $f: M^{n} \rightarrow \mathbb{R}^{n+p}$, with $p \leq 4$ and $n>2 p+2$, is conformally rigid if $\nu_{s}^{c}(x) \leq n-2 s-1$ for all $x \in M^{n}$ and all $1 \leq s \leq p$.

Proof: Endow $M^{n}$ with the metric induced by $f$. Let $g: M^{n} \rightarrow \mathbb{R}^{n+p}$ be a conformal immersion and let

$$
G=\mathcal{J}(g)=\varphi_{g}^{-1}(\Psi \circ g): M^{n} \rightarrow \mathbb{V}^{n+p+1} \subset \mathbb{L}^{n+p+2}
$$

be its isometric light-cone representative. Thus the position vector $G$ is a light-like normal vector field along $G$ that satisfies

$$
\begin{equation*}
\left\langle\alpha^{G}(,), G\right\rangle=-\langle,\rangle \tag{9.55}
\end{equation*}
$$

and the normal bundle of $G$ splits orthogonally as

$$
N_{G} M=\Psi_{*} N_{g} M \oplus \mathbb{L}^{2}
$$

where $\mathbb{L}^{2}$ is a Lorentzian plane bundle having $G$ as a section. Hence there exist unique sections $\xi$ and $\eta$ of $\mathbb{L}^{2}$ such that

$$
G=\xi+\eta
$$

with

$$
\langle\xi, \xi\rangle=-1, \quad\langle\xi, \eta\rangle=0 \text { and }\langle\eta, \eta\rangle=1 .
$$

At $x \in M^{n}$, endow the vector space $W=N_{f} M \oplus N_{G} M$ with the inner product of signature $(p+1, p+1)$ given by

$$
\langle\langle,\rangle\rangle_{N_{f} M \oplus N_{G} M}=\langle,\rangle_{N_{f} M}-\langle,\rangle_{N_{G} M} .
$$

Then the bilinear form

$$
\beta=\alpha^{f} \oplus \alpha^{G}: T_{x} M \times T_{x} M \rightarrow W
$$

is flat by the Gauss equations of $f$ and $G$. Moreover, $\mathcal{N}(\beta)=\{0\}$ by (9.55). It follows from Lemma 4.20 that $\mathcal{S}(\beta)$ is degenerate, that is, the isotropic vector subspace

$$
\Omega=\mathcal{S}(\beta) \cap \mathcal{S}(\beta)^{\perp}
$$

is nontrivial. We claim that $\Omega$ has rank $p+1$, that is, that $\beta$ is null.
Since $\langle\langle\rangle$,$\rangle is positive definite on W_{1}=N_{f} M \oplus \operatorname{span}\{\xi\}$ and negative definite on $W_{2}=\Psi_{*} N_{g} M \oplus \operatorname{span}\{\eta\}$, the orthogonal projections $P_{1}: W \rightarrow W_{1}$ and $P_{2}: W \rightarrow W_{2}$ map $\Omega$ isomorphically onto $P_{1}(\Omega)$ and $P_{2}(\Omega)$, respectively.

We consider separately the two possible cases:
Case $\xi \notin P_{1} \Omega$. This is equivalent to requiring the orthogonal projection $\Pi_{1}: W \rightarrow N_{f} M$ to map $\Omega$ isomorphically onto $\Pi_{1}(\Omega)$. In this case, we have the orthogonal splittings

$$
N_{f} M=\Gamma_{f} \oplus \Gamma_{f}^{\perp} \text { and } N_{G} M=\Gamma_{G} \oplus \Gamma_{G}{ }^{\perp}
$$

where $\Gamma_{f}^{\perp}=\Pi_{1}(\Omega)$ and $\Gamma_{G}^{\perp}$ is the image $\Pi_{2}(\Omega)$ of $\Omega$ by the orthogonal projection $\Pi_{2}: W \rightarrow N_{G} M$, and an isometry $\mathcal{L}: \Gamma_{f}^{\perp} \rightarrow \Gamma_{G}^{\perp}$ such that

$$
\mathcal{L} \circ \Pi_{1}=\Pi_{2}, \quad \alpha_{\Gamma_{G^{\perp}}}^{G}=\mathcal{L} \circ \alpha_{\Gamma_{f}^{\perp}}^{f}
$$

and

$$
\Omega=\left\{(\gamma, \mathcal{L} \gamma): \gamma \in \Gamma_{f}^{\perp}\right\} \subset \Gamma_{f}^{\perp} \oplus \Gamma_{G}^{\perp} .
$$

Define $\hat{\beta}: T_{x} M \times T_{x} M \rightarrow \Gamma_{f} \oplus \Gamma_{G}$ by

$$
\hat{\beta}=\alpha_{\Gamma_{f}}^{f} \oplus \alpha_{\Gamma_{G}}^{G}
$$

and a vector subspace $V \subset T_{x} M$ by $V=\mathcal{N}(\hat{\beta})$.
Let $\delta$ be the orthogonal projection of the position vector $G$ onto $\Gamma_{G}^{\perp}$. We have

$$
\begin{aligned}
-\langle T, X\rangle & =\left\langle\alpha^{G}(T, X), G\right\rangle \\
& =\left\langle\alpha^{G}(T, X), \delta\right\rangle
\end{aligned}
$$

for $T \in V$ and $X \in T_{x} M$. In particular, this implies that $\delta \neq 0$. The vector $\gamma \in \Gamma_{f}^{\perp}$ defined by $\gamma=-\mathcal{L}^{-1}(\delta)$ then satisfies $\left.A_{\gamma}^{f}\right|_{V}=I$.

Now set $U=\Gamma_{f} \oplus \operatorname{span}\{\gamma\}$. We have

$$
\left(\alpha^{f}(T, X)-\langle T, X\rangle \gamma\right)_{U}=0
$$

for all $T \in V$ and $X \in T_{x} M$. If $\operatorname{dim} \Omega=r$, then $s=\operatorname{dim} U=p-r+1$ and

$$
\begin{aligned}
\operatorname{dim} V & \geq n-(p-r)-(p+2-r) \\
& =n-2 s \\
& >n-2 s-1 .
\end{aligned}
$$

This is a contradiction with the hypothesis that $\nu_{c}^{s} \leq n-2 s-1$.
Case $\xi \in P_{1} \Omega$. Let $\zeta \in \Omega$ be such that $\xi=P_{1}(\zeta)$. Then $\zeta$ is an isotropic vector in $\mathcal{S}\left(\alpha^{G}\right)^{\perp}$. Since $G \notin \mathcal{S}\left(\alpha^{G}\right)^{\perp}$ by $(9.55)$, the vectors $\zeta$ and $G$ are linearly independent, hence we can assume that $\langle\zeta, G\rangle=1$. Therefore, setting $S=\operatorname{span}\{G, \zeta\}$, we obtain

$$
\alpha_{S}(X, Y)=-\langle X, Y\rangle \zeta
$$

for all $X, Y \in T_{x} M$. Moreover, we have orthogonal splittings

$$
N_{f} M=\Gamma_{f} \oplus \Gamma_{f}^{\perp} \text { and } S^{\perp}=\Gamma_{G} \oplus \Gamma_{G}{ }^{\perp},
$$

where $\Gamma_{f}^{\perp}=P_{1}(\Omega) \cap N_{f} M$ and $\Gamma_{G}{ }^{\perp}=P_{2}(\Omega) \cap S^{\perp}$, and an isometry $\mathcal{L}: \Gamma_{f}^{\perp} \rightarrow \Gamma_{G}^{\perp}$ such that

$$
\mathcal{L} \circ P_{1}=P_{2}, \quad \alpha_{\Gamma_{G^{\perp}}}^{G}=\mathcal{L} \circ \alpha_{\Gamma_{f}^{\perp}}^{f}
$$

and

$$
\Omega=\left\{(\gamma, \mathcal{L} \gamma): \gamma \in \Gamma_{f}^{\perp}\right\} \oplus \operatorname{span}\{\zeta\} \subset \Gamma_{f}^{\perp} \oplus \Gamma_{G}^{\perp} \oplus \operatorname{span}\{\zeta\}
$$

Define $\hat{\beta}: T M \times T M \rightarrow \Gamma_{f} \oplus \Gamma_{G}$ by

$$
\hat{\beta}=\alpha_{\Gamma_{f}}^{f} \oplus \alpha_{\Gamma_{G}}^{G} .
$$

Then $\hat{\beta}$ is flat and $\mathcal{S}(\hat{\beta})$ is nondegenerate. Denote $s=\operatorname{dim} \Gamma_{f}$. Then

$$
\begin{aligned}
\operatorname{dim} \mathcal{N}\left(\alpha_{\Gamma_{f}}\right) & \geq \operatorname{dim} \operatorname{ker} \hat{\beta} \\
& \geq n-2 s \\
& >n-(2 s+1) .
\end{aligned}
$$

This is a contradiction with the hypothesis that $\nu_{c}^{s} \leq n-2 s-1$, and proves the claim.
Since $\Omega$ has rank $p+1$, we always have $\xi \in P_{1}(\Omega)$, and arguing as in the second case we see that there exists an isotropic vector $\zeta \in \mathcal{S}\left(\alpha^{G}\right)^{\perp}$ with $\langle\zeta, G\rangle=1$ and an isometry $\mathcal{L}: N_{f} M \rightarrow S$ such that $\alpha_{S}^{G}=\mathcal{L} \circ \alpha^{f}$. Hence

$$
\begin{equation*}
\alpha^{G}=\mathcal{L} \circ \alpha^{f}-\langle,\rangle \zeta . \tag{9.56}
\end{equation*}
$$

Now consider $F=\Psi \circ f: M^{n} \rightarrow \mathbb{V}^{n+p+1} \subset \mathbb{L}^{n+p+2}$. The normal bundle $N_{F} M$ of $F$ splits orthogonally as

$$
N_{F} M=\Psi_{*} N_{f} M \oplus \mathbb{L}^{2},
$$

where $\mathbb{L}^{2}$ is a Lorentzian plane bundle having $F$ as a section. The second fundamental form of $F$ splits accordingly as

$$
\begin{equation*}
\alpha^{F}=\Psi_{*} \circ \alpha^{f}-\langle,\rangle \rho \tag{9.57}
\end{equation*}
$$

where $\rho, F$ is a pseudo-orthonormal frame of $\mathbb{L}^{2}$ with $\langle\rho, \rho\rangle=0$ and $\langle\rho, F\rangle=1$. Define a vector bundle isometry $\tau: N_{F} M \rightarrow N_{G} M$ by

$$
\tau(F)=G, \tau(\rho)=\zeta \text { and } \tau \circ \Psi_{*}=\mathcal{L} .
$$

Then $\alpha^{G}=\tau \circ \alpha^{F}$ by (9.56) and (9.57). Since Lemma 4.16 holds in the Lorentzian case, $\tau$ preserves the normal connections. Hence $F$ and $G$ are congruent by the Fundamental theorem of submanifolds, and the proof now follows from Proposition 9.18.

Corollary 9.25. A conformal immersion $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 5$, is conformally rigid if $f$ has no principal curvature with multiplicity greater than $n-3$ at any point of $M^{n}$.

### 9.10 Conformal immersions of products

In this section we prove a conformal version of the decomposition Theorem 8.4 on isometric immersions of Riemannian products.

Theorem 9.26. Let $f: M^{n} \rightarrow \mathbb{R}^{m}, n \geq 3$, be a conformal immersion of a Riemannian product $M^{n}=\Pi_{i=1}^{r} M_{i}^{n_{i}}$. If the second fundamental form of $f$ is adapted to the product net of $M^{n}$, then one of the following possibilities holds:
(i) There exist an extrinsic product $\tilde{f}: M^{n} \rightarrow \mathbb{R}^{m}$ of isometric immersions, a homothety of $\mathbb{R}^{m}$ and an inversion I in $\mathbb{R}^{m}$ with respect to a sphere of unit radius such that

$$
f=I \circ H \circ \tilde{f}
$$

(ii) After possibly relabeling factors, there exist a substantial isometric immersion $f_{1}: M_{1} \rightarrow \mathbb{H}_{-c}^{k}$ and an extrinsic product $\tilde{f}: \Pi_{i=2}^{r} M_{i} \rightarrow \mathbb{S}_{c}^{m-k}$ of isometric immersions such that

$$
f=\Theta \circ\left(f_{1} \times \tilde{f}\right)
$$

where $\Theta: \mathbb{H}_{-c}^{k} \times \mathbb{S}_{c}^{m-k} \rightarrow \mathbb{R}^{m}$ is the conformal diffeomorphism in Examples 9.10.
(iii) There exist an extrinsic product $\tilde{f}: M^{n} \rightarrow \mathbb{S}_{c}^{m}$ of isometric immersions and a conformal diffeomorphism $\tau: \mathbb{S}_{c}^{m} \rightarrow \mathbb{R}^{m}$ (with one point of $\mathbb{S}_{c}^{m}$ removed) such that

$$
f=\tau \circ \tilde{f}
$$

Proof: Let $F=\mathcal{J}_{v, w, C}(f): M^{n} \rightarrow \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$ be the isometric light-cone representative of $f$ and let $\mathcal{E}=\left(E_{i}\right)_{i=1, \ldots, r}$ be the product net of $M^{n}$. We first prove the following fact.

Lemma 9.27. The second fundamental form of $F$ is adapted to $\mathcal{E}$.
Proof: It suffices to consider the case $r=2$, and then, relabeling the factors if necessary, we may assume that $n_{1} \geq 2$. Fixed $x=\left(x_{1}, x_{2}\right) \in M^{n}$ and $\hat{X} \in E_{2}(x)$, denote $L=M_{1}^{n_{1}} \times\left\{x_{2}\right\}$, let $\bar{X}=\pi_{2 *}(x)(\hat{X}) \in T_{x_{2}} M_{2}$ and, for any $y \in L$, let $\hat{X}(y)$ be the unique vector in $E_{2}(y)$ that projects to $\bar{X}$ by $\pi_{2 *}(y)$. Then $\hat{X}$ is a parallel vector field along $L$ with respect to the induced connection on $j^{*} T M$, where $j: L \rightarrow M^{n}$ is the inclusion. Hence, denoting by $\tilde{\nabla}$ the connection of $\mathbb{L}^{m+2}$, we have

$$
\tilde{\nabla}_{X} F_{*} \hat{X}=\alpha^{F}(X, \hat{X})=\omega(X) F
$$

with

$$
\omega(X)=\varphi \operatorname{Hess} \varphi^{-1}(X, \hat{X})
$$

for any $X \in \mathfrak{X}(L)$, where the second equality follows from (9.32) and the assumption that the second fundamental form of $f$ is adapted to $\mathcal{E}$. Therefore

$$
\begin{aligned}
0 & =\tilde{R}(X, Y) F_{*} \hat{X} \\
& =\tilde{\nabla}_{Y} \tilde{\nabla}_{X} F_{*} \hat{X}-\tilde{\nabla}_{X} \tilde{\nabla}_{Y} F_{*} \hat{X}-\tilde{\nabla}_{[X, Y]} F_{*} \hat{X} \\
& =d \omega(X, Y) F+\omega(X) F_{*} Y-\omega(Y) F_{*} X
\end{aligned}
$$

for all $X, Y \in \mathfrak{X}(L)$, where $\tilde{R}$ is the curvature tensor of $\mathbb{L}^{m+2}$. Choosing $X, Y$ linearly independent, the vector fields $F, F_{*} X$ and $F_{*} Y$ are also linearly independent, because $F$ is an immersion and the position vector $F$ is a nonzero normal vector field. Hence the preceding equation implies that $\omega$ vanishes. Thus

$$
\alpha^{F}(X, \hat{X})=0
$$

for all $X \in \mathfrak{X}(L)$.
In view of Lemma 9.27, we can apply Theorem 8.7 to $F$. Assume first that the assertion in case $(i)$ of that result holds for $F$. Namely, there exist an orthogonal decomposition

$$
\mathbb{L}^{m+2}=\mathbb{L}^{m_{1}} \times \Pi_{i=2}^{r+1} \mathbb{R}^{m_{i}}
$$

with $\mathbb{R}^{m_{r+1}}$ possibly trivial, a vector $\bar{v} \in \mathbb{R}^{m_{r+1}}$ (in case $\mathbb{R}^{m_{r+1}}$ is nontrivial) and substantial isometric immersions $F_{1}: M_{1} \rightarrow \mathbb{L}^{m_{1}}$ and $F_{i}: M_{i} \rightarrow \mathbb{R}^{m_{i}}, 2 \leq i \leq r$, such that

$$
\begin{equation*}
F\left(x_{1}, \ldots, x_{r}\right)=\left(F_{1}\left(x_{1}\right), \ldots, F_{r}\left(x_{r}\right), \bar{v}\right) . \tag{9.58}
\end{equation*}
$$

From (9.58) and $\langle F, F\rangle=0$ we obtain

$$
\sum_{i=1}^{r}\left\langle F_{i} \circ \pi_{i}, F_{i} \circ \pi_{i}\right\rangle+\langle\bar{v}, \bar{v}\rangle=0 .
$$

It follows that $\left\langle F_{i}, F_{i}\right\rangle$ is constant for $1 \leq i \leq r$, say,

$$
\left\langle F_{1}, F_{1}\right\rangle=-r_{1}^{2} \text { and }\left\langle F_{i}, F_{i}\right\rangle=r_{i}^{2}, \quad 2 \leq i \leq r
$$

with

$$
\begin{equation*}
r_{1}^{2}=\sum_{i=2}^{r} r_{i}^{2}+\langle\bar{v}, \bar{v}\rangle \tag{9.59}
\end{equation*}
$$

Hence there exist isometric immersions $f_{1}: M_{1} \rightarrow \mathbb{H}^{m_{1}-1}\left(r_{1}\right)$ and $f_{j}: M_{j} \rightarrow \mathbb{S}^{m_{j}-1}\left(r_{j}\right)$, $2 \leq j \leq r$, such that

$$
F_{j}=i_{j} \circ f_{j}, \quad 1 \leq j \leq r,
$$

where $i_{1}: \mathbb{H}^{m_{1}-1}\left(r_{1}\right) \rightarrow \mathbb{L}^{m_{1}}$ and $i_{j}: \mathbb{S}^{m_{j}-1}\left(r_{j}\right) \rightarrow \mathbb{R}^{m_{j}}, 2 \leq j \leq r$, are umbilical inclusions. Set $k=m_{1}-1, c=1 / r_{1}^{2}$ and define $\tilde{f}: \Pi_{i=2}^{r} M_{i} \rightarrow \mathbb{R}^{m-m_{1}+2}$ by

$$
\tilde{f}\left(x_{2}, \ldots, x_{r}\right)=\left(f_{2}\left(x_{2}\right), \ldots, f_{r}\left(x_{r}\right), \bar{v}\right)
$$

for all $\left(x_{2}, \ldots, x_{r}\right) \in \Pi_{i=2}^{r} M_{i}$. In view of (9.59), the map $\tilde{f}$ is the extrinsic product of $f_{2}, \ldots, f_{r}$ into $\mathbb{S}_{c}^{m-k}$. We conclude that

$$
f=\mathcal{C}(F)=\Theta \circ\left(f_{1} \times \tilde{f}\right)
$$

where $\Theta: \mathbb{H}_{-c}^{k} \times \mathbb{S}_{c}^{m-k} \rightarrow \mathbb{R}^{m}$ is the conformal diffeomorphism defined in Examples 9.10 . Thus $f$ is as in part (ii) of the statement.

Now suppose that $F$ is given as in part (ii) of Theorem 8.7. Thus there exist an orthogonal decomposition

$$
\begin{equation*}
\mathbb{L}^{m+2}=\Pi_{i=1}^{r} \mathbb{R}^{m_{i}} \times \mathbb{L}^{m_{r+1}} \tag{9.60}
\end{equation*}
$$

a vector $\bar{v} \in \mathbb{L}^{m_{r+1}}$, and substantial isometric immersions $F_{i}: M_{i} \rightarrow \mathbb{R}^{m_{i}}, 1 \leq i \leq r$, such that

$$
\begin{equation*}
F\left(x_{1}, \ldots, x_{r}\right)=\left(F_{1}\left(x_{1}\right), \ldots, F_{r}\left(x_{r}\right), \bar{v}\right) . \tag{9.61}
\end{equation*}
$$

Using that $\langle F, F\rangle=0$, it follows from (9.61) that $\left\langle F_{i}, F_{i}\right\rangle=r_{i}^{2}, 1 \leq i \leq r$, with

$$
\sum_{i=1}^{r} r_{i}^{2}+\langle\bar{v}, \bar{v}\rangle=0
$$

Denote

$$
\mathbb{R}^{m_{r+1}-1}=\operatorname{span}\{\bar{v}\}^{\perp} \subset \mathbb{L}^{m_{r+1}}, \quad \mathbb{R}^{m+1}=\Pi_{i=1}^{r} \mathbb{R}^{m_{i}} \times \mathbb{R}^{m_{r+1}-1} \subset \mathbb{L}^{m+2}
$$

and write $F_{j}=i_{j} \circ f_{j}$, where $f_{j}: M_{j} \rightarrow \mathbb{S}^{m_{j}-1}\left(r_{j}\right)$ is an isometric immersion and $i_{j}: \mathbb{S}^{m_{j}-1}\left(r_{j}\right) \rightarrow \mathbb{R}^{m_{j}}$ is an umbilical inclusion for $1 \leq j \leq r$. Then 9.61 can be written as

$$
F=T_{B, \bar{v}} \circ \tilde{f},
$$

where $T_{B, \bar{v}}: \mathbb{S}_{c}^{m} \rightarrow \mathbb{L}^{m+2}$ is given by 9.6 for $c=-\langle\bar{v}, \bar{v}\rangle^{-1}$ and the inclusion map $B: \mathbb{R}^{m+1}=\operatorname{span}\{\bar{v}\}^{\perp} \rightarrow \mathbb{L}^{m+2}$, and $\tilde{f}: M^{n} \rightarrow \mathbb{S}_{c}^{m} \subset \mathbb{R}^{m+1}$ is the extrinsic product of $f_{1}, \ldots, f_{r}$ given by

$$
\tilde{f}\left(x_{1}, \ldots, x_{r}\right)=\left(F_{1}\left(x_{1}\right), \ldots, F_{r}\left(x_{r}\right), 0\right)
$$

Therefore

$$
f=\mathcal{C}(F)=\tau \circ \tilde{f}
$$

where $\tau=\mathcal{C}\left(T_{B, \bar{v}}\right)$, that is, $f$ is the composition of an extrinsic product of immersions into $\mathbb{S}_{c}^{m}$ with a conformal diffeomorphism of $\mathbb{S}_{c}^{m}$ (minus one point) onto $\mathbb{R}^{m}$.

Finally, suppose that $F$ satisfies the conclusion in part (iii) of Theorem 8.7. In this case, there exist $1 \leq s \leq r$, orthogonal decompositions

$$
\mathbb{L}^{m_{1}}=\Pi_{i=1}^{s} \mathbb{R}^{m_{i}} \times \mathbb{L}^{2} \text { and } \mathbb{L}^{m+2}=\mathbb{L}^{m_{1}} \times \Pi_{i=s+1}^{r+1} \mathbb{R}^{m_{i}}
$$

with $\mathbb{R}^{m_{r+1}}$ possibly trivial, a vector $\bar{v} \in \mathbb{R}^{m_{r+1}}$ (in case $\mathbb{R}^{m_{r+1}}$ is nontrivial), a function $\varphi \in C^{\infty}\left(M_{1} \times \cdots \times M_{s}\right)$, substantial isometric immersions $f_{i}: M_{i} \rightarrow \mathbb{R}^{m_{i}}, 1 \leq i \leq r$, a pseudo-orthonormal basis $\zeta, \bar{\zeta}$ of $\mathbb{L}^{2}$ with $\langle\zeta, \zeta\rangle=0=\langle\bar{\zeta}, \bar{\zeta}\rangle$ and $\langle\zeta, \bar{\zeta}\rangle=1$, and $\delta \in\{0,1\}$ such that

$$
F\left(x_{1}, \ldots, x_{r}\right)=\left(g\left(x_{1}, \ldots, x_{s}\right), f_{s+1}\left(x_{s+1}\right), \ldots, f_{r}\left(x_{r}\right), \bar{v}\right),
$$

where

$$
g\left(x_{1}, \ldots, x_{s}\right)=\left(f_{1}\left(x_{1}\right), \ldots, f_{s}\left(x_{s}\right), \varphi\left(x_{1}, \ldots, x_{s}\right) \zeta+\delta \bar{\zeta}\right)
$$

Since $\langle F, F\rangle=0$, the case $\delta=0$ is ruled out and we obtain

$$
2 \varphi+\sum_{i=1}^{r}\left\langle f_{i} \circ \pi_{i}, f_{i} \circ \pi_{i}\right\rangle+\langle\bar{v}, \bar{v}\rangle=0
$$

In particular, $\left\langle f_{i}, f_{i}\right\rangle=r_{i}^{2}$ is constant for $s+1 \leq i \leq r$, that is, $f_{i}$ takes values in $\mathbb{S}^{m_{i}-1}\left(r_{i}\right)$ for $s+1 \leq i \leq r$, and we can write

$$
F=\sum_{j=1}^{r} f_{j} \circ \pi_{j}+\bar{\zeta}-\frac{1}{2}\left(\sum_{j=1}^{r}\left\langle f_{j} \circ \pi_{j}, f_{j} \circ \pi_{j}\right\rangle+\langle\bar{v}, \bar{v}\rangle\right) \zeta+\bar{v} .
$$

Let $\Psi=\Psi_{v, w, C}$ and let $T \in O_{1}(m+2)$ be defined by

$$
T w=\zeta, \quad T v=\bar{\zeta}
$$

and by requiring that $T \circ C$ be an isometry of $\mathbb{R}^{m}$ onto $\prod_{i=1}^{r+1} \mathbb{R}^{m_{i}}$. Then

$$
F=T \circ \Psi \circ \tilde{f},
$$

where $\tilde{f}: M^{n} \rightarrow \mathbb{R}^{m}=\Pi_{i=1}^{r+1} \mathbb{R}^{m_{i}}$ is the extrinsic product of $f_{1}, \ldots, f_{r}$ given by

$$
\tilde{f}\left(x_{1}, \ldots, x_{r}\right)=\left(f_{1}\left(x_{1}\right), \ldots, f_{r}\left(x_{r}\right), \bar{v}\right)
$$

It follows from Proposition 9.12 that there exist a similarity $L$ and an inversion $I$ with respect to a hypersphere of unit radius such that

$$
f=\mathfrak{C}(F)=\mathcal{C}(T \circ \Psi \circ \tilde{f})=\mathcal{C}(T \circ \Psi) \circ \tilde{f}=I \circ L \circ \tilde{f}
$$

Remark 9.28. Had we not required the isometric immersion $f_{1}: M_{1} \rightarrow \mathbb{H}_{-c}^{k}$ in part (ii) of the statement of Theorem 9.26 to be substantial, then any conformal immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ as in part (iii) could also be given as in part (ii).

Indeed, for such a conformal immersion $f$, its isometric light-cone representative $F: M^{n} \rightarrow \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$ is given by 9.61 with respect to an orthogonal decomposition as in 9.60, with $\left\langle F_{i}, F_{i}\right\rangle=r_{i}^{2}, 1 \leq i \leq r$, and

$$
\sum_{i=1}^{r} r_{i}^{2}+\langle\bar{v}, \bar{v}\rangle=0
$$

Then one can write $f$ as in part (ii) of the statement in different ways. For instance, choose any $1 \leq i \leq r$, say, $i=1$, write

$$
\mathbb{L}^{k+1}=\mathbb{R}^{m_{1}} \oplus \mathbb{L}^{m_{r+1}}
$$

and define $\tilde{F}_{1}: M_{1} \rightarrow \mathbb{L}^{k+1}$ by

$$
\tilde{F}_{1}\left(x_{1}\right)=\left(F_{1}\left(x_{1}\right), \bar{v}\right) .
$$

Notice that $\tilde{F}_{1}\left(M_{1}\right) \subset \mathbb{H}^{k}\left(\tilde{r}_{1}\right)$, where

$$
\begin{aligned}
\tilde{r}_{1}^{2} & =-r_{1}^{2}-\langle\bar{v}, \bar{v}\rangle \\
& =\sum_{i=2}^{r} r_{i}^{2} .
\end{aligned}
$$

Thus we have an orthogonal decomposition

$$
\mathbb{L}^{m+2}=\mathbb{L}^{k+1} \times \Pi_{i=2}^{r} \mathbb{R}^{m_{i}}
$$

with respect to which $F$ decomposes as

$$
F\left(x_{1}, \ldots, x_{r}\right)=\left(\tilde{F}_{1}\left(x_{1}\right), F_{2}\left(x_{2}\right), \ldots, F_{r}\left(x_{r}\right)\right) .
$$

Moreover, there are isometric immersions $\tilde{f}_{1}: M_{1} \rightarrow \mathbb{H}^{k}\left(\tilde{r}_{1}\right), f_{j}: M_{j} \rightarrow \mathbb{S}^{m_{j}-1}\left(r_{j}\right)$, $2 \leq j \leq r$, such that

$$
\tilde{F}_{1}=i_{1} \circ \tilde{f}_{1} \text { and } F_{j}=i_{j} \circ f_{j}, \quad 2 \leq j \leq r
$$

where $i_{1}: \mathbb{H}^{k}\left(\tilde{r}_{1}\right) \rightarrow \mathbb{L}^{k+1}$ and $i_{j}: \mathbb{S}^{m_{j}-1}\left(r_{j}\right) \rightarrow \mathbb{R}^{m_{j}}, 2 \leq j \leq r$, are umbilical inclusions. Set $c=1 / \tilde{r}_{1}^{2}$ and let $\tilde{f}: \Pi_{i=2}^{r} M_{i} \rightarrow \mathbb{R}^{m-k+1}$ be defined by

$$
\tilde{f}\left(x_{2}, \ldots, x_{r}\right)=\left(f_{2}\left(x_{2}\right), \ldots, f_{r}\left(x_{r}\right)\right)
$$

for all $\left(x_{2}, \ldots, x_{r}\right) \in \prod_{i=2}^{r} M_{i}$. The map $\tilde{f}$ is then the extrinsic product of $f_{2}, \ldots, f_{r}$ into $\mathbb{S}_{c}^{m-k}$, and

$$
f=\mathcal{C}(F)=\Theta \circ\left(\tilde{f}_{1} \times \tilde{f}\right)
$$

Theorem 9.26 in case $n=m$ gives a classification of all conformal representations of Euclidean space of dimension $n \geq 3$ as a Riemannian product, that is, all conformal maps of a Riemannian product of dimension $n \geq 3$ into $\mathbb{R}^{n}$.

Corollary 9.29. Let $f: M^{n}=\Pi_{i=1}^{r} M_{i}^{n_{i}} \rightarrow \mathbb{R}^{n}, n \geq 3$, be a conformal map. Then one of the following possibilities holds:
(i) There exist an isometry $\Phi: \Pi_{i=1}^{r} \mathbb{R}^{n_{i}} \rightarrow \mathbb{R}^{n}$, local isometries $f_{i}: M_{i}^{n_{i}} \rightarrow \mathbb{R}^{n_{i}}$, an inversion I with respect to a sphere of unit radius and a homothety $H$ in $\mathbb{R}^{n}$ such that

$$
f=I \circ H \circ \Phi \circ\left(f_{1} \times \cdots \times f_{k}\right) .
$$

(ii) $r=2$ and, after relabeling the factors if necessary, there exist local isometries $f_{1}: M_{1}^{n_{1}} \rightarrow \mathbb{H}_{-c}^{n_{1}}$ and $f_{2}: M_{2}^{n_{2}} \rightarrow \mathbb{S}_{c}^{n_{2}}$ such that

$$
f=\Theta \circ\left(f_{1} \times f_{2}\right)
$$

Remark 9.30. Notice that Corollary 9.29 reduces to Corollary 9.15 when $r=n$.
When applying Theorem 9.26, one often needs first to show that a given Riemannian manifold is conformal to a Riemannian product. More precisely, given an orthogonal net $\mathcal{E}=\left(E_{i}\right)_{i=1, \ldots, r}$ on a Riemannian manifold, one must show that there exists, at least locally, a product representation $\psi: \Pi_{i=1}^{r} M_{i} \rightarrow U$ of $\mathcal{E}$ that is conformal with respect to a Riemannian product metric on $\Pi_{i=1}^{r} M_{i}$. To state a criterion for this to hold one needs the following notion of a conformal product net on a Riemannian manifold.

An orthogonal net $\mathcal{E}=\left(E_{i}\right)_{i=1, \ldots, r}$ on a Riemannian manifold is a conformal product net if

$$
E_{i} \text { and } E_{i}^{\perp} \text { are umbilical and }\left\langle\nabla_{X_{\perp_{i}}} \eta_{i}, X_{i}\right\rangle=\left\langle\nabla_{X_{i}} H_{i}, X_{\perp_{i}}\right\rangle
$$

for all $X_{i} \in \Gamma\left(E_{i}\right)$ and $X_{\perp_{i}} \in \Gamma\left(E_{i}^{\perp}\right), 1 \leq i \leq r$, where $H_{i}$ and $\eta_{i}$ are the mean curvature vector fields of $E_{i}$ and $E_{i}^{\perp}$, respectively.

Proposition 9.31. On a connected and simply connected product manifold $M=$ $\Pi_{i=1}^{r} M_{i}$ the product net $\mathcal{E}=\left(E_{i}\right)_{i=1, \ldots, r}$ is a conformal product net with respect to a Riemannian metric $\langle,\rangle^{\sim}$ on $M$ if and only if $\langle,\rangle^{\sim}$ is conformal to a Riemannian product metric.

We will not give the proof of Proposition 9.31, nor that of the following conformal version of the local de Rham's Theorem. A reference where both proofs can be found is provided in the Notes to this chapter.

Theorem 9.32. If a Riemannian manifold $M^{n}$ carries a conformal product net $\mathcal{E}=\left(E_{i}\right)_{i=1, \ldots, r}$, then there exists locally a product representation $\psi: \Pi_{i=1}^{r} M_{i} \rightarrow M^{n}$ of $\mathcal{E}$ which is conformal with respect to a Riemannian product metric on $\Pi_{i=1}^{r} M_{i}$.

The next consequence of Theorem 9.26 will be needed in the classification of conformally deformable hypersurfaces given in Chapter 17.

Corollary 9.33. Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be an isometric immersion that carries a Dupin principal normal vector field $\eta$ with multiplicity $k$. Assume that $E_{\eta}^{\perp}$ is an umbilical distribution. If $k=n-1$, suppose further that the integral curves of $E_{\eta}^{\perp}$ are extrinsic circles of $M^{n}$. Then $f(M)$ is, up to a conformal transformation of $\mathbb{R}^{m}$, an open subset of a submanifold of one of the following types:
(i) A k-cylinder.
(ii) A $(k-1)$-cylinder over an isometric immersion $G: M^{n-k+1} \rightarrow \mathbb{R}^{m-k+1}$ which is itself a cone over an isometric immersion $g: M^{n-k} \rightarrow \mathbb{S}^{m-k}$.
(iii) A rotation submanifold over an isometric immersion $h: M^{n-k} \rightarrow \mathbb{R}^{m-k}$.

Proof: Since $\eta$ is a Dupin principal normal vector field, the distribution $E_{\eta}$ is spherical by Proposition 1.22 . On the other hand, by Exercise 1.32 and the assumption for the case $k=n-1$, also $E_{\eta}^{\perp}$ is spherical. Thus $\left(E_{\eta}, E_{\eta}^{\perp}\right)$ is a conformal product net. By Theorem 9.32, for each $x \in M^{n}$ there exists a local product representation $\phi: M_{1} \times M_{2} \rightarrow W$ of $\left(E_{\eta}, E_{\eta}^{\perp}\right)$ onto an open neighborhood $W$ of $x$ which is a conformal diffeomorphism with respect to a product metric on $M_{1} \times M_{2}$. Applying Theorem 9.26 to $f \circ \phi$ implies that one of the following possibilities holds:
(a) There exist an orthogonal decomposition $\mathbb{R}^{m}=\mathbb{R}^{s} \times \mathbb{R}^{m-s}$, a conformal transformation $T$ of $\mathbb{R}^{m}$ and isometric immersions $g: M_{1} \rightarrow \mathbb{R}^{s}$ and $h: M_{2} \rightarrow \mathbb{R}^{m-s}$ such that

$$
f \circ \phi=T \circ \Phi \circ(g \times h) .
$$

(b) There exist isometric immersions $h: M_{1} \rightarrow \mathbb{H}^{m-s}$ and $g: M_{2} \rightarrow \mathbb{S}^{s}$ such that

$$
f \circ \phi=\Theta \circ(h \times g),
$$

where $\Theta: \mathbb{H}^{m-s} \times \mathbb{S}^{s} \rightarrow \mathbb{R}^{m}$ is the conformal diffeomorphism in Examples 9.10.
Moreover, since $f$ carries a Dupin principal normal vector field $\eta$ with multiplicity $k$, we can assume that either ( $a$ ) or (b) holds with $s=k$ and $g=$ id, or that (b) holds with $m-s=k$ and $h=\mathrm{id}$. Each of these possibilities implies that $f(W)$ is an open subset of a submanifold of one the three types in the statement. The proof now follows by applying Exercise 1.20 to the class of submanifolds that are of one of those three types.

### 9.11 Notes

The conformal (Moebius) geometry of submanifolds has been of great interest to differential geometers since the end of the nineteenth century. We refer the reader to books [4] and [220] for an account of several aspects of the subject.

The observation that, in the absence of umbilic points, there is a unique metric in the conformal class on the submanifold with respect to which the trace free second fundamental form has a given constant length goes back to Fialkow [180]. This observation allowed him to give a treatment of generic conformal submanifold geometry in purely Riemannian terms. Our treatment of the conformal Bonnet theorem in terms of the Moebius metric, Moebius second fundamental form, Moebius one-form and Blaschke tensor of a submanifold is based on the paper by Wang [342], where those concepts were introduced and the uniqueness assertion on hypersurfaces in Corollary 9.23 was proven. The existence and uniqueness Theorem 9.22 on immersions of arbitrary dimension and codimension can be regarded as the counterpart of Theorem 1.10 within the context of Moebius geometry.

A more general approach to the conformal Bonnet theorem, building upon the fundamental work by Cartan [68, was developed by Burstall-Calderbank [45], which in particular avoids the restriction on the umbilic points. Also see [45] for an account of previous conformal versions of the Gauss-Codazzi-Ricci equations and of the Bonnet theorem, where the corresponding references may be found.

The discussion in Section 9.5 on the equivalence between the rigidity of the paraboloid model of Euclidean space and Liouville's theorem on conformal mappings on open subsets of Euclidean space of dimension $n \geq 3$ is based on Tojeiro 333].

Theorem 9.24 on the conformal rigidity of Euclidean submanifolds was obtained by do Carmo-Dajczer in [59], where the notion of the conformal $s$-nullities of a conformal immersion was introduced. Corollary 9.25 was proved by Cartan [65] as part of his classification of conformally deformable hypersurfaces of dimension $n \geq 5$ of $\mathbb{R}^{n+1}$, which will be discussed in Chapter 17 .

Theorem 9.26 on conformal immersions of Riemannian products into Euclidean space and Theorem 9.32 on the conformal version of the local de Rham Theorem were obtained by Tojeiro [330], 332, respectively. Corollary 9.33 was first proved by Dajczer-Tojeiro 141 with different arguments.

The conformal version given in Exercise 9.8 of Tompkin's Corollary 4.12 is due to Moore [257].

The study of conformal submanifold geometry will be pursued further in Chapters 16 and 17, which will be devoted to two important families of Euclidean submanifolds that are invariant under Moebius transformations, namely, conformally flat submanifolds of dimension $n \geq 3$ and hypersurfaces of dimension $n \geq 5$ that admit nontrivial conformal deformations.

Another remarkable Moebius invariant family of Euclidean submanifolds consists of the so-called Wintgen ideal submanifolds. They arise in connection with the following pointwise inequality relating intrinsic and extrinsic invariants of a submanifold of a space form. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ be an isometric immersion of a Riemannian manifold. At any point $x \in M^{n}$, let $s(x)$ denote the scalar curvature of $M^{n}$ at $x$ and let $s_{N}(x)$ be given by

$$
n(n-1) s_{N}(x)=\left\|R^{\perp}(x)\right\|=2\left(\sum_{\substack{1 \leq i<j \leq n \\ 1 \leq r<s \leq p}}\left\langle R^{\perp}\left(X_{i}, X_{j}\right) \xi_{r}, \xi_{s}\right\rangle^{2}\right)^{1 / 2}
$$

where $X_{1}, \ldots, X_{n}$ and $\xi_{1}, \ldots, \xi_{p}$ are orthonormal bases of $T_{x} M$ and $N_{f} M(x)$, respectively. It was shown independently by Ge-Tang [199] and Lu [239], after work by several authors on special cases, that the inequality

$$
s(x) \leq c+\|\mathcal{H}(x)\|^{2}-s_{N}(x)
$$

holds at any $x \in M^{n}$. For $n=2=p$ and $c=0$, it reduces to the inequality

$$
K(x)+\left|K_{N}(x)\right| \leq\|\mathcal{H}(x)\|^{2}
$$

between the Gaussian curvature $K(x)$, the absolute value of the normal curvature $K_{N}(x)$ and the length of the mean curvature vector $\mathcal{H}(x)$ at $x$. This particular case of the inequality was proved by Wintgen [345], and for this reason, isometric immersions $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ for which the equality in the general inequality is attained at any point became known as Wintgen ideal submanifolds.

One class of Wintgen ideal submanifolds consists of minimal isometric immersions $f: M^{n} \rightarrow \mathbb{R}^{m}$ with index of relative nullity $\nu=n-2$, which have been studied by Dajczer-Florit 97 a and, more generally, of the compositions of minimal isometric immersions $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ with index of relative nullity $\nu=n-2$ with a conformal diffeomorphism between $\mathbb{Q}_{c}^{m}$ and $\mathbb{R}^{m}$. Other trivial examples of Wintgen ideal submanifolds are the umbilical ones.

Wintgen ideal submanifolds $f: M^{n} \rightarrow \mathbb{R}^{n+2}, n \geq 3$, that are free of minimal and umbilic points were classified by Dajczer-Tojeiro [146] as follows. Start with a simply connected minimal surface $g: M^{2} \rightarrow \mathbb{R}^{n+2}$, oriented by a global conformal diffeomorphism onto either the complex plane or the unit disk. Then consider its conjugate minimal surface $h: M^{2} \rightarrow \mathbb{R}^{n+2}$, each of whose components with respect to this global parameter is the harmonic conjugate of the corresponding component of $g$. Equivalently, $h_{*}=g_{*} \circ J$, where $J$ is the complex structure on $M^{2}$ compatible with its orientation. Now decompose the position vector of $h$ in its tangent and normal components with respect to $g$, that is,

$$
h=g_{*} h^{T}+h^{N} .
$$

Finally, on the complement of the subset of isolated points of $M$ where $h^{N}$ vanishes, let $\Lambda_{1}$ be the unit bundle of the vector subbundle $\Lambda$ of the normal bundle of $g$ that is orthogonal to $h^{N}$.

It was shown in [146] that the restriction of the map $\phi: \Lambda_{1} \rightarrow \mathbb{R}^{n+2}$, defined by

$$
\phi(y, w)=g(y)+g_{*} J h^{T}(y)+\left\|h^{N}(y)\right\| w,
$$

to the subset of its regular points, parametrizes an $n$-dimensional Wintgen ideal submanifold of $\mathbb{R}^{n+2}$. Conversely, any Wintgen ideal submanifold $f: M^{n} \rightarrow \mathbb{R}^{n+2}, n \geq 3$, free of umbilical and minimal points, can be parameterized in this way.

Wintgen ideal surfaces $f: M^{2} \rightarrow \mathbb{R}^{4}$ are precisely the surfaces in $\mathbb{R}^{4}$ whose ellipses of curvature

$$
E(x)=\left\{\alpha(X, X): X \in T_{x} M \text { and }\|X\|=1\right\}
$$

at all points $x \in M^{2}$ are circles, and are also known as superconformal surfaces. In this case, it was shown in [147] that if $g$ and $h$ are conjugate minimal surfaces as before, $\hat{J}_{+}$ and $\hat{J}_{-}$are the two possible complex structures on $N_{g} M$ and $\mathcal{J}_{+}, \mathcal{J}_{-}$are the complex structures on $g^{*} T \mathbb{R}^{4}$ given by

$$
\mathcal{J}_{ \pm} \circ g_{*}=g_{*} \circ J \text { and }\left.\mathcal{J}_{ \pm}\right|_{T_{g} \perp M}=\hat{J}_{ \pm}
$$

then each of the maps $\phi_{ \pm}: M^{2} \rightarrow \mathbb{R}^{4}$ defined by

$$
\phi_{ \pm}=g+\mathcal{J}_{ \pm} h
$$

parametrizes, at regular points, a superconformal surface. Conversely, any simply connected superconformal surface that is free of minimal and umbilical points can be constructed in this way.

We point out that an alternative description of $n$-dimensional Wintgen ideal submanifolds of $\mathbb{R}^{n+2}$, $n \geq 3$, was given by Li-Ma-Wang-Xie [237]. It is an open problem to classify Wintgen ideal submanifolds of arbitrary codimension. We refer to Xie-Li-Ma-Wang [351] and the references therein for significant contributions in this direction.

### 9.12 Exercises

Exercise 9.1. Let $g_{1}$ and $g_{2}$ be conformal (pseudo)-Riemannian metrics on a differentiable manifold $M^{n}$ and let $\lambda \in C^{\infty}(M)$ be the conformal factor of $g_{2}$ with respect to $g_{1}$, that is, $\lambda$ is a positive smooth function such that $g_{2}=\lambda^{2} g_{1}$. Show that the Levi-Civita connections $\nabla^{1}$ and $\nabla^{2}$ of $g_{1}$ and $g_{2}$, respectively, are related by

$$
\nabla_{X}^{2} Y=\nabla_{X}^{1} Y+\frac{1}{\lambda}\left(Y(\lambda) X+X(\lambda) Y-g_{1}(X, Y) \operatorname{grad}_{1} \lambda\right)
$$

for all $X, Y \in \mathfrak{X}(M)$.
Exercise 9.2. Let $f: M^{n} \rightarrow \tilde{M}^{m}$ be an immersion and let $g_{1}$ and $g_{2}$ be conformal metrics on $\tilde{M}^{m}$. Denote $f_{j}=f:\left(M^{n}, f^{*} g_{j}\right) \rightarrow\left(\tilde{M}^{m}, g_{j}\right), 1 \leq j \leq 2$.
(i) Show that the second fundamental forms of $f_{1}$ and $f_{2}$ are related by

$$
\alpha^{f_{2}}(X, Y)=\alpha^{f_{1}}(X, Y)-\frac{1}{\lambda} g_{1}(X, Y)\left(\operatorname{grad}_{1} \lambda\right)^{\perp}
$$

for all $x \in M^{n}$ and $X, Y \in T_{x} M$, where $\lambda \in C^{\infty}(\tilde{M})$ is the conformal factor of $g_{2}$ with respect to $g_{1}$ and $\operatorname{grad}_{1} \lambda$ denotes the gradient of $\lambda$ with respect to $g_{1}$.
(ii) If $\eta$ is a principal normal vector of $f_{1}$ at $x \in M^{n}$, conclude that

$$
\eta-\frac{1}{\lambda}\left(\operatorname{grad}_{1} \lambda\right)^{\perp}
$$

is a principal normal vector of $f_{2}$ at $x$.
(iii) Show that the normal connections $f_{1}$ and $f_{2}$ are related by

$$
\nabla_{X}^{2 \perp} \xi=\nabla_{X}^{1 \perp} \xi+\frac{X(\lambda)}{\lambda} \xi
$$

for all $X \in \mathfrak{X}(M)$ and $\xi \in \Gamma\left(N_{f} M\right)$.
(iv) Show that the normal curvature tensors $R_{1}^{\perp}$ and $R_{2}^{\perp}$ of $f_{1}$ and $f_{2}$, respectively, are related by

$$
R_{1}^{\perp}(X, Y) \xi=R_{2}^{\perp}(X, Y) \xi
$$

for all $x \in M^{n}, X, Y \in T_{x} M$ and $\xi \in N_{f} M(x)$.
Exercise 9.3. Let $f, g: M^{n} \rightarrow \mathbb{R}^{m}$ be immersions with the same generalized Gauss map, that is, there exists $\Phi \in \Gamma(\operatorname{End}(T M))$ such that $g_{*}=f_{*} \circ \Phi$ (see Exercise 1.25). Assume, in addition, that $f$ and $g$ are conformal, that is, there exists $\phi \in C^{\infty}(M)$ such that the metrics induced by $f$ and $g$ are related by

$$
\langle,\rangle_{g}=e^{2 \phi}\langle,\rangle_{f}
$$

(i) Show that the tensor $T \in \Gamma(\operatorname{End}(T M))$ defined by $T=e^{-\phi} \Phi$ is orthogonal and that the pair $(T, \phi)$ satisfies the differential equation

$$
\left(\nabla_{X} T\right) Y=\langle Y, \operatorname{grad} \phi\rangle T X-\langle X, Y\rangle T \operatorname{grad} \phi
$$

and the condition

$$
\alpha^{f}(X, T Y)=\alpha^{f}(T X, Y)
$$

for all $X, Y \in \mathfrak{X}(M)$.
(ii) Show that, conversely, if $f: M^{n} \rightarrow \mathbb{R}^{m}$ is an isometric immersion of a simply connected Riemannian manifold, then any pair $(T, \phi)$ satisfying the two preceding conditions gives rise to a conformal immersion $g: M^{n} \rightarrow \mathbb{R}^{m}$ with the same Gauss map as $f$.

Hint: Use Exercises 1.25 and 9.1 .
Exercise 9.4. Prove that Corollary 9.15 implies Corollary 9.14, and that this, in turn, implies Theorem 9.13 .

Exercise 9.5. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 2$, be a cyclide of Dupin of characteristic ( $m, n-m$ ), that is, $f$ has two distinct principal curvatures of multiplicities $m$ and $n-m$, respectively, which are constant along the corresponding eigenbundles. Show that there exists a conformal diffeomorphism $\psi$ of an open subset $W \subset \mathbb{Q}_{c}^{n-m} \times \mathbb{S}^{m}$, $c>-1$, onto $M^{n}$, such that

$$
f \circ \psi=\left.\Theta \circ\left(f_{1} \times i\right)\right|_{W},
$$

where $f_{1}: \mathbb{Q}_{c}^{n-m} \rightarrow \mathbb{H}^{n-m+1}$ is an umbilical inclusion (a unit-speed extrinsic circle if $n-m=1$ ), id : $\mathbb{S}^{m} \rightarrow \mathbb{S}^{m}$ is the identity map and $\Theta: \mathbb{H}^{m-s} \times \mathbb{S}^{s} \rightarrow \mathbb{R}^{m}$ is the conformal diffeomorphism defined in Examples 9.10 .

Exercise 9.6. Let $f: M^{n} \rightarrow \mathbb{R}^{m}, n \geq 4$, be an isometric immersion with nowhere flat normal bundle carrying a principal curvature normal vector field $\eta$ of multiplicity $n-2$ such that $E_{\eta}^{\perp}$ is integrable. Show that $f(M)$ is, up to a conformal transformation of $\mathbb{R}^{m}$, an open subset of a submanifold of one of the three types in Corollary 9.33.
Hint: Use the Codazzi equation to show that

$$
\begin{align*}
\left\langle C_{T} X, A_{\xi} Y\right\rangle= & \|\eta\|\left\langle\nabla_{X} Y, T\right\rangle-\left\langle A_{\xi} Y, \nabla_{T} X\right\rangle-\left\langle A_{\xi} X, \nabla_{T} Y\right\rangle \\
& +T\left\langle A_{\xi} X, Y\right\rangle-\left\langle A_{\nabla_{\frac{⿺}{T}} \xi} X, Y\right\rangle \tag{9.62}
\end{align*}
$$

for all $X, Y \in \Gamma\left(E_{\eta}^{\perp}\right)$ and $T \in \Gamma\left(E_{\eta}\right)$. Use the assumption on the integrability of $E_{\eta}^{\perp}$ to show that $C_{T}$ is symmetric for all $T \in \Gamma\left(E_{\eta}\right)$ and that the first term in the right-hand side of (9.62) is symmetric in $X$ and $Y$. Conclude that

$$
\begin{equation*}
\left[C_{T},\left.A_{\xi}\right|_{E_{\bar{\eta}}}\right]=0 . \tag{9.63}
\end{equation*}
$$

Then use that $E_{\eta}^{\perp}$ has rank 2 to prove that at any point of $M^{n}$ either there exists $T_{0} \in E_{\eta}$ such that $C_{T}=\left\langle T, T_{0}\right\rangle I$ for all $T \in \Gamma\left(E_{\eta}\right)$ or there exists $T_{1} \in \Gamma\left(E_{\eta}\right)$ such that $C_{T_{1}}$ (is symmetric and) has two distinct real eigenvalues. Notice that if the latter possibility holds at some point $x$, then it also holds in an open neighborhood $U$ of $x$. Use (9.63) to show that this implies that $f$ has flat normal bundle on $U$, contradicting the assumption. Conclude that the first possibility holds everywhere, and then use Corollary 9.33 .

Exercise 9.7. Let $M^{n}$ be a Riemannian manifold. Show that the following holds:
(i) Any conformal immersion $f: M^{n} \rightarrow \mathbb{S}_{c}^{m}$ with conformal factor $\varphi \in C^{\infty}(M)$ gives rise to an isometric immersion $\mathcal{J}(f): M^{n} \rightarrow \mathbb{V}_{+}^{m+1}$ given by

$$
\mathcal{J}(f)=\frac{1}{\varphi} T_{B, z} \circ f
$$

where $T_{B, z}$ is given by (9.6) for a vector $z \in \mathbb{L}^{m+2}$ with $\langle z, z\rangle=-1 / c$.
(ii) Any isometric immersion $F: M^{n} \rightarrow \mathbb{V}_{+}^{m+1}$ gives rise to a conformal immersion $\mathcal{C}(F): M^{n} \rightarrow \mathbb{S}_{c}^{m}$ given by

$$
T_{B, z} \circ \mathcal{C}(F)=\frac{1}{\langle F, z\rangle} F .
$$

(iii) For any conformal immersion $f: M^{n} \rightarrow \mathbb{S}_{c}^{m}$ and for any isometric immersion $F: M^{n} \rightarrow \mathbb{V}_{+}^{m+1}$ one has

$$
\mathcal{C}(\mathcal{J}(f))=f \quad \text { and } \mathcal{J}(\mathcal{C}(F))=F
$$

(iv) If $f, g: M^{n} \rightarrow \mathbb{S}_{c}^{m}$ are conformal immersions, then $f$ and $g$ are conformally congruent if and only if $\mathcal{J}(f), \mathcal{J}(g): M^{n} \rightarrow \mathbb{V}_{+}^{m+1} \subset \mathbb{L}^{m+2}$ are isometrically congruent.

State and prove similar assertions if $\mathbb{S}_{c}^{m}$ is replaced by $\mathbb{H}_{c}^{m}$.
Exercise 9.8. Show that a compact flat Riemannian manifold $M^{n}$ does not admit a conformal immersion into $\mathbb{R}^{2 n-2}$.
Hint: Suppose that there exists a conformal immersion $f: M^{n} \rightarrow \mathbb{R}^{2 n-2}$ and let $F=$ $\mathcal{J}(f): M^{n} \rightarrow \mathbb{V}^{2 n-1} \subset \mathbb{L}^{2 n}$ be its isometric light-cone representative. Given a futurepointing time-like vector $v=\left(v_{0}, \ldots, v_{2 n-1}\right) \in \mathbb{L}^{2 n}$, that is, $v_{0}>0$, let $x \in M^{n}$ be a point at which the height-function $h^{v}: M^{n} \rightarrow \mathbb{R}$ given by

$$
h^{v}(x)=-\langle F(x), v\rangle
$$

attains its maximum. Then $v \in N_{F} M(x)$ and

$$
\operatorname{Hess} h^{v}(x)(X, Y)=-\left\langle\alpha^{F}(X, Y), v\right\rangle
$$

for all $X, Y \in T_{x} M$ (see Corollary 1.3), and hence $\phi()=,\left\langle\alpha^{F}(), v,\right\rangle$ is positivedefinite. Show that this contradicts Exercise 5.1.

## Chapter 10

## Isometric immersions of warped products

In this chapter we discuss two other useful ways of constructing immersions of product manifolds from immersions of the factors, with an increasing degree of generality. Namely, we introduce the notions of (extrinsic) warped products of immersions and, more generally, of partial tubes over extrinsic products of immersions.

Both types of immersions share with extrinsic products of immersions the property that their second fundamental forms are adapted to the product structure of the manifold. Once this condition is satisfied, it is shown that immersions of each kind are characterized by the special types of metrics they induce on the product manifold. These are, respectively, warped product metrics and metrics called polar.

We then discuss sufficient conditions, in terms of the $s$-nullities, for the second fundamental form of an isometric immersion of a product manifold endowed with a warped product metric to be adapted to the product structure of the manifold.

### 10.1 Polar metrics on product manifolds

Our aim in this section is to introduce some classes of metrics on a product manifold and to characterize them in terms of the geometry of its product net.

The results in this section are stated without proofs, but references where the proofs may be found are provided in the Notes of this chapter. We use the notations and terminology introduced in Section 8.1.

A metric $g$ on a product manifold $M=\prod_{i=0}^{r} M_{i}$ is called polar if there exist a metric $g_{0}$ on $M_{0}$ and smooth maps $x_{0} \in M_{0} \mapsto g_{a}\left(x_{0}\right), 1 \leq a \leq r$, where each $g_{a}\left(x_{0}\right)$ is a metric on $M_{a}$, such that

$$
\begin{equation*}
g=\pi_{0}^{*} g_{0}+\sum_{a=1}^{r} \pi_{a}^{*}\left(g_{a} \circ \pi_{0}\right), \tag{10.1}
\end{equation*}
$$

that is,

$$
g(x)=\left(\pi_{0 *}(x)\right)^{*} g_{0}\left(x_{0}\right)+\sum_{a=1}^{r}\left(\pi_{a *}(x)\right)^{*}\left(g_{a}\left(x_{0}\right)\left(x_{a}\right)\right)
$$

for all $x=\left(x_{0}, \ldots, x_{r}\right) \in M$.
A special type of polar metric is the warped product of the metrics $g_{0}, \ldots, g_{r}$ on $M_{0}, \ldots, M_{r}$, respectively, with smooth warping functions $\rho_{a}: M_{0} \rightarrow \mathbb{R}_{+}, 1 \leq a \leq r$, that is, the metric given by

$$
g=\pi_{0}^{*} g_{0}+\sum_{a=1}^{r}\left(\rho_{a} \circ \pi_{0}\right)^{2} \pi_{a}^{*} g_{a}
$$

It is usual to denote a product manifold $M=\prod_{i=0}^{r} M_{i}$, endowed with a warped product metric with warping functions $\rho_{a}: M_{0} \rightarrow \mathbb{R}_{+}, 1 \leq a \leq r$, by

$$
M=M_{0} \times_{\rho_{1}} M_{1} \times \cdots \times_{\rho_{r}} M_{r} .
$$

In particular, the Riemannian product of $g_{0}, \ldots, g_{r}$ corresponds to the case in which the warping functions $\rho_{a}, 1 \leq a \leq r$, are identically one. Thus, warped (respectively, Riemannian) product metrics correspond to polar metrics for which all metrics $g_{a}\left(x_{0}\right)$ on $M_{a}, 1 \leq a \leq r, x_{0} \in M_{0}$, are homothetical (respectively, isometric) to a fixed Riemannian metric.

The next result characterizes polar metrics on a product manifold in terms of the geometry of its product net.

Proposition 10.1. A Riemannian metric on a product manifold $M=\Pi_{i=0}^{r} M_{i}$ is polar if and only if the product net $\mathcal{E}=\left(E_{i}\right)_{i=0, \ldots, r}$ of $M$ is an orthogonal net such that $E_{a}^{\perp}$ is totally geodesic for all $1 \leq a \leq r$.

The additional geometric properties that the product net of a product manifold must have with respect to a Riemannian metric $g$ in order that $g$ be a warped product metric are as follows.

Proposition 10.2. A Riemannian metric on a product manifold $M=\Pi_{i=0}^{r} M_{i}$ is a warped product metric if and only if the product net $\mathcal{E}=\left(E_{i}\right)_{i=0, \ldots, r}$ of $M$ is an orthogonal net such that $E_{a}$ is spherical and $E_{a}^{\perp}$ is totally geodesic for all $1 \leq a \leq r$.

The next result characterizes Riemannian manifolds that can be locally or globally decomposed as a product manifold endowed with a polar metric.

Theorem 10.3. Let $M$ be a Riemannian manifold carrying an orthogonal net $\mathcal{E}=$ $\left(E_{i}\right)_{i=0, \ldots, r}$ such that $E_{a}^{\perp}$ is totally geodesic for $1 \leq a \leq r$. Then there exists locally (globally, if $M$ is simply connected and the leaves of $E_{a}^{\perp}$ are complete) a product representation $\psi: \Pi_{i=0}^{r} M_{i} \rightarrow M$ of $\mathcal{E}$ which is an isometry with respect to a polar metric on $\Pi_{i=0}^{r} M_{i}$.

The preceding theorem can be regarded as a generalization of the theorem of de Rham as well as of its extension given next for warped product manifolds.

Theorem 10.4. Let $M$ be a Riemannian manifold carrying an orthogonal net $\mathcal{E}=$ $\left(E_{i}\right)_{i=0, \ldots, r}$ such that $E_{a}$ is spherical (respectively, totally geodesic) and $E_{a}^{\perp}$ is totally geodesic for $1 \leq a \leq r$. Then there exists locally (globally, if $M$ is simply connected and complete) a product representation $\psi: \prod_{i=0}^{r} M_{i} \rightarrow M$ of $\mathcal{E}$ which is an isometry with respect to a warped (respectively, Riemannian) product metric on $\prod_{i=0}^{r} M_{i}$.

### 10.2 Partial tubes

In this section we describe a general way of constructing immersions of product manifolds into space forms starting with isometric immersions of the factors. For simplicity of the presentation, we first consider immersions into Euclidean space of product manifolds with only two factors.

### 10.2.1 Partial tubes in Euclidean space

Let $f_{1}: M_{1} \rightarrow \mathbb{R}^{m}$ be an isometric immersion along which there is an orthonormal set $\left\{\xi_{1}, \ldots, \xi_{k}\right\}$ of normal vector fields that are parallel in the normal connection. The subbundle $\mathcal{L}=\operatorname{span}\left\{\xi_{1}, \ldots, \xi_{k}\right\}$ of $N_{f_{1}} M_{1}$ is thus parallel and flat. Hence the map $\phi: M_{1} \times \mathbb{R}^{k} \rightarrow \mathcal{L}$, defined by

$$
\phi_{x_{1}}(y)=\phi\left(x_{1}, y\right)=\sum_{i=1}^{k} y_{i} \xi_{i}\left(x_{1}\right)
$$

for all $x_{1} \in M_{1}$ and $y=\left(y_{1}, \ldots, y_{k}\right) \in \mathbb{R}^{k}$, is a parallel vector bundle isometry.
Given an isometric immersion $f_{0}: M_{0} \rightarrow \mathbb{R}^{k}$, denote $M=M_{0} \times M_{1}$ and let $f: M \rightarrow \mathbb{R}^{m}$ be defined by

$$
\begin{equation*}
f\left(x_{0}, x_{1}\right)=f_{1}\left(x_{1}\right)+\phi_{x_{1}}\left(f_{0}\left(x_{0}\right)\right) . \tag{10.2}
\end{equation*}
$$

In part (ii) of the next result we determine the condition for $f$ to be an immersion at a given point $\left(x_{0}, x_{1}\right) \in M$. If that condition is satisfied at any point of $M$, then $f$ is called the partial tube over $f_{1}$ with fiber $f_{0}$, or simply the partial tube determined by $\left(f_{0}, f_{1}, \phi\right)$.
Proposition 10.5. With $f_{0}, f_{1}, f$ and $\phi$ as above, the following assertions hold:
(i) The differential of $f$ at $x=\left(x_{0}, x_{1}\right)$ is given by

$$
\begin{equation*}
f_{*} \tau_{0^{*}}^{x} X_{0}=\phi_{x_{1}}\left(f_{0_{*}} X_{0}\right) \tag{10.3}
\end{equation*}
$$

for any $X_{0} \in T_{x_{0}} M_{0}$, and

$$
\begin{equation*}
f_{*} \tau_{1_{*}}^{x}=f_{1 *} P, \tag{10.4}
\end{equation*}
$$

where $P=P(x)$ is the endomorphism of $T_{x_{1}} M_{1}$ defined by

$$
\begin{equation*}
P=I-A_{\phi_{x_{1}}\left(f_{0}\left(x_{0}\right)\right)}^{f_{1}} \tag{10.5}
\end{equation*}
$$

(ii) The map $f$ is an immersion at $x=\left(x_{0}, x_{1}\right)$ if and only if $P(x)$ is invertible.
(iii) If $f$ is an immersion at $x=\left(x_{0}, x_{1}\right)$, then

$$
N_{f} M(x)=\mathcal{L}^{\perp}\left(x_{1}\right) \oplus \phi_{x_{1}}\left(N_{f_{0}} M_{0}\left(x_{0}\right)\right) \subset N_{f_{1}} M_{1}\left(x_{1}\right)
$$

where $\mathcal{L}^{\perp}\left(x_{1}\right)$ is the orthogonal complement of $\mathcal{L}\left(x_{1}\right)$ in $N_{f_{1}} M_{1}\left(x_{1}\right)$.
(iv) If $f$ is an immersion at $x=\left(x_{0}, x_{1}\right)$, then

$$
\begin{equation*}
A_{\xi}^{f}(x) \tau_{1 *}^{x}=\tau_{1 *}^{x} P^{-1} A_{\xi}^{f_{1}}\left(x_{1}\right) \tag{10.6}
\end{equation*}
$$

for any $\xi \in N_{f} M(x)$,

$$
\begin{equation*}
A_{\delta}^{f}(x) \tau_{0^{*}}^{x}=0 \tag{10.7}
\end{equation*}
$$

for any $\delta \in \mathcal{L}^{\perp}\left(x_{1}\right)$, and

$$
\begin{equation*}
A_{\phi_{x_{1}}(\zeta)}^{f}(x) \tau_{0^{*}}^{x}=\tau_{0 *}^{x} A_{\zeta}^{f_{0}}\left(x_{0}\right) \tag{10.8}
\end{equation*}
$$

for any $\zeta \in N_{f_{0}} M_{0}\left(x_{0}\right)$. Equivalently, if $\pi: N_{f_{1}} M_{1}\left(x_{1}\right) \rightarrow N_{f} M(x)$ denotes the orthogonal projection, then

$$
\begin{equation*}
\alpha^{f}\left(\tau_{1 *}^{x} X_{1}, \tau_{1 *}^{x} Y_{1}\right)=\pi\left(\alpha^{f_{1}}\left(P X_{1}, Y_{1}\right)\right) \tag{10.9}
\end{equation*}
$$

for all $X_{1}, Y_{1} \in T_{x_{1}} M_{1}$,

$$
\begin{equation*}
\alpha^{f}\left(\tau_{0 *}^{x} X_{0}, \tau_{1 *}^{x} X_{1}\right)=0 \tag{10.10}
\end{equation*}
$$

for all $X_{0} \in T_{x_{0}} M_{0}$ and $X_{1} \in T_{x_{1}} M_{1}$, and

$$
\begin{equation*}
\alpha^{f}\left(\tau_{0^{*}}^{x} X_{0}, \tau_{0^{*}}^{x} Y_{0}\right)=\phi_{x_{1}}\left(\alpha^{f_{0}}\left(X_{0}, Y_{0}\right)\right) \tag{10.11}
\end{equation*}
$$

for all $X_{0}, Y_{0} \in T_{x_{0}} M_{0}$.
Proof: The proofs of 10.3 and 10.4 are straightforward, and the assertions in parts (ii) and (iii) are immediate consequences of those formulas.

To prove 10.6), given $\xi \in N_{f} M(x)$ and $X_{1} \in T_{x_{1}} M_{1}$, let $\gamma: J \rightarrow M_{1}$ be a smooth curve, with $0 \in J$, such that $\gamma(0)=x_{1}$ and $\gamma^{\prime}(0)=X_{1}$. Let $\xi(t)$ be the parallel transport of $\xi$ along the curve $\tau_{1}^{x} \circ \gamma$ in the normal connection. Using (10.4) we obtain

$$
\begin{aligned}
f_{*}(x) A_{\xi}^{f}(x) \tau_{1 *}^{x} X_{1} & =-\tilde{\nabla}_{\tau_{1 *}^{x} X_{1}} \xi \\
& =-\left.\frac{d}{d t}\right|_{t=0} \xi\left(x_{0}, \gamma(t)\right) \\
& =f_{1 *}\left(x_{1}\right) A_{\xi}^{f_{1}}\left(x_{1}\right) X_{1} \\
& =f_{*}(x) \tau_{1 *}^{x} P^{-1} A_{\xi}^{f_{1}}\left(x_{1}\right) X_{1}
\end{aligned}
$$

where $\tilde{\nabla}$ is the Euclidean connection. The proofs of 10.7 and 10.8 are similar.
As a consequence of part (ii) of the preceding result, we obtain the following necessary and sufficient condition for $f$ to be an immersion.

Corollary 10.6. The map $f$ is an immersion if and only if $f_{0}\left(M_{0}\right) \subset \Omega\left(f_{1} ; \phi\right)$, where

$$
\Omega\left(f_{1} ; \phi\right)=\left\{Y \in \mathbb{R}^{k}: I-A_{\phi_{x_{1}}(Y)}^{f_{1}} \text { is nonsingular for any } x_{1} \in M_{1}\right\} .
$$

To provide a better description of the subset $\Omega\left(f_{1} ; \phi\right)$, let $\eta_{1}, \ldots, \eta_{s} \in \Gamma(\mathcal{L})$ be the distinct principal normal vector fields of $f_{1}$ with respect to $\mathcal{L}$ (see Exercise 1.35). Thus there exists an orthogonal decomposition $T M_{1}=\oplus_{i=1}^{s} E_{i}$ such that

$$
\left.A_{\zeta}^{f_{1}}\right|_{E_{i}}=\left\langle\zeta, \eta_{i}\right\rangle I
$$

for any $\zeta \in \Gamma(\mathcal{L})$. Therefore $I-A_{\phi_{x_{1}}(Y)}^{f_{1}}$ is nonsingular if and only if

$$
\left\langle\phi_{x_{1}}(Y), \eta_{i}\left(x_{1}\right)\right\rangle \neq 1
$$

for any $1 \leq i \leq s$, that is, if and only if $\phi_{x_{1}}(Y)$ does not belong to any of the focal hyperplanes

$$
H_{i}^{f_{1}}\left(x_{1}\right)=\left\{\zeta \in N_{f_{1}} M_{1}\left(x_{1}\right):\left\langle\zeta, \eta_{i}\left(x_{1}\right)\right\rangle=1\right\}, \quad 1 \leq i \leq s .
$$

For each $x_{1} \in M_{1}$, let $V_{i}\left(x_{1}\right) \in \mathbb{R}^{k}$ be such that

$$
\phi_{x_{1}}\left(V_{i}\left(x_{1}\right)\right)=\eta_{i}\left(x_{1}\right), \quad 1 \leq i \leq s .
$$

Then

$$
\Omega\left(f_{1} ; \phi\right)=\cap_{x_{1} \in M_{1}} \cap_{i=1}^{s}\left\{Y \in \mathbb{R}^{k}:\left\langle Y, V_{i}\left(x_{1}\right)\right\rangle \neq 1\right\} .
$$

We denote by $\Omega^{0}\left(f_{1} ; \phi\right)$ the connected component of $\Omega\left(f_{1} ; \phi\right)$ given by

$$
\Omega^{0}\left(f_{1} ; \phi\right)=\cap_{x_{1} \in M_{1}} \cap_{i=1}^{s}\left\{Y \in \mathbb{R}^{k}:\left\langle Y, V_{i}\left(x_{1}\right)\right\rangle>1\right\}
$$

and always assume that $f_{0}\left(M_{0}\right) \subset \Omega^{0}\left(f_{1} ; \phi\right)$.
Remark 10.7. Let $f$ be the partial tube determined by $\left(f_{0}, f_{1}, \phi\right)$ and let $v \in \Omega\left(f_{1} ; \phi\right)$. Since the map $\tilde{f}_{1}: M_{1} \rightarrow \mathbb{R}^{m}$ defined by

$$
\tilde{f}_{1}\left(x_{1}\right)=f_{1}\left(x_{1}\right)+\phi_{x_{1}}(v)
$$

satisfies

$$
\tilde{f}_{1 *}=f_{1 *}\left(I-A_{\phi_{x_{1}}(v)}^{f_{1}}\right),
$$

then $\tilde{f}_{1}$ is an immersion with the same normal bundle as $f_{1}$. In particular, $\xi_{1}, \ldots, \xi_{k}$ are also parallel normal vector fields along $\tilde{f}_{1}$, and thus $\mathcal{L}=\operatorname{span}\left\{\xi_{1}, \ldots, \xi_{k}\right\}$ is also a parallel flat normal subbundle of $N_{\tilde{f}_{1}} M_{1}$. The shape operator of $\tilde{f}_{1}$ is given by

$$
A_{\xi}^{\tilde{f}_{1}}=\left(I-A_{\phi_{x_{1}}(v)}^{f_{1}}\right)^{-1} A_{\xi}^{f_{1}}
$$

for any $\xi \in \Gamma\left(N_{\tilde{f}_{1}} M_{1}\right)$. It follows that

$$
I-A_{\phi_{x_{1}}(Y)}^{\tilde{f}_{1}}=\left(I-A_{\phi_{x_{1}}(v)}^{f_{1}}\right)^{-1}\left(I-A_{\phi_{x_{1}}(Y+v)}^{f_{1}}\right)
$$

for any $Y \in \mathbb{R}^{k}$. Thus

$$
Y \in \Omega\left(\tilde{f}_{1} ; \phi\right) \text { if and only if } Y+v \in \Omega\left(f_{1} ; \phi\right)
$$

Defining $\tilde{f}_{0}: M_{0} \rightarrow \mathbb{R}^{k}$ by $\tilde{f}_{0}=f_{0}-v$, it follows that $f_{0}\left(M_{0}\right) \subset \Omega\left(f_{1} ; \phi\right)$ if and only if $\tilde{f}_{0}\left(M_{0}\right) \subset \Omega\left(\tilde{f}_{1} ; \phi\right)$. Since

$$
f\left(x_{0}, x_{1}\right)=\tilde{f}_{1}\left(x_{1}\right)+\phi_{x_{1}}\left(\tilde{f}_{0}\left(x_{0}\right)\right),
$$

then the map $f$ is also the partial tube determined by $\left(\tilde{f}_{0}, \tilde{f}_{1}, \phi\right)$.
In particular, if $f$ is the partial tube determined by $\left(f_{0}, f_{1}, \phi\right)$, then one can always assume that $f_{0}$ is a substantial immersion, for if $f_{0}\left(M_{0}\right)$ is contained in the affine subspace $v+\mathbb{R}^{\ell} \subset \mathbb{R}^{k}$, one can replace $f_{0}$ by $\tilde{f}_{0}: M_{0} \rightarrow \mathbb{R}^{\ell}$ given by

$$
\tilde{f}_{0}\left(x_{0}\right)=f_{0}\left(x_{0}\right)-v
$$

replace $f_{1}: M_{1} \rightarrow \mathbb{R}^{m}$ by

$$
\tilde{f}_{1}\left(x_{1}\right)=f_{1}\left(x_{1}\right)+\phi_{x_{1}}(v)
$$

and then $\phi$ by its restriction to $M_{1} \times \mathbb{R}^{\ell}$.
Another consequence of Proposition 10.5 is the following.
Corollary 10.8. The metric $g$ induced by $f$ is the polar metric

$$
\begin{equation*}
g=\pi_{0}^{*} g_{0}+\pi_{1}^{*}\left(g_{1} \circ \pi_{0}\right) \tag{10.12}
\end{equation*}
$$

where $g_{0}$ is the metric of $M_{0}$ and, for any $x=\left(x_{0}, x_{1}\right) \in M_{0} \times M_{1}$, the metric $g_{1}\left(x_{0}\right)$ on $M_{1}$ is given, in terms of the metric $g_{1}$ of $M_{1}$ and the endomorphism $P$ defined by (10.5), by

$$
g_{1}\left(x_{0}\right)\left(X_{1}, Y_{1}\right)=g_{1}\left(P^{2} X_{1}, Y_{1}\right)
$$

for all $X_{1}, Y_{1} \in T_{x_{1}} M_{1}$. Moreover, the second fundamental form of $f$ is adapted to the product net of $M_{0} \times M_{1}$.

Examples 10.9. (i) Let $\mathbb{R}^{m}=\mathbb{R}^{m_{1}} \times \mathbb{R}^{m_{2}}$ be an orthogonal decomposition. Take an isometric immersion $f_{1}: M_{1} \rightarrow \mathbb{R}^{m_{1}}$, and let $\tilde{f}_{1}$ stand for $f_{1}$ regarded as a map into $\mathbb{R}^{m}$, that is,

$$
\begin{equation*}
\tilde{f}_{1}\left(x_{1}\right)=\left(f_{1}\left(x_{1}\right), 0\right) . \tag{10.13}
\end{equation*}
$$

Let $\mathcal{L}$ denote the vector subbundle of $N_{\tilde{f}_{1}} M_{1}$ whose fiber at any point $x_{1} \in M_{1}$ is $\mathbb{R}^{m_{2}}$, and consider the obvious parallel vector bundle isometry $\phi: M_{1} \times \mathbb{R}^{m_{2}} \rightarrow \mathcal{L}$. Notice that $\Omega\left(\tilde{f}_{1}, \phi\right)=\mathbb{R}^{m_{2}}$. Given any isometric immersion $f_{0}: M_{0} \rightarrow \mathbb{R}^{m_{2}}$, the partial tube $f: M_{0} \times M_{1} \rightarrow \mathbb{R}^{m}$ determined by $\left(f_{0}, \tilde{f}_{1}, \phi\right)$ is the extrinsic product of $f_{0}$ and $f_{1}$.
(ii) Let $\mathbb{R}^{m}=\mathbb{R}^{m_{1}} \times \mathbb{R}^{m_{2}}$ be an orthogonal decomposition. Consider an isometric immersion $f_{1}: M_{1} \rightarrow \mathbb{S}^{m_{1}-1} \subset \mathbb{R}^{m_{1}}$ and define $\tilde{f}_{1}$ as in 10.13. Let $\mathcal{L}$ be the flat parallel vector subbundle of rank $k=m_{2}+1$ of $N_{\tilde{f}_{1}} M_{1}$ whose fiber at $x_{1} \in M_{1}$ is

$$
\mathcal{L}\left(x_{1}\right)=\operatorname{span}\left\{\tilde{f}_{1}\left(x_{1}\right)\right\} \oplus \mathbb{R}^{m_{2}}
$$

and let $\phi: M_{1} \times \mathbb{R}^{k} \rightarrow \mathcal{L}$ be a parallel vector bundle isometry. Let $e \in \mathbb{R}^{k}$ be such that $\phi_{x_{1}}(e)=\tilde{f}_{1}\left(x_{1}\right)$ for all $x_{1} \in M_{1}$. Then

$$
A_{\phi_{x_{1}}(Y)}^{\tilde{f}_{1}}=-\langle Y, e\rangle I
$$

for any $Y \in \mathbb{R}^{k}$, where $I$ is the identity endomorphism of $T_{x_{1}} M_{1}$. In particular,

$$
\Omega^{0}\left(\tilde{f}_{1}, \phi\right)=\left\{Y \in \mathbb{R}^{k}:\langle Y, e\rangle+1>0\right\} .
$$

Given an isometric immersion $\tilde{f}_{0}: M_{0} \rightarrow \Omega^{0}\left(\tilde{f}_{1}, \phi\right) \subset \mathbb{R}^{k}$, let $f: M_{0} \times M_{1} \rightarrow \mathbb{R}^{m}$ be the partial tube determined by $\left(\tilde{f}_{0}, \tilde{f}_{1}, \phi\right)$. Thus

$$
\begin{aligned}
f\left(x_{0}, x_{1}\right) & =\tilde{f}_{1}\left(x_{1}\right)+\phi_{x_{1}}\left(\tilde{f}_{0}\left(x_{0}\right)\right) \\
& =\phi_{x_{1}}\left(f_{0}\left(x_{0}\right)\right)
\end{aligned}
$$

where $f_{0}: M_{0} \rightarrow \mathbb{R}^{k}$ is given by $f_{0}=\tilde{f}_{0}+e$. Note that the condition $\tilde{f}_{0}\left(M_{0}\right) \subset \Omega^{0}\left(\tilde{f}_{1}, \phi\right)$ reduces to

$$
\left\langle f_{0}\left(x_{0}\right), e\right\rangle>0
$$

for all $x_{0} \in M_{0}$.
The map $f$ is called the warped product of $f_{0}$ and $f_{1}$. If $f_{1}: \mathbb{S}^{m_{1}-1} \rightarrow \mathbb{S}^{m_{1}-1}$ is the identity map, then $f$ is said to be a rotational submanifold with $f_{0}: M_{0} \rightarrow \mathbb{R}^{k}$ as profile. On the other hand, if $f_{0}$ is the identity map on

$$
\Omega^{0}\left(\tilde{f}_{1}\right)=\left\{Y \in \mathbb{R}^{k}:\langle Y, e\rangle>0\right\}
$$

then $f$ coincides with the generalized cone over $f_{1}$, which in this case is the cylinder over the (standard) cone over $f_{1}$ in $\mathbb{R}^{m_{1}}$. In particular, for $m_{1}=m$ the immersion $f$ is the cone over $f_{1}: M_{1} \rightarrow \mathbb{S}^{m-1}$.

The next result summarizes several consequences of Proposition 10.5, as well as of Corollary 10.8, for $f_{0}, f_{1}, f$ and $\phi$ as in the second part of Examples 10.9.

Corollary 10.10. With $f_{0}, f_{1}, f$ and $\phi$ as in part (ii) of Examples 10.9, the following assertions hold:
(i) The map $f$ is an immersion whose induced metric $g$ is the warped product of the metrics $g_{0}$ and $g_{1}$ in $M_{0}$ and $M_{1}$, respectively, with warping function $\rho: M_{0} \rightarrow \mathbb{R}_{+}$ given by

$$
\rho\left(x_{0}\right)=\left\langle f_{0}\left(x_{0}\right), e\right\rangle .
$$

(ii) The normal space of $f$ at $x=\left(x_{0}, x_{1}\right)$ is

$$
N_{f} M(x)=i_{1 *} N_{f_{1}} M_{1}\left(x_{1}\right) \oplus \phi_{x_{1}}\left(N_{f_{0}} M_{0}\left(x_{0}\right)\right) \subset N_{\tilde{f}_{1}} M_{1}\left(x_{1}\right)
$$

where $i_{1}: \mathbb{S}^{m_{1}-1} \rightarrow \mathbb{R}^{m_{1}}$ is the inclusion.
(iii) The second fundamental form of $f$ at $x=\left(x_{0}, x_{1}\right)$ is given by

$$
\alpha^{f}\left(\tau_{1 *}^{x} X_{1}, \tau_{1 *}^{x} Y_{1}\right)=\left\langle f_{0}\left(x_{0}\right), e\right\rangle\left(i_{1 *} \alpha^{f_{1}}\left(X_{1}, Y_{1}\right)-g_{1}\left(X_{1}, Y_{1}\right) \phi_{x_{1}}\left(e^{\perp}\right)\right)
$$

for all $X_{1}, Y_{1} \in T_{x_{1}} M_{1}$, where $e^{\perp}$ is the orthogonal projection of e onto $N_{f_{0}} M_{0}\left(x_{0}\right)$,

$$
\alpha^{f}\left(\tau_{0 *}^{x} X_{0}, \tau_{1 *}^{x} X_{1}\right)=0
$$

for all $X_{0} \in T_{x_{0}} M_{0}$ and $X_{1} \in T_{x_{1}} M_{1}$, and

$$
\alpha^{f}\left(\tau_{0^{*}}^{x} X_{0}, \tau_{0^{*}}^{x} Y_{0}\right)=\phi_{x_{1}}\left(\alpha^{f_{0}}\left(X_{0}, Y_{0}\right)\right)
$$

for all $X_{0}, Y_{0} \in T_{x_{0}} M_{0}$.
Proof: Since

$$
A_{\phi_{x_{1}}\left(\tilde{f}_{0}\left(x_{0}\right)\right)}^{\tilde{f}_{1}}=-\left\langle\tilde{f}_{0}\left(x_{0}\right), e\right\rangle I
$$

for all $x=\left(x_{0}, x_{1}\right) \in M_{0} \times M_{1}$, the endomorphism

$$
P=P(x)=I-A_{\phi_{x_{1}}\left(\tilde{f_{0}}\left(x_{0}\right)\right)}^{\tilde{\tilde{f}_{1}}}
$$

reduces to $P=\left\langle f_{0}\left(x_{0}\right), e\right\rangle I$, and the assertion in part $(i)$ follows from Corollary 10.8 . The formulas in parts (ii) and (iii) are immediate consequences of the corresponding ones in Proposition 10.5.

### 10.2.2 Partial tubes in the sphere and the hyperbolic space

The definitions and results in the previous section can be easily extended to immersions into the sphere and the hyperbolic space.

Let $\mathbb{R}_{\mu}^{m+1}$ denote either Euclidean space $\mathbb{R}^{m+1}$ or Lorentzian space $\mathbb{L}^{m+1}$, depending on whether $\mu=0$ or 1 , respectively. Denote by $\mathbb{Q}_{\epsilon}^{m} \subset \mathbb{R}_{\mu}^{m+1}, \epsilon=1-2 \mu$, either the sphere $\mathbb{S}^{m}$ or the hyperbolic space $\mathbb{H}^{m}$. Let $\tilde{f}_{1}: M_{1} \rightarrow \mathbb{R}_{\mu}^{m+1}$ be an isometric immersion such that $\tilde{f}_{1}\left(M_{1}\right)$ is contained in $\mathbb{Q}_{\epsilon}^{m} \subset \mathbb{R}_{\mu}^{m+1}$, so that there exists an isometric immersion $f_{1}: M_{1} \rightarrow \mathbb{Q}_{\epsilon}^{m}$ such that $\tilde{f}_{1}=i \circ f_{1}$, where $i: \mathbb{Q}_{\epsilon}^{m} \rightarrow \mathbb{R}_{\mu}^{m+1}$ is the inclusion.

Assume that there exists an orthonormal set $\left\{\xi_{1}, \ldots, \xi_{k}\right\}$ of normal vector fields along $f_{1}$ that are parallel in the normal connection and let $\mathcal{L}$ denote the parallel and flat subbundle of $N_{f_{1}} M_{1}$ spanned by $\left\{\xi_{1}, \ldots, \xi_{k}\right\}$. The subbundle

$$
\tilde{\mathcal{L}}=i_{*} \mathcal{L} \oplus \operatorname{span}\left\{\tilde{f}_{1}\right\}
$$

of $N_{\tilde{f}_{1}} M_{1}$ is also parallel and flat; hence there exists a parallel vector bundle isometry $\phi: M_{1} \times \mathbb{R}_{\mu}^{k+1} \rightarrow \tilde{\mathcal{L}}$. Let $e \in \mathbb{R}_{\mu}^{k+1}$ be such that $\tilde{f}_{1}\left(x_{1}\right)=\phi_{x_{1}}(e)$ for all $x_{1} \in M_{1}$, and let $f_{0}: M_{0} \rightarrow \mathbb{R}_{\mu}^{k+1}$ be an isometric immersion such that

$$
\begin{equation*}
f_{0}\left(M_{0}\right) \subset \mathbb{Q}_{\epsilon}^{k} \cap\left(e+\Omega^{0}\left(\tilde{f}_{1}, \phi\right)\right) \subset \mathbb{R}_{\mu}^{k+1} \tag{10.14}
\end{equation*}
$$

Define a map $f: M_{0} \times M_{1} \rightarrow \mathbb{Q}_{\epsilon}^{m} \subset \mathbb{R}_{\mu}^{m+1}$ by

$$
f\left(x_{0}, x_{1}\right)=\phi_{x_{1}}\left(f_{0}\left(x_{0}\right)\right) .
$$

Then $f$ is called the partial tube over $f_{1}$ with fiber $f_{0}$. Note that we can write

$$
f\left(x_{0}, x_{1}\right)=\tilde{f}_{1}\left(x_{1}\right)+\phi_{x_{1}}\left(\tilde{f}_{0}\left(x_{0}\right)\right)
$$

where $\tilde{f}_{0}\left(x_{0}\right)=f_{0}\left(x_{0}\right)-e$. Notice also that

$$
I-A_{\phi_{x_{1}}(Y)}^{\tilde{f}_{1}}=-A_{\phi_{x_{1}}(e+Y)}^{\tilde{f}_{1}}
$$

for all $Y \in \mathbb{R}_{\mu}^{k+1}$, hence $\tilde{f}_{0}\left(M_{0}\right) \subset \Omega^{0}\left(\tilde{f}_{1}, \phi\right)$ by 10.14). Thus the map $f$ (regarded as a map into $\mathbb{R}_{\mu}^{m+1}$ ) is the partial tube over $\tilde{f}_{1}$ with fiber $\tilde{f}_{0}$.

Important special cases of partial tubes in the sphere and the hyperbolic space are the warped products of immersions defined next.

We start with the case of the sphere. Given an orthogonal decomposition

$$
\mathbb{R}^{m+1}=\mathbb{R}^{m_{1}} \times \mathbb{R}^{m_{2}}
$$

let $f_{1}: M_{1} \rightarrow \mathbb{S}^{m_{1}-1} \subset \mathbb{R}^{m_{1}}$ be an isometric immersion and let $\tilde{f}_{1}: M_{1} \rightarrow \mathbb{R}^{m+1}$ be defined by

$$
\tilde{f}_{1}\left(x_{1}\right)=\left(f_{1}\left(x_{1}\right), 0\right)
$$

Consider the flat parallel vector subbundle $\mathcal{L}$ of rank $k=m_{2}+1$ of $N_{\tilde{f}_{1}} M_{1}$ whose fiber at $x_{1} \in M_{1}$ is

$$
\mathcal{L}\left(x_{1}\right)=\operatorname{span}\left\{\tilde{f}_{1}\left(x_{1}\right)\right\} \oplus \mathbb{R}^{m_{2}} .
$$

Let $\phi: M_{1} \times \mathbb{R}^{k} \rightarrow \mathcal{L}$ be a parallel vector bundle isometry and let $e \in \mathbb{R}^{k}$ be such that $\phi_{x_{1}}(e)=\tilde{f}_{1}\left(x_{1}\right)$. Note that

$$
A_{\phi_{x_{1}}(Y)}^{\tilde{f}_{1}}=-\langle Y, e\rangle I
$$

for all $Y \in \mathbb{R}^{k}$. Hence

$$
e+\Omega^{0}\left(\tilde{f}_{1}, \phi\right)=\Omega^{0}\left(\tilde{f}_{1}\right)=\left\{Y \in \mathbb{R}^{k}:\langle Y, e\rangle>0\right\} \subset \mathbb{R}^{k}
$$

If $f_{0}: M_{0} \rightarrow \mathbb{S}^{k-1} \cap \Omega^{0}\left(\tilde{f}_{1}\right) \subset \mathbb{R}^{k}$ is an isometric immersion, then the partial tube $f: M_{0} \times M_{1} \rightarrow \mathbb{S}^{m} \subset \mathbb{R}^{m+1}$, given by

$$
f\left(x_{0}, x_{1}\right)=\phi_{x_{1}}\left(f_{0}\left(x_{0}\right)\right),
$$

is called the warped product of $f_{0}$ and $f_{1}$.
Warped products of immersions into the hyperbolic space are of three different types. First, for an orthogonal decomposition

$$
\mathbb{L}^{m+1}=\mathbb{L}^{m_{1}} \times \mathbb{R}^{m_{2}}
$$

let $f_{1}: M_{1} \rightarrow \mathbb{H}^{m_{1}-1} \subset \mathbb{L}^{m_{1}}$ be an isometric immersion and let $\tilde{f}_{1}: M_{1} \rightarrow \mathbb{L}^{m+1}$ be given by

$$
\tilde{f}_{1}\left(x_{1}\right)=\left(f_{1}\left(x_{1}\right), 0\right) .
$$

Consider the flat parallel vector subbundle $\mathcal{L}$ of $\operatorname{rank} k=m_{2}+1$ of $N_{\tilde{f}_{1}} M_{1}$ whose fiber at $x_{1} \in M_{1}$ is

$$
\mathcal{L}\left(x_{1}\right)=\operatorname{span}\left\{\tilde{f}_{1}\left(x_{1}\right)\right\} \oplus \mathbb{R}^{m_{2}} .
$$

Let $\phi: M_{\tilde{\sim}} \times \mathbb{L}^{k} \rightarrow \mathcal{L}$ be a parallel vector bundle isometry and let $e \in \mathbb{L}^{k}$ be such that $\phi_{x_{1}}(e)=\tilde{f}_{1}\left(x_{1}\right)$ for all $x_{1} \in M_{1}$. As in the case of warped products into the sphere we have

$$
e+\Omega^{0}\left(\tilde{f}_{1}, \phi\right)=\Omega^{0}\left(\tilde{f}_{1}\right)=\left\{Y \in \mathbb{L}^{k}:\langle Y, e\rangle>0\right\} .
$$

Note that $\mathbb{H}^{k-1} \subset \Omega^{0}\left(\tilde{f}_{1}\right)$ if $e=\left(e_{0}, \ldots, e_{k-1}\right) \in \mathbb{L}^{k}$ is chosen so that $e_{0}<0$.
If $f_{0}: M_{0} \rightarrow \mathbb{H}^{k-1} \subset \mathbb{L}^{k}$ is an isometric immersion, then the partial tube $f: M_{0} \times$ $M_{1} \rightarrow \mathbb{H}^{m} \subset \mathbb{L}^{m+1}$, given by

$$
f\left(x_{0}, x_{1}\right)=\phi_{x_{1}}\left(f_{0}\left(x_{0}\right)\right),
$$

is called the warped product of hyperbolic type of $f_{0}$ and $f_{1}$.
A similar construction can be done by starting with an orthogonal decomposition

$$
\mathbb{L}^{m+1}=\mathbb{R}^{m_{1}} \times \mathbb{L}^{m_{2}}
$$

and an isometric immersion $f_{1}: M_{1} \rightarrow \mathbb{S}^{m_{1}-1} \subset \mathbb{R}^{m_{1}}$. Define $\tilde{f}_{1}: M_{1} \rightarrow \mathbb{L}^{m+1}$ by

$$
\tilde{f}_{1}\left(x_{1}\right)=\left(f_{1}\left(x_{1}\right), 0\right)
$$

and consider the flat parallel vector subbundle $\mathcal{L}$ of rank $k=m_{2}+1$ of $N_{\tilde{f}_{1}} M_{1}$ whose fiber at $x_{1} \in M_{1}$ is

$$
\mathcal{L}\left(x_{1}\right)=\operatorname{span}\left\{\tilde{f}_{1}\left(x_{1}\right)\right\} \oplus \mathbb{L}^{m_{2}} .
$$

As before, let $\phi: M_{1} \times \mathbb{L}^{k} \rightarrow \mathcal{L}$ be a parallel vector bundle isometry and let $e \in \mathbb{L}^{k}$ be such that $\phi_{x_{1}}(e)=\tilde{f}_{1}\left(x_{1}\right)$ for all $x_{1} \in M_{1}$. If $f_{0}: M_{0} \rightarrow \mathbb{H}^{k-1} \cap \Omega^{0}\left(\tilde{f}_{1}\right) \subset \mathbb{L}^{k}$ is an isometric immersion, where

$$
\Omega^{0}\left(\tilde{f}_{1}\right)=e+\Omega^{0}\left(\tilde{f}_{1}, \phi\right)=\left\{Y \in \mathbb{L}^{k}:\langle Y, e\rangle>0\right\}
$$

then the partial tube $f: M_{0} \times M_{1} \rightarrow \mathbb{H}^{m} \subset \mathbb{L}^{m+1}$, given by

$$
f\left(x_{0}, x_{1}\right)=\phi_{x_{1}}\left(f_{0}\left(x_{0}\right)\right),
$$

is called the warped product of elliptic type of $f_{0}$ and $f_{1}$.
To define the notion of a warped product of immersions into the hyperbolic space of parabolic type, consider orthogonal decompositions

$$
\mathbb{L}^{m+1}=\mathbb{L}^{m_{1}+1} \times \mathbb{R}^{m_{2}} \text { and } \mathbb{L}^{m_{1}+1}=\mathbb{R}^{m_{1}-1} \times \mathbb{L}^{2}
$$

and choose a pseudo-orthonormal basis $v_{0}, v_{1}$ of $\mathbb{L}^{2}$ with

$$
\left\langle v_{0}, v_{0}\right\rangle=0=\left\langle v_{1}, v_{1}\right\rangle \text { and }\left\langle v_{0}, v_{1}\right\rangle=1 .
$$

Now let $f_{1}: M_{1} \rightarrow \mathbb{R}^{m_{1}-1}$ be an isometric immersion and define $\tilde{f}_{1}: M_{1} \rightarrow \mathbb{L}^{m+1}$ by

$$
\tilde{f}_{1}\left(x_{1}\right)=\left(\Psi\left(f_{1}\left(x_{1}\right)\right), 0\right)
$$

where $\Psi: \mathbb{R}^{m_{1}-1} \rightarrow \mathbb{V}^{m_{1}} \subset \mathbb{L}^{m_{1}+1}$ is the isometric embedding defined as in Section 9.1 by

$$
\begin{equation*}
\Psi(x)=v_{0}+C x-\frac{1}{2}\|x\|^{2} v_{1} \tag{10.15}
\end{equation*}
$$

in terms of a linear isometry $C: \mathbb{R}^{m_{1}-1} \rightarrow \operatorname{span}\left\{v_{0}, v_{1}\right\}^{\perp} \subset \mathbb{L}^{m_{1}+1}$.
Consider the flat parallel vector subbundle of rank $k=m_{2}+2$ of $N_{\tilde{f}_{1}} M_{1}$ whose fiber at $x_{1} \in M_{1}$ is

$$
\mathcal{L}\left(x_{1}\right)=\operatorname{span}\left\{v_{1}, \tilde{f}_{1}\left(x_{1}\right)\right\} \oplus \mathbb{R}^{m_{2}} .
$$

Write $\mathbb{L}^{k}=\mathbb{L}^{2} \times \mathbb{R}^{m_{2}}$ and let $\phi: M_{1} \times \mathbb{L}^{k} \rightarrow \mathcal{L}$ be a parallel vector bundle isometry such that

$$
\phi_{x_{1}}\left(v_{0}\right)=\tilde{f}_{1}\left(x_{1}\right) \text { and } \phi_{x_{1}}\left(v_{1}\right)=v_{1}
$$

for all $x_{1} \in M_{1}$. Then, if $v_{1}$ is chosen so that $v_{1}^{0}<0$, we have

$$
\mathbb{H}^{k-1} \subset \Omega^{0}\left(\tilde{f}_{1}\right)=\left\{Y \in \mathbb{L}^{k}:\left\langle Y, v_{1}\right\rangle>0\right\}=v_{0}+\Omega^{0}\left(\tilde{f}_{1}, \phi\right) .
$$

Given an isometric immersion $f_{0}: M_{0} \rightarrow \mathbb{H}^{k-1} \subset \mathbb{L}^{k}$, the map $f: M_{0} \times M_{1} \rightarrow \mathbb{H}^{m} \subset$ $\mathbb{L}^{m+1}$, given by

$$
f\left(x_{0}, x_{1}\right)=\phi_{x_{1}}\left(f_{0}\left(x_{0}\right)\right),
$$

is called the warped product of parabolic type of $f_{0}$ and $f_{1}$.
Notice that, although $\tilde{f}_{1}$ does not take values in $\mathbb{H}^{m}$, its parallel translate $\hat{f}_{1}$ does, where

$$
\hat{f}_{1}=\tilde{f}_{1}-\frac{1}{2} v_{1}
$$

and since $\tilde{f}_{1}$ and $\hat{f}_{1}$ have the same normal bundle, we can regard $f$ as the partial tube over $\hat{f}_{1}$ with fiber $f_{0}$.

In each of the preceding types of warped products of immersions, if $f_{1}$ is the identity map, then $f$ is called a rotational submanifold with $f_{0}$ as profile. On the other hand, the reader is asked to check that, if $f_{0}$ is the identity map, then $f$ coincides with the generalized cone over $f_{1}$.

The proof of the next corollary of Proposition 10.5 is also left to the reader.

Corollary 10.11. With $f_{0}, f_{1}, f$ and $\phi$ as above, the following assertions hold:
(i) The map $f$ is an immersion whose induced metric $g$ is the warped product of the metrics $g_{0}$ and $g_{1}$ in $M_{0}$ and $M_{1}$, respectively, with warping function $\rho: M_{0} \rightarrow \mathbb{R}_{+}$ given by

$$
\rho\left(x_{0}\right)=\left\langle f_{0}\left(x_{0}\right), e\right\rangle
$$

in the hyperbolic and elliptic cases, and by

$$
\rho\left(x_{0}\right)=\left\langle f_{0}\left(x_{0}\right), v_{1}\right\rangle
$$

in the parabolic case.
(ii) The normal space of $f$ at $x=\left(x_{0}, x_{1}\right)$ is given by

$$
i_{*} N_{f} M(x)=i_{1 *} N_{f_{1}} M_{1}\left(x_{1}\right) \oplus \phi_{x_{1}}\left(N_{f_{0}} M_{0}\left(x_{0}\right)\right) \subset N_{\tilde{f}_{1}} M_{1}\left(x_{1}\right)
$$

where $i$ is the inclusion of $\mathbb{Q}_{\epsilon}^{m}$ into $\mathbb{R}_{\mu}^{m+1}$ and $i_{1}$ stands either for the inclusion of $\mathbb{Q}_{\epsilon}^{m_{1}-1}$ into $\mathbb{R}_{\mu}^{m_{1}}$ in the elliptic and hyperbolic cases, or for the map $\Psi$ in the parabolic one.
(iii) The second fundamental form of $f$ at $x=\left(x_{0}, x_{1}\right)$ is given by

$$
i_{*} \alpha^{f}\left(\tau_{1 *}^{x} X_{1}, \tau_{1 *}^{x} Y_{1}\right)=\left\langle f_{0}\left(x_{0}\right), e_{1}\right\rangle\left(i_{1 *} \alpha^{f_{1}}\left(X_{1}, Y_{1}\right)-g_{1}\left(X_{1}, Y_{1}\right) \phi_{x_{1}}\left(e_{1}^{\perp}\right)\right),
$$

in the hyperbolic and elliptic cases, and by

$$
i_{*} \alpha^{f}\left(\tau_{1 *}^{x} X_{1}, \tau_{1 *}^{x} Y_{1}\right)=\left\langle f_{0}\left(x_{0}\right), v_{1}\right\rangle\left(i_{1 *} \alpha^{f_{1}}\left(X_{1}, Y_{1}\right)-g_{1}\left(X_{1}, Y_{1}\right) \phi_{x_{1}}\left(v_{1}^{\perp}\right)\right)
$$

in the parabolic case, where $e_{1}^{\perp}$ and $v_{1}^{\perp}$ are the orthogonal projections of $e_{1}$ and $v_{1}$ onto $N_{f_{0}} M_{0}\left(x_{0}\right)$,

$$
\alpha^{f}\left(\tau_{0 *}^{x} X_{0}, \tau_{1 *}^{x} X_{1}\right)=0
$$

and

$$
\alpha^{f}\left(\tau_{0 *}^{x} X_{0}, \tau_{0 *}^{x} Y_{0}\right)=\phi_{x_{1}}\left(\alpha^{f_{0}}\left(X_{0}, Y_{0}\right)\right)
$$

for all $X_{0}, Y_{0} \in T_{x_{0}} M_{0}$ and $X_{1}, Y_{1} \in T_{x_{1}} M_{1}$.

### 10.2.3 Partial tubes over extrinsic products

Next we consider partial tubes in $\mathbb{Q}_{\epsilon}^{m}, \epsilon \in\{-1,0,1\}$, over an extrinsic product $\tilde{f}: \tilde{M}=\Pi_{a=1}^{r} M_{a} \rightarrow \mathbb{Q}_{\epsilon}^{m}$. Assume as before that there exists a parallel vector bundle isometry $\phi: \tilde{M} \times \mathbb{R}_{\mu}^{k+|\epsilon|} \rightarrow \tilde{\mathcal{L}}, 2 \mu=1-\epsilon$, onto a flat parallel subbundle $\tilde{\mathcal{L}}$ of $N_{\tilde{f}} \tilde{M}$, with $\tilde{f}$ regarded as an isometric immersion into $\mathbb{R}_{\mu}^{m+1}$ and the subbundle $\tilde{\mathcal{L}}$ having the position vector as a section if $\epsilon \in\{-1,1\}$. If $\epsilon=0$ (respectively, $\epsilon \in\{-1,1\}$ ), consider an isometric immersion $f_{0}: M_{0} \rightarrow \Omega^{0}(\tilde{f} ; \phi) \subset \mathbb{R}^{k}$ (respectively, $f_{0}: M_{0} \rightarrow$ $\left(e+\Omega^{0}(\tilde{f} ; \phi)\right) \cap \mathbb{Q}_{\epsilon}^{k} \subset \mathbb{R}_{\mu}^{k+1}$, where $\phi_{\tilde{x}}(e)=\tilde{f}(\tilde{x})$ for all $\left.\tilde{x} \in \tilde{M}\right)$. Then the partial tube $f: M=\Pi_{i=0}^{r} M_{i} \rightarrow \mathbb{Q}_{\epsilon}^{m}$ determined by $\left(f_{0}, \tilde{f}, \phi\right)$ has the following properties.

Proposition 10.12. The metric induced on $M$ by $f$ is polar and the second fundamental form of $f$ is adapted to the product net of $M$.

Proof: We give the proof for immersions into Euclidean space, the other cases being similar. Thus there exist an orthogonal decomposition

$$
\mathbb{R}^{m}=\Pi_{j=0}^{r} \mathbb{R}^{m_{j}}
$$

with $\mathbb{R}^{m_{0}}$ possibly trivial, a vector $v_{0} \in \mathbb{R}^{m_{0}}$ (in case $\mathbb{R}^{m_{0}}$ is nontrivial) and isometric immersions $f_{a}: M_{a} \rightarrow \mathbb{R}^{n_{a}}, 1 \leq a \leq r$, such that

$$
\tilde{f}\left(x_{1}, \ldots, x_{r}\right)=\left(v_{0}, f_{1}\left(x_{1}\right), \ldots, f_{r}\left(x_{r}\right)\right)
$$

for all $\tilde{x}=\left(x_{1}, \ldots, x_{r}\right) \in \tilde{M}$.
Regard $M$ as the product $M=M_{0} \times \tilde{M}$ and denote by $\tilde{\pi}: M \rightarrow \tilde{M}$ the projection. By Corollary 10.8, the metric induced by $f$ is given by

$$
\begin{equation*}
g=\pi_{0}^{*} g_{0}+\tilde{\pi}^{*}\left(\tilde{g} \circ \pi_{0}\right), \tag{10.16}
\end{equation*}
$$

where $g_{0}$ is the metric on $M_{0}$ and, for all $x=\left(x_{0}, \tilde{x}\right) \in M=M_{0} \times \tilde{M}$, the metric $\tilde{g}\left(x_{0}\right)$ on $\tilde{M}$ is given, in terms of the product metric $\tilde{g}$ of $\tilde{M}$, by

$$
\tilde{g}\left(x_{0}\right)(\tilde{X}, \tilde{Y})=\tilde{g}\left(P^{2} \tilde{X}, \tilde{Y}\right)
$$

for all $\tilde{X}, \tilde{Y} \in T_{\tilde{x}} \tilde{M}$, where

$$
P=P\left(x_{0}, \tilde{x}\right)=I-A_{\phi_{\tilde{x}}\left(f_{0}\left(x_{0}\right)\right)}^{\tilde{f}} .
$$

Since $\mathcal{L}$ is a flat parallel subbundle of $N_{\tilde{f}} \tilde{M}$, so are its projections $\mathcal{L}_{a}$ onto $N_{f_{a}} M_{a}$ for $1 \leq a \leq r$. Thus there exist parallel vector bundle isometries $\phi^{a}: M_{a} \times \mathbb{R}^{m_{a}} \rightarrow \mathcal{L}_{a}$, $1 \leq a \leq r$, such that $\phi$ is the restriction to $\tilde{M} \times \mathbb{R}^{k}$ of the parallel vector bundle isometry $\tilde{\phi}: \tilde{M} \times \mathbb{R}^{\ell} \rightarrow \oplus_{a=1}^{r} \mathcal{L}_{a}$, with $\mathbb{R}^{\ell}=\Pi_{a=1}^{r} \mathbb{R}^{m_{a}}$, given by

$$
\begin{equation*}
\tilde{\phi}_{\tilde{x}} \sum_{a=1}^{r} v_{a}=\sum_{a=1}^{r} \phi_{\tilde{x}_{a}}^{a} v_{a} \tag{10.17}
\end{equation*}
$$

for all $\tilde{x}=\left(\tilde{x}_{1}, \ldots, \tilde{x}_{r}\right) \in \tilde{M}$. We denote by $\tilde{\pi}_{a}$ either of the projections $\oplus_{a=1}^{r} \mathcal{L}_{a} \mapsto \mathcal{L}_{a}$, $\Pi_{a=1}^{r} \mathbb{R}^{m_{a}} \mapsto \mathbb{R}^{m_{a}}$ or $\tilde{M} \mapsto M_{a}$. Also, for all $\tilde{x}=\left(\tilde{x}_{1}, \ldots, \tilde{x}_{r}\right) \in \tilde{M}$, we denote by $\tilde{\tau}_{a}^{\tilde{x}}: M_{a} \rightarrow \tilde{M}$ the inclusion given by

$$
\tilde{\tau}_{a}^{\tilde{x}}\left(x_{a}\right)=\left(\tilde{x}_{1}, \ldots, x_{a}, \ldots, \tilde{x}_{r}\right) .
$$

By part (iv) of Proposition 8.3,

$$
\begin{equation*}
\tilde{f}_{*} A_{\xi}^{\tilde{f}} \tilde{\tau}_{a *}^{\tilde{x}}=f_{a_{*}} A_{\tilde{\pi}_{a} \xi}^{f_{a}} \tag{10.18}
\end{equation*}
$$

for all $\xi \in \Gamma\left(N_{\tilde{f}} \tilde{M}\right)$. Therefore

$$
\begin{aligned}
\tilde{f}_{*} A_{\phi_{\bar{x}}\left(f_{0}\left(x_{0}\right)\right)}^{\tilde{f}} \tilde{\tau}_{a *}^{\tilde{x}} & =f_{a_{*}} A_{\tilde{\pi}_{a} \phi_{\bar{x}}\left(f_{0}\left(x_{0}\right)\right)}^{f_{0}} \\
& =f_{a_{*}} A_{\phi_{\tilde{x}_{a}}^{a}\left(\tilde{\pi}_{a}\left(f_{0}\left(x_{0}\right)\right)\right)}^{f_{0}}
\end{aligned}
$$

for all $1 \leq a \leq r$. Given $\tilde{X} \in T_{\tilde{x}} \tilde{M}$, let $\tilde{X}=\sum_{a=1}^{r} \tilde{X}_{a}$ be its decomposition with respect to the product net of $\tilde{M}$. Then

$$
\begin{aligned}
\tilde{f}_{*} A_{\phi_{\bar{x}}\left(f_{0}\left(x_{0}\right)\right)}^{\tilde{f}} \tilde{X} & =\sum_{a=1}^{r} \tilde{f}_{*} A_{\phi_{\bar{x}}\left(f_{0}\left(x_{0}\right)\right)}^{\tilde{f}} \tilde{X}^{a} \\
& =\sum_{a=1}^{r} \tilde{f}_{*} A_{\phi_{\bar{x}}\left(f_{0}\left(x_{0}\right)\right)}^{\tilde{f}} \tilde{\tau}_{a *}^{\tilde{x}} \tilde{\pi}_{a *} \tilde{X}^{a} \\
& =\sum_{a=1}^{r} f_{a_{*}} A_{\phi_{\tilde{x}_{a}}^{a}\left(\tilde{\pi}_{a}\left(f_{0}\left(x_{0}\right)\right)\right)}^{f_{a}} \tilde{\pi}_{a *} \tilde{X}^{a} .
\end{aligned}
$$

It follows that

$$
\begin{equation*}
\tilde{g}\left(x_{0}\right)=\sum_{a=1}^{r} \tilde{\pi}_{a}^{*} g_{a}\left(x_{0}\right), \tag{10.19}
\end{equation*}
$$

where $g_{a}\left(x_{0}\right)$ is the metric on $M_{a}$ given, in terms of the metric $g_{a}$ of $M_{a}$, by

$$
\begin{equation*}
g_{a}\left(x_{0}\right)\left(X_{a}, Y_{a}\right)=g_{a}\left(\left(I-A_{\phi_{\tilde{x}_{a}}\left(\tilde{\pi}_{a}\left(f_{0}\left(x_{0}\right)\right)\right)}^{f_{a}}\right)^{2} X_{a}, Y_{a}\right) \tag{10.20}
\end{equation*}
$$

for all $X_{a}, Y_{a} \in T_{\tilde{x}_{a}} M_{a}$. Since $\tilde{\pi}_{a} \circ \tilde{\pi}=\pi_{a}$ for $1 \leq a \leq r$, we conclude from (10.16) and (10.19) that $g$ has the form (10.1), with $g_{a}\left(x_{0}\right)$ as in 10.20). The assertion on the second fundamental form of $f$ is a consequence of (10.18) and part (iv) of Proposition 10.5.

### 10.2.4 The decomposition theorem

In this section we prove a converse of Proposition 10.12 that provides a general decomposition theorem for immersions of product manifolds. The proof relies on the following lemma.

Lemma 10.13. Let $f: M \rightarrow \mathbb{R}_{\mu}^{m}$ be an isometric immersion and let $D$ be a vector subbundle of TM. Then the following conditions on $D$ are equivalent:
(i) $D$ is totally geodesic and $\alpha^{f}$ is adapted to the net $\left(D, D^{\perp}\right)$.
(ii) $D$ is integrable and $f_{*} D^{\perp}$ is constant in $\mathbb{R}_{\mu}^{m}$ along each leaf of $D$.

Proof: If $D$ is integrable, then the subbundle $f_{*} D^{\perp}$ is constant in $\mathbb{R}_{\mu}^{m}$ along each leaf of $D$ if and only if $\tilde{\nabla}_{X} f_{*} Y \in f_{*} D^{\perp}$ for all $X \in \Gamma(D)$ and $Y \in \Gamma\left(D^{\perp}\right)$. Since

$$
\tilde{\nabla}_{X} f_{*} Y=f_{*} \nabla_{X} Y+\alpha^{f}(X, Y)
$$

this is the case if and only if $\nabla_{X} Y \in \Gamma\left(D^{\perp}\right)$ and $\alpha^{f}(X, Y)=0$ for all $X \in \Gamma(D)$ and $Y \in \Gamma\left(D^{\perp}\right)$, that is, if and only if the conditions in part ( $i$ ) hold.

Theorem 10.14. Let $f: M=\prod_{i=0}^{r} M_{r} \rightarrow \mathbb{Q}_{c}^{m}$ be an isometric immersion of a product manifold endowed with a polar metric. If the second fundamental form of $f$ is adapted to the product net of $M$, then $f$ is a partial tube over an extrinsic product $\tilde{f}: \tilde{M}=$ $\Pi_{a=1}^{r} M_{r} \rightarrow \mathbb{Q}_{c}^{m}$ of immersions.

Assuming, for simplicity, that $c=\epsilon \in\{-1,0,1\}$, then the statement is, more precisely, that there exist an extrinsic product $\tilde{f}: \tilde{M}=\prod_{a=1}^{r} M_{r} \rightarrow \mathbb{Q}_{\epsilon}^{m} \subset \mathbb{R}_{\mu}^{m+1}$ of immersions, a parallel vector bundle isometry $\phi: \tilde{M} \times \mathbb{R}_{\mu}^{k+|\epsilon|} \rightarrow \tilde{\mathcal{L}}, 2 \mu=1-\epsilon$, onto a flat parallel subbundle of $N_{\tilde{f}} \tilde{M}$ (with $\tilde{f}$ regarded as an isometric immersion into $\mathbb{R}_{\mu}^{m+1}$ and the subbundle $\tilde{\mathcal{L}}$ having the position vector as a section if $\epsilon \in\{-1,1\}$ ), and an isometric immersion $f_{0}: M_{0} \rightarrow \mathbb{R}_{\mu}^{k+|\epsilon|}$ (with $f_{0}\left(M_{0}\right) \subset \Omega^{0}(\tilde{f} ; \phi)$ if $\epsilon=0$ and $f_{0}\left(M_{0}\right) \subset\left(e+\Omega^{0}(\tilde{f} ; \phi)\right) \cap \mathbb{Q}_{\epsilon}^{k} \subset \mathbb{R}_{\mu}^{k+1}$ if $\epsilon \in\{-1,1\}$, where $\phi_{\tilde{x}}(e)=\tilde{f}(\tilde{x})$ for all $\left.\tilde{x} \in \tilde{M}\right)$, such that $f$ is the partial tube determined by $\left(f_{0}, \tilde{f}, \phi\right)$.
Proof: First we give the proof for the case in which $r=1$ and $\epsilon=0$. For a fixed $\bar{x}_{0} \in M_{0}$, let $\mu_{\bar{x}_{0}}$ be the inclusion of $M_{1}$ into $M=M_{0} \times M_{1}$ given by

$$
\mu_{\bar{x}_{0}}\left(x_{1}\right)=\left(\bar{x}_{0}, x_{1}\right)
$$

and define $f_{1}: M_{1} \rightarrow \mathbb{R}^{m}$ by $f_{1}=f \circ \mu_{\bar{x}_{0}}$. Let $\mathcal{E}=\left(E_{0}, E_{1}\right)$ be the product net of M. By Proposition 10.1, $\mathcal{E}$ is an orthogonal net and $E_{0}$ is totally geodesic. Given $x_{1} \in M_{1}$, it follows from Lemma 10.13 that the image by $f$ of the leaf $M_{0} \times\left\{x_{1}\right\}$ of $E_{0}$ is contained in the affine normal space of $f_{1}$ at $x_{1}$, that is,

$$
f\left(x_{0}, x_{1}\right) \in f_{1}\left(x_{1}\right)+N_{f_{1}} M_{1}\left(x_{1}\right)
$$

for all $x_{0} \in M_{0}$. Hence, for each $x_{0} \in M_{0}$, we can regard

$$
x_{1} \in M_{1} \mapsto \xi^{x_{0}}\left(x_{1}\right)=f\left(x_{0}, x_{1}\right)-f_{1}\left(x_{1}\right)
$$

as a normal vector field along $f_{1}$. Let $X_{1} \in T_{x_{1}} M_{1}$. Again from Lemma 10.13 we obtain

$$
\tilde{\nabla}_{X_{1}} \xi^{x_{0}}=f_{*}\left(x_{0}, x_{1}\right) \mu_{x_{0} *} X_{1}-f_{*}\left(\bar{x}_{0}, x_{1}\right) \mu_{\bar{x}_{0} *} X_{1} \in f_{*}\left(\bar{x}_{0}, x_{1}\right) E_{1}\left(\bar{x}_{0}, x_{1}\right)=f_{1_{*}} T_{x_{1}} M_{1} .
$$

Hence $\xi^{x_{0}}$ is a parallel normal vector field along $f_{1}$. For a fixed $x_{1} \in M_{1}$, set

$$
\mathcal{L}\left(x_{1}\right)=\operatorname{span}\left\{\xi^{x_{0}}\left(x_{1}\right): x_{0} \in M_{0}\right\} .
$$

Then, for any pair of points $x_{1}, \tilde{x}_{1} \in M_{1}$, parallel transport in the normal connection of $f_{1}$ along any curve joining $x_{1}$ and $\tilde{x}_{1}$ takes $\mathcal{L}\left(x_{1}\right)$ onto $\mathcal{L}\left(\tilde{x}_{1}\right)$. Thus such subspaces define a parallel flat normal subbundle $\mathcal{L}$ of $N_{f_{1}} M_{1}$, and there exists a parallel vector bundle isometry $\phi: M_{1} \times \mathbb{R}^{k} \rightarrow \mathcal{L}$ such that

$$
\phi_{x_{1}}^{-1}\left(\xi^{x_{0}}\left(x_{1}\right)\right)=\phi_{\tilde{x}_{1}}^{-1}\left(\xi^{x_{0}}\left(\tilde{x}_{1}\right)\right)
$$

for all $x_{0} \in M_{0}$ and $x_{1}, \tilde{x}_{1} \in M_{1}$. Therefore there is a well-defined map $f_{0}: M_{0} \rightarrow \mathbb{R}^{k}$ such that

$$
\phi_{x_{1}}\left(f_{0}\left(x_{0}\right)\right)=\xi^{x_{0}}\left(x_{1}\right)
$$

for all $x_{1} \in M_{1}$, and hence $f$ is given by (10.2). Moreover, from

$$
f_{*}(x) \tau_{0 *}^{x} X_{0}=\phi_{x_{1}}\left(f_{0 *} X_{0}\right)
$$

for all $x=\left(x_{0}, x_{1}\right) \in M$ and $X_{0} \in T_{x_{0}} M_{0}$, it follows that $f_{0}$ is an isometric immersion.
Suppose now that ( $r=1$ and) $\epsilon \in\{-1,1\}$, and apply the preceding argument to $f$, regarded as a map into $\mathbb{R}_{\mu}^{m+1} \supset \mathbb{Q}_{\epsilon}^{m}$. Then $f_{1}\left(M_{1}\right) \subset \mathbb{Q}_{\epsilon}^{m} \subset \mathbb{R}_{\mu}^{m+1}$ and we may assume that the vector subbundle $\mathcal{L}$ has the position vector of $f_{1}$ in $\mathbb{R}_{\mu}^{m+1}$ as a section. We obtain a parallel vector bundle isometry $\phi: M_{1} \times \mathbb{R}_{\mu}^{k+1} \rightarrow \mathcal{L}$ and an isometric immersion $\tilde{f}_{0}: M_{0} \rightarrow \mathbb{R}_{\mu}^{k+1}$ such that

$$
f\left(x_{0}, x_{1}\right)=f_{1}\left(x_{1}\right)+\phi_{x_{1}}\left(\tilde{f}_{0}\left(x_{0}\right)\right)
$$

for all $x_{0} \in M_{0}$ and $x_{1} \in M_{1}$. Let $e_{1} \in \mathbb{R}_{\mu}^{k+1}$ be such that

$$
\phi_{x_{1}}\left(e_{1}\right)=f_{1}\left(x_{1}\right)
$$

for all $x_{1} \in M_{1}$. Defining

$$
f_{0}\left(x_{0}\right)=\tilde{f}_{0}\left(x_{0}\right)+e_{1}
$$

for all $x_{0} \in M_{0}$, we see that

$$
f\left(x_{0}, x_{1}\right)=\phi_{x_{1}}\left(f_{0}\left(x_{0}\right)\right),
$$

hence $f_{0}\left(M_{0}\right) \subset \mathbb{Q}_{\epsilon}^{k} \subset \mathbb{R}_{\mu}^{k+1}$. It follows that $f$ is the partial tube over $f_{1}$ with fiber $f_{0}$.
Suppose now that $r$ is arbitrary. As before, fix $\bar{x}_{0} \in M_{0}$ and set $\tilde{f}=f \circ \mu_{\bar{x}_{0}}$, where $\mu_{\bar{x}_{0}}$ is the inclusion of $\tilde{M}=\prod_{a=1}^{r} M_{a}^{n_{a}}$ into $M$ given by (8.15). Then the metric $\tilde{g}$ induced by $\tilde{f}$ is the product metric

$$
\tilde{g}=\mu_{\bar{x}_{0}}^{*} g=\sum_{a=1}^{r} \tilde{\pi}_{a}^{*} g_{a}\left(\bar{x}_{0}\right),
$$

where $\tilde{\pi}_{a}: \Pi_{a=1}^{r} M_{a}^{n_{a}} \rightarrow M_{a}^{n_{a}}$ is the projection. On the other hand, the second fundamental form $\alpha^{\tilde{f}}$ of $\tilde{f}$ is given by

$$
\begin{equation*}
\alpha^{\tilde{f}}(X, Y)=\alpha^{f}\left(\mu_{\bar{x}_{0} *} X, \mu_{\bar{x}_{0} *} Y\right)+f_{*} \alpha^{\mu_{\bar{x}_{0}}}(X, Y) . \tag{10.21}
\end{equation*}
$$

By Proposition 10.1, the product net $\mathcal{E}=\left(E_{i}\right)_{i=0, \ldots, r}$ of $M$ is an orthogonal net such that $E_{a}^{\perp}$ is totally geodesic for $1 \leq a \leq r$. In particular, this implies that

$$
\left\langle\nabla_{X_{b}} X_{a}, X_{0}\right\rangle=0
$$

for all $X_{a} \in \Gamma\left(E_{a}\right), X_{b} \in \Gamma\left(E_{b}\right), 1 \leq a \neq b \leq r$, and $X_{0} \in \Gamma\left(E_{0}\right)$, and hence $\alpha^{\mu_{\bar{x}_{0}}}$ is adapted to the product net $\overline{\mathcal{E}}=\left(\bar{E}_{a}\right)_{a=1, \ldots, r}$ of $\tilde{M}$. Using this and the fact that $\alpha^{f}$ is adapted to $\mathcal{E}$, it follows from (10.21 that $\alpha^{\tilde{f}}$ is adapted to $\overline{\mathcal{E}}$. Hence $\tilde{f}$ is an extrinsic product of isometric immersions by Theorem 8.4, Corollary 8.6 and Corollary 8.8.

Finally, we apply the case $r=1$ just proved to $f: M \rightarrow \mathbb{Q}_{\epsilon}^{m}$, regarding $M$ as the product of $M_{0}$ and $\tilde{M}$. We conclude that there exist a parallel vector bundle isometry $\phi: \tilde{M} \times \mathbb{R}_{\mu}^{k+|\epsilon|} \rightarrow \mathcal{L}$ onto a flat parallel subbundle of $N_{\tilde{f}} \tilde{M}$ (with $\tilde{f}$ regarded as an isometric immersion into $\mathbb{R}_{\mu}^{m+|\epsilon|}$ ), and an isometric immersion $f_{0}: M_{0} \rightarrow \Omega(\tilde{f} ; \phi) \cap \mathbb{Q}_{\epsilon}^{k} \subset$ $\mathbb{R}_{\mu}^{k+|\epsilon|}$ such that $f$ is the partial tube determined by $\left(f_{0}, \tilde{f}, \phi\right)$.

Let $g: L^{n-\nu} \rightarrow \mathbb{Q}_{c}^{m}$ be an isometric immersion that carries a parallel flat normal subbundle $\mathcal{V}$ of rank $\nu$. The $n$-dimensional submanifold parametrized, on the open subset of regular points, in terms of the exponential map of $\mathbb{Q}_{c}^{m}$ by

$$
\gamma \in \mathcal{V} \mapsto \exp _{g(\pi(\gamma))}(\gamma)
$$

where $\pi: \mathcal{V} \rightarrow L^{n-\nu}$ is the projection, is called the generalized cylinder in $\mathbb{Q}_{c}^{m}$ over $g$ determined by $\mathcal{V}$. We leave to the reader to check that any such submanifold carries a relative nullity distribution $\Delta$ of rank $\nu$, whose leaves are the fibers of $\mathcal{V}$, and that the conullity distribution $\Delta^{\perp}$ is integrable, its leaves being given by the parallel sections of $\nu$. The following consequence of Theorem 10.14 shows that generalized cylinders are the only submanifolds that have a relative nullity distribution with integrable conullity.

Proposition 10.15. Let $f: M \rightarrow \mathbb{Q}_{\epsilon}^{m}, \epsilon \in\{-1,0,1\}$, be an isometric immersion with constant index of relative nullity $\nu$. Assume that the conullity distribution $\Delta^{\perp}$ is integrable. Then $f$ is locally (globally if $M$ is simply connected and the leaves of the relative nullity $\Delta$ are complete) a generalized cylinder over an isometric immersion $f_{1}: M_{1}^{n-\nu} \rightarrow \mathbb{Q}_{\epsilon}^{m}$.

Proof: Since $\Delta$ is a totally geodesic distribution, by Theorem 10.3 there exists locally (globally if $M$ is simply connected and the leaves of $\Delta$ are complete) a product representation $\psi: M_{0} \times M_{1} \rightarrow M$ of the orthogonal net $\left(\Delta, \Delta^{\perp}\right)$ which is an isometry with respect to a polar metric on $M_{0} \times M_{1}$.

The second fundamental form of $f \circ \psi$ is clearly adapted to the product net $\mathcal{E}=\left(E_{0}, E_{1}\right)$ of $M_{0} \times M_{1}$, for $E_{0}$ is the relative nullity distribution of $f \circ \psi$. It follows from Theorem 10.14 that there exists an isometric immersion $f_{1}: M_{1}^{n-\nu} \rightarrow \mathbb{Q}_{\epsilon}^{m}$, a parallel vector bundle isometry $\phi: M_{1} \times \mathbb{R}_{\mu}^{\nu+|\epsilon|} \rightarrow \tilde{\mathcal{L}}, 2 \mu=1-\epsilon$, onto a flat parallel subbundle of $N_{f_{1}} M_{1}$ (with $f_{1}$ regarded as an isometric immersion into $\mathbb{R}_{\mu}^{m+1} \supset \mathbb{Q}_{\epsilon}^{m}$ and the subbundle $\tilde{\mathcal{L}}$ having the position vector as a section if $\epsilon \in\{-1,1\}$ ), and an isometric immersion $f_{0}: M_{0} \rightarrow \mathbb{R}_{\mu}^{\nu+|\epsilon|}$ (with $f_{0}\left(M_{0}\right) \subset \Omega^{0}\left(f_{1} ; \phi\right)$ if $\epsilon=0$ and $f_{0}\left(M_{0}\right) \subset$ $\left(e+\Omega^{0}\left(f_{1} ; \phi\right)\right) \cap \mathbb{Q}_{\epsilon}^{\nu} \subset \mathbb{R}_{\mu}^{\nu+1}$ if $\epsilon \in\{-1,1\}$, where $\phi_{x_{1}}(e)=f_{1}\left(x_{1}\right)$ for all $\left.x_{1} \in M_{1}\right)$, such that $f \circ \psi$ is the partial tube determined by $\left(f_{0}, f_{1}, \phi\right)$.

Since $E_{0}$ is the relative nullity distribution of $f \circ \psi$, it follows that $f_{0}$ must be an inclusion of an open subset $M_{0}$ of either $\Omega^{0}\left(f_{1} ; \phi\right)$ or $e+\Omega^{0}\left(f_{1} ; \phi\right)$, according to whether $\epsilon=0$ or $\epsilon \in\{-1,1\}$, respectively. If $\mathcal{V}$ stands for the flat parallel normal subbundle of $N_{f_{1}} M_{1}$ given by $i_{*} \mathcal{V} \oplus \operatorname{span}\left\{f_{1}\right\}=\tilde{\mathcal{L}}$ if $\epsilon \in\{-1,1\}$, where $i: \mathbb{Q}_{\epsilon}^{\nu} \rightarrow \mathbb{R}_{\mu}^{\nu+1}$ is the inclusion, and $\mathcal{V}=\tilde{\mathcal{L}}$ if $\epsilon=0$, we conclude that $f$ is locally (globally if $M$ is simply connected and the leaves of $\Delta$ are complete) a generalized cylinder over $f_{1}$ determined by $\nu$.

### 10.3 Isometric immersions of warped products

The main result of this section is a decomposition theorem for isometric immersions of warped product manifolds. We first introduce the general notion of a warped product of immersions into space forms, extending the case of two factors discussed in part (ii) of Examples 10.9 and in Section 10.2.2.

### 10.3.1 Warped products of immersions into Euclidean space

The general notion of a warped product of immersions into Euclidean space is defined as follows. Start with an orthogonal decomposition

$$
\mathbb{R}^{m}=\Pi_{j=0}^{r} \mathbb{R}^{m_{j}},
$$

with $\mathbb{R}^{m_{0}}$ possibly trivial, and isometric immersions $\tilde{f}_{a}: M_{a} \rightarrow \mathbb{R}^{m_{a}}, 1 \leq a \leq r$. Assume that there exists $1 \leq s \leq r$ such that $\tilde{f}_{a}\left(M_{a}\right) \subset \mathbb{S}^{m_{a}-1}$ for $1 \leq a \leq s$, so there are isometric immersions $f_{a}: M_{a} \rightarrow \mathbb{S}^{m_{a}-1}$ such that $\tilde{f}_{a}=i_{a} \circ f_{a}, 1 \leq a \leq s$, where $i_{a}: \mathbb{S}^{m_{a}-1} \rightarrow \mathbb{R}^{m_{a}}$ is the inclusion. Set $f_{a}=\tilde{f}_{a}$ and let $i_{a}: \mathbb{R}^{m_{a}} \rightarrow \mathbb{R}^{m_{a}}$ be the identity map for $s+1 \leq a \leq r$ if $s<r$.

Now let $\tilde{f}: \tilde{M}=\Pi_{a=1}^{r} M_{a} \rightarrow \mathbb{R}^{m}$ be the extrinsic product of $f_{1}, \ldots, f_{r}$ given by

$$
\tilde{f}\left(x_{1}, \ldots, x_{r}\right)=\left(0, \tilde{f}_{1}\left(x_{1}\right), \ldots, \tilde{f}_{r}\left(x_{r}\right)\right)
$$

for all $\tilde{x}=\left(x_{1}, \ldots, x_{r}\right) \in \tilde{M}$. The subbundle $\mathcal{L}$ of $N_{\tilde{f}} \tilde{M}$ whose fiber at $\tilde{x}$ is

$$
\mathcal{L}(\tilde{x})=\mathbb{R}^{m_{0}} \oplus \operatorname{span}\left\{\tilde{f}_{1}\left(x_{1}\right), \ldots, \tilde{f}_{s}\left(x_{s}\right)\right\}
$$

is then parallel and flat, so there is a parallel vector bundle isometry $\phi: \tilde{M} \times \mathbb{R}^{k} \rightarrow \mathcal{L}$, where $k=s+m_{0}$.

Let $e_{1}, \ldots, e_{s} \in \mathbb{R}^{k}$ be such that $\phi_{\tilde{x}}\left(e_{a}\right)=\tilde{f}_{a}\left(x_{a}\right)$ for $1 \leq a \leq s$ and define

$$
\Omega^{0}(\tilde{f})=\left\{Y \in \mathbb{R}^{k}:\left\langle Y, e_{a}\right\rangle>0 \text { for all } 1 \leq a \leq s\right\} .
$$

Given an isometric immersion $f_{0}: M_{0} \rightarrow \Omega^{0}(\tilde{f}) \subset \mathbb{R}^{k}$, the map $f: M=\Pi_{i=0}^{r} M_{i} \rightarrow \mathbb{R}^{m}$ given by

$$
\begin{equation*}
f\left(x_{0}, x_{1}, \ldots, x_{r}\right)=f\left(x_{0}, \tilde{x}\right)=\left(0, \ldots, 0, \tilde{f}_{s+1}\left(x_{s+1}\right), \ldots, \tilde{f}_{r}\left(x_{r}\right)\right)+\phi_{\tilde{x}}\left(f_{0}\left(x_{0}\right)\right) \tag{10.22}
\end{equation*}
$$

is called the warped product of $f_{0}, f_{1}, \ldots, f_{r}$.
Proposition 10.16. The following assertions hold:
(i) The map $f$ is an immersion whose induced metric $g$ is the warped product of the metrics $g_{0}, \ldots, g_{r}$ of $M_{0}, \ldots, M_{r}$ with warping functions $\rho_{a}: M_{0} \rightarrow \mathbb{R}_{+}$given by

$$
\rho_{a}\left(x_{0}\right)=\left\{\begin{array}{l}
\left\langle f_{0}\left(x_{0}\right), e_{a}\right\rangle \text { if } 1 \leq a \leq s \\
1 \quad \text { if } s+1 \leq a \leq r .
\end{array}\right.
$$

(ii) The normal space of $f$ at $x=\left(x_{0}, x_{1}, \ldots, x_{r}\right)=\left(x_{0}, \tilde{x}\right)$ is

$$
N_{f} M(x)=\oplus_{a=1}^{r} i_{a *} N_{f_{a}} M_{a}\left(x_{a}\right) \oplus \phi_{\tilde{x}}\left(N_{f_{0}} M_{0}\left(x_{0}\right)\right) \subset N_{\tilde{f}} \tilde{M}(\tilde{x}) .
$$

(iii) The second fundamental form of $f$ at $x$ is given by

$$
\alpha^{f}\left(\tau_{a *}^{x} X_{a}, \tau_{a^{*}}^{x} Y_{a}\right)=\left\{\begin{array}{l}
\rho_{a}\left(x_{0}\right)\left(i_{a *} \alpha^{f_{a}}\left(X_{a}, Y_{a}\right)-g_{a}\left(X_{a}, Y_{a}\right) \phi_{\tilde{x}}\left(e_{a}^{\perp}\right)\right) \text { if } 1 \leq a \leq s  \tag{10.23}\\
i_{a *} \alpha^{f_{a}}\left(X_{a}, Y_{a}\right) \text { if } s+1 \leq a \leq r
\end{array}\right.
$$

for all $X_{a}, Y_{a} \in T_{x_{a}} M_{a}$, where $e_{a}^{\perp}$ denotes the orthogonal projection of $e_{a}$ onto $N_{f_{0}} M_{0}\left(x_{0}\right)$,

$$
\begin{equation*}
\alpha^{f}\left(\tau_{0 *}^{x} X_{0}, \tau_{0 *}^{x} Y_{0}\right)=\phi_{\tilde{x}}\left(\alpha^{f_{0}}\left(X_{0}, Y_{0}\right)\right) \tag{10.24}
\end{equation*}
$$

for all $X_{0}, Y_{0} \in T_{x_{0}} M_{0}$, and

$$
\begin{equation*}
\alpha^{f}\left(\tau_{i *}^{x} X_{i}, \tau_{j *}^{x} X_{j}\right)=0 \tag{10.25}
\end{equation*}
$$

for all $X_{i} \in T_{x_{i}} M_{i}$ and $X_{j} \in T_{x_{j}} M_{j}, 0 \leq i \neq j \leq r$.
Proof: Given $\bar{x}=\left(\bar{x}_{0}, \bar{x}_{1}, \ldots, \bar{x}_{r}\right)=\left(\bar{x}_{0}, \hat{x}\right)$, where $\hat{x}=\left(\bar{x}_{1}, \ldots, \bar{x}_{r}\right)$, we denote by $\hat{\tau}_{1}^{\bar{x}}: \tilde{M} \rightarrow M$ the inclusion given by $\tilde{x} \mapsto\left(\bar{x}_{0}, \tilde{x}\right)$. Notice that $\tau_{a}^{\bar{x}}=\hat{\tau}_{1}^{\bar{x}} \circ \tilde{\tau}_{a}^{\hat{x}}$. From part (iv) of Proposition 8.3 we obtain

$$
\begin{equation*}
A_{\phi_{\tilde{x}}(Y)}^{\tilde{f}} \tilde{\tau}_{a^{*}}^{\tilde{x}}=-\left\langle Y, e_{a}\right) \tilde{\tau}_{a^{*}}^{\tilde{x}} \tag{10.26}
\end{equation*}
$$

for $1 \leq a \leq s$, and

$$
\begin{equation*}
A_{\phi_{\tilde{x}}(Y)}^{\tilde{f}} \tilde{\tau}_{a^{*}}^{\tilde{x}}=0 \tag{10.27}
\end{equation*}
$$

for $s+1 \leq a \leq r$. It follows that $Y \in \Omega^{0}(\tilde{f})$ if and only if

$$
Y-\sum_{a=1}^{s} e_{a} \in \Omega^{0}(\tilde{f} ; \phi) .
$$

Define $\tilde{f}_{0}: M_{0} \rightarrow \mathbb{R}^{k}$ by

$$
\tilde{f}_{0}=f_{0}-\sum_{a=1}^{s} e_{a} .
$$

Then $\tilde{f}_{0}\left(M_{0}\right) \subset \Omega^{0}(\tilde{f} ; \phi)$ whenever $f_{0}\left(M_{0}\right) \subset \Omega^{0}(\tilde{f})$, and

$$
f\left(x_{0}, \tilde{x}\right)=\tilde{f}(\tilde{x})+\phi_{\tilde{x}}\left(\tilde{f}_{0}\left(x_{0}\right)\right)
$$

for all $x_{0} \in M_{0}$ and $\tilde{x} \in \tilde{M}$. Thus $f$ is the partial tube determined by $\left(\tilde{f}_{0}, \tilde{f}, \phi\right)$. In particular, $f$ is an immersion.

By (10.26), for all $x_{0} \in M_{0}$ and $\tilde{x} \in \tilde{M}$ the endomorphism

$$
P\left(x_{0}, \tilde{x}\right)=I-A_{\phi_{\tilde{x}}\left(\tilde{f}_{0}\left(x_{0}\right)\right)}^{\tilde{\tilde{r}}}
$$

of $T_{\tilde{x}} \tilde{M}$ satisfies

$$
\begin{align*}
P\left(x_{0}, \tilde{x}\right) \tilde{\tau}_{a^{*}}^{\tilde{x}} & =\left(1+\left\langle\tilde{f}_{0}\left(x_{0}\right), e_{a}\right\rangle\right) \tilde{\tau}_{a^{*}}^{\tilde{x}} \\
& =\left\langle f_{0}\left(x_{0}\right), e_{a}\right\rangle \tilde{\tau}_{a^{*}}^{\tilde{x}} \tag{10.28}
\end{align*}
$$

if $1 \leq a \leq s$, whereas

$$
\begin{equation*}
P\left(x_{0}, \tilde{x}\right) \tilde{\tau}_{a^{*}}^{\tilde{x}}=\tilde{\tau}_{a^{*}}^{\tilde{x}} \tag{10.29}
\end{equation*}
$$

if $s+1 \leq a \leq r$ by (10.27). Therefore, by (10.4 we have

$$
\begin{align*}
f_{*} \tau_{a^{*}}^{x} & =f_{*} \hat{\tau}_{1 *}^{x} \tilde{\tau}_{a^{*}}^{\tilde{x}} \\
& =\tilde{f}_{*} P\left(x_{0}, \tilde{x}\right) \tilde{\tau}_{a^{*}}^{\tilde{x}} \\
& =\rho_{a}\left(x_{0}\right) \tilde{f}_{*} \tilde{\tau}_{a^{*}} \\
& =\rho_{a}\left(x_{0}\right) \tilde{f}_{a *} \tag{10.30}
\end{align*}
$$

for all $1 \leq a \leq r$. On the other hand,

$$
\begin{equation*}
f_{*} \tau_{0^{*}}^{x} X_{0}=\phi_{\tilde{x}}\left(f_{0_{*}} X_{0}\right) \tag{10.31}
\end{equation*}
$$

for all $X_{0} \in T_{x_{0}} M_{0}$.
Equations (10.30) and (10.31) imply that the normal space of $f$ is as stated in part (ii). The assertion in part (i) on the metric $g$ induced by $f$ can also be derived from these equations as follows. We have

$$
\begin{aligned}
g(x)(X, Y) & =g(x)\left(\sum_{i=0}^{r} \tau_{i *}^{x} \pi_{i *} X, \sum_{j=0}^{r} \tau_{j *}^{x} \pi_{j *} Y\right) \\
& =\left\langle f_{*} \sum_{i=0}^{r} \tau_{i *}^{x} \pi_{i *} X, f_{*} \sum_{j=0}^{r} \tau_{j *}^{x} \pi_{j *} Y\right\rangle \\
& =\left\langle\phi_{\tilde{x}}\left(f_{0 *} \pi_{0 *} X_{0}\right), \phi_{\tilde{x}}\left(f_{0 *} \pi_{0 *} Y_{0}\right)\right\rangle+\sum_{a=1}^{r} \rho_{a}^{2}\left(x_{0}\right)\left\langle f_{a *} \pi_{a *} X_{a}, f_{a *} \pi_{a *} Y_{a}\right\rangle \\
& =g_{0}\left(x_{0}\right)\left(\pi_{0 *} X_{0}, \pi_{0 *} Y_{0}\right)+\sum_{a=1}^{r} \rho_{a}^{2}\left(x_{0}\right) g_{a}\left(x_{a}\right)\left(\pi_{a *} X_{a}, \pi_{a *} Y_{a}\right) \\
& =\left(\pi_{0}^{*} g_{0}+\sum_{a=1}^{r}\left(\rho_{a} \circ \pi_{0}\right)^{2} \pi_{a}^{*} g_{a}\right)(x)(X, Y)
\end{aligned}
$$

for all $x=\left(x_{0}, x_{1}, \ldots, x_{r}\right) \in M$ and $X, Y \in T_{x} M$.
We now compute the second fundamental form of $f$. From (10.9), 10.28) and (10.29) we have

$$
\begin{align*}
\alpha^{f}\left(\tau_{a^{*}}^{x} X_{a}, \tau_{a^{*}}^{x} Y_{a}\right) & =\alpha^{f}\left(\hat{\tau}_{1 *}^{x} \tilde{\tau}_{a^{*}}^{\tilde{x}} X_{a}, \hat{\tau}_{1 *}^{x} \tilde{\tau}_{\tau^{*}}^{\tilde{x}} Y_{a}\right) \\
& =\pi\left(\alpha^{\tilde{f}}\left(P\left(x_{0}, \tilde{x}\right) \tilde{\tau}_{a^{*}}^{a^{*}} X_{a}, \tilde{\tau}_{a^{*}}^{\tilde{x}} Y_{a}\right)\right) \\
& =\rho_{a}\left(x_{0}\right) \pi\left(\alpha^{\tilde{f}}\left(\tilde{\tau}_{a^{*}}^{\tilde{x}} X_{a}, \tilde{\tau}_{a^{*}} Y_{a}\right)\right) \tag{10.32}
\end{align*}
$$

for all $X_{a}, Y_{a} \in T_{x_{a}} M_{a}$, where $\pi$ denotes the orthogonal projection of $N_{\tilde{f}} \tilde{M}(\tilde{x})$ onto $N_{f} M(x)$. On the other hand,

$$
\alpha^{\tilde{f}}\left(\tilde{\tau}_{a^{*}}^{\tilde{x}} X_{a}, \tilde{\tau}_{a^{*}}^{\tilde{x}} Y_{a}\right)=\alpha^{\tilde{f_{a}}}\left(X_{a}, Y_{a}\right)
$$

for all $X_{a}, Y_{a} \in T_{x_{a}} M_{a}$ by part (iv) of Proposition 8.3, and

$$
\alpha^{\tilde{f}_{a}}\left(X_{a}, Y_{a}\right)=i_{a *} \alpha^{f_{a}}\left(X_{a}, Y_{a}\right)-g_{a}\left(X_{a}, Y_{a}\right) \tilde{f}_{a}\left(x_{a}\right)
$$

if $1 \leq a \leq s$, whereas

$$
\alpha^{\tilde{f}_{a}}\left(X_{a}, Y_{a}\right)=i_{a *} \alpha^{f_{a}}\left(X_{a}, Y_{a}\right)
$$

if $s+1 \leq a \leq r$. Thus

$$
\pi\left(\alpha^{\tilde{f}}\left(\tilde{\tau}_{a *}^{\tilde{x}} X_{a}, \tilde{\tau}_{a *}^{\tilde{x}} Y_{a}\right)\right)=i_{a *} \alpha^{f_{a}}\left(X_{a}, Y_{a}\right)-g_{a}\left(X_{a}, Y_{a}\right) \phi_{\tilde{x}}\left(e_{a}^{\perp}\right)
$$

if $1 \leq a \leq s$, and

$$
\pi\left(\alpha^{\tilde{f}}\left(\tilde{\tau}_{a^{*}}^{\tilde{x}} X_{a}, \tilde{\tau}_{a^{*}}^{\tilde{x}} Y_{a}\right)\right)=i_{a *} \alpha^{f_{a}}\left(X_{a}, Y_{a}\right)
$$

if $s+1 \leq a \leq r$. Substituting into (10.32) yields (10.23). Now

$$
\begin{aligned}
\alpha^{f}\left(\tau_{a^{*}}^{x} X_{a}, \tau_{b *}^{x} X_{b}\right) & =\alpha^{f}\left(\hat{\tau}_{12}^{x} \tilde{\tau}_{a^{*}}^{\tilde{x}} X_{a}, \hat{\tau}_{1 *}^{x} \tilde{\tau}_{b^{\tilde{x}}}^{\tilde{x}} X_{b}\right) \\
& =\pi\left(\alpha^{\tilde{f}}\left(P \tilde{\tau}_{a^{*}}^{\tilde{x}} X_{a}, \tilde{\tau}_{b *}^{\tilde{x}} X_{b}\right)\right) \\
& =\rho_{a}\left(x_{0}\right) \pi\left(\alpha^{\tilde{f}}\left(\tilde{\tau}_{a^{*}}^{\tilde{x}} X_{a}, \tilde{\tau}_{b *}^{\tilde{x}} X_{b}\right)\right) \\
& =0
\end{aligned}
$$

for all $X_{a} \in T_{x_{a}} M_{a}$ and $X_{b} \in T_{x_{b}} M_{b}$ with $1 \leq a \neq b \leq r$. Formula (10.25) then follows from the preceding one and (10.10), whereas (10.24) is a consequence of 10.11).

If all factors $f_{a}$ of the extrinsic product $\tilde{f}$ are identity maps, that is, if for all $1 \leq a \leq s$ (respectively, $s+1 \leq a \leq r$ ) the immersion $f_{a}: M_{a} \rightarrow \mathbb{S}^{m_{a}-1}$ (respectively, $f_{a}: M_{a} \rightarrow \mathbb{R}^{m_{a}}$ ) is the identity map of $\mathbb{S}^{m_{a}-1}$ (respectively, $\mathbb{R}^{m_{a}}$ ), then the map $f$ is called the multi-rotational submanifold determined by $\tilde{f}$ with $f_{0}$ as profile. If, in addition, also $f_{0}: \Omega^{0}(\tilde{f}) \rightarrow \Omega^{0}(\tilde{f})$ is the identity map, then $f$ is called the warped product representation determined by $\tilde{f}$.

Corollary 10.17. Let $\tilde{f}: \Pi_{a=1}^{r} N_{a} \rightarrow \mathbb{R}^{m}=\Pi_{j=0}^{r} \mathbb{R}^{m_{j}}$ be an extrinsic product of identity maps $i d_{a}$ : $N_{a} \rightarrow N_{a}, 1 \leq a \leq r$, where $N_{a}=\mathbb{S}^{m_{a}-1}$ for $1 \leq a \leq s \leq r$ and $N_{a}=\mathbb{R}^{m_{a}}$ for $s+1 \leq a \leq r$. Then the warped product representation

$$
\psi: \Pi_{j=0}^{r} N_{j} \rightarrow \mathbb{R}^{m}
$$

determined by $\tilde{f}$, where $N_{0}=\Omega^{0}(\tilde{f}) \subset \mathbb{R}^{k}, k=s+m_{0}$, is an isometry onto $\mathbb{R}^{m} \backslash$ $\cup_{a=1}^{s}\left(\mathbb{R}^{m_{a}}\right)^{\perp}$ with respect to a warped product metric on $\prod_{j=0}^{r} N_{j}$ of the metrics on $N_{j}$, $0 \leq j \leq r$, with warping functions $\rho_{a}: N_{0} \rightarrow \mathbb{R}_{+}$given by

$$
\left\{\begin{array}{l}
\rho_{a}(Y)=\left\langle Y, e_{a}\right\rangle \text { if } 1 \leq a \leq s \\
1 \text { if } s+1 \leq a \leq r .
\end{array}\right.
$$

For the extrinsic product $\tilde{f}$ defined in the beginning of this section, assume that $s=r$, that is, there exist isometric immersions $f_{a}: M_{a} \rightarrow \mathbb{S}^{m_{a}-1}$ such that $\tilde{f}_{a}=i_{a} \circ f_{a}$, $1 \leq a \leq r$, where $i_{a}: \mathbb{S}^{m_{a}-1} \rightarrow \mathbb{R}^{m_{a}}$ is the inclusion. Suppose that $f_{0}: M_{0} \rightarrow \Omega^{0}(\tilde{f})$ takes values in $\mathbb{S}^{k-1} \cap \Omega^{0}(\tilde{f}) \subset \mathbb{R}^{k}$. Then the map 10.22 , which in this case reduces to

$$
f\left(x_{0}, \tilde{x}\right)=\phi_{\tilde{x}}\left(f_{0}\left(x_{0}\right)\right)
$$

for all $\tilde{x}=\left(x_{1}, \ldots, x_{r}\right) \in \tilde{M}=\prod_{a=1}^{r} M_{a}$, gives rise to an immersion

$$
f: M=\Pi_{i=0}^{r} M_{i} \rightarrow \mathbb{S}^{m-1}
$$

also called the warped product of $f_{0}, \ldots, f_{r}$. Let $i: \mathbb{S}^{m-1} \rightarrow \mathbb{R}^{m}$ and $i_{0}: \mathbb{S}^{k-1} \rightarrow \mathbb{R}^{k}$ denote the umbilical inclusions. One has the following consequence of Proposition 10.16 . Corollary 10.18. All the assertions in Proposition 10.16 hold with $N_{f} M, N_{f_{0}} M_{0}, \alpha^{f}$ and $\alpha^{f_{0}}$ replaced by $i_{*} N_{f} M, i_{0 *} N_{f_{0}} M_{0}, i_{*} \alpha^{f}$ and $i_{0 *} \alpha^{f_{0}}$, respectively.
Proof: We only give the proof of the formula for $\alpha^{f}$ correspondent to 10.23 , the proofs of the other assertions being straightforward. Denote $\hat{f}=i \circ f$ and $f_{0}=i_{0} \circ f_{0}$. On the one hand,

$$
\begin{aligned}
\alpha^{\hat{f}}\left(\tau_{a^{*}}^{x} X_{a}, \tau_{a^{*}}^{x} Y_{a}\right) & =i_{*} \alpha^{f}\left(\tau_{a^{*}}^{x} X_{a}, \tau_{a^{*}}^{x} Y_{a}\right)-g\left(\tau_{a^{*}}^{x} X_{a}, \tau_{a^{*}}^{x} Y_{a}\right) \hat{f} \\
& =i_{*} \alpha^{f}\left(\tau_{a^{*}}^{x} X_{a}, \tau_{a^{*}}^{x} Y_{a}\right)-\rho_{a}^{2}\left(x_{0}\right) g_{a}\left(X_{a}, Y_{a}\right) \phi_{\tilde{x}}\left(f_{0}\left(x_{0}\right)\right) .
\end{aligned}
$$

On the other hand, by 10.23 ) we have

$$
\alpha^{\hat{f}}\left(\tau_{a^{*}}^{x} X_{a}, \tau_{a^{*}}^{x} Y_{a}\right)=\rho_{a}\left(x_{0}\right)\left(i_{a *} \alpha^{f_{a}}\left(X_{a}, Y_{a}\right)-g_{a}\left(X_{a}, Y_{a}\right) \phi_{\tilde{x}}\left(e_{a}^{\perp}+\left\langle e_{a}, f_{0}\left(x_{0}\right)\right\rangle f_{0}\left(x_{0}\right)\right),\right.
$$

bearing in mind that

$$
N_{\hat{f}_{0}} M_{0}=i_{0 *} N_{f_{0}} M_{0} \oplus \operatorname{span}\left\{f_{0}\left(x_{0}\right)\right\} .
$$

Using that $\rho_{a}\left(x_{0}\right)=\left\langle e_{a}, f_{0}\left(x_{0}\right)\right\rangle$, it follows that

$$
i_{*} \alpha^{f}\left(\tau_{a^{*}}^{x} X_{a}, \tau_{a^{*}}^{x} Y_{a}\right)=\rho_{a}\left(x_{0}\right)\left(i_{a *} \alpha^{f_{a}}\left(X_{a}, Y_{a}\right)-g_{a}\left(X_{a}, Y_{a}\right) \phi_{\tilde{x}}\left(e_{a}^{\perp}\right)\right),
$$

and this completes the proof.
As before, if in Corollary 10.18 all factors $f_{a}: M_{a} \rightarrow \mathbb{S}^{m_{a}-1}, 1 \leq a \leq r$, of the extrinsic product $\tilde{f}$ are identity maps $\operatorname{id}_{a}: \mathbb{S}^{m_{a}-1} \rightarrow \mathbb{S}^{m_{a}-1}$, then $f$ is called the multi-rotational submanifold determined by $\tilde{f}$ with $f_{0}$ as profile. If, in addition, also $f_{0}: \Omega^{0}(\tilde{f}) \cap \mathbb{S}^{k-1} \rightarrow \Omega^{0}(\tilde{f}) \cap \mathbb{S}^{k-1}$ is the identity map, then $f$ is called the warped product representation of $\mathbb{S}^{m-1}$ determined by $\tilde{f}$.
Corollary 10.19. Let $\tilde{f}: \prod_{a=1}^{r} N_{a} \rightarrow \mathbb{S}^{m-1} \subset \mathbb{R}^{m}=\Pi_{j=0}^{r} \mathbb{R}^{m_{j}}$ be an extrinsic product of the identity maps id $d_{a}: N_{a} \rightarrow N_{a}$ of $N_{a}=\mathbb{S}^{m_{a}-1}, 1 \leq a \leq r$. Then the warped product representation

$$
\psi: \prod_{j=0}^{r} N_{j} \rightarrow \mathbb{S}^{m-1}
$$

determined by $\tilde{f}$, where $N_{0}=\Omega^{0}(\tilde{f}) \cap \mathbb{S}^{k-1} \subset \mathbb{R}^{k}, k=r+m_{0}$, is an isometry onto $\mathbb{S}^{m-1} \cap\left(\mathbb{R}^{m} \backslash \cup_{a=1}^{r}\left(\mathbb{R}^{m_{a}}\right)^{\perp}\right)$ with respect to a warped product metric on $\prod_{j=0}^{r} N_{j}$ of the metrics on $N_{j}, 0 \leq j \leq r$, with warping functions $\rho_{a}: N_{0} \rightarrow \mathbb{R}_{+}, 1 \leq a \leq r$, given by $\rho_{a}(Y)=\left\langle Y, e_{a}\right\rangle$.

### 10.3.2 Warped products of immersions into the hyperbolic space

Warped products of immersions into the hyperbolic space are of three different types, which are introduced next separately.

Consider first an orthogonal decomposition

$$
\begin{equation*}
\mathbb{L}^{m+1}=\mathbb{L}^{m_{1}} \times \prod_{i=2}^{r+1} \mathbb{R}^{m_{i}} \tag{10.33}
\end{equation*}
$$

with $\mathbb{R}^{m_{r+1}}$ possibly trivial, and isometric immersions

$$
f_{1}: M_{1} \rightarrow \mathbb{H}^{m_{1}-1} \text { and } f_{a}: M_{a} \rightarrow \mathbb{S}^{m_{a}-1}, \quad 2 \leq a \leq r
$$

Denote by

$$
i_{1}: \mathbb{H}^{m_{1}-1} \rightarrow \mathbb{L}^{m_{1}} \text { and } i_{a}: \mathbb{S}^{m_{a}-1} \rightarrow \mathbb{R}^{m_{a}}, \quad 2 \leq a \leq r
$$

the umbilical inclusions and set $\tilde{f}_{1}=i_{1} \circ f_{1}$ and $\tilde{f}_{a}=i_{a} \circ f_{a}, 2 \leq a \leq r$.
Let $\tilde{f}: \tilde{M}=\Pi_{a=1}^{r} M_{a} \rightarrow \mathbb{L}^{m+1}$ be given by

$$
\tilde{f}(\tilde{x})=\left(\tilde{f}_{1}\left(x_{1}\right), \ldots, \tilde{f}_{r}\left(x_{r}\right), 0\right)
$$

for all $\tilde{x}=\left(x_{1}, \ldots, x_{r}\right) \in \tilde{M}$. Let $\mathcal{L}$ be the flat parallel vector subbundle of $N_{\tilde{f}} \tilde{M}$ whose fiber at $\tilde{x}$ is the subspace of

$$
N_{\tilde{f}} \tilde{M}(\tilde{x})=N_{\tilde{f}_{1}} M_{1}\left(x_{1}\right) \oplus \cdots \oplus N_{\tilde{f}_{r}} M_{r}\left(x_{r}\right) \oplus \mathbb{R}^{m_{r+1}}
$$

given by

$$
\mathcal{L}(\tilde{x})=\operatorname{span}\left\{\tilde{f}_{1}\left(x_{1}\right), \ldots, \tilde{f}_{r}\left(x_{r}\right)\right\} \oplus \mathbb{R}^{m_{r+1}}
$$

Let $\phi: \tilde{M} \times \mathbb{L}^{k} \rightarrow \mathcal{L}, k=r+m_{r+1}$, be a parallel vector bundle isometry, and let $e_{1}, \ldots, e_{r} \in \mathbb{L}^{k}$ be such that $e_{1}^{0}<0$ and

$$
\phi_{\tilde{x}}\left(e_{a}\right)=\tilde{f}_{a}\left(x_{a}\right) \text { for } 1 \leq a \leq r .
$$

Finally, define

$$
\begin{equation*}
\Omega^{0}(\tilde{f})=\left\{Y \in \mathbb{L}^{k}:\left\langle Y, e_{a}\right\rangle>0 \text { for all } 2 \leq a \leq r\right\} \tag{10.34}
\end{equation*}
$$

and let $f_{0}: M_{0} \rightarrow \Omega^{0}(\tilde{f}) \cap \mathbb{H}^{k-1} \subset \mathbb{L}^{k}$ be an isometric immersion. The map

$$
f: \Pi_{i=0}^{r} M_{i}=M_{0} \times \tilde{M} \rightarrow \mathbb{H}^{m} \subset \mathbb{L}^{m+1}
$$

defined by

$$
f\left(x_{0}, x_{1}, \ldots, x_{r}\right)=f\left(x_{0}, \tilde{x}\right)=\phi_{\tilde{x}}\left(f_{0}\left(x_{0}\right)\right)
$$

is called the warped product (of hyperbolic type) of $f_{0}, f_{1}, \ldots, f_{r}$.
In a similar way one defines warped products of immersions of elliptic type. Namely, start with an orthogonal decomposition

$$
\begin{equation*}
\mathbb{L}^{m+1}=\Pi_{a=1}^{r} \mathbb{R}^{m_{a}} \times \mathbb{L}^{m_{r+1}} \tag{10.35}
\end{equation*}
$$

and isometric immersions $f_{a}: M_{a} \rightarrow \mathbb{S}^{m_{a}-1}, 1 \leq a \leq r$. As before, denote by $i_{a}: \mathbb{S}^{m_{a}-1} \rightarrow \mathbb{R}^{m_{a}}$ the umbilical inclusion and set $\tilde{f}_{a}=i_{a} \circ f_{a}$ for $1 \leq a \leq r$. Define

$$
\tilde{f}: \tilde{M}=\Pi_{a=1}^{r} M_{a} \rightarrow \mathbb{H}^{m} \subset \mathbb{L}^{m+1}
$$

by

$$
\tilde{f}(\tilde{x})=\left(\tilde{f}_{1}\left(x_{1}\right), \ldots, \tilde{f}_{r}\left(x_{r}\right), 0\right)
$$

for all $\tilde{x}=\left(x_{1}, \ldots, x_{r}\right) \in \tilde{M}$. Consider the flat parallel vector subbundle $\mathcal{L}$ of rank $k=r+m_{r+1}$ of $N_{\tilde{f}} \tilde{M}$ whose fiber at $\tilde{x}$ is

$$
\mathcal{L}(\tilde{x})=\operatorname{span}\left\{\tilde{f}_{1}\left(x_{1}\right), \ldots, \tilde{f}_{r}\left(x_{r}\right)\right\} \oplus \mathbb{L}^{m_{r+1}} .
$$

Let $\phi: \tilde{M} \times \mathbb{L}^{k} \rightarrow \mathcal{L}$ be a parallel vector bundle isometry and let $e_{1}, \ldots, e_{r} \in \mathbb{L}^{k}$ be such that

$$
\phi_{\tilde{x}}\left(e_{a}\right)=\tilde{f}_{a}\left(x_{a}\right) \text { for } 1 \leq a \leq r .
$$

Define

$$
\begin{equation*}
\Omega^{0}(\tilde{f})=\left\{Y \in \mathbb{L}^{k}:\left\langle Y, e_{a}\right\rangle>0 \text { for all } 1 \leq a \leq r\right\} \tag{10.36}
\end{equation*}
$$

and let $f_{0}: M_{0} \rightarrow \Omega^{0}(\tilde{f}) \cap \mathbb{H}^{k-1} \subset \mathbb{L}^{k}$ be an isometric immersion. The map

$$
f: \Pi_{i=0}^{r} M_{i}=M_{0} \times \tilde{M} \rightarrow \mathbb{H}^{m} \subset \mathbb{L}^{m+1}
$$

given by

$$
f\left(x_{0}, x_{1}, \ldots, x_{r}\right)=f\left(x_{0}, \tilde{x}\right)=\phi_{\tilde{x}}\left(f_{0}\left(x_{0}\right)\right)
$$

is called the warped product (of elliptic type) of $f_{0}, f_{1}, \ldots, f_{r}$.
Finally we define warped products of parabolic type. Start with orthogonal decompositions

$$
\begin{equation*}
\mathbb{R}^{\ell-1}=\Pi_{b=1}^{s} \mathbb{R}^{m_{j}}, \mathbb{L}^{\ell+1}=\mathbb{R}^{\ell-1} \oplus \mathbb{L}^{2} \text { and } \mathbb{L}^{m+1}=\mathbb{L}^{\ell+1} \times \Pi_{a=s+1}^{r+1} \mathbb{R}^{m_{i}} \tag{10.37}
\end{equation*}
$$

with $\mathbb{R}^{m_{r+1}}$ possibly trivial, and isometric immersions

$$
f_{b}: M_{b} \rightarrow \mathbb{R}^{m_{b}}, 1 \leq b \leq s, \text { and } f_{a}: M_{a} \rightarrow \mathbb{S}^{m_{a}-1}, s+1 \leq a \leq r,
$$

if $s<r$. Denote $\tilde{f}_{a}=i_{a} \circ f_{a}, s+1 \leq a \leq r$, where $i_{a}: \mathbb{S}^{m_{a}-1} \rightarrow \mathbb{R}^{m_{a}}$ is the umbilical inclusion.

Let $\hat{f}: \hat{M}=\Pi_{b=1}^{s} M_{b} \rightarrow \mathbb{R}^{\ell-1}$ be the extrinsic product of $f_{1}, \ldots, f_{s}$, given by

$$
\hat{f}(\hat{x})=\left(f_{1}\left(x_{1}\right), \ldots, f_{s}\left(x_{s}\right)\right)
$$

for all $\hat{x}=\left(x_{1}, \ldots, x_{s}\right) \in \hat{M}$, and define $\tilde{f}: \tilde{M}=\Pi_{j=1}^{r} M_{j} \rightarrow \mathbb{L}^{m+1}$ by

$$
\tilde{f}(\tilde{x})=\left(\Psi(\hat{f}(\hat{x})), \tilde{f}_{s+1}\left(x_{s+1}\right), \ldots, \tilde{f}_{r}\left(x_{r}\right), 0\right)
$$

for all $\tilde{x}=\left(x_{1}, \ldots, x_{r}\right) \in \tilde{M}$, where $\Psi: \mathbb{R}^{\ell-1} \rightarrow \mathbb{V}^{\ell} \subset \mathbb{L}^{\ell+1}$ is the isometric embedding of $\mathbb{R}^{\ell-1}$ into $\mathbb{V}^{\ell} \subset \mathbb{L}^{\ell+1}=\mathbb{R}^{\ell-1} \oplus \mathbb{L}^{2}$ given by 10.15 in terms of a pseudo-orthonormal basis $v_{0}, v_{1}$ of $\mathbb{L}^{2}$ with

$$
\begin{equation*}
v_{1}^{0}<0,\left\langle v_{0}, v_{0}\right\rangle=0=\left\langle v_{1}, v_{1}\right\rangle \text { and }\left\langle v_{0}, v_{1}\right\rangle=1 \tag{10.38}
\end{equation*}
$$

Now consider the flat parallel vector subbundle $\mathcal{L}$ of rank $k=m_{r+1}+r-s+2$ of $N_{\tilde{f}} \tilde{M}$ whose fiber at $\tilde{x}$ is the subspace of

$$
N_{\tilde{f}} \tilde{M}(\tilde{x})=\Psi_{*} N_{\hat{f}} \hat{M}(\hat{x}) \oplus N_{\Psi} \mathbb{R}^{\ell-1}(\hat{f}(\hat{x})) \oplus_{a=s+1}^{r} N_{\tilde{f}_{a}} M_{a}\left(x_{a}\right) \oplus \mathbb{R}^{m_{r+1}}
$$

given by

$$
\mathcal{L}(\tilde{x})=\operatorname{span}\left\{v_{1}, \Psi(\hat{f}(\hat{x})), \tilde{f}_{s+1}\left(x_{s+1}\right), \ldots, \tilde{f}_{r}\left(x_{r}\right)\right\} \oplus \mathbb{R}^{m_{r+1}}
$$

Write $\mathbb{L}^{k}=\mathbb{L}^{2} \times \mathbb{R}^{k-2}=\mathbb{L}^{2} \times \mathbb{R}^{r-s} \times \mathbb{R}^{m_{r+1}}$ and let $\phi: \tilde{M} \times \mathbb{L}^{k} \rightarrow \mathcal{L}$ be a parallel vector bundle isometry such that

$$
\phi_{\tilde{x}}\left(v_{0}\right)=\Psi(\hat{f}(\hat{x})), \quad \phi_{\tilde{x}}\left(v_{1}\right)=v_{1} \text { and } \phi_{\tilde{x}}\left(e_{a}\right)=\tilde{f}_{a}\left(x_{a}\right), \quad s+1 \leq a \leq r,
$$

where $e_{s+1}, \ldots, e_{r}$ is an orthonormal basis of $\mathbb{R}^{r-s}$.
Finally, define

$$
\begin{equation*}
\Omega^{0}(\tilde{f})=\left\{Y \in \mathbb{L}^{k}:\left\langle Y, e_{a}\right\rangle>0 \text { for all } s+1 \leq a \leq r\right\} \tag{10.39}
\end{equation*}
$$

and let $f_{0}: M_{0} \rightarrow \Omega^{0}(\tilde{f}) \cap \mathbb{H}^{k-1} \subset \mathbb{L}^{k}$ be an isometric immersion. Then the map

$$
f: \Pi_{i=0}^{r} M_{i}=M_{0} \times \tilde{M} \rightarrow \mathbb{H}^{m} \subset \mathbb{L}^{m+1}
$$

given by

$$
f\left(x_{0}, x_{1}, \ldots, x_{r}\right)=f\left(x_{0}, \tilde{x}\right)=\phi_{\tilde{x}}\left(f_{0}\left(x_{0}\right)\right)
$$

is called the warped product (of parabolic type) of $f_{0}, f_{1}, \ldots, f_{s}, f_{s+1}, \ldots, f_{r}$.
The proof of the next result is left to the reader.
Proposition 10.20. The following assertions hold:
(i) The map $f$ is an immersion whose induced metric $g$ is the warped product of the metrics $g_{0}, \ldots, g_{r}$ of $M_{0}, \ldots, M_{r}$ with warping functions $\rho_{a}: M_{0} \rightarrow \mathbb{R}_{+}$given by

$$
\rho_{a}\left(x_{0}\right)=\left\langle f_{0}\left(x_{0}\right), e_{a}\right\rangle, \quad 1 \leq a \leq r,
$$

in the hyperbolic and elliptic cases, and by

$$
\rho_{a}\left(x_{0}\right)=\left\{\begin{array}{l}
\left\langle f_{0}\left(x_{0}\right), v_{1}\right\rangle \text { if } 1 \leq a \leq s \\
\left\langle f_{0}\left(x_{0}\right), e_{a}\right\rangle \text { if } s+1 \leq a \leq r
\end{array}\right.
$$

in the parabolic case.
(ii) The normal space of $f$ at $x=\left(x_{0}, x_{1}, \ldots, x_{r}\right)=\left(x_{0}, \tilde{x}\right)$ is given by

$$
i_{*} N_{f} M(x)=\oplus_{a=1}^{r} i_{a *} N_{f_{a}} M_{a}\left(x_{a}\right) \oplus \phi_{\tilde{x}}\left(i_{0 *} N_{f_{0}} M_{0}\left(x_{0}\right)\right) \subset N_{\tilde{f}} \tilde{M}(\tilde{x})
$$

in the hyperbolic and elliptic cases, and by

$$
i_{*} N_{f} M(x)=\Psi_{*} \oplus_{b=1}^{s} N_{f_{b}} M_{b}\left(x_{b}\right) \oplus_{a=s+1}^{r} i_{a *} N_{f_{a}} M_{a}\left(x_{a}\right) \oplus \phi_{\tilde{x}}\left(i_{0 *} N_{f_{0}} M_{0}\left(x_{0}\right)\right)
$$

in the parabolic case, where $i_{0}: \mathbb{H}^{k-1} \rightarrow \mathbb{L}^{k}$ and $i: \mathbb{H}^{m} \rightarrow \mathbb{L}^{m+1}$ are inclusions.
(iii) The second fundamental form of $f$ at $x$ is given by

$$
i_{*} \alpha^{f}\left(\tau_{a^{*}}^{x} X_{a}, \tau_{a *}^{x} Y_{a}\right)=\rho_{a}\left(x_{0}\right)\left(i_{a *} \alpha^{f_{a}}\left(X_{a}, Y_{a}\right)-g_{a}\left(X_{a}, Y_{a}\right) \phi_{\tilde{x}}\left(e_{a}^{\perp}\right)\right)
$$

for all $X_{a}, Y_{a} \in T_{x_{a}} M_{a}$ and $1 \leq a \leq r$ (respectively, $s+1 \leq a \leq r$ ) in the hyperbolic and elliptic cases (respectively, parabolic case), and by

$$
i_{*} \alpha^{f}\left(\tau_{b *}^{x} X_{b}, \tau_{b *}^{x} Y_{b}\right)=\rho_{b}\left(x_{0}\right)\left(\Psi_{*} \alpha^{f_{b}}\left(X_{b}, Y_{b}\right)-g_{b}\left(X_{b}, Y_{b}\right) \phi_{\tilde{x}}\left(v_{1}^{\perp}\right)\right)
$$

for all $X_{b}, Y_{b} \in T_{x_{b}} M_{b}$ and $1 \leq b \leq s$ in the parabolic case, where $v_{1}^{\perp}$ and $e_{a}^{\perp}$ are the orthogonal projections of $v_{1}$ and $e_{a}$ onto $i_{0 *} N_{f_{0}} M_{0}\left(x_{0}\right)$, respectively. Moreover, in all cases we have

$$
i_{*} \alpha^{f}\left(\tau_{0^{*}}^{x} X_{0}, \tau_{0^{*}}^{x} Y_{0}\right)=\phi_{\tilde{x}}\left(i_{0 *} \alpha^{f_{0}}\left(X_{0}, Y_{0}\right)\right)
$$

for all $X_{0}, Y_{0} \in T_{x_{0}} M_{0}$, and

$$
\alpha^{f}\left(\tau_{i *}^{x} X_{i}, \tau_{j *}^{x} X_{j}\right)=0
$$

for all $X_{i} \in T_{x_{i}} M_{i}$ and $X_{j} \in T_{x_{j}} M_{j}, 0 \leq i \neq j \leq r$.
In the three types of warped products of immersions into the hyperbolic space just described, if all factors $f_{1}, \ldots, f_{r}$ of the correspondent extrinsic product $\tilde{f}$ are identity maps, then the warped product $f$ of $f_{0}, f_{1}, \ldots, f_{r}$ is called, as in the Euclidean and spherical cases, the multi-rotational submanifold determined by $\tilde{f}$ with $f_{0}$ as profile. If, in addition, also $f_{0}$ is the identity map, then $f$ is called the warped product representation determined by $\tilde{f}$.

Therefore, in the hyperbolic case the extrinsic product $\tilde{f}$ reduces, in terms of an orthogonal decomposition of $\mathbb{L}^{m+1}$ as in 10.33 , to the map

$$
\tilde{f}: \tilde{N}=\mathbb{H}^{m_{1}-1} \times \Pi_{a=2}^{r} \mathbb{S}^{m_{a}-1} \rightarrow \mathbb{H}^{m} \subset \mathbb{L}^{m+1}
$$

given by

$$
\tilde{f}\left(x_{1}, \ldots, x_{r}\right)=\left(i_{1}\left(x_{1}\right), \ldots, i_{r}\left(x_{r}\right), 0\right)
$$

where $i_{1}: \mathbb{H}^{m_{1}-1} \rightarrow \mathbb{L}^{m_{1}}$ and $i_{a}: \mathbb{S}^{m_{a}-1} \rightarrow \mathbb{R}^{m_{a}}, 2 \leq a \leq r$, are umbilical inclusions.

Let $\phi: \tilde{N} \times \mathbb{L}^{k} \rightarrow N_{\tilde{f}} \tilde{N}, k=r+m_{r+1}$, be a parallel vector bundle isometry, and let $e_{1}, \ldots e_{r} \in \mathbb{L}^{k}$ be an orthonormal set such that $e_{1}^{0}<0$ and $\phi_{\tilde{x}}\left(e_{a}\right)=i_{a}\left(x_{a}\right)$ for all $\tilde{x}=\left(x_{1}, \ldots, x_{r}\right) \in \tilde{N}$ and $1 \leq a \leq r$. Set

$$
N_{0}=\Omega^{0}(\tilde{f}) \cap \mathbb{H}^{k-1},
$$

with $\Omega^{0}(\tilde{f})$ given by 10.34). Then the map

$$
f: N_{0} \times \tilde{N} \rightarrow \mathbb{H}^{m} \subset \mathbb{L}^{m+1}
$$

defined by

$$
f\left(x_{0}, \tilde{x}\right)=\phi_{\tilde{x}}\left(i_{0}\left(x_{0}\right)\right),
$$

where $i_{0}: N_{0} \rightarrow \mathbb{L}^{k}$ is an umbilical inclusion, becomes an isometry of the warped product manifold

$$
\left(N_{0} \subset \mathbb{H}^{k-1}\right) \times_{\sigma_{1}} \mathbb{H}^{m_{1}-1} \times_{\sigma_{2}} \mathbb{S}^{m_{2}-1} \times \cdots \times_{\sigma_{r}} \mathbb{S}^{m_{r}-1}
$$

with warping functions $\sigma_{a}: N_{0} \subset \mathbb{L}^{k} \rightarrow \mathbb{R}_{+}$given by

$$
\sigma_{a}(Y)=\left\langle Y, e_{a}\right\rangle, \quad 1 \leq a \leq r
$$

onto the open dense subset

$$
\mathbb{H}^{m} \cap\left(\mathbb{L}^{m+1} \backslash \cup_{a=2}^{r}\left(\mathbb{R}^{m_{a}}\right)^{\perp}\right)
$$

It is called the warped product representation of hyperbolic type of $\mathbb{H}^{m}$.
In the elliptic case, the extrinsic product $\tilde{f}$ reduces, in terms of an orthogonal decomposition of $\mathbb{L}^{m+1}$ as in (10.35), to the map

$$
\tilde{f}: \tilde{N}=\Pi_{a=1}^{r} \mathbb{S}^{m_{a}-1} \rightarrow \mathbb{H}^{m} \subset \mathbb{L}^{m+1}
$$

given by

$$
\tilde{f}\left(x_{1}, \ldots, x_{r}\right)=\left(i_{1}\left(x_{1}\right), \ldots, i_{r}\left(x_{r}\right), 0\right)
$$

where $i_{a}: \mathbb{S}^{m_{a}-1} \rightarrow \mathbb{R}^{m_{a}}, 1 \leq a \leq r$, are umbilical inclusions.
Let $\phi: \tilde{N} \times \mathbb{L}^{k} \rightarrow N_{\tilde{f}} \tilde{N}, k=r+m_{r+1}$, be a parallel vector bundle isometry, and let $e_{\tilde{1}}, \ldots, e_{r} \in \mathbb{L}^{k}$ be an orthonormal set such that $\phi_{\tilde{x}}\left(e_{a}\right)=i_{a}\left(x_{a}\right)$ for $\tilde{x}=\left(x_{1}, \ldots, x_{r}\right) \in$ $\tilde{N}$ and $1 \leq a \leq r$. Set

$$
N_{0}=\Omega^{0}(\tilde{f}) \cap \mathbb{H}^{k-1}
$$

with $\Omega^{0}(\tilde{f})$ given by 10.34. Then the map

$$
f: N_{0} \times \tilde{N} \rightarrow \mathbb{H}^{m} \subset \mathbb{L}^{m+1}
$$

defined by

$$
f\left(x_{0}, \tilde{x}\right)=\phi_{\tilde{x}}\left(i_{0}\left(x_{0}\right)\right),
$$

where $i_{0}: N_{0} \rightarrow \mathbb{L}^{k}$ is an umbilical inclusion, becomes an isometry of the warped product manifold

$$
\left(N_{0} \subset \mathbb{H}^{k-1}\right) \times_{\sigma_{1}} \mathbb{S}^{m_{1}-1} \times \cdots \times_{\sigma_{r}} \mathbb{S}^{m_{r}-1}
$$

with warping functions $\sigma_{a}: N_{0} \subset \mathbb{L}^{k} \rightarrow \mathbb{R}_{+}$given by

$$
\sigma_{a}(Y)=\left\langle Y, e_{a}\right\rangle, \quad 1 \leq a \leq r,
$$

onto the open dense subset

$$
\mathbb{H}^{m} \cap\left(\mathbb{L}^{m+1} \backslash \cup_{a=1}^{r}\left(\mathbb{R}^{m_{a}}\right)^{\perp}\right) .
$$

It is called the warped product representation of elliptic type of $\mathbb{H}^{m}$.
Finally, in the parabolic case the extrinsic product $\tilde{f}$ reduces, in terms of the orthogonal decompositions as in (10.37), to the map

$$
\tilde{f}: \tilde{N}=\Pi_{b=1}^{s} \mathbb{R}^{m_{b}} \times \Pi_{a=s+1}^{r} \mathbb{S}^{m_{a}-1} \rightarrow \mathbb{H}^{m} \subset \mathbb{L}^{m+1}
$$

given by

$$
\tilde{f}\left(x_{1}, \ldots, x_{r}\right)=\left(\Psi(\hat{x}), i_{s+1}\left(x_{s+1}\right), \ldots, i_{r}\left(x_{r}\right)\right)
$$

where $\hat{x}=\left(x_{1}, \ldots, x_{s}\right) \in \Pi_{b=1}^{s} \mathbb{R}^{m_{b}}=\mathbb{R}^{\ell-1}$ and $\Psi: \mathbb{R}^{\ell-1} \rightarrow \mathbb{L}^{\ell+1}$ is the isometric embedding of $\mathbb{R}^{\ell-1}$ into $\mathbb{V}^{\ell} \subset \mathbb{L}^{\ell+1}=\mathbb{R}^{\ell-1} \oplus \mathbb{L}^{2}$ defined by (10.15) in terms of a pseudo-orthonormal basis $v_{0}, v_{1}$ of $\mathbb{L}^{2}$ as in 10.38 .

Write $\mathbb{L}^{k}=\mathbb{L}^{2} \times \mathbb{R}^{k-2}=\mathbb{L}^{2} \times \mathbb{R}^{r-s} \times \mathbb{R}^{m_{r+1}}$ and let $\phi: \tilde{N} \times \mathbb{L}^{k} \rightarrow N_{\tilde{f}} \tilde{N}, k=$ $m_{r+1}+r-s+2$, be a parallel vector bundle isometry such that

$$
\phi_{\tilde{x}}\left(v_{0}\right)=\Psi(\hat{x}), \quad \phi_{\tilde{x}}\left(v_{1}\right)=v_{1} \text { and } \phi_{\tilde{x}}\left(e_{a}\right)=\tilde{f}_{a}\left(x_{a}\right), \quad s+1 \leq a \leq r,
$$

for all $\tilde{x}=\left(x_{1}, \ldots, x_{r}\right) \in \tilde{N}$, where $e_{s+1}, \ldots, e_{r}$ is an orthonormal basis of $\mathbb{R}^{r-s}$.
Let $\Omega^{0}(\tilde{f})$ be given as in 10.39) and set

$$
N_{0}=\Omega^{0}(\tilde{f}) \cap \mathbb{H}^{k-1}
$$

Then the map

$$
f: N_{0} \times \tilde{N} \rightarrow \mathbb{H}^{m} \subset \mathbb{L}^{m+1}
$$

given by

$$
f\left(x_{0}, \tilde{x}\right)=\phi_{\tilde{x}}\left(i_{0}\left(x_{0}\right)\right)
$$

where $i_{0}: N_{0} \rightarrow \mathbb{L}^{k}$ is an umbilical inclusion, becomes an isometry of the warped product manifold

$$
\left(N_{0} \subset \mathbb{H}^{k-1}\right) \times_{\sigma_{1}} \mathbb{R}^{m_{1}} \times \cdots \times_{\sigma_{s}} \mathbb{R}^{m_{s}} \times_{\sigma_{s+1}} \mathbb{S}^{m_{s+1}-1} \times_{\sigma_{r}} \mathbb{S}^{m_{r}-1}
$$

with warping functions $\sigma_{a}: N_{0} \subset \mathbb{L}^{k} \rightarrow \mathbb{R}_{+}$given by

$$
\sigma_{b}(Y)=\left\langle Y, v_{1},\right\rangle, \quad 1 \leq b \leq s, \quad \text { and } \sigma_{a}(Y)=\left\langle Y, e_{a},\right\rangle, \quad s+1 \leq a \leq r,
$$

onto the open dense subset

$$
\mathbb{H}^{m} \cap\left(\mathbb{L}^{m+1} \backslash \cup_{a=s+1}^{r}\left(\mathbb{R}^{m_{a}}\right)^{\perp}\right)
$$

It is called the warped product representation of parabolic type of $\mathbb{H}^{m}$.

### 10.3.3 Nölker's theorem

We are now in a position to state and prove a decomposition theorem for isometric immersions of warped product manifolds into space forms.

Theorem 10.21. Any isometric immersion $f: M \rightarrow \mathbb{Q}_{\epsilon}^{m}$ of a warped product manifold whose second fundamental form is adapted to the product net of $M$ is a warped product of immersions.

Proof: We give the proof for isometric immersions into Euclidean space, the other cases being similar. First we assume that $M=M_{0} \times M_{1}$ is a product manifold with only two factors, endowed with a warped product metric

$$
g=\pi_{0}^{*} g_{0}+\left(\rho \circ \pi_{0}\right)^{2} \pi_{1}^{*} g_{1}
$$

where $\pi_{i}: M \rightarrow M_{i}, 0 \leq i \leq 1$, are the projections, $g_{0}$ and $g_{1}$ are the metrics on $M_{0}$ and $M_{1}$, respectively, and $\rho \in C^{\infty}\left(M_{0}\right)$ is the warping function.

If $M$ is a Riemannian product, that is, if the warping function $\rho$ is constant, then $f$ is an extrinsic product of immersions by Theorem 8.4. Thus, from now on we assume that the warping function $\rho$ is not constant.

For a fixed $\bar{x}=\left(\bar{x}_{0}, \bar{x}_{1}\right) \in M$, let $f_{1}: M_{1} \rightarrow \mathbb{R}^{m}$ be given by $f_{1}=f \circ \tau_{1}^{\bar{x}}$. Notice that the metric induced by $f_{1}$ is $g_{1}\left(\bar{x}_{0}\right)=\rho^{2}\left(\bar{x}_{0}\right) g_{1}$, where $g_{1}$ is the metric on $M_{1}$. Thus, replacing $g_{1}$ by $\rho^{2}\left(\bar{x}_{0}\right) g_{1}$ and the warping function $\rho$ by $\rho / \rho\left(\bar{x}_{0}\right)$, we may assume that $f_{1}$ is an isometric immersion.

By Theorem 10.14, there exist a parallel vector bundle isometry $\phi: M_{1} \times \mathbb{R}^{s} \rightarrow \mathcal{L}$ onto a flat parallel subbundle of $N_{f_{1}} M_{1}$ and an isometric immersion

$$
f_{0}: M_{0} \rightarrow \Omega^{0}\left(f_{1} ; \phi\right) \subset \mathbb{R}^{s}
$$

such that $f$ is the partial tube determined by $\left(f_{0}, f_{1}, \phi\right)$. In view of Remark 10.7, we may also assume that $f_{0}$ is substantial in $\mathbb{R}^{s}$.

By Corollary 10.8, the metric induced by $f$ is given by 10.12, where $g_{0}$ is the metric on $M_{0}$ and, for all $x=\left(x_{0}, x_{1}\right) \in M$, the metric $g_{1}\left(x_{0}\right)$ on $M_{1}$ is given by

$$
g_{1}\left(x_{0}\right)\left(X_{1}, Y_{1}\right)=g_{1}\left(P^{2} X_{1}, Y_{1}\right)
$$

for all $X_{1}, Y_{1} \in T_{x_{1}} M_{1}$. Here $P=P\left(x_{0}, x_{1}\right)$ is the endomorphism of $T_{x_{1}} M_{1}$ given by (10.5). Therefore we must have

$$
\begin{aligned}
\rho^{2}\left(x_{0}\right) g_{1}\left(X_{1}, Y_{1}\right) & =g_{1}\left(x_{0}\right)\left(X_{1}, Y_{1}\right) \\
& =g_{1}\left(P^{2} X_{1}, Y_{1}\right)
\end{aligned}
$$

for all for all $\left(x_{0}, x_{1}\right) \in M$ and all $X_{1}, Y_{1} \in T_{x_{1}} M_{1}$. Hence

$$
\begin{equation*}
\left(I-A_{\phi_{x_{1}}\left(f_{0}\left(x_{0}\right)\right)}^{f_{1}}\right)^{2}=\rho^{2}\left(x_{0}\right) I \tag{10.40}
\end{equation*}
$$

for all $\left(x_{0}, x_{1}\right) \in M$. Notice that if $\mathcal{L} \subset N_{1}^{\perp}\left(f_{1}\right)$ then the preceding equation would imply $\rho$ to be identically one, in contradiction with our assumption.

We claim that $\mathcal{L}$ is an umbilical subbundle of $N_{f_{1}} M_{1}$. Let $\eta_{1}, \ldots, \eta_{k} \in \Gamma(\mathcal{L})$ be the distinct principal normal vector fields of $f_{1}$ with respect to $\mathcal{L}$ (see Exercise 1.35). Thus there exists an orthogonal decomposition $T M_{1}=\oplus_{i=1}^{k} E_{i}$ such that

$$
\left.A_{\zeta}^{f_{1}}\right|_{E_{i}}=\left\langle\zeta, \eta_{i}\right\rangle I
$$

for all $\zeta \in \Gamma(\mathcal{L})$. We must show that $k=1$.
Write $\eta_{i}\left(x_{1}\right)=\phi_{x_{1}}\left(V_{i}\left(x_{1}\right)\right)$ for all $x_{1} \in M_{1}$. Then 10.40 can be written as

$$
\left\langle V_{i}\left(x_{1}\right), f_{0}\left(x_{0}\right)\right\rangle=1+\rho\left(x_{0}\right)
$$

for all $\left(x_{0}, x_{1}\right) \in M$ and $1 \leq i \leq k$. If $k \geq 2$, then

$$
\left\langle V_{i}\left(x_{1}\right)-V_{j}\left(x_{1}\right), f_{0}\left(x_{0}\right)\right\rangle=0
$$

for all $\left(x_{0}, x_{1}\right) \in M$ and $1 \leq i \neq j \leq k$, which contradicts the fact that $f_{0}$ is substantial in $\mathbb{R}^{s}$. Thus $k=1$ and our claim is proved.

If $M_{1}$ has dimension at least two, Exercise 2.14 implies that $f_{1}\left(M_{1}\right)$ is contained in an $(m-s)$-dimensional sphere $\mathbb{S}^{m-s}, 1 \leq s \leq m-1$, which we can assume to be of unit radius and centered at the origin of a subspace $\mathbb{R}^{m-s+1} \subset \mathbb{R}^{m}$, and that $\mathcal{L}$ is the vector subbundle of $N_{f_{1}} M_{1}$ whose fiber at $x_{1} \in M_{1}$ is spanned by the position vector $f_{1}\left(x_{1}\right)$ and the orthogonal complement $\mathbb{R}^{s-1}$ of $\mathbb{R}^{m-s+1}$.

Now assume that $f_{1}: M_{1} \rightarrow \mathbb{R}^{m}$ is a unit-speed curve $\gamma: I \rightarrow \mathbb{R}^{m}$. We claim that $\gamma(I)$ is contained in a sphere of $\mathbb{R}^{m}$, which we can assume to be a hypersphere $\mathbb{S}^{m-s}$ of unit radius centered at the origin of a subspace $\mathbb{R}^{m-s+1} \subset \mathbb{R}^{m}, 1 \leq s \leq m-1$, and that the fiber at $t \in I=M_{1}$ of the vector subbundle $\mathcal{L}$ of $N_{\gamma} I$ is spanned by the position vector $\gamma(t)$ in $\mathbb{R}^{m-s+1}$ and the orthogonal complement $\mathbb{R}^{s-1}$ of $\mathbb{R}^{m-s+1}$ in $\mathbb{R}^{m}$.

Eq. 10.40) implies that the function $\rho: M \rightarrow \mathbb{R}_{+}$, given by

$$
\begin{equation*}
\rho\left(x_{0}, t\right)=1-\left\langle\gamma^{\prime \prime}(t), \phi_{t}\left(f_{0}\left(x_{0}\right)\right)\right\rangle, \tag{10.41}
\end{equation*}
$$

does not depend on $t$. Differentiating with respect to $t$ and using that $f_{0}$ is substantial yields $\left\langle\gamma^{\prime \prime \prime}(t), \xi\right\rangle=0$ for all $t \in I$ and for all $\xi \in \mathcal{L}(t)$. Observe also that one cannot have $\gamma^{\prime \prime}(t) \in \mathcal{L}^{\perp}(t)$ at any $t \in I$, for this and (10.41) would imply $\rho$ to be identically one. The claim then follows from Exercise 2.8. In either case we conclude that $f$ is the warped product of $f_{0}$ and $f_{1}$.

Suppose now that $M=\Pi_{i=0}^{r} M_{i}$ is an arbitrary product manifold endowed with a warped product metric

$$
g=\pi_{0}^{*} g_{0}+\sum_{a=1}^{r}\left(\rho_{a} \circ \pi_{0}\right)^{2} \pi_{a}^{*} g_{a}
$$

for some $\rho_{a} \in C^{\infty}\left(M_{0}\right)$ with $\rho_{a}>0,1 \leq a \leq r$. Assume that there exists $1 \leq k \leq r$ such that $\rho_{a}$ is identically one for $k+1 \leq a \leq r$. Therefore the metric $g$ is a polar metric as in (10.1), with

$$
\begin{equation*}
g_{a}\left(x_{0}\right)=\rho_{a}^{2}\left(x_{0}\right) g_{a} \tag{10.42}
\end{equation*}
$$

for all $x_{0} \in M_{0}, 1 \leq a \leq r$. For a fixed $\bar{x}=\left(\bar{x}_{0}, \ldots, \bar{x}_{r}\right) \in M$, let

$$
\tilde{f}: \tilde{M}=\Pi_{a=1}^{r} M_{a} \rightarrow \mathbb{R}^{m}
$$

be given by

$$
\tilde{f}=f \circ \hat{\tau}_{1}^{\bar{x}},
$$

where $\hat{\tau}_{1}^{\bar{x}}: \tilde{M} \rightarrow M$ is given by $\hat{\tau}_{1}^{\tilde{x}}(\tilde{x})=\left(x_{0}, \tilde{x}\right)$ for all $\tilde{x}=\left(x_{1}, \ldots, x_{r}\right) \in \tilde{M}$. Then the metric induced by $\tilde{f}$ is

$$
\hat{\tau}_{1}^{\bar{x} *} g=\sum_{a=1}^{r} \rho_{a}^{2}\left(\bar{x}_{0}\right) \tilde{\pi}_{a}^{*} g_{a} .
$$

Hence we may replace each $g_{a}$ by $\rho_{a}^{2}\left(\bar{x}_{0}\right) g_{a}$, and each warping function $\rho_{a}$ by $\rho_{a} / \rho_{a}\left(\bar{x}_{0}\right)$, so as to make $\tilde{f}$ into an isometric immersion with respect to the product metric of $g_{1}, \ldots, g_{r}$ on $\tilde{M}$.

By Theorem 8.4, the isometric immersion $\tilde{f}: \tilde{M} \rightarrow \mathbb{R}^{m}$ is an extrinsic product of isometric immersions $f_{a}: M_{a} \rightarrow \mathbb{R}^{m_{a}}, 1 \leq a \leq r$, with respect to an orthogonal decomposition

$$
\mathbb{R}^{m}=\Pi_{j=0}^{r} \mathbb{R}^{m_{j}},
$$

with $\mathbb{R}^{m_{0}}$ possibly trivial. Thus there exist $v_{0} \in \mathbb{R}^{m_{0}}$ (in case $\mathbb{R}^{m_{0}}$ is nontrivial) and isometric immersions $f_{i}: M_{i} \rightarrow \mathbb{R}^{m_{i}}, 1 \leq i \leq r$, such that

$$
\tilde{f}\left(x_{1}, \ldots, x_{r}\right)=\left(v_{0}, f_{1}\left(x_{1}\right), \ldots, f_{r}\left(x_{r}\right)\right)
$$

for all $\left(x_{1}, \ldots, x_{r}\right) \in \tilde{M}$. Moreover, by Theorem 10.14 there exist a parallel vector bundle isometry $\phi: \tilde{M} \times \mathbb{R}^{s} \rightarrow \mathcal{L}$ onto a flat parallel subbundle $\mathcal{L}$ of $N_{\tilde{f}} \tilde{M}$, and a substantial isometric immersion $\tilde{f}_{0}: M_{0} \rightarrow \Omega^{0}(\tilde{f} ; \phi) \subset \mathbb{R}^{s}$ such that $f: M \rightarrow \mathbb{R}^{m}$ is the partial tube determined by $\left(\tilde{f}_{0}, \tilde{f}, \phi\right)$.

As in the proof of Proposition 10.12, let $\mathcal{L}_{a}$ be the projection of $\mathcal{L}$ onto $N_{f_{a}} M_{a}$ and let $\phi^{a}: M_{a} \times \mathbb{R}^{n_{a}} \rightarrow \mathcal{L}_{a}, 1 \leq a \leq r$, be a parallel vector bundle isometry so that $\phi$ is the restriction of the parallel vector bundle isometry

$$
\tilde{\phi}: \mathbb{R}^{s}=\prod_{a=1}^{r} \mathbb{R}^{n_{a}} \rightarrow \prod_{a=1}^{r} \mathcal{L}_{a}
$$

defined by 10.17). Then the metric $g_{a}\left(x_{0}\right)$ is given by 10.20 . Comparing with 10.42) yields

$$
\left(I-A_{\phi_{x_{a}}^{a}\left(\tilde{\pi}_{a}\left(f_{0}\left(x_{0}\right)\right)\right)}^{f_{a}}\right)^{2}=\rho_{a}^{2}\left(x_{0}\right) I .
$$

Arguing as in the preceding case, we see that $\mathcal{L}_{a}$ is an umbilical subbundle of $N_{f_{a}} M_{a}$ for $1 \leq a \leq k$ and belongs to $N_{1}^{\perp}\left(f_{a}\right)$ for $k+1 \leq a \leq r$. The proof is now completed as before.

Theorem 10.21 implies, in particular, that any isometry of a warped product manifold onto an open subset of $\mathbb{Q}_{\epsilon}^{n}, \epsilon \in\{-1,0,1\}$, is essentially the restriction of a warped product representation of $\mathbb{Q}_{\epsilon}^{n}$ determined by an extrinsic product of identity maps.

Corollary 10.22. Let $f: M \rightarrow U \subset \mathbb{Q}_{\epsilon}^{n}, \epsilon \in\{-1,0,1\}$, be an isometry of a warped product manifold $M=M_{0} \times_{\rho_{1}} M_{1} \times \cdots \times_{\rho_{r}} M_{r}$ onto an open subset $U \subset \mathbb{Q}_{\epsilon}^{n}$. Then there exists a warped product representation

$$
\psi: N=N_{0} \times_{\sigma_{1}} N_{1} \times \cdots \times_{\sigma_{r}} N_{r} \rightarrow \mathbb{Q}_{\epsilon}^{n}
$$

either as in Corollary 10.17, Corollary 10.19 or of one of the three types described after Proposition 10.20, depending on whether $\epsilon=0,1$ or -1 , respectively, and isometries $f_{j}: M_{j} \subset U_{j}$ onto open subsets $U_{j} \subset N_{j}, 0 \leq j \leq r$, such that $U \subset \psi(N), \rho_{a}=\sigma_{a} \circ f_{0}$ for $1 \leq a \leq r$ and

$$
f=\psi \circ\left(f_{0} \times \cdots \times f_{r}\right) .
$$

It is convenient to restate Theorem 10.21 in terms of the warped product representations of $\mathbb{Q}_{\epsilon}^{m}$.

Corollary 10.23. Let $f: M \rightarrow \mathbb{Q}_{\epsilon}^{m}, \epsilon \in\{-1,0,1\}$, be an isometric immersion of a warped product manifold $M=M_{0} \times_{\rho_{1}} M_{1} \times \cdots \times_{\rho_{r}} M_{r}$ whose second fundamental form is adapted to the product net of $M$. Then there exist a warped product representation

$$
\psi: N_{0} \times_{\sigma_{1}} N_{1} \times \cdots \times_{\sigma_{r}} N_{r} \rightarrow \mathbb{Q}_{\epsilon}^{m}
$$

either as in Corollary 10.17, Corollary 10.19 or of one of the three types described after Proposition 10.20, depending on whether $\epsilon=0$, 1 or -1 , respectively, and isometric immersions $f_{j}: M_{j} \rightarrow N_{j}, 0 \leq j \leq r$, such that $\rho_{a}=\sigma_{a} \circ f_{0}$ for $1 \leq a \leq r$ and

$$
f=\psi \circ\left(f_{0} \times \cdots \times f_{r}\right)
$$

### 10.4 Immersions of warped products and s-nullities

In this section Lemma 8.15 is applied to isometric immersions of warped product manifolds.

Theorem 10.24. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ be an isometric immersion of a warped product manifold whose warping functions are pairwise linearly independent on any open subset. If $2 p<n$ and at any point the $s$-nullities satisfy $\nu_{s}<n-2 s$ for $1 \leq s \leq p$, then $f$ is a warped product of isometric immersions.

Proof: We use the fact that the curvature tensor $R$ of a warped product metric

$$
g=\pi_{0}^{*} g_{0}+\sum_{a=1}^{r}\left(\rho_{a} \circ \pi_{0}\right)^{2} \pi_{a}^{*} g_{a}
$$

on the product manifold $M^{n}=\Pi_{i=0}^{r} M_{i}$, with smooth warping functions $\rho_{a}: M_{0} \rightarrow \mathbb{R}_{+}$, $1 \leq a \leq r$, is related to the curvature tensor $\tilde{R}$ of the product metric $\tilde{g}=\sum_{i=0}^{r} \pi_{i}^{*} g_{i}$ by

$$
\begin{align*}
R(X, Y)= & \tilde{R}(X, Y)-\sum_{a, b=1}^{r}\left\langle\eta_{a}, \eta_{b}\right\rangle X^{a} \wedge Y^{b}  \tag{10.43}\\
& +\sum_{a=1}^{r}\left[\left(\nabla_{X^{0}} \eta_{a}-\left\langle\eta_{a}, X\right\rangle \eta_{a}\right) \wedge Y^{a}+X^{a} \wedge\left(\nabla_{Y^{0}} \eta_{a}-\left\langle\eta_{a}, Y\right\rangle \eta_{a}\right)\right]
\end{align*}
$$

for all $x \in M^{n}$ and $X, Y \in T_{x} M$, where

$$
\eta_{a}=-\operatorname{grad} \log \left(\rho_{a} \circ \pi_{0}\right), \quad 1 \leq a \leq r,
$$

and $X=\sum_{i=0}^{r} X^{i}$ is the orthogonal decomposition of $X \in T_{x} M$ with respect to the product net $\mathcal{E}=\left(E_{i}\right)_{i=0, \ldots, r}$ of $M^{n}$ at $x$ (see Exercise 10.8).

It follows from (10.43) that

$$
R(X, Y, Z, U)=R(X, Y, U, V)=R(X, U, V, W)=0
$$

for all $X, Y, Z \in E_{a}(x)$ and $U, V, W \in E_{a}^{\perp}(x), 1 \leq a \leq r$. Hence Lemma 8.15 applies to the second fundamental form $\alpha$ of $f$ at any point $x \in M^{n}$, with respect to the orthogonal splitting

$$
T_{x} M=E_{a}(x) \oplus E_{a}^{\perp}(x), \quad 1 \leq a \leq r .
$$

Therefore $\alpha$ is adapted to $\mathcal{E}$, and the statement follows from Theorem 10.21 .
Remark 10.25. The assumption in Theorem 10.24 that the warping functions be pairwise linearly independent should not be seen as a restriction. In fact, if two warping functions are linearly dependent, one can scale the metric of one of the factors by a constant in such a way that the two warping functions coincide, and then replace both factors by their product.

The next result restricts the codimension to the number of factors and assumes a curvature condition.

Theorem 10.26. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+r+1}, 2 r+2<n$, be an isometric immersion of a warped product manifold

$$
M^{n}=M_{0}^{n_{0}} \times{ }_{\rho_{1}} M_{1}^{n_{1}} \times \cdots \times_{\rho_{r}} M_{r}^{n_{r}} .
$$

Assume that no factor of $M^{n}$ has an open subset where the sectional curvature is constant, and that the warping functions are not constant on any open subset of $M_{0}$. If at any point the $s$-nullities satisfy $\nu_{s}<n-2 s$ for $1 \leq s \leq r+1$, then $f$ is a warped product of hypersurfaces.

Proof: If the number $k$ of pairwise linearly independent warping functions is $k=r$, the result follows from Theorem 10.24 and the curvature assumption, which forces the immersions of all factors to have positive codimension. Thus we may assume that $k<r$ and let $\rho_{i_{1}}, \ldots, \rho_{i_{k}}, 1 \leq i_{1}<\cdots<i_{k} \leq r$, be the pairwise linearly independent warping functions. Hence we may view $M^{n}$ as a warped product

$$
\begin{equation*}
M^{n}=M_{0}^{\ell_{0}} \times_{\rho_{i_{1}}} \hat{M}_{1}^{\ell_{1}} \times \cdots \times_{\rho_{i_{k}}} \hat{M}_{k}^{\ell_{k}} \tag{10.44}
\end{equation*}
$$

where $M_{0}^{\ell_{0}}=M_{0}^{n_{0}}$ and the factors $\hat{M}_{a}^{\ell_{a}}, 1 \leq a \leq k$, are the Riemannian products

$$
\hat{M}_{a}^{\ell_{a}}=\Pi_{i \in I_{a}} M_{i}^{n_{i}}, \quad 1 \leq a \leq k,
$$

and $I_{a}$ denotes the set of all indices $1 \leq b \leq r$ that correspond to factors with the same associated warping function $\rho_{i_{a}}$ after homotheties.

We apply Theorem 10.24 to $M^{n}$ with the warped product structure (10.44). By Corollary 10.23 , there exist a warped product representation

$$
\psi: N=N_{0}^{m_{0}} \times_{\sigma_{1}} N_{1}^{m_{1}} \times \cdots \times_{\sigma_{r}} N_{k}^{m_{k}} \rightarrow \mathbb{Q}_{c}^{n+k+1}
$$

and isometric immersions $\hat{f}_{i}: \hat{M}_{i}^{\ell_{i}} \rightarrow N_{i}^{m_{i}}, 0 \leq i \leq k$, such that

$$
\begin{equation*}
f=\psi \circ\left(\hat{f}_{0} \times \hat{f}_{1} \times \cdots \times \hat{f}_{k}\right) . \tag{10.45}
\end{equation*}
$$

We show next that at any point of $\hat{M}_{a}^{\ell_{a}}, 1 \leq a \leq k$, the $s$-nullities of $\hat{f}_{a}$ for $1 \leq s \leq \operatorname{cod}\left(\hat{f}_{a}\right)=m_{a}-\ell_{a}$ satisfy

$$
\begin{equation*}
\nu_{s}^{\hat{f}_{a}}<\ell_{a}-2 s \tag{10.46}
\end{equation*}
$$

In view of (10.45) we have

$$
\begin{equation*}
\alpha^{f}=\psi_{*} \alpha^{\hat{f}} \tag{10.47}
\end{equation*}
$$

where $\hat{f}=\hat{f}_{0} \times \cdots \times \hat{f}_{k}$. On the other hand, by Exercise 10.9 we have

$$
\begin{align*}
& \bar{\pi}_{0 *} \alpha^{\hat{f}}(X, Y)=\alpha^{\hat{f}_{0}}\left(\pi_{0 *} X, \pi_{0 *} Y\right) \\
& \quad-\sum_{a=1}^{k} \rho_{a}\left(x_{0}\right)\left\langle\pi_{a *} X, \pi_{a *} Y\right\rangle\left(\left(\operatorname{grad} \sigma_{a}\right)\left(f_{0}\left(x_{0}\right)\right)-f_{0 *} \operatorname{grad} \rho_{a}\left(x_{0}\right)\right) \tag{10.48}
\end{align*}
$$

and

$$
\begin{equation*}
\bar{\pi}_{a *} \alpha^{\hat{f}}(X, Y)=\alpha^{\hat{f}_{a}}\left(\pi_{a *} X, \pi_{a *} Y\right), \quad 1 \leq a \leq k, \tag{10.49}
\end{equation*}
$$

for all $x \in M$ and $X, Y \in T_{x} M$, where $\pi_{i}: M \rightarrow M_{i}$ and $\bar{\pi}_{i}: N \rightarrow N_{i}$ denote the canonical projections, $0 \leq i \leq k$.

We argue by contradiction. Assume that there exist $1 \leq a_{0} \leq k, \bar{x}_{a_{0}} \in \hat{M}_{a_{0}}^{\ell_{0}}$ and $1 \leq s_{0} \leq \operatorname{cod}\left(\hat{f}_{a_{0}}\right)$ such that $\nu_{s_{0}}^{\hat{f}_{a_{0}}} \geq \ell_{a_{0}}-2 s_{0}$ at $\bar{x}_{a_{0}}$. Then there exists a subspace $U_{a_{0}}^{s_{0}} \subset N_{\hat{f}_{a_{0}}} \hat{M}_{a_{0}}\left(\bar{x}_{a_{0}}\right)$ such that

$$
\operatorname{dim} W_{a_{0}}=\left\{Y \in T_{\bar{x}_{a_{0}}} M_{a_{0}}: \alpha_{U_{a_{0}}^{\delta_{0}^{0}}}^{\hat{f}_{0_{0}}}(Y, Z)=0 \text { for all } Z \in T_{\bar{x}_{a_{0}}} M_{a_{0}}\right\} \geq \ell_{a_{0}}-2 s_{0} .
$$

Let $\mathcal{E}=\left(E_{i}\right)_{i=0, \ldots, k}$ be the product net of $M^{n}$ with respect to its decomposition (10.44) and let $\bar{\varepsilon}=\left(\bar{E}_{i}\right)_{i=0, \ldots, k}$ be the product net of $N$. Given any $x \in \pi_{a_{0}}^{-1}\left(\bar{x}_{a_{0}}\right)$, let $U^{s_{0}} \subset$ $\bar{E}_{a_{0}}(\hat{f}(x)) \subset T_{\hat{f}(x)} N$ be such that $\bar{\pi}_{a_{0} *} U^{s_{0}}=U_{a_{0}}^{s_{0}}$.

By (10.48) and 10.49) we have

$$
\bar{\pi}_{a_{0} *} \alpha_{U^{s_{0}}}^{\hat{f}}(Y, Z)=\alpha^{\hat{f}_{a}}\left(\pi_{a_{0} *} Y, \pi_{a_{0} *} Z\right)
$$

and

$$
\bar{\pi}_{a *} \alpha_{U^{s_{0}}}^{\hat{f}}(Y, Z)=0
$$

for all $Y, Z \in T_{\bar{x}_{a_{0}}} M$ and $1 \leq a \neq a_{0} \leq k$. Thus

$$
\alpha_{U^{s_{0}}}^{\hat{f}}(Y, Z)=0
$$

if $Y \in W_{a_{0}} \oplus_{a \neq a_{0}} E_{a}(x)$ and $Z \in T_{x} M$, and hence

$$
\nu_{s_{0}}(\hat{f}) \geq \ell_{a_{0}}-2 s_{0}+\sum_{a \neq a_{0}}^{k} \ell_{a}=n-2 s_{0} .
$$

In view of (10.47), this is a contradiction and proves 10.46).
Assume that $\left|I_{a}\right|>1$ for some $1 \leq a \leq k$. From (10.46) and Theorem 8.14 it follows that $\hat{f}_{a}$ is an extrinsic product of isometric immersions $g_{1}^{a}, \ldots, g_{\left|I_{a}\right|}^{a}$. By the curvature assumption, $\operatorname{cod}\left(g_{i}^{a}\right) \geq 1$ for all $i \in I_{a}$. Therefore

$$
\operatorname{cod}\left(\hat{f}_{a}\right) \geq\left|I_{a}\right| \text { if } N_{a}^{m_{a}}=\mathbb{R}^{m_{a}} \text { and } \operatorname{cod}\left(\hat{f}_{a}\right)>\left|I_{a}\right| \text { otherwise. }
$$

The curvature assumption implies that $\operatorname{cod}\left(\hat{f}_{i}\right) \geq 1$ if either $i=0$ or $\left|I_{i}\right|=1$. Hence

$$
r+1=\sum_{i=0}^{k} \operatorname{cod}\left(\hat{f}_{i}\right) \geq \sum_{i=0}^{k}\left|I_{i}\right|=r+1 .
$$

Therefore $\operatorname{cod}\left(\hat{f}_{i}\right)=\left|I_{i}\right|$ for all $0 \leq i \leq k$. In particular, if $\left|I_{i}\right|>1$ then $\hat{f}_{i}$ is a product of Euclidean hypersurfaces. We conclude that each factor in the initial product decomposition of $M^{n}$ must be a hypersurface.

In the case of warped products $N^{p+n}=L^{p} \times{ }_{\rho} M^{n}$ with only two factors, under the assumptions that $n \geq 3$ and that $N^{p+n}$ is free of points with constant sectional curvature $c$, a complete description of their isometric immersions into $\mathbb{Q}_{c}^{p+n+k}$ with codimension $k \leq 2$ is given below without proof. Notice that all the assumptions in this result are of purely intrinsic nature.

Theorem 10.27. Assume that a warped product $N^{p+n}=L^{p} \times_{\rho} M^{n}$ with $n \geq 3$ is free of points with constant sectional curvature c. Then, for any isometric immersion $f: N^{p+n} \rightarrow \mathbb{Q}_{c}^{p+n+2}$, there exists an open dense subset of $N^{p+n}$ each of whose points lies in an open product neighborhood $U=L_{0}^{p} \times M_{0}^{n} \subset L^{p} \times M^{n}$ such that one of the following possibilities holds:
(i) $\left.f\right|_{U}$ is a warped product of isometric immersions with respect to a warped product representation $\psi: V^{p+k_{1}} \times{ }_{\sigma} \mathbb{Q}_{\tilde{c}}^{n+k_{2}} \rightarrow \mathbb{Q}_{c}^{p+n+2}, k_{1}+k_{2}=2$.

(ii) $\left.f\right|_{U}$ is a composition $H \circ g$ of isometric immersions where $g$ is a warped product of isometric immersions $g=\psi \circ\left(h_{1} \times h_{2}\right)$ determined by a warped product representation $\psi: V^{p+k_{1}} \times_{\sigma} \mathbb{Q}_{\tilde{c}}^{n+k_{2}} \rightarrow \mathbb{Q}_{c}^{p+n+1}$ with $k_{1}+k_{2}=1$, and $H: W \rightarrow \mathbb{Q}_{c}^{p+n+2}$ is an isometric immersion of an open subset $W \supset g(U)$ of $\mathbb{Q}_{c}^{p+n+1}$.

(iii) There exist open intervals $I, J \subset \mathbb{R}$ such that $L_{0}^{p}, M_{0}^{n}, U$ split as $L_{0}^{p}=L_{0}^{p-1} \times_{\rho_{1}} I$, $M_{0}^{n}=J \times{ }_{\rho_{2}} M_{0}^{n-1}$ and

$$
U=L_{0}^{p-1} \times_{\rho_{1}}\left(\left(I \times_{\rho_{3}} J\right) \times_{\bar{\rho}} M_{0}^{n-1}\right),
$$

where $\rho_{1} \in C^{\infty}\left(L_{0}^{p-1}\right), \rho_{2} \in C^{\infty}(J), \rho_{3} \in C^{\infty}(I)$ and $\bar{\rho} \in C^{\infty}(I \times J)$ satisfy

$$
\rho=\left(\rho_{1} \circ \pi_{L_{0}^{p-1}}\right)\left(\rho_{3} \circ \pi_{I}\right) \text { and } \bar{\rho}=\left(\rho_{3} \circ \pi_{I}\right)\left(\rho_{2} \circ \pi_{J}\right) \text {, }
$$

and there exist warped product representations

$$
\psi_{1}: V^{p-1} \times_{\sigma_{1}} \mathbb{Q}_{\tilde{c}}^{n+3} \rightarrow \mathbb{Q}_{c}^{p+n+2} \text { and } \psi_{2}: W^{4} \times_{\sigma_{2}} \mathbb{Q}_{\bar{c}}^{n-1} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+3} \text {, }
$$

an isometric immersion $g: I \times_{\rho_{3}} J \rightarrow W^{4}$ and isometries $i_{1}: L_{0}^{p-1} \rightarrow W^{p-1} \subset$ $V^{p-1} \subset \mathbb{Q}_{c}^{p-1}$ and $i_{2}: M^{n-1} \rightarrow W^{n-1} \subset \mathbb{Q}_{\bar{c}}^{n-1}$ onto open subsets such that

$$
\left.f\right|_{U}=\psi_{1} \circ\left(i_{1} \times\left(\psi_{2} \circ\left(g \times i_{2}\right)\right)\right), \quad \bar{\rho}=\sigma_{2} \circ g \text { and } \rho_{1}=\sigma_{1} \circ i_{1} .
$$

Moreover, $L_{0}^{p}$ has constant sectional curvature c if $p \geq 2$.


Cases (i) to (iii) are disjoint. Moreover, in case (iii) the isometric immersion $g: I \times_{\rho_{3}} J \rightarrow W^{4}$ is neither a warped product $g=\psi_{3} \circ(\alpha \times \beta)$, where $\psi_{3}: V^{1+k_{1}} \times_{\sigma_{3}}$ $\mathbb{Q}_{\hat{c}}^{1+k_{2}} \rightarrow \mathbb{Q}_{\hat{c}}^{4}$ is a warped product representation with $k_{1}+k_{2}=2$ and $\alpha: I \rightarrow V^{1+k_{1}}$ and $\beta: J \rightarrow \mathbb{Q}_{\hat{c}}^{1+k_{2}}$ are unit-speed curves with $\rho_{3}=\sigma_{3} \circ \alpha$, nor a composition $H \circ G$ of such a warped product $G=\psi_{3} \circ(\alpha \times \beta)$, determined by a warped product representation $\psi_{3}: V^{1+k_{1}} \times_{\sigma_{3}} \mathbb{Q}_{\hat{c}}^{1+k_{2}} \rightarrow \mathbb{Q}_{\tilde{c}}^{3}$ as before with $k_{1}+k_{2}=1$, with an isometric immersion $H$ of an open subset $W \supset G(I \times J)$ into $\mathbb{Q}_{\tilde{c}}^{4}$.

### 10.5 Notes

The de Rham-type characterization in Theorem 10.4 of warped product manifolds is due to Hiepko [214]. Polar metrics were defined by Tojeiro [331], where Theorem 10.3 was obtained.

Partial tubes were introduced by Carter-West [72]. In fact, the definition of partial tube in [72] is more general than the one considered here. Given a submanifold $f: M^{n} \rightarrow \mathbb{R}^{m}$, the partial tube over $f$ with a smooth submanifold $S \subset \mathbb{R}^{k}$ as fiber was defined in [72] as follows. Let $B(f)$ be a smooth subbundle of the normal bundle $N_{f} M$ of $f$, with type fiber $S$, which is invariant under parallel transport along any curve in $M^{n}$. Equivalently, the fiber $S$ of $B(f)$ at any point is a union of orbits, at that point, of the normal holonomy group of $f$, that is, the holonomy group of the normal bundle of $f$. It is also assumed that no point of $B(f)$ is a critical point of the endpoint map $\eta: N_{f} M \rightarrow \mathbb{R}^{m}$ given by $\eta(x, \xi)=f(x)+\xi$. The restriction of $\eta$ to $B(f)$ is then the partial tube over $f$ with $S$ as fiber. The definition considered here corresponds to the case in which each point of $S$ is itself an orbit of the normal holonomy group of $f$. The general definition includes the important special case of a holonomy tube, which is a partial tube whose fiber at any point is a single orbit of the normal holonomy group of $f$ at that point. For an interesting application of this notion for the classification of submanifolds with constant principal curvatures, among other applications, see Heintze-Olmos-Thorbergsson [215] or the book [34]. See also [71] for further applications of partial tubes.

The decomposition Theorem 10.14 for isometric immersions of product manifolds endowed with polar metrics was proved by Tojeiro [331]. The notion of warped product of immersions, as well as the decomposition Theorem 10.21 for isometric immersions of warped product manifolds into space forms, is due to Nölker [267].

Theorem 10.27 has been taken from Dajczer-Tojeiro [144]. The sufficient conditions in Theorems 10.24 and 10.26 for the second fundamental form of an isometric immersion of a warped product to be adapted to its product structure in terms of the $s$-nullities were obtained by Dajczer-Vlachos [151].

A conformal version of 10.21 , which also extends Theorem 9.26, was obtained by Tojeiro [334]. The result in Exercise 10.3 is due to do Carmo-Dajczer [56], where rotation hypersurfaces in the sphere and in the hyperbolic space were first studied, whereas those in Exercises 10.6 and 10.7 have been obtained by Chen 83 .

### 10.6 Exercises

Exercise 10.1. (a) Show that any rotation hypersurface in $\mathbb{R}^{n+1}$ can be locally parametrized by

$$
f\left(t_{1}, \ldots, t_{n-1}, s\right)=\left(x_{1}(s) \varphi_{1}, \ldots, x_{1}(s) \varphi_{n}, x_{n+1}(s)\right)
$$

where $\varphi\left(t_{1}, \ldots, t_{n-1}\right)=\left(\varphi_{1}, \ldots, \varphi_{n}\right)$ is a parametrization of the unit sphere in $\mathbb{R}^{n}$ and $x(s)=\left(x_{1}(s), x_{n+1}(s)\right), x_{1}>0$, is a parametrization by arc-length of the profile curve.
(b) Show that a rotation hypersurface in $\mathbb{S}^{n+1}$ can be locally parametrized by

$$
f\left(t_{1}, \ldots, t_{n-1}, s\right)=\left(x_{1}(s) \varphi_{1}, \ldots, x_{1}(s) \varphi_{n}, x_{n+1}(s), x_{n+2}(s)\right)
$$

where $\varphi\left(t_{1}, \ldots, t_{n-1}\right)=\left(\varphi_{1}, \ldots, \varphi_{n}\right)$ is a parametrization of the unit sphere in $\mathbb{R}^{n}$ and $x(s)=\left(x_{1}(s), x_{n+1}(s), x_{n+2}(s)\right), x_{1}>0$, is a parametrization by arc-length of the profile curve in $\mathbb{S}^{2} \subset \mathbb{R}^{3}=\operatorname{span}\left\{e_{1}, e_{n+1}, e_{n+2}\right\}$.
(c) (i) Show that a rotation hypersurface in $\mathbb{H}^{n+1}$ of spherical (respectively, hyperbolic) type can be locally parametrized by

$$
f\left(t_{1}, \ldots, t_{n-1}, s\right)=\left(x_{1}(s) \varphi_{1}, \ldots, x_{1}(s) \varphi_{n}, x_{n+1}(s), x_{n+2}(s)\right)
$$

where $e_{1}, \ldots, e_{n+2}$ is an orthonormal basis of $\mathbb{L}^{n+2}$ such that $e_{n+2}$ (respectively, $e_{1}$ ) is time-like, $\varphi\left(t_{1}, \ldots, t_{n-1}\right)=\left(\varphi_{1}, \ldots, \varphi_{n}\right)$ is a parametrization of the unit sphere in $\mathbb{R}^{n}=$ $\operatorname{span}\left\{e_{1}, \ldots, e_{n}\right\}$ (respectively, the "unit" hyperbolic space of $\mathbb{L}^{n}=\operatorname{span}\left\{e_{1}, \ldots, e_{n}\right\}$ ) and $x(s)=\left(x_{1}(s), x_{n+1}(s), x_{n+2}(s)\right), x_{1}>0$, is a parametrization by arc-length of the profile curve in $\mathbb{S}^{2} \subset \mathbb{R}^{3}=\operatorname{span}\left\{e_{1}, e_{n+1}, e_{n+2}\right\}$ (respectively, $\mathbb{H}^{2} \subset \mathbb{L}^{3}=\operatorname{span}\left\{e_{1}, e_{n+1}, e_{n+2}\right\}$ ).
(ii) Show that a rotation hypersurface in $\mathbb{H}^{n+1}$ of parabolic type can be parametrized by

$$
\begin{equation*}
f\left(t_{2}, \ldots, t_{n}, s\right)=x_{1}(s) \Psi\left(t_{2}, \ldots, t_{n}\right)+x_{n+1}(s) e_{n+1}+x_{n+2}(s) e_{n+2} \tag{10.50}
\end{equation*}
$$

where $e_{1}, \ldots, e_{n+2}$ is a pseudo-orthonormal basis of $\mathbb{L}^{n+2}$ such that

$$
\begin{gathered}
\left\langle e_{1}, e_{1}\right\rangle=0=\left\langle e_{n+1}, e_{n+1}\right\rangle, \quad\left\langle e_{1}, e_{n+1}\right\rangle=1, \\
\left\langle e_{k}, e_{j}\right\rangle=\delta_{k j}, \quad k=1, \ldots, n+2, \quad j=2, \ldots, n+2, \quad j \neq n+1, \\
\Psi\left(t_{2}, \ldots, t_{n}\right)=e_{1}+\sum_{i=2}^{n} t_{i} e_{i}-\frac{1}{2} \sum_{i=2}^{n} t_{i}^{2} e_{n+1}
\end{gathered}
$$

and $x(s)=\left(x_{1}(s), x_{n+1}(s), x_{n+2}(s)\right), x_{1}>0$, is a parametrization by arc-length of the profile curve in $\mathbb{H}^{2} \subset \mathbb{L}^{3}=\operatorname{span}\left\{e_{1}, e_{n+1}, e_{n+2}\right\}$. Use that

$$
2 x_{1} x_{n+1}+x_{n+2}^{2}=-1
$$

to show that 10.50 can also be written as

$$
f\left(t_{2}, \ldots, t_{n}, s\right)=\left(x_{1}(s), x_{1}(s) t_{2}, \ldots, x_{1}(s) t_{n},-\frac{1+x_{n+2}^{2}(s)+x_{1}^{2}(s) \sum_{i=2}^{n} t_{i}^{2}}{2 x_{1}(s)}, x_{n+2}(s)\right)
$$

Exercise 10.2. Let $f: M_{\delta}^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ be a rotation hypersurface, where $\delta$ is either 1,0 or -1 if $c<0$, depending on whether $f$ is of spherical, parabolic or hyperbolic type, respectively, and $\delta=1$ if $c \geq 0$. Show that the principal curvatures of $f$ are given by

$$
\lambda=-\frac{\sqrt{\delta-c x_{1}^{2}-\dot{x}_{1}^{2}}}{x_{1}} \text { and } \mu=\frac{\ddot{x}_{1}+c x_{1}}{\sqrt{\delta-c x_{1}^{2}-\dot{x}_{1}^{2}}}
$$

with $\lambda$ corresponding to all of the $t_{i}$-coordinate curves and $\mu$ to the $s$-coordinate curve.
Exercise 10.3. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}, n \geq 3$, be a hypersurface. Assume that the principal curvatures $k_{1}, \ldots, k_{n}$ of $f$ satisfy

$$
k_{1}=\cdots=k_{n-1}=\lambda \text { and } k_{n}=\mu=\mu(\lambda) \neq \lambda .
$$

Show that $f(M)$ is an open subset of a rotation hypersurface.
Hint: Use Exercise 1.18 and the assumption that $\mu=\mu(\lambda)$ to show that the onedimensional distribution $E_{\mu}=\operatorname{ker}(\mu I-A)$, where $A$ is the shape operator of $f$, is totally geodesic. Use also part (ii) of Proposition 1.22 to show that $E_{\lambda}=\operatorname{ker}(\lambda I-A)$ is a spherical distribution, and conclude from Theorem 10.4 that $M^{n}$ is locally a warped product $I \times{ }_{\rho} \mathbb{Q}_{\tilde{c}}^{n-1}$. Then use Theorem 10.21 .

Exercise 10.4. Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be an isometric immersion of a Riemannian manifold that carries an orthogonal net $\mathcal{E}=\left(E_{0}, E_{1}\right)$ with $E_{0}$ totally geodesic. Assume that the second fundamental form of $f$ is adapted to $\mathcal{E}$.
(i) Show that there exist locally (globally, if $M^{n}$ is simply connected and the leaves of $E_{0}$ are complete) a product representation $\psi: M_{0} \times M_{1} \rightarrow M$ of $\mathcal{E}$, an immersion $f_{1}: M_{1} \rightarrow \mathbb{R}^{m}$, a parallel vector bundle isometry $\phi: M_{1} \times \mathbb{R}^{s} \rightarrow \mathcal{L}$ onto a flat parallel subbundle of $N_{f_{1}} M_{1}$ and an immersion $f_{0}: M_{0} \rightarrow \Omega\left(f_{1} ; \phi\right) \subset \mathbb{R}^{s}$ such that $f \circ \psi$ is the partial tube determined by $\left(f_{0}, f_{1}, \phi\right)$.
(ii) If, in addition, the distribution $E_{1}$ is spherical, show that $f$ is a warped product of immersions $f_{1}: M_{1} \rightarrow \mathbb{S}^{m-s} \subset \mathbb{R}^{m-s+1}$ and $f_{0}: M_{0} \rightarrow \mathbb{R}^{s}$.

Exercise 10.5. Let $f: M^{2} \rightarrow \mathbb{R}^{m}$ be a surface with flat normal bundle free of umbilical points. Let $\mathcal{E}=\left(E_{0}, E_{1}\right)$ be the orthogonal net on $M^{2}$ determined by its curvature lines. Assume that those correspondent to $E_{0}$ are geodesics.
(i) Show that there exist locally (globally, if $M^{2}$ is simply connected and the geodesic integral curves of $E_{0}$ are complete) a product representation $\psi: I \times J \rightarrow M^{2}$ of $\mathcal{E}$, where $I, J \subset \mathbb{R}$ are open intervals and $I=\mathbb{R}$ under the global assumptions, a smooth curve $\beta: J \rightarrow \mathbb{R}^{m}$, a parallel vector bundle isometry $\phi: J \times \mathbb{R}^{s} \rightarrow \mathcal{L}$ onto a flat parallel subbundle $\mathcal{L}$ of $N_{\beta} J$ and a smooth curve $\alpha: I \rightarrow \Omega(\beta ; \phi) \subset \mathbb{R}^{s}$ such that $f \circ \psi$ is the partial tube determined by $(\alpha, \beta, \phi)$.
(ii) If, in addition, the curvature lines correspondent to $E_{1}$ have constant geodesic curvature, show that $f$ is a warped product of curves $\alpha: I \rightarrow \mathbb{S}^{m-s} \subset \mathbb{R}^{m-s+1}$ and $\beta: J \rightarrow \mathbb{R}^{s}$.
(iii) Conclude that a surface in $\mathbb{R}^{3}$ with no umbilic points such that the curvature lines correspondent to one of the principal curvatures are geodesics, and those correspondent to the other have constant geodesic curvature, must be an open subset of a cylinder over a plane curve, a cone over a curve in the sphere, or a rotation surface.

Exercise 10.6. Let $f: M_{1}^{n_{1}} \times_{\rho} M_{2}^{n_{1}} \rightarrow \mathbb{Q}_{c}^{m}$, with $c \leq 0$ and $n_{1}, n_{2} \geq 2$, be a minimal isometric immersion of a warped product of Riemannian manifolds. If $\rho$ is harmonic, show that $c=0$ and that $f$ is a warped product of immersions.
Hint: Let $\mathcal{E}=\left(E_{1}, E_{2}\right)$ be the product net of $M=M_{1}^{n_{1}} \times M_{2}^{n_{2}}$ and, at $x=\left(x_{1}, x_{2}\right) \in M$, consider an orthonormal basis $u_{1}, \ldots, u_{n_{1}}, v_{1}, \ldots, v_{n_{2}}$ of $T_{x} M$ with $u_{1}, \ldots, u_{n_{1}} \in E_{1}(x)$ and $v_{1}, \ldots, v_{n_{2}} \in E_{2}(x)$. Use the Gauss equation of $f$ to show that

$$
\left\langle\nabla_{u_{i}} \eta, u_{i}\right\rangle-\left\langle\eta, u_{i}\right\rangle^{2}=c+\left\langle\alpha\left(u_{i}, u_{i}\right), \alpha\left(v_{j}, v_{j}\right)\right\rangle-\left\|\alpha\left(u_{i}, v_{j}\right)\right\|^{2}
$$

where $\eta=-\operatorname{grad} \log \rho$. Sum in both indices and use the minimality condition to obtain

$$
\begin{equation*}
n_{2} \frac{\Delta \rho}{\rho}+n_{1} n_{2} c=\left\|\sum_{i} \alpha\left(u_{i}, u_{i}\right)\right\|^{2}+\sum_{i, j}\left\|\alpha\left(u_{i}, v_{j}\right)\right\|^{2}, \tag{10.51}
\end{equation*}
$$

and then conclude using Theorem 10.21
Exercise 10.7. Prove that there exists no minimal isometric immersion of a warped product $M_{1}^{n_{1}} \times_{\rho} M_{2}^{n_{1}}$ into $\mathbb{Q}_{c}^{m}$ if $M_{1}^{n_{1}}$ is compact, $c \leq 0$ and $n_{1}, n_{2} \geq 2$.
Hint: Let $f: M_{1}^{n_{1}} \times{ }_{\rho} M_{2}^{n_{1}} \rightarrow \mathbb{Q}_{c}^{m}$ be a minimal isometric immersion. From (10.51), $c \leq 0$ and $\rho>0$ obtain $\Delta \rho \geq 0$. Use the compactness of $M_{1}^{n_{1}}$ to conclude that $\rho$ is constant by means of Hopf's theorem (see part (ii) of Exercise 3.5). Now use (10.51) again to show that $c=0$, and conclude from Exercise 8.7 that $f$ is an extrinsic product of minimal isometric immersions $f_{1}: M_{1}^{n_{1}} \rightarrow \mathbb{R}^{m_{1}}$ and $f_{2}: M_{2}^{n_{2}} \rightarrow \mathbb{R}^{m_{2}}$, with the metric of $M_{1}^{n_{1}}$ divided by $\rho$. Then obtain a contradiction with Corollary 3.7.

Exercise 10.8. Let $M^{n}=M_{0} \times_{\rho_{1}} M_{1} \times \cdots \times_{\rho_{r}} M_{r}$ be a warped product manifold, let $\mathcal{E}=\left(E_{i}\right)_{i=0, \ldots, r}$ be its product net, and let $\pi_{i}: M \rightarrow M_{i}, 0 \leq i \leq r$, denote the projection.
(i) Show that the Levi-Civita connection $\nabla$ of $M^{n}$ and the Levi-Civita connection $\hat{\nabla}$ of the Riemannian product manifold $M_{0} \times M_{1} \times \cdots \times M_{r}$ are related by

$$
\begin{equation*}
\nabla_{X} Y-\hat{\nabla}_{X} Y=\sum_{a=1}^{r}\left(\left\langle X^{a}, Y^{a}\right\rangle \eta_{a}-\left\langle\eta_{a}, X\right\rangle Y^{a}-\left\langle\eta_{a}, Y\right\rangle X^{a}\right) \tag{10.52}
\end{equation*}
$$

where $X \mapsto X^{i}$ denotes the orthogonal projection onto $E_{i}, 0 \leq i \leq r$, and $\eta_{a}=-\operatorname{grad}\left(\log \circ \rho_{a} \circ \pi_{0}\right), 1 \leq a \leq r$.
(ii) Prove that $E_{a}$ is spherical with mean curvature normal vector field $\eta_{a}$ and $E_{a}^{\perp}$ is totally geodesic for $1 \leq a \leq r$.
(iii) Show that the curvature tensors $R$ and $\hat{R}$ of $\nabla$ and $\hat{\nabla}$, respectively, are related by

$$
\begin{aligned}
R(X, Y)= & \hat{R}(X, Y)-\sum_{a, b=1}^{r}\left\langle\eta_{a}, \eta_{b}\right\rangle X^{a} \wedge Y^{b} \\
& +\sum_{a=1}^{r}\left[\left(\nabla_{X^{0}} \eta_{a}-\left\langle\eta_{a}, X\right\rangle \eta_{a}\right) \wedge Y^{a}+X^{a} \wedge\left(\nabla_{Y^{0}} \eta_{a}-\left\langle\eta_{a}, Y\right\rangle \eta_{a}\right)\right]
\end{aligned}
$$

Hint for part ( $i$ ): First use the fact that the tensor $S \in \Gamma\left(\operatorname{Hom}^{2}(T M, T M ; T M)\right)$ defined by the right-hand side of 10.52 is symmetric to show that $\tilde{\nabla}=\hat{\nabla}+S$ is a torsion-free connection on $T M$. Then use that $\left(\hat{\nabla}_{X} Y\right)^{i}=\hat{\nabla}_{X} Y^{i}, 0 \leq i \leq r$, as follows from the fact that $E_{i}$ is totally geodesic with respect to $\hat{\nabla}$, to prove that $\tilde{\nabla}$ is compatible with $g$. Conclude that $\tilde{\nabla}=\nabla$.

Exercise 10.9. Let $M^{n}=M_{0} \times_{\rho_{1}} M_{1} \times \cdots \times_{\rho_{r}} M_{r}$ and $N^{m}=N_{0} \times_{\sigma_{1}} N_{1} \times \cdots \times_{\sigma_{r}} N_{r}$ be warped product manifolds, and let $\pi_{i}: M^{n} \rightarrow M_{i}$ and $\bar{\pi}_{i}: N^{m} \rightarrow N_{i}$ denote the canonical projections. Let $f_{i}: M_{i} \rightarrow N_{i}$ be an isometric immersion for $0 \leq i \leq r$ and suppose that $\rho_{a}=\sigma_{a} \circ f_{0}$ for $1 \leq a \leq r$. Show that $f=f_{0} \times \cdots \times f_{r}: M^{n} \rightarrow N^{m}$ is an isometric immersion and that the following assertions hold:
(i) $\bar{\pi}_{i *} f_{*} T_{x} M=f_{i *} T_{x_{i}} M_{i}$ and $\bar{\pi}_{i *} N_{f} M(x)=N_{f_{i}} M_{i}\left(x_{i}\right)$ for all $0 \leq i \leq r$ and $x=$ $\left(x_{0}, \ldots, x_{r}\right) \in M^{n}$.
(ii) The second fundamental form of $f$ satisfies

$$
\begin{aligned}
\bar{\pi}_{0 *} \alpha^{f}(X, Y)= & \alpha^{f_{0}}\left(\pi_{0 *} X, \pi_{0 *} Y\right) \\
& -\sum_{a=1}^{r} \rho_{a}\left(x_{0}\right)\left\langle\pi_{a *} X, \pi_{a *} Y\right\rangle\left(\left(\operatorname{grad} \sigma_{a}\right)\left(f_{0}\left(x_{0}\right)\right)-f_{0 *} \operatorname{grad} \rho_{a}\left(x_{0}\right)\right)
\end{aligned}
$$

and

$$
\bar{\pi}_{a *} \alpha^{f}(X, Y)=\alpha^{f_{a}}\left(\pi_{a *} X, \pi_{a *} Y\right), \quad 1 \leq a \leq r,
$$

for all $x \in M^{n}$ and $X, Y \in T_{x} M$.
(iii) Derive the formulas in part (iii) of Proposition 10.16, as well as those in part (iii) of Proposition 10.20, from those in part (ii) above.

Hint for part (ii): use part (i) of the preceding exercise.

## Chapter 11

## The Sbrana-Cartan hypersurfaces

By the classical Beez-Killing theorem, a hypersurface $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ is rigid if it has type number $\tau \geq 3$ at any point. Therefore, if $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ is an isometric immersion such that $M^{n}$ admits another isometric immersion $\tilde{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ that is not congruent to $f$ on any open subset of $M^{n}$, then $f$ must have type number $\tau \leq 2$ at any point. Notice that $f$ has type number $\tau \leq 1$ at a point of $M^{n}$ if and only if all sectional curvatures of $M^{n}$ at that point are equal to $c$, as follows from the Gauss equation. Totally geodesic hypersurfaces have already been classified in Chapter 1, whereas hypersurfaces of constant type number $\tau=1$ can locally be explicitly parametrized by means of the Gauss parametrization; see Corollaries 7.20 and 7.23 .

The study of Euclidean hypersurfaces that allow isometric deformations has been carried out by Sbrana in 1909 and Cartan in 1916. For this reason, an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ is called a Sbrana-Cartan hypersurface if $M^{n}$ is a Riemannian manifold of dimension $n \geq 3$ free of points where all sectional curvatures are equal to $c$ that admits an isometric immersion $\tilde{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ not congruent to $f$ on any open subset. By the discussion above, a Sbrana-Cartan hypersurface must have constant type number $\tau=2$.

Examples of Sbrana-Cartan hypersurfaces of $\mathbb{R}^{n+1}$ are cylinders over surfaces $g: L^{2} \rightarrow \mathbb{R}^{3}$ that are free of flat points and admit isometric deformations that are not congruent to $g$ on any open subset of $L^{2}$. Similar examples in any space form $\mathbb{Q}_{c}^{n+1}$ are generalized cones over surfaces in an umbilical submanifold $\mathbb{Q}_{\tilde{c}}^{3} \subset \mathbb{Q}_{c}^{n+1}, \tilde{c}>c$. These are called surface-like Sbrana-Cartan hypersurfaces. Another class of examples consists of ruled hypersurfaces, that is, hypersurfaces carrying a foliation of codimension one by totally geodesic submanifolds of the ambient space. These will be shown to admit locally as many isometric deformations as smooth real functions on an open interval, all of them ruled with the same rulings.

Since a Sbrana-Cartan hypersurface $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ with $n \geq 3$ has constant type number $\tau=2$, it can be locally parametrized, in terms of the Gauss parametrization, by a surface in the unit sphere of either $\mathbb{R}^{n+1}, \mathbb{R}^{n+2}$ or $\mathbb{L}^{n+2}$, corresponding to $c=0$, $c>0$ or $c<0$, respectively, and a smooth function on the surface if $c=0$. The main purpose of this chapter is to give a proof of the parametric description of Sbrana-Cartan
hypersurfaces by determining which of such surfaces (and functions on them if $c=0$ ) give rise to Sbrana-Cartan hypersurfaces that are neither surface-like nor ruled.

An alternative description due to Cartan of the Sbrana-Cartan hypersurfaces of Euclidean space will be provided in Chapter 14. It shows that they all arise as envelopes of certain two-parameter congruences of affine hyperplanes.

### 11.1 The reduction

If $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ is a Sbrana-Cartan hypersurface and $\tilde{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ is an isometric immersion that is not congruent to $f$ on any open subset of $M^{n}$, then $f$ and $\tilde{f}$ share a common relative nullity distribution $\Delta$ of rank $n-2$ by Corollary 4.15. Therefore the shape operator $\tilde{A}$ of $\tilde{f}$ is a Codazzi tensor on $M^{n}$ having $\Delta$ as its kernel, and which is not a constant multiple of $A$ on any open subset. Next we study the restrictions that are imposed on $f$ by the existence of a Codazzi tensor $\tilde{A}$ on $M^{n}$ such that $\Delta \subset \operatorname{ker} \tilde{A}$.

The results of this section will also be used in the parametric description of infinitesimally bendable Euclidean hypersurfaces in Chapter 14 . For this reason, we do not assume initially that $\tilde{A}$ also satisfies $\left.\operatorname{det} \tilde{A}\right|_{\Delta^{\perp}}=\left.\operatorname{det} A\right|_{\Delta^{\perp}}$, as follows from the Gauss equations of $f$ and $\tilde{f}$. Instead, we postpone the use of this additional condition until Section 11.3.

### 11.1.1 Hyperbolic, parabolic and elliptic hypersurfaces

Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}, n \geq 3$, be a hypersurface that carries a relative nullity distribution $\Delta$ of rank $n-2$, and let $C: \Gamma(\Delta) \rightarrow \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ denote its splitting tensor. Then $f$ is said to be hyperbolic (respectively, parabolic or elliptic) if there exists $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ satisfying the following conditions:
(i) $J^{2}=I$ (respectively, $J^{2}=0$, with $J \neq 0$, or $J^{2}=-I$ ),
(ii) $\nabla_{T} J=0$ for all $T \in \Gamma(\Delta)$,
(iii) $C_{T} \in \operatorname{span}\{I, J\}$ for all $T \in \Gamma(\Delta)$.

Lemma 11.1. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}, n \geq 3$, be a hypersurface that carries a relative nullity distribution $\Delta$ of rank $n-2$. Assume that there exists a symmetric Codazzi tensor $B \in \Gamma(\operatorname{End}(T M))$ that is not a constant multiple of $A$ on any open subset of $M^{n}$ and such that $\Delta \subset \operatorname{ker} B$. Then $f$ is either hyperbolic, parabolic or elliptic with respect to a tensor $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ on each connected component of an open dense subset $\mathcal{U} \subset M^{n}$, depending on whether the tensor $D \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ defined as

$$
D=\left.\left(\left.A\right|_{\Delta^{\perp}}\right)^{-1} B\right|_{\Delta^{\perp}}
$$

has two distinct real eigenvalues, one real eigenvalue of multiplicity two or a pair of complex conjugate eigenvalues, respectively. Moreover, on $\mathcal{U}$ the tensor $D$ satisfies:
(i) $D \in \operatorname{span}\{I, J\}$ and $D \notin \operatorname{span}\{I\}$,
(ii) $\nabla_{T} D=0$ for all $T \in \Gamma(\Delta)$.

Conversely, assume that $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}, n \geq 3$, is hyperbolic, parabolic or elliptic with respect to $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ and is not surface-like on any open subset of $M^{n}$. Suppose also that there exists $D \in \Gamma\left(E n d\left(\Delta^{\perp}\right)\right)$ satisfying the conditions in parts ( $i$ ) and (ii) above, and such that the tensor $B \in \Gamma(E n d(T M))$, defined by

$$
\begin{equation*}
\left.B\right|_{\Delta^{\perp}}=A D \text { and } \Delta \subset \operatorname{ker} B \tag{11.1}
\end{equation*}
$$

satisfies

$$
\begin{equation*}
\left(\nabla_{X} B\right) Y=\left(\nabla_{Y} B\right) X \tag{11.2}
\end{equation*}
$$

for all $X, Y \in \Gamma\left(\Delta^{\perp}\right)$. Then $B$ is a symmetric Codazzi tensor on $M^{n}$.
Proof: By Proposition 7.3 we have

$$
\begin{equation*}
\nabla_{T} A=A C_{T} \tag{11.3}
\end{equation*}
$$

and

$$
\begin{equation*}
A C_{T}=C_{T}^{t} A \tag{11.4}
\end{equation*}
$$

for any $T \in \Gamma(\Delta)$. These equations rely only on the Codazzi equation for $A$ applied to vector fields $T \in \Gamma(\Delta)$ and $X \in \Gamma\left(\Delta^{\perp}\right)$. Therefore they also hold for $B$, that is,

$$
\begin{equation*}
\nabla_{T} B=B C_{T} \tag{11.5}
\end{equation*}
$$

and

$$
\begin{equation*}
B C_{T}=C_{T}^{t} B \tag{11.6}
\end{equation*}
$$

for any $T \in \Gamma(\Delta)$. Eqs. (11.4) and (11.6) yield

$$
A D C_{T}=C_{T}^{t} A D=A C_{T} D
$$

Thus $A\left[D, C_{T}\right]=0$, and hence

$$
\begin{equation*}
\left[D, C_{T}\right]=0 \tag{11.7}
\end{equation*}
$$

for any $T \in \Gamma(\Delta)$. On the other hand, from (11.3) and (11.5) we obtain

$$
\begin{aligned}
A C_{T} D & =\left(\nabla_{T} A\right) D \\
& =\nabla_{T} A D-A \nabla_{T} D \\
& =\nabla_{T} B-A \nabla_{T} D \\
& =A D C_{T}-A \nabla_{T} D .
\end{aligned}
$$

It follows from 11.7) that

$$
A \nabla_{T} D=A\left[D, C_{T}\right]=0
$$

which implies part (ii).

Since $B$ is not a constant multiple of $A$ on any open subset of $M^{n}$, there is an open dense subset $\mathcal{U}$ of $M^{n}$ where $D \notin \operatorname{span}\{I\}$. Let $U \subset \mathcal{U}$ be an open subset where $D$ has either two smooth distinct real eigenvalues, a single real eigenvalue of multiplicity two or a pair of smooth complex conjugate eigenvalues. By looking at the Jordan form of $D$ one can write

$$
D=a I+b J
$$

where $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ satisfies $J^{2}=\epsilon I$, with $\epsilon=1,0$ or -1 , respectively. Here $a, b \in C^{\infty}(U)$, with $b$ nowhere vanishing and $b=1$ if $\epsilon=0$.

Let $S \subset \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ be the subspace of all elements that commute with $D$, or equivalently, commute with $J$. It is easily seen that $S=\operatorname{span}\{I, J\}$, and it follows from (11.7) that $C(\Gamma(\Delta)) \subset S$. To complete the proof of the direct statement, it remains to show that $\nabla_{T} J=0$ for any $T \in \Gamma(\Delta)$. From part (ii) we obtain

$$
T(a) I+T(b) J+b \nabla_{T} J=0
$$

for any $T \in \Gamma(\Delta)$. Hence

$$
T(a) J+\epsilon T(b) I+b\left(\nabla_{T} J\right) J=0 \text { and } T(a) J+\epsilon T(b) I+b J\left(\nabla_{T} J\right)=0
$$

Adding the two equations yields $T(a)=T(b)=0$, and hence $\nabla_{T} J=0$.
We conclude that $f$ is either hyperbolic, parabolic or elliptic with respect to $J$, corresponding to $\epsilon=1,0$ or -1 , respectively.

We now prove the converse. Since $f$ is not surface-like on any open subset of $M^{n}$, by Propositions 7.4 and 7.6 there exists an open dense subset of $M^{n}$ where $C(\Gamma(\Delta))$ is not contained in span $\{I\}$. It follows from condition (iii) and (11.4) that

$$
\begin{equation*}
A J=J^{t} A, \tag{11.8}
\end{equation*}
$$

and hence $A D=D^{t} A$, for $D \in \operatorname{span}\{I, J\}$. Thus $B$ is symmetric.
The Codazzi equation

$$
\left(\nabla_{X} B\right) Y=\left(\nabla_{Y} B\right) X
$$

for $X, Y \in \Gamma\left(\Delta^{\perp}\right)$ holds by assumption. It is trivially satisfied if both $X, Y \in \Gamma(\Delta)$. Finally, if $Y=T \in \Gamma(\Delta)$ and $X \in \Gamma\left(\Delta^{\perp}\right)$ the equation reduces to

$$
\begin{equation*}
\nabla_{T} A D=A D C_{T} \tag{11.9}
\end{equation*}
$$

From condition (iii) and part (i) it follows that (11.7) holds. On the other hand, in view of part (ii) we have

$$
\nabla_{T} A D=\left(\nabla_{T} A\right) D,
$$

thus we obtain (11.9) from (11.3) and 11.7).
A hypersurface $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ is said to be ruled if it carries a totally geodesic distribution $L$ of rank $n-1$ and the restriction of $f$ to each leaf of $L$ is also totally geodesic.

Proposition 11.2. Under the assumptions of Lemma 11.1, if $D$ has one constant real eigenvalue of multiplicity two, then $f$ is ruled.

Conversely, any simply connected ruled hypersurface $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ that is not surface-like on any open subset of $M^{n}$ and is free of points where all sectional curvatures are equal to $c$ is parabolic. Moreover, the set of symmetric Codazzi tensors $B \in \Gamma(E n d(T M))$ such that $\Delta \subset \operatorname{ker} B$ and

$$
\left.\operatorname{det} B\right|_{\Delta^{\perp}}=\left.\delta^{2} \operatorname{det} A\right|_{\Delta^{\perp}}
$$

for a given $\delta \in \mathbb{R}$ is in one-to-one correspondence with the set of smooth real functions on an open interval.

Proof: The assumptions are that there exists $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ such that $J \neq 0, J^{2}=0$, $\nabla_{T} J=0, C_{T} \in \operatorname{span}\{I, J\}$ for all $T \in \Gamma(\Delta)$, and that

$$
D=\delta I+J
$$

for some $\delta \in \mathbb{R}$.
Let $Y$ be a unit-length vector field spanning ker $J$, and let $X \in \Gamma\left(\Delta^{\perp}\right)$ be orthogonal to $Y$ and such that $J X=Y$. In particular, $D Y=\delta Y$. The condition that $\nabla_{T} J=0$ for all $T \in \Gamma(\Delta)$ is easily seen to be equivalent to

$$
\begin{equation*}
\nabla_{T} Y=0=\nabla_{T} X \tag{11.10}
\end{equation*}
$$

for all $T \in \Gamma(\Delta)$. Hence, replacing $J$ by $\|X\| J$, one can assume that also $X$ has unit length and that

$$
D=\delta I+\theta J
$$

where $\theta \in C^{\infty}(M)$.
We prove next that the distribution

$$
L=\Delta \oplus \operatorname{span}\{Y\}
$$

is totally geodesic and that the restriction of $f$ to each leaf of $L$ is also totally geodesic.
Since $B$ is symmetric then $A J$ is also symmetric, because $D \in \operatorname{span}\{I, J\}$ and $D \notin \operatorname{span}\{I\}$. Hence

$$
\begin{align*}
\langle A Y, Y\rangle & =\langle A J X, Y\rangle \\
& =\langle X, A J Y\rangle \\
& =0 . \tag{11.11}
\end{align*}
$$

Moreover, from $C(\Gamma(\Delta)) \subset \operatorname{span}\{I, J\}$ and $J Y=0$ we obtain

$$
\left\langle C_{T} Y, X\right\rangle=0
$$

for all $T \in \Gamma(\Delta)$. Thus

$$
\begin{equation*}
\left\langle\nabla_{Y} T, X\right\rangle=-\left\langle C_{T} Y, X\right\rangle=0 \tag{11.12}
\end{equation*}
$$

for all $T \in \Gamma(\Delta)$. Now observe that

$$
A D=\delta A+\theta A J
$$

Since $B$ and any constant multiple of $A$ are Codazzi tensors, then the same holds for the tensor $\Phi$ defined by

$$
\left.\Phi\right|_{\Delta^{\perp}}=\theta A J \text { and } \Delta \subset \operatorname{ker} \Phi .
$$

Hence

$$
\begin{equation*}
\nabla_{X} \Phi Y-\Phi \nabla_{X} Y=\nabla_{Y} \Phi X-\Phi \nabla_{Y} X \tag{11.13}
\end{equation*}
$$

Writing $\mu=\langle A Y, X\rangle$, it follows using (11.11) that

$$
\Phi X=\theta \mu X \text { and } \Phi Y=0 .
$$

Substituting in (11.13) and taking the inner product of both sides with $Y$ yield

$$
\theta \mu\left\langle\nabla_{Y} Y, X\right\rangle=0,
$$

hence

$$
\begin{equation*}
\left\langle\nabla_{Y} Y, X\right\rangle=0 \tag{11.14}
\end{equation*}
$$

because $D \neq \delta I$ and $\left.A\right|_{\Delta^{\perp}}$ is invertible. We see that $L$ is totally geodesic from the first equality in (11.10), (11.12), (11.14) and the fact that $\Delta$ is totally geodesic. Finally, from (11.11) and that $\Delta$ is the relative nullity distribution of $f$, it follows that the restriction of $f$ to each leaf of $L$ is totally geodesic. Therefore $f$ is ruled.

We now prove the converse. Let $L$ be a totally geodesic distribution of rank $n-1$ on $M^{n}$ such that the restriction of $f$ to each of its leaves is also totally geodesic. Then, at each point $x \in M^{n}$, the subspace $A L(x)$ is contained in the one-dimensional subspace $L^{\perp}(x)$; hence the relative nullity subspace $\Delta(x)$ has dimension at least $n-2$. Since $\Delta(x)$ cannot have dimension greater than $n-2$ by the assumption that $M^{n}$ does not have points where all sectional curvatures are equal to $c$, it follows that $f$ carries a relative nullity distribution $\Delta$ of constant rank $n-2$. Since $M^{n}$ is simply connected, there is a global orthonormal frame $\{X, Y\}$ of $\Delta^{\perp}$ such that $X \in \Gamma\left(L^{\perp}\right)$.

Define $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ by setting

$$
\begin{equation*}
J X=Y \text { and } J Y=0 \tag{11.15}
\end{equation*}
$$

We first prove that $f$ is parabolic with respect to $J$. Since the distributions $\Delta$ and $L$ are both totally geodesic, it follows that $\nabla_{T} Y=0$, which is equivalent to $\nabla_{T} J=0$.

To show that the splitting tensor satisfies $C(\Gamma(\Delta)) \subset \operatorname{span}\{I, J\}$, it suffices to prove that

$$
C_{T} J=J C_{T}
$$

for all $T \in \Gamma(\Delta)$. This is easily seen to be equivalent to

$$
\left\langle\nabla_{X} X, T\right\rangle=\left\langle\nabla_{Y} Y, T\right\rangle
$$

for all $T \in \Gamma(\Delta)$. To prove this fact, write

$$
\mu=\langle A X, Y\rangle
$$

so that $A Y=\mu X$. Then take the $Y$-component of (11.3) applied to $X$, and the $X$-component of the same equation applied to $Y$, to obtain

$$
\left\langle\nabla_{X} X, T\right\rangle=T(\log \mu)=\left\langle\nabla_{Y} Y, T\right\rangle .
$$

To complete the proof, it remains to show that the set of tensors $D \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ satisfying the conditions in parts $(i)$ and (ii) of Lemma 11.1, with $\operatorname{det} D=\delta^{2} \in \mathbb{R}$ and such that $B \in \Gamma(\operatorname{End}(T M))$, defined by (11.1), satisfies (11.2), is in one-to-one correspondence with the set of smooth real functions on an open interval.

Any tensor $D \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ as in part $(i)$ such that $\operatorname{det} D=\delta^{2}$ is given by

$$
D=\delta I+\theta J
$$

for some $\theta \in C^{\infty}(M)$. Then the condition in part (ii) is satisfied if and only if $T(\theta)=0$ for all $T \in \Gamma(\Delta)$. On the other hand, in view of the Codazzi equation of $f$, condition (11.2) holds if and only if the tensor $\Phi=\theta A J$ satisfies (11.13). As shown in the proof of the direct statement, the $Y$-component of that equation is equivalent to

$$
\left\langle\nabla_{Y} Y, X\right\rangle=0
$$

which is satisfied because the distribution $L=\Delta \oplus \operatorname{span}\{Y\}$ is totally geodesic. On the other hand, the $X$-component of 11.13 ) is equivalent to the equation

$$
\begin{equation*}
Y(\log (\theta \mu))=\left\langle\nabla_{X} X, Y\right\rangle \tag{11.16}
\end{equation*}
$$

Choosing an arbitrary smooth function as initial condition along a fixed maximal integral curve of $X$, there exists a unique $\theta \in C^{\infty}(M)$, with

$$
\begin{equation*}
T(\theta)=0 \tag{11.17}
\end{equation*}
$$

for all $T \in \Gamma(\Delta)$, such that $\theta \mu$ is a solution of 11.16).
Proposition 11.2 gives rise to the first class of Sbrana-Cartan hypersurfaces that are not surface-like.

Corollary 11.3. Any simply connected ruled hypersurface $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ that is not surface-like on any open subset of $M^{n}$ and is free of points where all sectional curvatures are equal to $c$ is a (parabolic) Sbrana-Cartan hypersurface. Moreover, all isometric immersions of $M^{n}$ into $\mathbb{Q}_{c}^{n+1}$ are ruled with the same rulings, and their congruence classes are in one-to-one correspondence with the smooth functions on an open interval.

Proof: By Proposition 11.2, the set of symmetric Codazzi tensors $\tilde{A} \in \Gamma(\operatorname{End}(T M))$ such that $\Delta \subset \operatorname{ker} \tilde{A}$ and $\left.\operatorname{det} \tilde{A}\right|_{\Delta^{\perp}}=\left.\operatorname{det} A\right|_{\Delta^{\perp}}$ is in one-to-one correspondence with the set of smooth real functions on an open interval. More precisely, let $X, Y$ be an orthonormal frame of $\Delta^{\perp}$ with $X$ orthogonal to the rulings, and let $J \in \Gamma(\operatorname{End}(T M))$ be given by 11.15 ). Then any Codazzi tensor $\tilde{A}$ on $M^{n}$ with $\Delta \subset \operatorname{ker} \tilde{A}$ such that $\left.\operatorname{det} \tilde{A}\right|_{\Delta^{\perp}}=\left.\operatorname{det} A\right|_{\Delta^{\perp}}$ is given by $\tilde{A}=A D$, where $D=I+\theta J$ and $\theta \in C^{\infty}(M)$ is arbitrarily prescribed along an integral curve of $X$ and required to satisfy (11.16) and (11.17).

For each such Codazzi tensor $\tilde{A}$ on $M^{n}$, the Gauss and Codazzi equations for an isometric immersion of $M^{n}$ into $\mathbb{Q}_{c}^{n+1}$ are trivially satisfied. Since $M^{n}$ is simply connected, by Theorem 1.11 there exists an isometric immersion $\tilde{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ with $\tilde{A}$ as its shape operator. That $\tilde{f}$ is also ruled with the same rulings follows from

$$
\begin{aligned}
\langle\tilde{A} Y, Y\rangle & =\langle A D Y, Y\rangle \\
& =\langle A Y, Y\rangle+\theta\langle A J Y, Y\rangle \\
& =0
\end{aligned}
$$

It remains to show that these are all isometric immersions of $M^{n}$ into $\mathbb{Q}_{c}^{n+1}$. This follows from the fact that, if $\tilde{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ is an isometric immersion, then $\Delta$ is also the relative nullity distribution of $\tilde{f}$ by Corollary 4.15, and hence the shape operator $\tilde{A}$ of $\tilde{f}$ is a Codazzi tensor on $M^{n}$ such that $\operatorname{ker} \tilde{A}=\Delta$. Moreover, the Gauss equations for $f$ and $\tilde{f}$ yield $\left.\operatorname{det} A\right|_{\Delta^{\perp}}=\left.\operatorname{det} \tilde{A}\right|_{\Delta^{\perp}}$.

### 11.1.2 Projectable vector fields and tensors

In order to determine which hyperbolic and elliptic hypersurfaces $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ carry a tensor $D$ satisfying all the conditions in the converse statement of Lemma 11.1 , the strategy is to parametrize $f$ in terms of the Gauss parametrization, and then to reduce the problem to an equivalent one for its Gauss image, as well as its support function if $c=0$. To show that the tensor $D$ on the hypersurface can be "projected down" to a tensor $\bar{D}$ on its Gauss image, we discuss next some general criteria for vector fields and tensors to be projectable under a given submersion.

If $\pi: M \rightarrow L$ is a submersion between differentiable manifolds, then $X \in \mathfrak{X}(M)$ is said to be projectable if it is $\pi$-related to some $\bar{X} \in \mathfrak{X}(L)$, that is, if there exists $\bar{X} \in \mathfrak{X}(L)$ such that $\pi_{*} X=\bar{X} \circ \pi$.

Proposition 11.4. Let $\Delta$ be an integrable distribution on a differentiable manifold $M$ and let $\pi: M \rightarrow L$ be the projection onto the (local) quotient space $L=M / \Delta$ of leaves of $\Delta$. Then $X \in \mathfrak{X}(M)$ is projectable if and only if $[X, T] \in \Gamma(\Delta)$ for any $T \in \Gamma(\Delta)$.

Proof: Suppose that $\pi_{*} X=\bar{X} \circ \pi$ for some $\bar{X} \in \mathfrak{X}(L)$. Given $T \in \Gamma(\Delta)$, we have

$$
\pi_{*}[X, T]=\left[\pi_{*} X, \pi_{*} T\right]=[\bar{X}, 0] \circ \pi=0,
$$

hence $[X, T] \in \Gamma(\Delta)$.

For the converse, in order to prove that $X$ is projectable, one must show that, for each $y \in L$, the map $\psi: F=\pi^{-1}(y) \rightarrow T_{y} L$ defined by

$$
\psi(x)=\pi_{*}(x) X_{x}
$$

is constant. Given $x \in F$ and $v \in T_{x} F$, choose $T \in \Delta$ with $T(x)=v$ and let $g_{t}$ be the local one-parameter group of diffeomorphisms generated by $T$. By the assumption, and since $\pi \circ g_{t}=\pi$, we have

$$
\begin{aligned}
0 & =\pi_{*}[X, T](x) \\
& =\lim _{t \mapsto 0} \frac{1}{t}\left(\pi_{*} X\left(g_{t}(x)\right)-\pi_{*} g_{t *} X(x)\right) \\
& =\lim _{t \mapsto 0} \frac{1}{t}\left(\pi_{*} X\left(g_{t}(x)\right)-\pi_{*} X(x)\right) \\
& =\psi_{*}(x) v,
\end{aligned}
$$

and this concludes the proof.
For a vector field $X \in \Gamma\left(\Delta^{\perp}\right)$, the conclusion of Proposition 11.4 can be expressed in terms of the splitting tensor $C$ of $\Delta$.

Corollary 11.5. Let $\Delta$ be an integrable distribution on a Riemannian manifold $M$ and let $L=M / \Delta$ be the (local) quotient space of leaves of $\Delta$. Then $X \in \Gamma\left(\Delta^{\perp}\right)$ is projectable if and only if

$$
\nabla_{T}^{h} X+C_{T} X=0
$$

for any $T \in \Gamma(\Delta)$.
Proof: Since

$$
[X, T]=\nabla_{X}^{v} T-C_{T} X-\nabla_{T}^{v} X-\nabla_{T}^{h} X
$$

for any $T \in \Gamma(\Delta)$, then $[X, T] \in \Gamma(\Delta)$ if and only if

$$
\nabla_{T}^{h} X+C_{T} X=0
$$

and the statement follows from Proposition 11.4 .
Let $M$ be a Riemannian manifold and let $\pi: M \rightarrow L$ be a submersion. Consider the vertical distribution $\Delta$ of $\pi$, that is, for any $x \in M$ the subspace $\Delta(x)$ is the tangent space to the fiber $\pi^{-1}(\pi(x))$ through $x$. Given $\bar{D} \in \Gamma(\operatorname{End}(T L))$, its horizontal lift $D$ is the element of $\Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ such that, for any $x \in M$ and $v \in \Delta^{\perp}(x)$, the vector $D v$ is the unique one in $\Delta^{\perp}(x)$ that is projected to $\bar{D} \pi_{*} v$ under $\pi_{*}$. In other words, $D$ is the unique element of $\Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ such that $\bar{D} \circ \pi_{*}=\pi_{*} \circ D$.

A tensor $D \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ is said to be projectable with respect to $\pi$ if it is the horizontal lift of some tensor $\bar{D}$ on $L$. Clearly, $D$ is projectable with respect to $\pi$ if and only if for all $\bar{x} \in L, x, y \in \pi^{-1}(\bar{x}), v \in \Delta^{\perp}(x)$ and $w \in \Delta^{\perp}(y)$ with $\pi_{*} v=\pi_{*} w$, one has that $\pi_{*} D v=\pi_{*} D w$.

Proposition 11.6. Let $M$ be a Riemannian manifold and let $\pi: M \rightarrow L$ be a submersion with $\Delta$ as its vertical distribution. Then a tensor $D \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ is projectable with respect to $\pi$ if and only if $D X$ is projectable whenever $X \in \Gamma\left(\Delta^{\perp}\right)$ is projectable.

Proof: Suppose that $D$ is projectable, that is, that there is a tensor $\bar{D}$ in $L$ such that

$$
\bar{D} \circ \pi_{*}=\pi_{*} \circ D .
$$

Given a projectable vector field $X$ in $M$, let $\bar{X}$ be the vector field in $L$ such that $\pi_{*} X=\bar{X} \circ \pi$. Then

$$
\pi_{*} D X=\bar{D} \circ \pi_{*} X=\bar{D} \circ \bar{X} \circ \pi,
$$

thus $D X$ is projectable.
Conversely, suppose there exist $\bar{x} \in L, x, y \in \pi^{-1}(\bar{x}), v \in \Delta^{\perp}(x)$ and $w \in \Delta^{\perp}(y)$ such that

$$
\pi_{*} v=\pi_{*} w \text { and } \pi_{*} D v \neq \pi_{*} D w .
$$

Let $\bar{v}=\pi_{*} v=\pi_{*} w$ and let $\bar{X} \in \mathfrak{X}(L)$ be any vector field such that $\bar{X}(\bar{x})=\bar{v}$. Let $X$ be the horizontal lift of $\bar{X}$. Then $X(x)=v$ and $X(y)=w$. Since

$$
\pi_{*} D X(x)=\pi_{*} D v \neq \pi_{*} D w=\pi_{*} D X(y),
$$

the vector field $D X$ is not projectable.
Corollary 11.7. Let $\Delta$ be an integrable distribution on a Riemannian manifold $M$ and let $L=M / \Delta$ be the (local) quotient space of leaves of $\Delta$. A tensor $D \in \Gamma\left(E n d\left(\Delta^{\perp}\right)\right)$ is projectable if and only if

$$
\begin{equation*}
\nabla_{T}^{h} D=\left[D, C_{T}\right] \tag{11.18}
\end{equation*}
$$

for all $T \in \Gamma(\Delta)$.
Proof: Assume that $D \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ is projectable. By Proposition 11.6, if $X \in$ $\Gamma\left(\Delta^{\perp}\right)$ is projectable, then so is $D X$. Since

$$
\begin{equation*}
\nabla_{T}^{h} D X+C_{T} D X=\left(\nabla_{T}^{h} D-\left[D, C_{T}\right]\right) X+D\left(\nabla_{T}^{h} X+C_{T} X\right) \tag{11.19}
\end{equation*}
$$

for any $X \in \Gamma\left(\Delta^{\perp}\right)$, it follows from Corollary 11.5 that $\nabla_{T}^{h} D-\left[D, C_{T}\right]$ vanishes on projectable vector fields. Thus it vanishes, for this is a tensorial property.

Conversely, if (11.18) holds then 11.19 and Corollary 11.5 imply that $D X$ is projectable whenever $X \in \Gamma\left(\Delta^{\perp}\right)$ is projectable. Thus $D$ is projectable by Proposition 11.6 .

### 11.1.3 Hyperbolic and elliptic surfaces

Let $g: L^{2} \rightarrow \mathbb{S}_{1, \mu}^{n}, n \geq 4$, be a surface in the Euclidean or Lorentzian sphere. Assume that the first normal spaces $N_{1}^{g}$ of $g$ have dimension two everywhere. Then,
given $x \in L^{2}$ and a basis $X, Y$ of $T_{x} L$, there exist $a, b, c \in \mathbb{R}$ with $a^{2}+b^{2}+c^{2} \neq 0$ such that the second fundamental form $\alpha^{g}$ of $g$ satisfies

$$
a \alpha^{g}(X, X)+2 c \alpha^{g}(X, Y)+b \alpha^{g}(Y, Y)=0 .
$$

The surface $g$ is said to be elliptic (respectively, hyperbolic or parabolic) at $x \in L^{2}$ if $a b-c^{2}>0$ (respectively, $<0$ or $=0$ ). In Exercise 11.2, the reader is asked to show that this condition is independent of the given basis, and that it is equivalent to the existence of an endomorphism $J$ on $T_{x} L$ satisfying $J^{2}=\epsilon I$ with $\epsilon=-1$ (respectively, $\epsilon=1$ or $\epsilon=0$ ) and

$$
\begin{equation*}
\alpha^{g}(J X, Y)=\alpha^{g}(X, J Y) \tag{11.20}
\end{equation*}
$$

for all $X, Y \in T_{x} L$. Moreover, in the elliptic and hyperbolic cases the endomorphism $J$ is unique up to sign.

The surface $g$ is said to be elliptic (respectively, hyperbolic or parabolic) if it is elliptic (respectively, hyperbolic or parabolic) at every point of $L^{2}$. In this case, the endomorphisms $J$ on each tangent space give rise to a tensor $J$ on $L^{2}$ such that 11.20) holds for all $X, Y \in \mathfrak{X}(L)$.

When the first normal spaces of $g$ have dimension less than two, the surface $g$ is still called elliptic, hyperbolic or parabolic with respect to such a tensor $J$ if the condition (11.20) is satisfied.

Proposition 11.8. Given a surface $g: L^{2} \rightarrow \mathbb{S}_{1, \mu}^{n}$, set $h=i \circ g: L^{2} \rightarrow \mathbb{R}_{\mu}^{n+1}$, where $i: \mathbb{S}_{1, \mu}^{n} \rightarrow \mathbb{R}_{\mu}^{n+1}$ is the inclusion map. Then 11.20) holds if and only if any height function $h^{v}=\langle h, v\rangle$, for $v \in \mathbb{R}_{\mu}^{n+1}$, satisfies

$$
\left(\text { Hess }^{v}+h^{v} I\right) \circ J=J^{t} \circ\left(\text { Hess } h^{v}+h^{v} I\right)
$$

where Hess $h^{v}$ denotes the endomorphism of TL associated with the Hessian of $h^{v}$ with respect to the induced metric.

Proof: By Corollary 1.3, the Hessian of $h^{v}$ as a symmetric bilinear form satisfies

$$
\begin{aligned}
\text { Hess } h^{v}(X, Y) & =\left\langle\alpha^{h}(X, Y), v\right\rangle \\
& =\left\langle i_{*} \alpha^{g}(X, Y)-\langle X, Y\rangle h, v\right\rangle
\end{aligned}
$$

for all $X, Y \in \mathfrak{X}(L)$. Hence

$$
\left\langle i_{*} \alpha^{g}(X, Y), v\right\rangle=\operatorname{Hess} h^{v}(X, Y)+\langle X, Y\rangle h^{v}
$$

for all $X, Y \in \mathfrak{X}(L)$. Thus the endomorphism of $T L$ associated with Hess $h^{v}$ satisfies

$$
\left\langle i_{*} \alpha^{g}(J X, Y)-i_{*} \alpha^{g}(X, J Y), v\right\rangle=\left\langle\left(\left(\operatorname{Hess} h^{v}+h^{v} I\right) J-J^{t}\left(\operatorname{Hess} h^{v}+h^{v} I\right)\right) X, Y\right\rangle
$$

for all $X, Y \in \mathfrak{X}(L)$, and the conclusion follows.

For a given surface $g: L^{2} \rightarrow \mathbb{S}^{n}$ and $\gamma \in C^{\infty}(L)$, we say that the pair $(g, \gamma)$ is elliptic (respectively, hyperbolic or parabolic) with respect to a tensor $J$ on $L^{2}$ satisfying $J^{2}=\epsilon I$ with $\epsilon=-1$ (respectively, $\epsilon=1$ or $\epsilon=0$ ) if $g$ is elliptic (respectively, hyperbolic or parabolic) with respect to $J$ and $\gamma$ satisfies the same condition as any height function of $h=i \circ g$, namely, if

$$
(\operatorname{Hess} \gamma+\gamma I) \circ J=J^{t} \circ(\operatorname{Hess} \gamma+\gamma I) .
$$

We say that a local system of coordinates $(u, v)$ on $L^{2}$ is real conjugate for a surface $g: L^{2} \rightarrow \mathbb{S}_{1, \mu}^{n}$ if the condition

$$
\alpha^{g}\left(\partial_{u}, \partial_{v}\right)=0
$$

holds for $\partial_{u}=\partial / \partial u$ and $\partial_{v}=\partial / \partial v$. The coordinate system $(u, v)$ is said to be complex conjugate for $g$ if

$$
\alpha^{g}\left(\partial_{z}, \partial_{\bar{z}}\right)=0
$$

where $z=u+i v$ and $\partial_{z}=(1 / 2)\left(\partial_{u}-i \partial_{v}\right)$, that is, if

$$
\alpha^{g}\left(\partial_{u}, \partial_{u}\right)+\alpha^{g}\left(\partial_{v}, \partial_{v}\right)=0 .
$$

In the following result, in the case of real conjugate coordinates (respectively, complex conjugate coordinates) we denote $F=\left\langle\partial_{u}, \partial_{v}\right\rangle$ (respectively, $F=\left\langle\partial_{z}, \partial_{\bar{z}}\right\rangle$, where $\langle$,$\rangle also stands for the \mathbb{C}$-bilinear extension of the metric of $L^{2}$ ) and $\Gamma^{1}, \Gamma^{2}$ (respectively, $\Gamma$ ) are the Christoffel symbols defined by

$$
\begin{equation*}
\nabla_{\partial_{u}} \partial_{v}=\Gamma^{1} \partial_{u}+\Gamma^{2} \partial_{v} \tag{11.21}
\end{equation*}
$$

(respectively,

$$
\begin{equation*}
\nabla_{\partial_{z}} \partial_{\bar{z}}=\Gamma \partial_{z}+\bar{\Gamma} \partial_{\bar{z}} \tag{11.22}
\end{equation*}
$$

where $\nabla$ also denotes the $\mathbb{C}$-bilinear extension of $\nabla$ ).
Proposition 11.9. Given a surface $g: L^{2} \rightarrow \mathbb{S}_{1, \mu}^{n}$, set $h=i \circ g$, where $i: \mathbb{S}_{1, \mu}^{n} \rightarrow \mathbb{R}_{\mu}^{n+1}$ is the inclusion. The following assertions are equivalent:
(i) The coordinates $(u, v)$ are either real conjugate or complex conjugate for $g$.
(ii) The position vector of $h$ satisfies

$$
\begin{equation*}
h_{u v}-\Gamma^{1} h_{u}-\Gamma^{2} h_{v}+F h=0 \tag{11.23}
\end{equation*}
$$

in the case of real conjugate coordinates, or

$$
\begin{equation*}
h_{z \bar{z}}-\Gamma h_{z}-\bar{\Gamma} h_{\bar{z}}+F h=0 \tag{11.24}
\end{equation*}
$$

for complex conjugate coordinates.

Proof: The condition $\alpha^{g}\left(\partial_{u}, \partial_{v}\right)=0$ is equivalent to

$$
\alpha^{h}\left(\partial_{u}, \partial_{v}\right)+F h=0
$$

whereas $\alpha^{g}\left(\partial_{z}, \partial_{\bar{z}}\right)=0$ is equivalent to

$$
\alpha^{h}\left(\partial_{z}, \partial_{\bar{z}}\right)+F h=0 .
$$

The preceding equations can also be written as (11.23) and (11.24), respectively.
Proposition 11.10. If the surface $g: L^{2} \rightarrow \mathbb{S}_{1, \mu}^{n}$ is hyperbolic (respectively, elliptic), then there exist locally real conjugate (respectively, complex conjugate) coordinates on $L^{2}$ for $g$. Conversely, if there exist real conjugate (respectively, complex conjugate) coordinates on $L^{2}$, then $g: L^{2} \rightarrow \mathbb{S}_{1, \mu}^{n}$ is hyperbolic (respectively, elliptic).
Proof: Assume that $g$ is hyperbolic and let $J$ be a tensor on $L^{2}$ such that $J^{2}=I$ and

$$
\alpha^{g}(J X, Y)=\alpha^{g}(X, J Y)
$$

for all $X, Y \in \mathfrak{X}(L)$. Let $X, Y$ be a frame of eigenvectors of $J$ associated with the eigenvalues 1 and -1 , respectively. Then there exist local coordinates $(u, v)$ in $L^{2}$ such that the coordinate vector fields $\partial_{u}$ and $\partial_{v}$ are collinear with $X$ and $Y$, respectively. Hence

$$
\alpha^{g}\left(\partial_{u}, \partial_{v}\right)=\alpha^{g}\left(J \partial_{u}, \partial_{v}\right)=\alpha^{g}\left(\partial_{u}, J \partial_{v}\right)=-\alpha^{g}\left(\partial_{u}, \partial_{v}\right)
$$

and, consequently, $\alpha^{g}\left(\partial_{u}, \partial_{v}\right)=0$.
Conversely, if $(u, v)$ are real conjugate coordinates on $L^{2}$ for $g$, let $J$ be the tensor defined by $J \partial_{u}=\partial_{u}$ and $J \partial_{v}=-\partial_{v}$. Then $J^{2}=I$ and 11.20 holds, since this equation is satisfied for $X, Y \in\left\{\partial_{u}, \partial_{v}\right\}$. Thus $g$ is hyperbolic with respect to $J$. The proof for the elliptic case is similar.

### 11.1.4 From hypersurfaces to surfaces and backwards

In this section we relate the notions of hyperbolic and elliptic hypersurfaces $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}, c \neq 0$ (respectively, $c=0$ ) with those of hyperbolic and elliptic surfaces $g: L^{2} \rightarrow \mathbb{S}_{1, \mu}^{n+1}$ (respectively, pairs $(g, \gamma)$, where $g: L^{2} \rightarrow \mathbb{S}^{n}$ and $\left.\gamma \in C^{\infty}(L)\right)$. The results are stated and proved for the case $c=0$. The statements and proofs for $c \neq 0$ require a few minor modifications that are left to the reader.

Proposition 11.11. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be a hypersurface carrying a relative nullity distribution $\Delta$ of rank $n-2$. Assume that $f$ is not surface-like on any open subset, and let $g: L^{2} \rightarrow \mathbb{S}^{n}$ and $\gamma \in C^{\infty}(L)$ parametrize $f$ in terms of the Gauss parametrization. If $f$ is hyperbolic (respectively, elliptic) with respect to $J \in \Gamma\left(E n d\left(\Delta^{\perp}\right)\right)$, then $J$ is the horizontal lift of a tensor $\bar{J} \in \Gamma(E n d(T L))$ and the pair $(g, \gamma)$ is hyperbolic (respectively, elliptic) with respect to $\bar{J}$.

Conversely, if the pair $(g, \gamma)$ is hyperbolic (respectively, elliptic) with respect to $\bar{J} \in$ $\Gamma(\operatorname{End}(T L))$, then $f$ is hyperbolic (respectively, elliptic) with respect to the horizontal lift $J \in \Gamma\left(E n d\left(\Delta^{\perp}\right)\right)$ of $\bar{J}$.

Proof: Since $\nabla_{T} J=0$ and $\left[J, C_{T}\right]=0$ for all $T \in \Gamma(\Delta)$, it follows from Corollary 11.7 that $J$ is projectable, that is, there exists a tensor $\bar{J}$ on $L^{2}$ such that

$$
\bar{J} \circ \pi_{*}=\pi_{*} \circ J
$$

That $\bar{J}^{2}=\bar{I}$ (respectively, $\bar{J}^{2}=-\bar{I}$ ) follows from the similar property of $J$. It remains to prove that

$$
\begin{equation*}
A_{w} \bar{J}=\bar{J}^{t} A_{w} \tag{11.25}
\end{equation*}
$$

for all $w \in N_{g} L$, and that

$$
\begin{equation*}
(\operatorname{Hess} \gamma+\gamma \bar{I}) \bar{J}=\bar{J}^{t}(\operatorname{Hess} \gamma+\gamma \bar{I}) \tag{11.26}
\end{equation*}
$$

Given $\bar{X}, \bar{Y} \in \mathfrak{X}(L)$, let $X, Y \in \Gamma\left(\Delta^{\perp}\right)$ be their horizontal lifts to $M^{n}$. Set $h=i \circ g$, where $i: \mathbb{S}^{n} \rightarrow \mathbb{R}^{n+1}$ is the inclusion, and let $\eta: M^{n} \rightarrow \mathbb{S}^{n}$ be the Gauss map of $f$, so that $h$ and the Gauss map $\eta: M^{n} \rightarrow \mathbb{S}^{n}$ of $f$ are related by $i \circ \eta=h \circ \pi$. Then

$$
\begin{equation*}
f_{*} A X=-i_{*} \eta_{*} X=-h_{*} \pi_{*} X=-h_{*} \bar{X} \tag{11.27}
\end{equation*}
$$

hence

$$
\begin{equation*}
f_{*} A J X=-h_{*} \pi_{*} J X=-h_{*} \bar{J} \pi_{*} X=-h_{*} \bar{J} \bar{X} \tag{11.28}
\end{equation*}
$$

Let $\hat{\pi}: \Lambda=N_{g} L \rightarrow L^{2}$ be the canonical projection. By Theorem 7.18, there exists a diffeomorphism $\theta: U \subset \Lambda \rightarrow M^{n}$ from an open neighborhood of the zero section of $\Lambda$ such that $\pi \circ \theta=\hat{\pi}$ and

$$
f \circ \theta(y, w)=\gamma(y) h(y)+h_{*} \nabla \gamma(y)+i_{*} w
$$

for all $(y, w) \in \Lambda$. Let $j: T_{y} L \rightarrow T_{(y, w)} \Lambda$ be the map given by Proposition 7.19. Using (7.24), (7.27) and (11.28), we obtain

$$
\begin{align*}
-\left\langle A J \theta_{*} j \bar{X}, \theta_{*} j \bar{Y}\right\rangle & =-\left\langle f_{*} A J \theta_{*} j \bar{X}, f_{*} \theta_{*} j \bar{Y}\right\rangle \\
& =\left\langle h_{*} \bar{J} \pi_{*} \theta_{*} j \bar{X}, h_{*} \bar{Y}\right\rangle \\
& =\left\langle\bar{J} \hat{\pi}_{*} j \bar{X}, \bar{Y}\right\rangle^{\prime} \\
& =\left\langle\bar{J} P_{w}^{-1} \bar{X}, \bar{Y}\right\rangle^{\prime} \tag{11.29}
\end{align*}
$$

for all $\bar{X}, \bar{Y} \in T_{y} L$. Since $A C_{T}$ is symmetric for all $T \in \Gamma(\Delta)$ by Proposition 7.3, and the splitting tensor satisfies $C(\Gamma(\Delta)) \subset \operatorname{span}\{I, J\}$ and $C(\Gamma(\Delta)) \not \subset \operatorname{span}\{I\}$, it follows that $A J$ is symmetric. By 11.29 this implies that $\bar{J} P_{w}^{-1}=P_{w}^{-1} \bar{J}^{t}$, or equivalently, that

$$
\begin{equation*}
P_{w} \bar{J}=\bar{J}^{t} P_{w} . \tag{11.30}
\end{equation*}
$$

In particular, for $w=0$ this gives (11.26), and then (11.26) and (11.30) imply (11.25).
Conversely, suppose that the pair $(g, \gamma)$ is hyperbolic (respectively, elliptic) with respect to $\bar{J} \in \Gamma(\operatorname{End}(T L))$, and let $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ be the horizontal lift of $\bar{J}$. We prove next that $f$ is hyperbolic (respectively, elliptic) with respect to $J$.

Since the pair $(g, \gamma)$ is hyperbolic (respectively, elliptic) with respect to $\bar{J}$, then 11.25 and 11.26 hold, hence so does (11.30). It follows from (11.29) that $A J$ is symmetric.

By Corollary 11.7 we have

$$
\begin{equation*}
\nabla_{T} J=\left[J, C_{T}\right] \tag{11.31}
\end{equation*}
$$

for all $T \in \Gamma(\Delta)$. On the other hand, an easy computation yields

$$
\nabla_{T} A J-A J C_{T}=\left(\nabla_{T} A-A C_{T}\right) J+A\left(\nabla_{T} J-\left[J, C_{T}\right]\right)
$$

Hence (11.31) and Proposition 7.3 give

$$
\nabla_{T} A J=A J C_{T}
$$

In particular, this implies that $A J C_{T}$ is symmetric, and hence

$$
A J C_{T}=C_{T}^{t} J^{t} A=C_{T}^{t} A J=A C_{T} J
$$

Therefore

$$
\begin{equation*}
\left[J, C_{T}\right]=0 \tag{11.32}
\end{equation*}
$$

for all $T \in \Gamma(\Delta)$. In view of (11.31), this implies that

$$
\nabla_{T} J=0
$$

for all $T \in \Gamma(\Delta)$. It also follows from (11.32) that

$$
C(\Gamma(\Delta)) \subset \operatorname{span}\{I, J\} .
$$

Thus $f$ is hyperbolic (respectively, elliptic) with respect to $J$.
The following result is needed for the parametric descriptions of Sbrana-Cartan hypersurfaces and infinitesimally bendable hypersurfaces to be given in Chapter 14 .

Proposition 11.12. For a hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ under the assumptions of Proposition 11.11, let $D \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ satisfy conditions ( $i$ ) and (ii) in Lemma 11.1 and be such that (11.2) holds for $B \in \Gamma(\operatorname{End}(T M))$ given by $\sqrt{11.1)}$. Then $D$ is the horizontal lift of a Codazzi tensor $\bar{D} \in \Gamma(E n d(T L))$ such that $\bar{D} \in \operatorname{span}\{\bar{I}, \bar{J}\}$ and $\bar{D} \notin \operatorname{span}\{\bar{I}\}$.

Conversely, if $\bar{D} \in \Gamma(E n d(T L))$ is a Codazzi tensor such that $\bar{D} \in \operatorname{span}\{\bar{I}, \bar{J}\}$ and $\bar{D} \notin \operatorname{span}\{\bar{I}\}$, then its horizontal lift $D \in \Gamma\left(E n d\left(\Delta^{\perp}\right)\right)$ satisfies conditions (i) and (ii) in Lemma 11.1 and the tensor $B \in \Gamma(\operatorname{End}(T M))$, defined by (11.1), satisfies (11.2).

Proof: Since $C(\Gamma(\Delta)) \subset \operatorname{span}\{I, J\}$ and $D \in \operatorname{span}\{I, J\}$, then (11.7) holds. By Corollary 11.7, this and condition (ii) in Lemma 11.1 imply that $D$ is projectable, that is, there exists $\bar{D} \in \Gamma(\operatorname{End}(T L))$ such that

$$
\bar{D} \circ \pi_{*}=\pi_{*} \circ D
$$

In particular, $\bar{D} \in \operatorname{span}\{\bar{I}, \bar{J}\}$, for $D \in \operatorname{span}\{I, J\}$ by condition $(i)$ in Lemma 11.1. Moreover, $\bar{D} \notin \operatorname{span}\{\bar{I}\}$ by the similar property of $D$. We now prove that $\bar{D}$ is a Codazzi tensor on $L^{2}$.

Given $\bar{X}, \bar{Y} \in \mathfrak{X}(L)$, let $X, Y \in \Gamma\left(\Delta^{\perp}\right)$ be their horizontal lifts to $M^{n}$. As before $h=i \circ g$, where $i: \mathbb{S}^{n} \rightarrow \mathbb{R}^{n+1}$ is the inclusion, so that $i \circ \eta=h \circ \pi$, where $\eta: M^{n} \rightarrow \mathbb{S}^{n}$ is the Gauss map of $f$. Since

$$
f_{*} A X=-h_{*} \bar{X}
$$

and

$$
\begin{equation*}
f_{*} A D X=-h_{*} \bar{D} \bar{X} \tag{11.33}
\end{equation*}
$$

then

$$
\begin{align*}
f_{*} A D[X, Y] & =-h_{*} \bar{D} \pi_{*}[X, Y] \\
& =-h_{*} \bar{D}\left[\pi_{*} X, \pi_{*} Y\right] \\
& =-h_{*} \bar{D}[\bar{X}, \bar{Y}] . \tag{11.34}
\end{align*}
$$

Using (11.27) and (11.33), the Gauss formulas of $f$ and $g$ yield

$$
\begin{align*}
f_{*} \nabla_{X} A D Y & =\tilde{\nabla}_{X} f_{*} A D Y-\langle A X, A D Y\rangle i \circ \eta \\
& =-\tilde{\nabla}_{\bar{X}} h_{*} \bar{D} \bar{Y}-\left\langle h_{*} \bar{X}, h_{*} \bar{D} \bar{Y}\right\rangle h \circ \pi \\
& =-h_{*} \nabla_{\bar{X}}^{\prime} \bar{D} \bar{Y}-\alpha^{h}(\bar{X}, \bar{D} \bar{Y})-\langle\bar{X}, \bar{D} \bar{Y}\rangle^{\prime} h \circ \pi \\
& =-h_{*} \nabla_{\bar{X}}^{\prime} \bar{D} \bar{Y}-\alpha^{g}(\bar{X}, \bar{D} \bar{Y}) . \tag{11.35}
\end{align*}
$$

That $\bar{D}$ is a Codazzi tensor on $L^{2}$ now follows from (11.34), (11.35) and the fact that (11.2) holds for $B \in \Gamma(\operatorname{End}(T M))$ given by (11.1).

Conversely, let $\bar{D} \in \Gamma(\operatorname{End}(T L))$ be a Codazzi tensor such that $\bar{D} \in \operatorname{span}\{\bar{I}, \bar{J}\}$ and $\bar{D} \notin \operatorname{span}\{\bar{I}\}$. Let $D \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ be the horizontal lift of $\bar{D}$. Clearly, $D$ satisfies condition $(i)$ in Lemma 11.1. By Corollary 11.7 we have

$$
\nabla_{T} D=\left[D, C_{T}\right]
$$

for all $T \in \Gamma(\Delta)$. On the other hand, since $D \in \operatorname{span}\{I, J\}$ and $C(\Gamma(\Delta)) \subset \operatorname{span}\{I, J\}$, it follows that

$$
\left[D, C_{T}\right]=0
$$

for all $T \in \Gamma(\Delta)$. Thus condition (ii) in Lemma 11.1 is also satisfied. Finally, if $B \in \Gamma(\operatorname{End}(T M))$ is defined by (11.1), that $B$ satisfies (11.2) follows from (11.34), 11.35) and the fact that $\bar{D}$ being a Codazzi tensor on $L^{2}$.

### 11.2 Surfaces of first and second species

Before we turn to the description of the Sbrana-Cartan hypersurfaces in terms of the Gauss parametrization, we characterize in this section the hyperbolic and elliptic surfaces $g: L^{2} \rightarrow \mathbb{S}_{1, \mu}^{n}$ with respect to $\bar{J} \in \Gamma(\operatorname{End}(T L))$ that carry a Codazzi tensor
$\bar{D} \in \Gamma(\operatorname{End}(T L))$ satisfying $\bar{D} \bar{J}=\bar{J}^{t} \bar{D}$ and such that $\operatorname{det} \bar{D}=1$. In the next section, it is shown that these surfaces are precisely the Gauss maps of the Sbrana-Cartan hypersurfaces.

In the following result, $\Gamma^{1}, \Gamma^{2}$ (respectively, $\Gamma$ ) are the Christoffel symbols defined in (11.21) (respectively, (11.22)).

Proposition 11.13. For a surface $g: L^{2} \rightarrow \mathbb{S}_{1, \mu}^{n}$, the following two assertions are equivalent:
(i) The surface $g$ is hyperbolic (respectively, elliptic) with respect to $\bar{J} \in \Gamma(\operatorname{End}(T L))$ and there exists $\bar{D} \in \operatorname{span}\{\bar{I}, \bar{J}\}$ such that $\bar{D} \neq \pm \bar{I}$ and
(a) $\operatorname{det} \bar{D}=1$,
(b) $\left(\nabla_{\bar{X}}^{\prime} \bar{D}\right) \bar{Y}-\left(\nabla_{\bar{Y}}^{\prime} \bar{D}\right) \bar{X}=0$ for all $\bar{X}, \bar{Y} \in \mathfrak{X}(L)$.
(ii) $L^{2}$ carries local real conjugate (respectively, complex conjugate) coordinates ( $u, v$ ) for $g$, and the system of equations

$$
\left\{\begin{align*}
\tau_{u} & =2 \Gamma^{2} \tau(1-\tau)  \tag{11.36}\\
\tau_{v} & =2 \Gamma^{1}(1-\tau)
\end{align*}\right.
$$

has positive solutions other than the trivial one $\tau=1$ (respectively, the equation

$$
\begin{equation*}
\rho_{\bar{z}}+\Gamma(\rho-\bar{\rho})=0 \tag{11.37}
\end{equation*}
$$

admits a solution $\rho=\rho(z, \bar{z})$ that takes values in the unit circle, other than the
trivial one $\rho=1$ ).
Proof: Suppose first that $g$ is hyperbolic with respect to a tensor $\bar{J}$ on $L^{2}$ satisfying $\bar{J}^{2}=\bar{I}$, and that there exists $\bar{D} \in \operatorname{span}\{\bar{I}, \bar{J}\}$ such that $(a)$ and (b) hold. With respect to local real conjugate coordinates $(u, v)$ in $L^{2}$ given by Proposition 11.10, the equation in part (b) reduces to

$$
\begin{equation*}
\nabla_{\partial_{u}}^{\prime} \bar{D} \partial_{v}=\nabla_{\partial_{v}}^{\prime} \bar{D} \partial_{u} . \tag{11.38}
\end{equation*}
$$

Since $\bar{D} \in \operatorname{span}\{\bar{I}, \bar{J}\}$ and $\operatorname{det} \bar{D}=1$, there exists $\theta \in C^{\infty}(L)$ such that

$$
\begin{equation*}
\bar{D} \partial_{u}=\theta \partial_{u} \text { and } \bar{D} \partial_{v}=\theta^{-1} \partial_{v} . \tag{11.39}
\end{equation*}
$$

Hence

$$
\begin{aligned}
\nabla_{\partial_{u}}^{\prime} \bar{D} \partial_{v}-\nabla_{\partial_{v}}^{\prime} \bar{D} \partial_{u} & =\nabla_{\partial_{u}}^{\prime} \theta^{-1} \partial_{v}-\nabla_{\partial_{v}}^{\prime} \theta \partial_{u} \\
& =\left(\left(\theta^{-1}-\theta\right) \Gamma^{1}-\theta_{v}\right) \partial_{u}+\left(\left(\theta^{-1}-\theta\right) \Gamma^{2}+\left(\theta^{-1}\right)_{u}\right) \partial_{v}
\end{aligned}
$$

and 11.38 is equivalent to the system of partial differential equations

$$
\left\{\begin{array}{l}
\theta_{v}=\Gamma^{1}\left(\theta^{-1}-\theta\right) \\
\left(\theta^{-1}\right)_{u}=-\Gamma^{2}\left(\theta^{-1}-\theta\right)
\end{array}\right.
$$

Multiplying the first equation by $2 \theta$ and the second by $2 \theta^{3}$ yields

$$
\left(\theta^{2}\right)_{v}=2 \Gamma^{1}\left(1-\theta^{2}\right) \text { and }\left(\theta^{2}\right)_{u}=2 \Gamma^{2} \theta^{2}\left(1-\theta^{2}\right)
$$

Hence, setting $\tau=\theta^{2}$, the preceding equations take the form (11.36).
Assume now that $g$ is elliptic with respect to a tensor $\bar{J}$ in $L^{2}$ with $\bar{J}^{2}=-\bar{I}$, and suppose that $(a)$ and $(b)$ hold for $\bar{D} \in \operatorname{span}\{\bar{I}, \bar{J}\}$. Let $(u, v)$ be the local complex conjugate coordinates in $L^{2}$ given by Proposition 11.10. Then the complex coordinate vector fields $\partial_{z}$ and $\partial_{\bar{z}}$ are eigenvectors of the complexified tensor

$$
\bar{J}^{\mathbb{C}}: T L^{2} \otimes \mathbb{C} \rightarrow T L^{2} \otimes \mathbb{C}
$$

Therefore, in terms of the complexified tensor

$$
\bar{D}^{\mathbb{C}}: T L^{2} \otimes \mathbb{C} \rightarrow T L^{2} \otimes \mathbb{C}
$$

of $\bar{D}$, condition (b) is equivalent to

$$
\begin{equation*}
\nabla_{\partial_{z}}^{\prime} \bar{D}^{\mathbb{C}} \partial_{\bar{z}}=\nabla_{\partial_{\bar{z}}}^{\prime} \bar{D}^{\mathbb{C}} \partial_{z} \tag{11.40}
\end{equation*}
$$

where $\nabla^{\prime}$ also stands for the complex-bilinear extension of the connection of $L^{2}$.
Since $\bar{D}^{\mathbb{C}} \in \operatorname{span}\left\{\bar{I}, \bar{J}^{\mathbb{C}}\right\}$ and $\operatorname{det} \bar{D}^{\mathbb{C}}=1$, there is a smooth function $\rho: L^{2} \rightarrow \mathbb{S}^{1} \subset$ $\mathbb{C}$ such that

$$
\begin{equation*}
\bar{D}^{\mathbb{C}} \partial_{z}=\rho \partial_{z} \text { and } \bar{D}^{\mathbb{C}} \partial_{\bar{z}}=\bar{\rho} \partial_{\bar{z}} . \tag{11.41}
\end{equation*}
$$

Therefore

$$
\begin{aligned}
\nabla_{\partial_{z}}^{\prime} \bar{D}^{\mathbb{C}} \partial_{\bar{z}}-\nabla_{\partial_{\bar{z}}}^{\prime} \bar{D}^{\mathbb{C}} \partial_{z} & =\nabla_{\partial_{z}}^{\prime} \bar{\rho} \partial_{\bar{z}}-\nabla_{\partial_{\bar{z}}}^{\prime} \rho \partial_{z} \\
& =\left(\Gamma(\bar{\rho}-\rho)-\rho_{\bar{z}}\right) \partial_{z}+\left(\bar{\Gamma}(\bar{\rho}-\rho)+\bar{\rho}_{z}\right) \partial_{\bar{z}}
\end{aligned}
$$

and hence 11.40 is equivalent to (11.37).
Conversely, assume that $L^{2}$ carries real conjugate coordinates $(u, v)$, and that (11.36) admits a nontrivial solution $\tau=\theta^{2}$. Define a tensor $\bar{D}$ in $L^{2}$ by 11.39, and let $\bar{J}$ be given by

$$
\bar{J} \partial_{u}=\partial_{u} \text { and } \bar{J} \partial_{v}=-\partial_{v} .
$$

Then the surface $g$ is hyperbolic with respect to $\bar{J}$, and $\operatorname{det} \bar{D}=1$. Moreover, the proof of the direct statement shows that $\bar{D}$ satisfies (b).

Similarly, assume that $L^{2}$ carries complex conjugate coordinates $(u, v)$, and that $\rho: L^{2} \rightarrow \mathbb{S}^{1} \subset \mathbb{C}$ is a nontrivial solution of 11.37 ). Then $g$ is elliptic with respect to the tensor $\bar{J}$ on $L^{2}$ such that

$$
\bar{J}^{\mathbb{C}} \partial_{z}=i \partial_{z} \text { and } \bar{J}^{\mathbb{C}} \partial_{\bar{z}}=-i \partial_{\bar{z}}
$$

and the tensor $\bar{D}$ in $L^{2}$ defined by (11.41) satisfies 11.40, that is, the equation in part (b) holds for $\bar{D}$.

### 11.2.1 Surfaces of first and second species of real type

We first look for surfaces $g: L^{2} \rightarrow \mathbb{S}_{1, \mu}^{n}$, endowed with real conjugate coordinates $(u, v)$, for which the system of equations (11.36) has positive solutions other than the trivial one $\tau=1$. The integrability condition for (11.36) is

$$
\begin{equation*}
\left(\Gamma_{v}^{2}-2 \Gamma^{1} \Gamma^{2}\right) \tau-\Gamma_{u}^{1}+2 \Gamma^{1} \Gamma^{2}=0 \tag{11.42}
\end{equation*}
$$

The surface $g$ is said to be of first species of real type if 11.42 is trivially satisfied, that is, if

$$
\begin{equation*}
\Gamma_{u}^{1}=\Gamma_{v}^{2}=2 \Gamma^{1} \Gamma^{2} \tag{11.43}
\end{equation*}
$$

and, in addition, an everywhere positive solution of 11.42) exists (which is always the case locally; see Exercise 11.6). The surface $g$ is called of second species of real type if it is not of first species and the function

$$
\tau=\frac{\Gamma_{u}^{1}-2 \Gamma^{1} \Gamma^{2}}{\Gamma_{v}^{2}-2 \Gamma^{1} \Gamma^{2}}
$$

is positive, not identically one and a (necessarily unique) solution of 11.36).
The reader is asked in Exercise 11.6 to compute the general solution of (11.43). Putting this together with Proposition 11.9 yields the following characterization of surfaces of first species of real type.

Proposition 11.14. A surface $g: L^{2} \rightarrow \mathbb{S}_{1, \mu}^{n}$ is of first species of real type if and only if there exist coordinates $(u, v)$ on $L^{2}$ and smooth functions $U=U(u), V=V(v)$ and $F=F(u, v)$ such that the position vector of $h=i \circ g$, where $i: \mathbb{S}_{1, \mu}^{n} \rightarrow \mathbb{R}_{\mu}^{n+1}$ is the inclusion map, satisfies the differential equation

$$
h_{u v}+\frac{V_{v}}{2(U+V)} h_{u}+\frac{U_{u}}{2(U+V)} h_{v}+F h=0 .
$$

### 11.2.2 Surfaces of first and second species of complex type

Consider now a surface $g: L^{2} \rightarrow \mathbb{S}_{1, \mu}^{n}$, endowed with complex conjugate coordinates, for which the differential equation (11.37) admits a solution $\rho=\rho(z, \bar{z})$ that takes values in the unit circle and is not the trivial one $\rho=1$. Differentiating $\rho \bar{\rho}=1$ and using (11.37) yield

$$
\begin{equation*}
\rho_{z}=-\rho^{2} \bar{\rho}_{z}=\rho^{2} \bar{\Gamma}(\bar{\rho}-\rho) . \tag{11.44}
\end{equation*}
$$

Differentiating (11.37) and (11.44) with respect to $z$ and $\bar{z}$, respectively, and using again both equations, we obtain

$$
\bar{\rho}\left(\Gamma_{z}-2 \Gamma \bar{\Gamma}\right)=\rho\left(\bar{\Gamma}_{\bar{z}}-2 \Gamma \bar{\Gamma}\right)
$$

or equivalently,

$$
\begin{equation*}
\operatorname{Im}\left(\bar{\rho}\left(\Gamma_{z}-2 \Gamma \bar{\Gamma}\right)\right)=0 \tag{11.45}
\end{equation*}
$$

The surface $g$ is called of first species of complex type when (11.45) is trivially satisfied, that is, if

$$
\begin{equation*}
\Gamma_{z}=2 \Gamma \bar{\Gamma}, \tag{11.46}
\end{equation*}
$$

which is the complex analogue of (11.43). It is said to be of second species of complex type if it is not of first species and (11.37) has a (necessarily unique) solution determined by (11.45).

For the general solution of $(11.46)$ the reader is referred to Exercise 11.7. Together with Proposition 11.9, it implies the following characterization of surfaces of first species of complex type.

Proposition 11.15. A surface $g: L^{2} \rightarrow \mathbb{S}_{1, \mu}^{n}$ is of first species of complex type if and only if there exist coordinates $(u, v)$ on $L^{2}$ and smooth functions $F=F(u, v)$ and $\phi=\phi(u, v)$, with

$$
\phi_{u u}+\phi_{v v}=0,
$$

such that the position vector of $h=i \circ g$, where $i: \mathbb{S}_{1, \mu}^{n} \rightarrow \mathbb{R}_{\mu}^{n+1}$ is the inclusion map, satisfies the differential equation

$$
h_{u u}+h_{v v}+\frac{\phi_{u}}{\phi} h_{u}+\frac{\phi_{v}}{\phi} h_{v}+F h=0 .
$$

### 11.3 The parametric description

We are now in a position to state and prove the parametric description of the Sbrana-Cartan hypersurfaces in terms of the Gauss parametrization.

Theorem 11.16. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}, c \neq 0$ (respectively, $c=0$ ), be a Sbrana-Cartan hypersurface that is neither surface-like nor ruled on any open subset of $M^{n}$. Then, on each connected component of an open dense subset of $M^{n}$, $f$ is parametrized in terms of the Gauss parametrization by either a hyperbolic or an elliptic surface $g: L^{2} \rightarrow \mathbb{S}_{1, \mu}^{n+1}$ (respectively, hyperbolic or elliptic pair $(g, \gamma)$, where $g: L^{2} \rightarrow \mathbb{S}^{n}$ is a surface and $\left.\gamma \in C^{\infty}(L)\right)$, with $g$ of first or second species of real or complex type.

Conversely, any simply connected hypersurface parametrized in terms of the Gauss parametrization by such a surface $g$ (respectively, pair $(g, \gamma)$ ) is a Sbrana-Cartan hypersurface that admits either a one-parameter family of isometric deformations (continuous class) or a single one (discrete class), according to whether $g$ is of first or second species, respectively.

Proof: Let $\tilde{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ be an isometric immersion that is not congruent to $f$ on any open subset of $M^{n}$. By Corollary 4.15, the hypersurfaces $f$ and $\tilde{f}$ share a common relative nullity distribution $\Delta$ of $\operatorname{rank} n-2$. Therefore the shape operator $\tilde{A}$ of $\tilde{f}$ is a Codazzi tensor on $M^{n}$ such that $\operatorname{ker} \tilde{A}=\Delta$. The Gauss equations for $f$ and $\tilde{f}$ yield

$$
\begin{equation*}
\left.\operatorname{det} A\right|_{\Delta^{\perp}}=\left.\operatorname{det} \tilde{A}\right|_{\Delta^{\perp}} . \tag{11.47}
\end{equation*}
$$

Since $\tilde{f}$ is not congruent to $f$ on any open subset of $M^{n}$, by Theorem 1.11 one cannot have $\tilde{A}= \pm A$ on any open subset of $M^{n}$. Thus $\tilde{A}$ is not a constant multiple of $A$ on any open subset of $M^{n}$.

It follows from Lemma 11.1 that $f$ is either hyperbolic, parabolic or elliptic with respect to a tensor $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ on each connected component of an open dense subset $\mathcal{U}$ of $M^{n}$, depending on whether the tensor

$$
D=\left.\left(\left.A\right|_{\Delta^{\perp}}\right)^{-1} \tilde{A}\right|_{\Delta^{\perp}} \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)
$$

has two distinct real eigenvalues, one real eigenvalue of multiplicity two or a pair of complex conjugate eigenvalues, respectively. The second case cannot occur by Proposition 11.2 and the assumption that $f$ is not ruled on any open subset of $M^{n}$. Thus $f$ is either hyperbolic or elliptic with respect to $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ on each connected component of $\mathcal{U}$. Moreover, the tensor $D$ satisfies conditions (i) and (ii) in Lemma 11.1, and $\operatorname{det} D=1$ in view of 11.47 ).

If $c \neq 0$ (respectively, $c=0$ ), let $f$ be parametrized, in terms of the Gauss parametrization, by the surface $g: L^{2} \rightarrow \mathbb{S}_{1, \mu}^{n+1}$ (respectively, the pair $(g, \gamma)$, where $g: L^{2} \rightarrow \mathbb{S}^{n}$ is a surface and $\left.\gamma \in C^{\infty}(L)\right)$ on some connected component $U$ of $\mathcal{U}$. By Proposition 11.11, $J$ is the horizontal lift of a tensor $\bar{J} \in \Gamma(\operatorname{End}(T L))$ and the surface $g$ (respectively, the pair $(g, \gamma)$ ) is either hyperbolic or elliptic with respect to $\bar{J}$, depending on whether $f$ is hyperbolic or elliptic on $U$ with respect to $J$.

By Proposition 11.12, also the tensor $D$ is the horizontal lift of a tensor $\bar{D} \in$ $\Gamma(\operatorname{End}(T L))$, which satisfies $\bar{D} \in \operatorname{span}\{\bar{I}, \bar{J}\}$ and is a Codazzi tensor on $L^{2}$. Moreover, $\operatorname{det} \bar{D}=1$, for $\operatorname{det} D=1$. Finally, it follows from Proposition 11.13 that the surface $g$ (respectively, the pair $(g, \gamma)$ ) is of first or second species of real or complex type.

Conversely, let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}, c \neq 0$ (respectively, $c=0$ ), be a simply connected hypersurface parametrized, in terms of the Gauss parametrization, by either a hyperbolic or elliptic surface $g: L^{2} \rightarrow \mathbb{S}_{1, \mu}^{n+1}$ (respectively, hyperbolic or elliptic pair $(g, \gamma)$, where $g: L^{2} \rightarrow \mathbb{S}^{n}$ is a surface and $\left.\gamma \in C^{\infty}(L)\right)$ with respect to $\bar{J} \in \Gamma(\operatorname{End}(T L))$, with $g$ of first or second species of real or complex type.

By Proposition 11.11 the hypersurface $f$ is either hyperbolic or elliptic with respect to the horizontal lift $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ of $\bar{J}$, depending on whether $g$ (respectively, the pair $(g, \gamma))$ is hyperbolic or elliptic with respect to $\bar{J}$. On the other hand, since $g$ is of first or second species of real or complex type, by Proposition 11.13 there exists a Codazzi tensor $\bar{D} \in \operatorname{span}\{\bar{I}, \bar{J}\}$ such that $\bar{D} \neq \pm \bar{I}$ and $\operatorname{det} \bar{D}=1$. It now follows from Proposition 11.12 that the horizontal lift $D \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ of $\bar{D}$ satisfies conditions $(i)$ and (ii) in Lemma 11.1, and that 11.2) holds for the tensor $\tilde{A} \in \Gamma(\operatorname{End}(T M))$ defined by

$$
\begin{equation*}
\left.\tilde{A}\right|_{\Delta^{\perp}}=A D \text { and } \Delta=\operatorname{ker} \tilde{A} . \tag{11.48}
\end{equation*}
$$

Moreover, $\operatorname{det} D=1$. Thus $\tilde{A}$ is a symmetric Codazzi tensor on $M^{n}$ by Lemma 11.1, and $\left.\operatorname{det} A\right|_{\Delta^{\perp}}=\left.\operatorname{det} \tilde{A}\right|_{\Delta^{\perp}}$. It follows that $\tilde{A}$ satisfies the Gauss and Codazzi equations for an isometric immersion of $M^{n}$ into $\mathbb{Q}_{c}^{n+1}$.

By Theorem 1.11 and the assumption that $M^{n}$ is simply connected, there exists an isometric immersion $\tilde{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ with $\tilde{A}$ as its shape operator. Since $\tilde{A} \neq \pm A$
on any open subset of $M^{n}$, for $D \neq \pm I$ on any such subset, the hypersurfaces $f$ and $\tilde{f}$ are not congruent on any open subset of $M^{n}$. Thus $f$ is a Sbrana-Cartan hypersurface.

For the last assertion on how many isometric deformations does the hypersurface $f$ have, notice that the above proof has shown that these are in one-to-one correspondence with the tensors $D \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right.$, with $\operatorname{det} D=1$, that satisfy conditions (i) and (ii) in Lemma 11.1 and are such that 11.2 ) holds for the tensor $\tilde{A} \in \Gamma(\operatorname{End}(T M))$ defined by (11.48). The set of all such tensors is, in turn, in one-to-one correspondence with the set of Codazzi tensors $\bar{D} \in \Gamma(\operatorname{End}(T L))$ such that $\operatorname{det} \bar{D}=1, \bar{D} \in \operatorname{span}\{\bar{I}, \bar{J}\}$ and $\bar{D} \neq \pm \bar{I}$. By Proposition 11.13 there exist as many such tensors as positive solutions of either (11.36) or (11.37), according to whether the Gauss map $g$ is of real or complex type. By (11.49) and (11.50) in Exercise 11.6, there is a one-parameter family of such solutions if $g$ is of first species, and a single one if $g$ is of second species.

### 11.4 Notes

The parametric description of the Sbrana-Cartan hypersurfaces of Euclidean space in terms of the Gauss parametrization was given by Sbrana [312] in 1909 after earlier works by Schur [314] in 1886 and Bianchi [35] in 1905, who only considered the three-dimensional case. An alternative description of these hypersurfaces as envelopes of certain two-parameter congruences was obtained by Cartan [64] in 1916, who gave a more precise statement of the classification. Cartan's description will be presented in the last section of Chapter 14. The classification of Sbrana-Cartan hypersurfaces was extended to the case of nonflat ambient space forms by Dajczer-Florit-Tojeiro [103]. We point out that the claims made in [163] and reproduced in [317] are incorrect.

The problem of determining whether Sbrana-Cartan hypersurfaces that allow a single deformation do exist was addressed neither by Sbrana nor by Cartan. An affirmative answer was obtained in [103]. More precisely, it was shown that the transversal intersection in $\mathbb{Q}_{c}^{n+2}$ of two hypersurfaces $N_{j}^{n+1}(c), 1 \leq j \leq 2$, with constant sectional curvature $c$, generically gives rise, by isometrically deforming $N_{j}^{n+1}(c)$ into $\mathbb{Q}_{c}^{n+1}$, $1 \leq j \leq 2$, to a hypersurface in $\mathbb{Q}_{c}^{n+1}$ of this type, together with its (unique) isometric deformation. Moreover, a parametric description was provided, in terms of the Gauss parametrization, of all Sbrana-Cartan hypersurfaces that can be obtained in this way, referred to in the sequel as Sbrana-Cartan hypersurfaces of intersection-type. These are the only known examples so far of Sbrana-Cartan hypersurfaces that admit a unique isometric deformation.

The study of Sbrana-Cartan hypersurfaces of intersection-type in Euclidean space was pursued further by Florit-Freitas [184], who obtained a nice characterization of them among the class of hyperbolic $n$-dimensional submanifolds of $\mathbb{R}^{n+2}$ that carry a relative nullity distribution $\Delta$ of rank $n-2$. The latter can be characterized as those $n$-dimensional submanifolds of $\mathbb{R}^{n+2}$ carrying a relative nullity distribution of rank $n-2$ that admit at any point a pair of linearly independent normal vectors whose shape operators have rank one. It was proved in [184] that for any hyperbolic submanifold $f: M^{n} \rightarrow \mathbb{R}^{n+2}$ there always exists a surface $g: L^{2} \rightarrow \mathbb{R}^{n+2}$ such that $g_{*} T_{\pi(x)} L=$
$N_{f} M(x)$ for all $x \in M^{n}$, where $L^{2}$ is the space of leaves of $\Delta$ and $\pi: M^{n} \rightarrow L^{2}$ is the quotient map. The surface $g$ is called the polar surface of $f$, and it was shown that $f$ can be recovered from $g$ by means of a Gauss-like parametrization. The transversal intersections in $\mathbb{R}^{n+2}$ of two flat hypersurfaces $N_{j}^{n+1}, 1 \leq j \leq 2$, were then characterized as those nowhere flat hyperbolic $n$-dimensional submanifolds whose polar surfaces are given by

$$
g(u, v)=\alpha_{1}(u)+\alpha_{2}(v)
$$

for some curves $\alpha_{j}: I_{j} \subset \mathbb{R} \rightarrow \mathbb{R}^{n+2}, 1 \leq j \leq 2$. This enabled the authors to derive a simple criterion, in terms of an invariant related to a pair of curves called their shared dimension, to decide whether a given Sbrana-Cartan hypersurface of intersection-type belongs to the discrete or to the continuous class.

Already in Dajczer-Florit-Tojeiro [103], the observation that Sbrana-Cartan hypersurfaces of intersection-type can belong either to the discrete or to the continuous class, or even be surface-like, was used to construct explicit examples where SbranaCartan hypersurfaces of different types are smoothly attached, thus showing the local nature of their classification.

The Gauss image of any Sbrana-Cartan hypersurface of intersection-type is therefore, generically, a surface of second species of real type. The existence of deformable hypersurfaces whose Gauss images are surfaces of second species of complex type was proved later by Dajczer-Florit [100], where a procedure to obtain explicit parametrized examples was also given.

A transformation that assigns to any $n$-dimensional deformable hypersurface belonging to the continuous real case, a family of new such hypersurfaces, was constructed by Bianchi for $n=3$, and then extended by Sbrana [312] to any dimension. The transformation actually acts on surfaces of first species of real type. The family of transformed surfaces depends on $n-1$ parameters and is obtained through an integration involving the solutions of a completely integrable first order linear system of differential equations. A permutability theorem for the transformation was also given in [312].

The characterization of the Sbrana-Cartan hypersurfaces whose isometric deformations have isometric Gauss maps is due to Dajczer-Gromoll [111; see Exercise 11.9.

A parametric description of the hypersurfaces $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ for which $M^{n}$ also admits an isometric immersion into the Lorentz space $\mathbb{L}^{n+1}$ was obtained by DajczerFlorit [94]. They are given, as the Sbrana-Cartan hypersurfaces, by means of the Gauss parametrization in terms of surfaces of first and second species of real type, but now making use of a negative solution of system 11.36).

### 11.5 Exercises

Exercise 11.1. Show that the last assertion in Corollary 11.3 is not true for surfacelike ruled hypersurfaces. More precisely, argue that a cylinder over a ruled surface in $\mathbb{R}^{3}$ may admit nonruled isometric deformations.

Exercise 11.2. Let $V$ and $W$ be vector spaces of dimensions 2 and $p \geq 2$, respectively, and let $\alpha: V \times V \rightarrow W$ be a symmetric bilinear form. Assume that there exist a basis
$X, Y$ of $V$ and $a, b, c \in \mathbb{R}$ such that $a^{2}+b^{2}+c^{2} \neq 0$ and

$$
a \alpha(X, X)+2 c \alpha(X, Y)+b \alpha(Y, Y)=0 .
$$

Show that $a b-c^{2}$ being positive, negative or zero is independent of the basis $X, Y$, and that it is equivalent to the existence of an endomorphism $J$ of $V$ such that $J^{2}=\epsilon I$ with $\epsilon=-1$ (respectively, $\epsilon=1$ or $\epsilon=0$, with $J \neq 0$ if $\epsilon=0$ ) and

$$
\alpha(J T, S)=\alpha(T, J S)
$$

for all $T, S \in V$.
Hint: Given any other basis $\tilde{X}, \tilde{Y}$ of $V$, write

$$
X=a_{11} \tilde{X}+a_{12} \tilde{Y} \text { and } Y=a_{21} \tilde{X}+a_{22} \tilde{Y}
$$

and denote

$$
P=\left(\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right) .
$$

Show that

$$
\tilde{a} \alpha(\tilde{X}, \tilde{X})+2 \tilde{c} \alpha(\tilde{X}, \tilde{Y})+\tilde{b} \alpha(\tilde{Y}, \tilde{Y})=0,
$$

where the matrices

$$
A=\left(\begin{array}{cc}
a & c \\
c & b
\end{array}\right) \text { and } \tilde{A}=\left(\begin{array}{cc}
\tilde{a} & \tilde{c} \\
\tilde{c} & \tilde{b}
\end{array}\right)
$$

are related by $\tilde{A}=P^{t} A P$. Conclude that $a b-c^{2}$ being positive, negative or zero is independent of the basis $X, Y$, and that there exists a basis $X, Y$ of $V$ for which the matrix $A$ is either

$$
\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right),\left(\begin{array}{rr}
1 & 0 \\
0 & -1
\end{array}\right) \text { or }\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right) .
$$

In other words, there exists a basis $X, Y$ of $V$ such that one of the following conditions hold:
(i) $\alpha(X, X)=-\alpha(Y, Y)$,
(ii) $\alpha(X, X)=\alpha(Y, Y)$,
(iii) $\alpha(X, X)=0$.

Define an endomorphism $J$ of $V$ by asking that $J X=Y$ and $J Y=-X$ in case $(i)$, $J X=Y$ and $J Y=X$ in case (ii), and $J X=0$ and $J Y=X$ in case (iii).

Exercise 11.3. Show that minimal surfaces endowed with isothermal coordinates are surfaces of first species of complex type.

Exercise 11.4. Given curves $c_{j}: I_{j} \subset \mathbb{R} \rightarrow \mathbb{R}^{n_{j}}, c_{j}=c_{j}\left(t_{j}\right), 1 \leq j \leq 2$, with Frenet frames $c_{j}^{\prime}=e_{1}^{j}, e_{2}^{j}, \ldots, e_{n_{j}}^{j}$ and first curvature functions $k^{j}$, let $\theta_{j}$ be given by $\theta_{j}^{\prime}=k^{j}$, $1 \leq j \leq 2$. Show that the map $g: I_{1} \times I_{2} \rightarrow \mathbb{R}^{n_{1}+n_{2}}$ defined by

$$
g=\frac{1}{\sqrt{\cos ^{2} \theta_{1}+\cos ^{2} \theta_{2}}}\left(\cos \theta_{2} e_{2}^{1}, \cos \theta_{1} e_{2}^{2}\right)
$$

parametrizes a surface of first species of real type.
Exercise 11.5. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ be a minimal simply connected isometric immersion with constant index of relative nullity $\nu=n-2$. For a smooth function $\theta: M^{n} \rightarrow[0, \pi)$, consider the tensor field $R(\theta)$ which acts pointwise as the identity on the relative nullity distribution $\Delta$ and as a rotation through an angle $\theta$ on $\Delta^{\perp}$. The traceless symmetric tensor $A(\theta)=R(\theta) \circ A$ clearly satisfies the Gauss equation. Show the following facts:
(i) The tensor $A(\theta)$ satisfies the Codazzi equation if and only if $\theta$ is constant.
(ii) The associated family $f_{\theta}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}, \theta \in[0, \pi)$, of minimal immersions given by Theorem 1.11 is a one-parameter family of non-congruent maps.
(iii) Any other minimal isometric immersion $\tilde{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ is congruent to some element in the associated family.

Exercise 11.6. (i) If $g: L^{2} \rightarrow \mathbb{S}_{1, \mu}^{n+1}$ is a surface of first species of real type, show that

$$
\omega=\Gamma^{2} d u+\Gamma^{1} d v
$$

is a closed one-form. Conclude that there exists locally $\varphi \in C^{\infty}(L)$ such that

$$
d \varphi+2 \varphi \omega=0
$$

Prove that

$$
\varphi=U+V
$$

for some smooth functions $U=U(u)$ and $V=V(v)$, and conclude that

$$
\Gamma^{1}=\frac{-V_{v}}{2(U+V)}, \quad \Gamma^{2}=\frac{-U_{u}}{2(U+V)}
$$

(ii) Show that the general solution of 11.36 is

$$
\begin{equation*}
\tau(u, v)=\frac{c-V(v)}{c+U(u)} \tag{11.49}
\end{equation*}
$$

where $c \in \mathbb{R}$. Conclude that, locally, there always exist positive solutions of 11.36) other than $\tau=1$.

Exercise 11.7. Show that the general solutions of (11.46) and 11.37) are, respectively,

$$
\Gamma=-\frac{1}{2 \phi}\left(\phi_{u}+i \phi_{v}\right)
$$

and

$$
\begin{equation*}
\rho=e^{i \theta} \text { with } \cot \theta=\frac{1}{\phi}(\lambda+\mu), \tag{11.50}
\end{equation*}
$$

where $\phi=\phi(u, v)$ satisfies

$$
\begin{equation*}
\phi_{u u}+\phi_{v v}=0, \tag{11.51}
\end{equation*}
$$

$\mu$ is any particular solution of $\mu_{v}=\phi_{u}$ and $\lambda=\lambda(u)$ is determined by

$$
\lambda_{u}+\mu_{u}+\phi_{v}=0
$$

up to a real constant.
Exercise 11.8. Prove that a Sbrana-Cartan hypersurface of continuous or discrete class with sectional curvature $K_{M} \geq 0$ is always of real type.

Exercise 11.9. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be a Sbrana-Cartan hypersurface. Assume that there is an isometric deformation $g: M^{n} \rightarrow \mathbb{R}^{n+1}$ such that $f$ and $g$ have isometric Gauss maps. Show that $f$ is minimal and that $g$ belongs to its associated family.
Hint: Having isometric Gauss maps means that the shape operators of $f$ and $g$ satisfy

$$
\left(A^{f}\right)^{2}=\left(A^{g}\right)^{2}=\left(A^{f} D\right)^{2}
$$

This easily implies the statement for surface-like hypersurfaces, and excludes ruled and hypersurfaces of real type and continuous or discrete class. If $f$ is of complex type in the continuous or discrete class, and $Z$ is the complex eigenfield of $D$ with $D Z=\rho Z$ for some smooth function $\rho \neq 1$ taking values in $\mathbb{S}^{1} \subset \mathbb{C}$, then, the complex coordinate vector $\partial_{z}=(1 / 2)\left(\partial_{u}-i \partial_{v}\right)$ induced on the Gauss image by $Z$ satisfies

$$
\left\langle\partial_{z}, \partial_{z}\right\rangle=\left\langle A^{f} Z, A^{f} Z\right\rangle=\left\langle A^{f} D Z, A^{f} D Z\right\rangle=\rho^{2}\left\langle A^{f} Z, A^{f} Z\right\rangle=\rho^{2}\left\langle\partial_{z}, \partial_{z}\right\rangle,
$$

hence $\left\langle\partial_{z}, \partial_{z}\right\rangle=0$. Verify that this means that the coordinates $(u, v)$ are isothermal. Use the condition that $(u, v)$ are complex conjugate to conclude that the Gauss image of $f$ is minimal. Moreover, using that the solutions of (11.51) associated to minimal surfaces are the constant ones, verify that the corresponding solutions of (11.37) given by Exercise 11.6 are $\rho=e^{i \theta}, \theta \in \mathbb{R}$. Conclude that the one-parameter family of isometric deformations of a minimal hypersurface coincides with its associated family.

## Chapter 12

## Genuine deformations

In order to find necessary conditions for a submanifold in a space form with codimension greater than one to admit isometric deformations, one has to take into account that any submanifold of a deformable submanifold already possesses the isometric deformations induced by the latter. Therefore, when studying the isometric deformations of a submanifold, one should look for the "genuine" ones, that is, those which are not induced by isometric deformations of an "extended" submanifold of higher dimension. Besides, it is also of interest to consider isometric deformations of a submanifold that take place in a possibly different codimension.

To make precise the preceding discussion we need to introduce some new concepts.

A pair of isometric immersions $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+q}$ is said to extend isometrically when there exist an isometric embedding $j: M^{n} \hookrightarrow N^{n+\ell}$ into a Riemannian manifold $N^{n+\ell}, 0<\ell \leq \min \{p, q\}$, and a pair of isometric immersions $F: N^{n+\ell} \rightarrow \mathbb{Q}_{c}^{n+p}$ and $\hat{F}: N^{n+\ell} \rightarrow \mathbb{Q}_{c}^{n+q}$ such that $f=F \circ j$ and $\hat{f}=\hat{F} \circ j$, that is, such that the following diagram commutes:


An isometric immersion $\hat{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+q}$ is called a genuine deformation of a given isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ if there exists no open subset $U \subset M^{n}$ along which the restrictions $\left.f\right|_{U}$ and $\left.\hat{f}\right|_{U}$ extend isometrically. Since in this case $f$ is also a genuine deformation of $\hat{f}$, we refer to $\{f, \hat{f}\}$ simply as a genuine pair.

Notice that a pair $\{f, \hat{f}\}$ being genuine for $p=1=q$ just means that there exists no open subset $U \subset M^{n}$ such that the restrictions $\left.f\right|_{U}$ and $\left.\hat{f}\right|_{U}$ are congruent. Therefore, the problem of determining the hypersurfaces of dimension $n \geq 3$ of $\mathbb{Q}_{c}^{n+1}$ that admit genuine deformations in $\mathbb{Q}_{c}^{n+1}$ is precisely the one studied in Chapter 11 .

The concept of a genuine deformation leads naturally to a weaker notion of rigidity than that considered in Chapter 4. Namely, an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ is said to be genuinely rigid in $\mathbb{Q}_{c}^{n+q}$, for a given $q$, if for any isometric immersion $\hat{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+q}$ there exists an open dense subset of $M^{n}$ such that $f$ and $\hat{f}$ extend isometrically along each connected component of $M^{n}$.

For instance, by the Beez-Killing theorem, the isometric inclusion $i: U \rightarrow \mathbb{R}^{n+1}$ of an open subset $U \subset \mathbb{S}^{n}, n \geq 3$, is rigid in the usual sense. On the other hand, one can produce many isometric immersions $f: U \rightarrow \mathbb{R}^{n+p}, p \geq 2$, just by composing $i$ with an isometric immersion $h: V \rightarrow \mathbb{R}^{n+p}$ of an open subset $V$ containing $i(U)$. It is a natural problem whether an isometric immersion $f: U \rightarrow \mathbb{R}^{n+p}$, for a given $p \geq 2$, must necessarily be given in this way, at least when restricted to the connected components of an open dense subset of $U$. In terms of the concept of genuine rigidity, this amounts to asking whether $i$ is genuinely rigid in $\mathbb{R}^{n+p}$.

One of the main results of this chapter shows that, in low codimension, there are strong restrictions on the metric and the geometry of a submanifold that admits genuine deformations. Several applications of that result are discussed. In particular, simple sufficient conditions for genuine rigidity of an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ are derived. For instance, it is shown that $f$ must be genuinely rigid in $\mathbb{Q}_{c}^{n+q}$ if the Ricci curvature of $M^{n}$ is everywhere greater than $c$ and the codimensions $p$ and $q$ satisfy some restrictions. As a very particular case, this implies the genuine rigidity of the inclusion $i: U \subset \mathbb{S}^{n} \rightarrow \mathbb{R}^{n+1}$ in $\mathbb{R}^{n+p}$ for $p \leq n-2$. More generally, sufficient conditions are given on an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ which assure that any isometric immersion $\hat{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+q}$ with $q \geq p$ must be, locally on an open dense subset of $M^{n}$, a composition of $f$ with an isometric immersion $h: U \rightarrow \mathbb{Q}_{c}^{n+q}$ of an open subset $U \subset \mathbb{Q}_{c}^{n+p}$ containing $f(M)$. As another application, isometric immersions $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ for which $M^{n}$ also admits an isometric immersion $\hat{f}: M^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+q}$ with $\tilde{c} \neq c$ are studied. A notion of a genuine pair $\{f, \hat{f}\}$ is introduced for this case, and it is shown that, under some assumptions on $p$ and $q$, such a pair can always be produced by means of a genuine pair of isometric immersions into space forms with the same constant sectional curvature.

When studying the possible isometric deformations of a compact Euclidean submanifold with codimension greater than one, one is naturally led to consider isometric extensions as in (1) of a pair of isometric immersions $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ that may have singular points, but only on $j(M)$. This leads to a weaker notion of genuine isometric deformations, called genuine isometric deformations in the singular sense, and hence to a stronger version of genuine rigidity. We devote a section to discussing these concepts, which will play a key role in the study of isometric deformations of compact Euclidean submanifolds with codimension greater than one in Chapter 13. In fact, the necessity of considering isometric extensions that may have singular points arises already in the study of local isometric deformations. Examples of Euclidean submanifolds with codimension two that are genuinely rigid but not genuinely rigid in the singular sense are provided.

The final section of this chapter is devoted to the description of submanifolds
that have a nonparallel first normal bundle of low rank. As seen in Chapter 2, the codimension of an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ can be reduced to $q<p$ whenever its first normal spaces form a parallel subbundle $N_{1}$ of rank $q$ of its normal bundle. It is then a natural problem to study what happens when $N_{1}$ is a proper subbundle of the normal bundle that is not parallel. Although apparently unrelated to the main subject of this chapter, the proofs of the results of that section use similar techniques, namely, the study of conditions under which a given submanifold can be isometrically extended to a ruled submanifold.

Finally, we point out that, for reasons of simplicity, several statements and most proofs in this chapter are given for Euclidean submanifolds, but can easily be adapted to submanifolds of the sphere and the hyperbolic space.

### 12.1 Ruled extensions

As a first step towards finding necessary conditions for a submanifold in a space form to admit genuine isometric deformations, in this section we derive restrictions imposed on a submanifold that admits an isometric extension to a ruled submanifold with certain properties.

An isometric immersion $F: N \rightarrow \mathbb{Q}_{c}^{m}$ is said to be $R^{r}$-ruled, or simply r-ruled, if there exists an $r$-dimensional smooth integrable distribution $R^{r} \subset T N$ whose leaves are mapped diffeomorphically by $F$ onto open subsets of totally geodesic submanifolds of the ambient space. The leaves of $R^{r}$, as well as their images by $F$, are called the rulings of $F$.

If $F: N \rightarrow \mathbb{R}^{m}$ is an $R^{r}$-ruled isometric immersion, then at each point $x \in N$ the normal space $N_{F} N(x)$ splits orthogonally as

$$
N_{F} N(x)=L_{R}(x) \oplus L_{R}^{\perp}(x),
$$

where

$$
L_{R}(x)=\operatorname{span}\left\{\alpha^{F}(Z, X): Z \in R^{r}(x) \text { and } X \in T_{x} N\right\} .
$$

Assume that $\ell_{R}=\operatorname{dim} L_{R}$ is constant, and hence that $L_{R}$ is a smooth normal subbundle. Clearly, it may happen that $L_{R}^{\perp}$ be trivial, that is, $L_{R}$ may coincide with the whole normal bundle $N_{F} N$. If otherwise, then the following important observation holds.

Proposition 12.1. The normal subbundle $L_{R}^{\perp}$ is parallel along $R^{r}$ in $\mathbb{R}^{m}$.
Proof: Since $R \subset \mathcal{N}\left(\alpha_{L_{R}^{\prime}}^{F}\right)$, it follows from the Codazzi equation

$$
\nabla_{X} A_{\eta} T-A_{\eta} \nabla_{X} T-A_{\nabla_{\frac{1}{x} \eta} T} T=\nabla_{T} A_{\eta} X-A_{\eta} \nabla_{T} X-A_{\nabla_{\frac{1}{T} \eta}} X
$$

that

$$
\left\langle A_{\nabla_{T} \eta} X, S\right\rangle=0
$$

for all $\eta \in \Gamma\left(L_{R}^{\perp}\right), S, T \in \Gamma(R)$ and $X \in \mathfrak{X}(N)$. This shows that $L_{R}^{\perp}$ is parallel along $R^{r}$ in the normal connection. Therefore

$$
\begin{aligned}
\tilde{\nabla}_{T} \xi & =-f_{*} A_{\xi} T+\nabla_{T}^{\perp} \xi \\
& =\nabla_{T}^{\perp} \xi \in \Gamma\left(L_{R}^{\perp}\right)
\end{aligned}
$$

for all $\xi \in \Gamma\left(L_{R}^{\perp}\right)$, where $\tilde{\nabla}$ stands for the connection of $\mathbb{R}^{m}$.
Let $j: M^{n} \rightarrow N$ be an isometric immersion and set $f=F \circ j: M^{n} \rightarrow \mathbb{R}^{m}$. The normal bundle of $f$ splits orthogonally as

$$
N_{f} M=F_{*} N_{j} M \oplus j^{*} N_{F} N
$$

and we have

$$
\begin{equation*}
\alpha^{f}(X, Y)=F_{*} \alpha^{j}(X, Y)+\alpha^{F}\left(j_{*} X, j_{*} Y\right) \tag{12.1}
\end{equation*}
$$

for all and $X, Y \in \mathfrak{X}(M)$ (see Exercise 1.6).
Assume that the subspaces

$$
R(j(x)) \cap j_{*} T_{x} M
$$

have constant dimension $d$ and let $D \subset T M$ be the distribution defined by

$$
j_{*} D(x)=R(j(x)) \cap j_{*} T_{x} M .
$$

Denote by $P$ the subbundle $L_{R}^{\perp}$ regarded as a subbundle of $N_{f} M$, that is, $P=j^{*} L_{R}^{\perp}$. It follows from Proposition 12.1 that $P$ is parallel along $D$ in $\mathbb{R}^{m}$. Moreover, from (12.1) we obtain

$$
\left\langle\alpha^{f}(T, X), \zeta\right\rangle=\left\langle\alpha^{F}\left(j_{*} T, j_{*} X\right), \zeta\right\rangle=0
$$

for all $T \in \Gamma(D), X \in \mathfrak{X}(M)$ and $\zeta \in \Gamma(P)$. Therefore $D \subset \mathcal{N}\left(\alpha_{P}^{f}\right)$.
Notice that if $R=\mathcal{N}\left(\alpha_{L_{\bar{R}}^{\prime}}^{F}\right)$, that is, if

$$
R^{\perp}(y)=\operatorname{span}\left\{A_{\xi} X: X \in T_{y} N \text { and } \xi \in L_{R}^{\perp}(y)\right\}
$$

for all $y \in N$, then also $D=\mathcal{N}\left(\alpha_{P}^{f}\right)$ if

$$
R^{\perp}(j(x))=\operatorname{span}\left\{A_{\xi} j_{*} X: X \in T_{x} M \text { and } \xi \in L_{R}^{\perp}(j(x))\right\}
$$

for all $x \in M^{n}$, in particular if $j$ is transversal to the rulings.
In the sequel we prove that, conversely, any isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ for which there exist subbundles $D$ of $T M$ and $P$ of $N_{f} M$ such that $D=\mathcal{N}\left(\alpha_{P}^{f}\right)$ and $P$ is parallel along $D$ in the normal connection (hence in $\mathbb{R}^{m}$ ) admits a (possibly trivial) isometric extension to a $R$-ruled isometric immersion $F: N \rightarrow \mathbb{R}^{m}$ with a nontrivial subbundle $L \frac{\perp}{R}$. As a first step, under these assumptions we prove the following.

Lemma 12.2. The distribution $D$ is integrable.

Proof: First notice that

$$
0=\widetilde{R}(Y, Z) \mu=\tilde{\nabla}_{Y} \tilde{\nabla}_{Z} \mu-\tilde{\nabla}_{Z} \tilde{\nabla}_{Y} \mu-\tilde{\nabla}_{[Y, Z]} \mu
$$

for all $Z, Y \in \Gamma(D)$ and $\mu \in \Gamma(P)$, where $\tilde{R}$ is the curvature tensor of $\mathbb{R}^{m}$. Since $P$ is parallel along $D$ in $\mathbb{R}^{m}$, it follows that

$$
\widetilde{\nabla}_{[Y, Z]} \mu \in \Gamma(P)
$$

and, in particular, that

$$
A_{\mu}[Y, Z]=0
$$

Thus $[Y, Z] \in \Gamma(D)$, and hence $D$ is integrable.
Now consider the orthogonal splittings

$$
T M=D \oplus E, \quad N_{f} M=L \oplus P
$$

and define $\gamma: \Gamma(E) \times \Gamma(P) \rightarrow \Gamma(E \oplus L)$ by

$$
\begin{align*}
\gamma(Y, \mu) & =\left(\tilde{\nabla}_{Y} \mu\right)_{E \oplus L} \\
& =-f_{*} A_{\mu} Y+\left(\nabla_{Y}^{\perp} \mu\right)_{L} . \tag{12.2}
\end{align*}
$$

It is easily seen that $\gamma$ is $C^{\infty}(M)$-linear in both variables, hence we can regard $\gamma$ as a section of $\operatorname{Hom}^{2}(E, P ; E \oplus L)$. Observe that

$$
E(x)=\operatorname{span}\left\{A_{\mu} Y: \mu \in P(x) \text { and } Y \in E(x)\right\}
$$

for all $x \in M^{n}$. Hence, the subspace $\Gamma(x) \subset f_{*} E(x) \oplus L(x)$ given by

$$
\begin{equation*}
\Gamma(x)=\mathcal{S}(\gamma)(x) \tag{12.3}
\end{equation*}
$$

satisfies

$$
\begin{equation*}
n-d=\operatorname{dim} E(x) \leq \operatorname{dim} \Gamma(x) \leq n-d+\ell \tag{12.4}
\end{equation*}
$$

where $\ell$ is the rank of $L$.
In the sequel, we assume that

$$
\begin{equation*}
\operatorname{dim} \Gamma(x)=k \tag{12.5}
\end{equation*}
$$

is constant on $M^{n}$. Then the affine vector bundle $\pi: \Lambda \rightarrow M^{n}$ of rank $r=n-d+\ell-k$ defined by the orthogonal splitting

$$
\begin{equation*}
\Gamma^{k} \oplus \Lambda^{r}=f_{*} E^{n-d} \oplus L^{\ell} \tag{12.6}
\end{equation*}
$$

satisfies

$$
\Lambda(x) \cap f_{*} T_{x} M=\{0\}
$$

for all $x \in M^{n}$. In fact, if $Z \in T_{x} M$ is such that $f_{*} Z \in \Lambda(x)$, then $Z \in E(x)$ and

$$
0=\left\langle f_{*} Z, \tilde{\nabla}_{X} \mu\right\rangle=-\left\langle A_{\mu} Z, X\right\rangle
$$

for all $\mu \in \Gamma(P)$ and $X \in T_{x} M$. Thus $Z \in D(x)$, and hence $Z=0$.
By the above, the affine subspaces

$$
R(x)=D(x) \oplus \Lambda(x)
$$

form an affine vector bundle over $M^{n}$ of rank $d+r=n+\ell-k$.
Lemma 12.3. The affine vector bundle $R$ is parallel in $\mathbb{R}^{m}$ along the leaves of $D$.
Proof: It suffices to show that the orthogonal complement $\Gamma \oplus P$ of $R$ in $\mathbb{R}^{m}$ is parallel in $\mathbb{R}^{m}$ along the leaves of $D$. First observe that

$$
\Gamma \oplus P=\operatorname{span}\left\{\tilde{\nabla}_{X} \mu: X \in T M \text { and } \mu \in \Gamma(P)\right\}
$$

Thus, from $\widetilde{R}(Y, Z) \mu=0$ we obtain

$$
\tilde{\nabla}_{Y} \tilde{\nabla}_{X} \mu=\tilde{\nabla}_{X} \tilde{\nabla}_{Y} \mu+\tilde{\nabla}_{[Y, X]} \mu \in \Gamma \oplus P
$$

for all $\mu \in \Gamma(P), Y \in \Gamma(D)$ and $X \in \mathfrak{X}(M)$.
Define $F: N^{n+r} \rightarrow \mathbb{R}^{m}$ as the restriction of the map

$$
\lambda \in \Lambda \mapsto f(\pi(\lambda))+\lambda
$$

to a tubular neighborhood $N^{n+r}$ of the 0 -section $j: M^{n} \hookrightarrow N^{n+r}$ of $\Lambda$ where that map is an embedding. Hence $f=F \circ j$ and

$$
\begin{equation*}
T_{j(x)} N=j_{*} T_{x} M \oplus \Lambda(x) \tag{12.7}
\end{equation*}
$$

for any $x \in M^{n}$.
By Lemma 12.3, the immersion $F$ is $R$-ruled, where $R(\lambda)=R(\pi(\lambda))$. From

$$
\left\langle\tilde{\nabla}_{X} \lambda, \mu\right\rangle=-\left\langle\lambda, \tilde{\nabla}_{X} \mu\right\rangle=0
$$

for all $\lambda \in \Gamma(\Lambda), \mu \in \Gamma(P)$ and $X \in T M$, it follows that $\mathcal{P} \subset N_{F} N$, where

$$
\mathcal{P}(\lambda)=P(\pi(\lambda)) .
$$

Moreover,

$$
R=\mathcal{N}\left(\alpha_{\mathcal{P}}^{F}\right)
$$

In fact, the inclusion $R \subset \mathcal{N}\left(\alpha_{\mathcal{P}}^{F}\right)$ holds because $\mathcal{P}$ is constant along $R$. For the opposite inclusion, observe that

$$
\left.\alpha_{\mathcal{P}}^{F}\right|_{T M \times T M}=\alpha_{P} .
$$

From (12.7) we see that $R=\mathcal{N}\left(\alpha_{\mathcal{P}}^{F}\right)$ on $j(M)$. Finally, observe that the dimension of $\mathcal{N}\left(\alpha_{\mathcal{P}}^{F}\right)$ can only decrease along $R \subset N^{n+r}$ from its value on $j(M)$ if $N^{n+r}$ is taken small enough.

Taking into account $\sqrt{12.4}$, the preceding facts can be summarized as follows.

Proposition 12.4. Assume that 12.5) holds. Then $F: N^{n+r} \rightarrow \mathbb{R}^{m}$ is a $R$-ruled extension of $f$ and

$$
j_{*} D(x)=R^{d+r}(j(x)) \cap j_{*} T_{x} M
$$

for all $x \in M^{n}$. Moreover, there is an orthogonal splitting

$$
N_{F} N=\mathcal{L} \oplus \mathcal{P}
$$

such that rank $\mathcal{L}=\ell-r, R=\mathcal{N}\left(\alpha_{\mathcal{P}}^{F}\right)$ and $\mathcal{P}$ is parallel along $R$ in $\mathbb{R}^{m}$. Furthermore, the following assertions hold:
(i) If $k=n-d+\ell(r=0)$ then $f$ is $D$-ruled and does not extend.
(ii) If $k=n-d(r=\ell)$ then $R$ is the relative nullity distribution of $F$.

### 12.2 Pairs of ruled extensions

The aim of this section is to provide sufficient conditions for a pair of isometric immersions to admit simultaneous $R$-ruled isometric extensions satisfying some additional properties. For that, we adapt the extension procedure developed in the previous section to this situation.

First we find necessary conditions.
Let $F: N^{n+r} \rightarrow \mathbb{R}^{n+p}$ and $\hat{F}: N^{n+r} \rightarrow \mathbb{R}^{n+q}$ be $R$-ruled isometric immersions, where $R \subset N^{n+r}$ is a smooth integrable distribution. By the Gauss equations of $F$ and $\hat{F}$,

$$
\left\langle\alpha^{F}(T, X), \alpha^{F}(S, Y)\right\rangle=\left\langle\alpha^{\hat{F}}(T, X), \alpha^{\hat{F}}(S, Y)\right\rangle
$$

for all $T, S \in \Gamma(R)$ and $X, Y \in \mathfrak{X}(N)$. Thus the map $\mathcal{T}_{R}: \Gamma\left(L_{R}^{F}\right) \rightarrow \Gamma\left(L_{R}^{\hat{F}}\right)$, given by

$$
\begin{equation*}
\mathcal{T}_{R}\left(\alpha^{F}(T, X)\right)=\alpha^{\hat{F}}(T, X) \tag{12.8}
\end{equation*}
$$

for all $T \in \Gamma(R)$ and $X \in \mathfrak{X}(M)$, determines a vector bundle isometry between $L_{R}^{F}$ and $L_{R}^{\hat{F}}$. Notice that (12.8) implies that

$$
\left.A_{\zeta}^{F}\right|_{R}=\left.A_{\mathcal{T}_{R} \zeta}^{\hat{F}}\right|_{R}
$$

for all $\zeta \in \Gamma\left(L_{R}\right)$.
Let us assume that the stronger condition

$$
A_{\zeta}^{F}=A_{\mathcal{T}_{R} \zeta}^{\hat{F}^{\prime}}
$$

is satisfied for all $\zeta \in \Gamma\left(L_{R}\right)$, that is,

$$
\mathcal{T}_{R}\left(\alpha_{L_{R}}^{F}(X, Y)\right)=\alpha_{\hat{L}_{R}}^{\hat{F}}(X, Y)
$$

for all $X, Y \in \mathfrak{X}(M)$. Using the Codazzi equations for $F$ and $\hat{F}$, and the fact that $L_{R}$ and $\hat{L}_{R}$ are parallel in the normal connection along $R$, one can easily verify that

$$
\left\langle{ }^{F} \nabla_{Z}^{\perp} \alpha^{F}(T, X), \alpha^{F}(S, Y)\right\rangle=\left\langle{ }^{\hat{F}} \nabla_{Z}^{\perp} \alpha^{\hat{F}}(T, X), \alpha^{\hat{F}}(S, Y)\right\rangle
$$

for all $T, S \in \Gamma(R)$ and $X, Y, Z \in \mathfrak{X}(M)$, that is,

$$
\mathcal{T}_{R}\left(\left({ }^{F} \nabla_{X}^{\perp} \zeta\right)_{L_{R}}\right)=\left({ }^{\hat{F}} \nabla_{X}^{\perp} \mathcal{T}_{R} \zeta\right)_{\hat{L}_{R}}
$$

for all $\zeta \in \Gamma\left(L_{R}\right)$ and $X \in \mathfrak{X}(M)$.
Now let $j: M^{n} \rightarrow N^{n+r}$ be an isometric immersion and let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ be defined by $f=F \circ j$ and $\hat{f}=\hat{F} \circ j$. Then $L_{j}=F_{*} N_{j} M$ and $\hat{L}_{j}=\hat{F}_{*} N_{j} M$ are subbundles of $N_{f} M$ and $N_{\hat{f}} M$, respectively, and $\mathcal{T}_{j}: \Gamma\left(L_{j}\right) \rightarrow \Gamma\left(\hat{L}_{j}\right)$, given by

$$
\mathfrak{T}_{j} F_{*} \xi=\hat{F}_{*} \xi
$$

for all $\xi \in \Gamma\left(N_{j} M\right)$, defines a vector bundle isometry between $L_{j}$ and $\hat{L}_{j}$.
Consider the orthogonal splittings of the normal bundles of $f$ and $\hat{f}$ as

$$
N_{f} M=L_{j} \oplus j^{*} L_{R}^{F} \oplus P \text { and } N_{\hat{f}} M=\hat{L}_{j} \oplus j^{*} L_{R}^{\hat{F}} \oplus \hat{P}
$$

where

$$
P=j^{*}\left(L_{R}^{F}\right)^{\perp} \text { and } \hat{P}=j^{*}\left(L_{R}^{\hat{F}}\right)^{\perp}
$$

Denote

$$
L=L_{j} \oplus j^{*} L_{R}^{F} \text { and } \hat{L}=\hat{L}_{j} \oplus j^{*} L_{R}^{\hat{F}}
$$

and let $\mathcal{T}: \Gamma(L) \rightarrow \Gamma(\hat{L})$ be given by

$$
\left.\mathcal{T}\right|_{L_{j}}=\mathcal{T}_{j} \text { and }\left.\mathcal{T}\right|_{j^{*} L_{R}^{F}}=\mathcal{T}_{R}
$$

Then $\mathcal{T}$ is a vector bundle isometry between $L$ and $\hat{L}$ such that

$$
\begin{aligned}
\hat{\alpha}_{\hat{L}}(X, Y) & =\hat{F}_{*} \alpha^{j}(X, Y)+\alpha_{\hat{L}_{R}}^{\hat{F}}\left(j_{*} X, j_{*} Y\right) \\
& =\mathcal{T}_{j} F_{*} \alpha^{j}(X, Y)+\mathcal{T}_{R} \alpha_{L_{R}}^{F}\left(j_{*} X, j_{*} Y\right) \\
& =\mathcal{T}\left(F_{*} \alpha^{j}(X, Y)+\alpha_{L_{R}}^{F}\left(j_{*} X, j_{*} Y\right)\right) \\
& =\mathcal{T} \alpha_{L}(X, Y)
\end{aligned}
$$

where $\alpha=\alpha^{f}, \hat{\alpha}=\alpha^{\hat{f}}$ and $X, Y \in \mathfrak{X}(M)$. On the other hand, from

$$
{ }^{f} \nabla_{X}^{\perp} F_{*} \xi=F_{*}^{j} \nabla_{X}^{\perp} \xi+\alpha^{F}\left(j_{*} X, \xi\right)
$$

and

$$
{ }^{\hat{f}} \nabla_{X}^{\perp} \hat{F}_{*} \xi=\hat{F}_{*}^{j} \nabla_{X}^{\perp} \xi+\alpha^{\hat{F}}\left(j_{*} X, \xi\right)
$$

for all $X \in \mathfrak{X}(M)$ and $\xi \in \Gamma\left(N_{j} M\right)$ (see Exercise 1.6), it follows that

$$
\begin{aligned}
\left({ }^{\hat{f}} \nabla_{X}^{\perp} \mathcal{T} F_{*} \xi\right)_{\hat{L}} & =\left({ }^{\hat{f}} \nabla_{X}^{\perp} \hat{F}_{*} \xi\right)_{\hat{L}} \\
& =\hat{F}_{*}^{j} \nabla_{X}^{\perp} \xi+\alpha_{\hat{L}_{R}}\left(j_{*} X, \xi\right) \\
& =\mathcal{T}_{j} F_{*}^{j} \nabla_{X}^{\perp} \xi+\mathcal{T}_{R} \alpha_{L_{R}}^{F}\left(j_{*} X, \xi\right) \\
& =\mathcal{T}\left({ }^{f} \nabla_{X}^{\perp} F_{*} \xi\right)_{L} .
\end{aligned}
$$

Similarly, from

$$
{ }^{f} \nabla_{X}^{\perp} \zeta=-F_{*}\left(A_{\zeta}^{F} j_{*} X\right)_{N_{j} M}+{ }^{F} \nabla_{j_{*} X}^{\perp} \zeta
$$

for all $X \in \mathfrak{X}(M)$ and $\zeta \in \Gamma\left(N_{F} N\right)$, and the corresponding formula for $\hat{f}$, we obtain

$$
\begin{aligned}
\left({ }^{\hat{f}} \nabla_{X}^{\perp} \mathcal{T} \zeta\right)_{\hat{L}} & =-\hat{F}_{*}\left(A_{\mathcal{T} \zeta}^{\hat{F}} j_{*} X\right)_{N_{j} M}+\left({ }^{\hat{F}} \nabla_{j_{*} X}^{\perp} \mathcal{T}_{R} \zeta\right)_{\hat{L}_{R}} \\
& =-\mathcal{T}_{j} F_{*}\left(A_{\zeta}^{F} j_{*} X\right)_{N_{j} M}+\mathcal{T}_{R}\left({ }^{F} \nabla_{j_{*} X}^{\perp} \zeta\right)_{L_{R}} \\
& =\mathcal{T}\left({ }^{f} \nabla_{X}^{\perp} \zeta\right)_{L}
\end{aligned}
$$

for all $X \in \mathfrak{X}(M)$ and $\zeta \in \Gamma\left(L_{R}^{F}\right)$. Thus $\mathcal{T}$ is a parallel vector bundle isometry with respect to the induced connections.

Assume as in the previous section that the subspaces

$$
R(j(x)) \cap j_{*} T_{x} M
$$

have constant dimension $d$ and let $D \subset T M$ be the distribution defined by

$$
j_{*} D(x)=R(j(x)) \cap j_{*} T_{x} M
$$

As before,

$$
D \subset \mathcal{N}\left(\alpha_{P}\right) \cap \mathcal{N}\left(\hat{\alpha}_{\hat{P}}\right)
$$

and both $L$ and $\hat{L}$ are parallel in the normal connection along $D$. Moreover, we actually have

$$
D=\mathcal{N}\left(\alpha_{P}\right) \cap \mathcal{N}\left(\hat{\alpha}_{\hat{P}}\right)
$$

if we assume that

$$
R=\mathcal{N}\left(\alpha_{L_{\hat{R}}}^{F}\right) \cap \mathcal{N}\left(\alpha_{\hat{L}_{\vec{R}}}^{\hat{F}}\right)
$$

and that $j$ is transversal to the rulings.
Conversely, let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ be isometric immersions. Assume that there exist a vector bundle isometry $\mathcal{T}: L^{\ell} \rightarrow \hat{L}^{\ell}$ between normal subbundles $L^{\ell} \subset N_{f} M$ and $\hat{L}^{\ell} \subset N_{\hat{f}} M$ of rank $\ell$ such that the tangent subspaces

$$
D(x)=\mathcal{N}\left(\alpha_{L^{\perp}}(x)\right) \cap \mathcal{N}\left(\hat{\alpha}_{\hat{L}^{\perp}}(x)\right)
$$

form a tangent subbundle of rank $d$ and the pair ( $\mathcal{T}, D^{d}$ ) satisfies the following two conditions:
$\left\{\begin{array}{l}\left(\mathcal{C}_{1}\right) \text { The vector bundle isometry } \mathcal{T} \text { is parallel with respect to the } \\ \quad \text { induced connections and preserves the second fundamental forms. } \\ \left(\mathcal{C}_{2}\right) \text { The vector subbundles } L^{\ell} \text { and } \hat{L}^{\ell} \text { are parallel along } D^{d} \text { in the } \\ \quad \text { normal connections. }\end{array}\right.$
Notice that condition $\left(\mathcal{C}_{1}\right)$ is equivalent to the vector bundle isometry

$$
\hat{\mathcal{T}}: f_{*} T M \oplus L \rightarrow \hat{f_{*}} T M \oplus \hat{L},
$$

given by

$$
\hat{\mathfrak{T}}\left(f_{*} X+\xi\right)=\hat{f}_{*} X+\mathfrak{T} \xi
$$

being parallel with respect to the connections on $f_{*} T M \oplus L$ and $\hat{f}_{*} T M \oplus \hat{L}$ induced from those on $f^{*} T \mathbb{R}^{n+p}$ and $\hat{f}^{*} T \mathbb{R}^{n+q}$, respectively.

Consider the orthogonal splitting

$$
T M=D \oplus E
$$

and, for each $x \in M^{n}$, let $\Gamma(x) \subset E(x) \oplus L(x)$ and $\hat{\Gamma}(x) \subset E(x) \oplus \hat{L}(x)$ be given by 12.3) for $f$ and $\hat{f}$, respectively. Denote by $\Lambda(x)$ the maximal subspace of $E(x) \oplus L(x)$ such that $\Lambda(x)$ is orthogonal to $\Gamma(x)$ and $\hat{\mathcal{T}}(\Lambda(x))$ is orthogonal to $\hat{\Gamma}(x)$. Finally, assume that the subspaces $\Lambda(x)$ have constant dimension and thus form a vector subbundle of rank $r$ of $T M \oplus L$.

As in the previous section, define isometric extensions $F: N^{n+r} \rightarrow \mathbb{R}^{n+p}$ of $f$ and $\hat{F}: N^{n+r} \rightarrow \mathbb{R}^{n+q}$ of $\hat{f}$, both ruled by $R=D \oplus \Lambda$, up to parallel identification along $\Lambda$.

Proposition 12.5. The immersions $F$ and $\hat{F}$ are isometric $R$-ruled extensions of $f$ and $\hat{f}$, respectively. Moreover, there are smooth orthogonal splittings

$$
N_{F} N=\mathcal{L} \oplus \mathcal{P} \text { and } N_{\hat{F}} N=\hat{\mathcal{L}} \oplus \hat{\mathcal{P}}
$$

and a vector bundle isometry $T: \mathcal{L} \rightarrow \hat{\mathcal{L}}$ such that:
(i) $\operatorname{rank} \mathcal{L}=\operatorname{rank} L-r$.
(ii) $R=\mathcal{N}\left(\alpha_{\mathcal{P}}^{F}\right) \cap \mathcal{N}\left(\hat{\alpha}_{\hat{\mathcal{P}}}^{\hat{F}}\right)$.
(iii) $\mathcal{P}$ and $\hat{\mathcal{P}}$ are parallel along the rulings in $\mathbb{R}^{m}$.
(iv) The vector bundle isometry $T$ is parallel with respect to the induced connections and preserves the second fundamental forms.

Proof: We will only prove that $F$ and $\hat{F}$, given by

$$
F(\lambda)=f(\pi(\lambda))+\lambda \text { and } \hat{F}(\lambda)=\hat{f}(\pi(\lambda))+\hat{\mathscr{T}}(\lambda)
$$

induce the same metric on $N^{n+r}$. The remaining assertions follow from the arguments in the proof of Proposition 12.4 .

Let $X$ denote both a tangent vector at $\pi(\lambda) \in M^{n}$ and its horizontal lift to $T_{\lambda} N$. Choose a section of $\Lambda$ through $\lambda$ along a curve in $M^{n}$ through $\pi(\lambda)$ and tangent to $X$, also denoted by $\lambda$. Then

$$
\begin{aligned}
\left\|F_{*}(\lambda) X\right\| & =\left\|f_{*} X+\tilde{\nabla}_{X} \lambda\right\| \\
& =\left\|f_{*} X+\left(\tilde{\nabla}_{X} \lambda\right)_{f_{*} T M \oplus L}\right\| \\
& =\left\|\hat{f}_{*} X+\left(\tilde{\nabla}_{X} \hat{\mathcal{T}}(\lambda)\right)_{\hat{f}_{*} T M \oplus \hat{L}}\right\| \\
& =\left\|\hat{f}_{*} X+\tilde{\nabla}_{X} \hat{\mathcal{T}}(\lambda)\right\| \\
& =\left\|\hat{F}_{*}(\lambda) X\right\|
\end{aligned}
$$

where in the second and fourth equalities we have used that $\tilde{\nabla}_{X} \lambda \in \Gamma\left(f_{*} T M \oplus L\right)$ and $\tilde{\nabla}_{X} \hat{\mathcal{T}}(\lambda) \in \Gamma\left(\hat{f}_{*} T M \oplus \hat{L}\right)$ for all $\lambda \in \Gamma(\Lambda)$, by the definition of $\Lambda$. On the other hand, the equality

$$
\left\|F_{*}(\lambda) V\right\|=\left\|\hat{F}_{*}(\lambda) V\right\|
$$

holds trivially if $V$ is a vertical vector at $\lambda \in \Lambda$.

### 12.2.1 Constructing pairs of ruled extensions

We show next how to construct a pair $\left(\mathcal{T}, D^{d}\right)$ satisfying conditions 12.9) in the previous section for a pair of isometric immersions into Euclidean spaces.

In the sequel, several vector subspaces will be pointwise defined either as images or kernels of certain tensor fields on a submanifold. To avoid cumbersome repetition, whenever necessary we agree that we are always working restricted to a connected component of an open dense subset of the submanifold where these subspaces have constant dimensions, and hence form smooth vector subbundles.

Given isometric immersions $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ with second fundamental forms $\alpha$ and $\hat{\alpha}$, respectively, endow the vector bundle $N_{f} M \oplus N_{\hat{f}} M$ with the indefinite metric of signature $(p, q)$ given by

$$
\langle\langle(\xi, \hat{\xi}),(\eta, \hat{\eta})\rangle\rangle_{N_{f} M \oplus N_{\hat{f}} M}=\langle\xi, \eta\rangle_{N_{f} M}-\left\langle\hat{\xi}, \hat{\eta}_{N_{\hat{f}} M},\right.
$$

and its vector subbundle

$$
\mathcal{S}(\alpha) \oplus \mathcal{S}(\hat{\alpha}) \subset N_{f} M \oplus N_{\hat{f}} M
$$

with the induced metric. Then consider the bilinear map

$$
\alpha \oplus \hat{\alpha}: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \Gamma(\mathcal{S}(\alpha) \oplus \mathcal{S}(\hat{\alpha})),
$$

which we also regard as a section of $\operatorname{Hom}^{2}(T M, T M ; \mathcal{S}(\alpha) \oplus \mathcal{S}(\hat{\alpha}))$.
Let $\Omega \subset \mathcal{S}(\alpha) \oplus \mathcal{S}(\hat{\alpha}))$ be the vector subbundle with isotropic fibers defined by

$$
\Omega=\mathcal{S}(\alpha \oplus \hat{\alpha}) \cap \mathcal{S}(\alpha \oplus \hat{\alpha})^{\perp} .
$$

The argument in the proof of Lemma 4.21 shows that there are orthogonal splittings

$$
\begin{equation*}
\mathcal{S}(\alpha)=\Gamma \oplus \Sigma \text { and } \mathcal{S}(\hat{\alpha})=\hat{\Gamma} \oplus \hat{\Sigma} \tag{12.10}
\end{equation*}
$$

and a smooth vector bundle isometry $\mathcal{L}: \Sigma \rightarrow \hat{\Sigma}$ such that

$$
\Omega=\{(\eta, \mathcal{L} \eta): \eta \in \Sigma\}
$$

and $\hat{\alpha}_{\hat{\Sigma}}=\mathcal{L} \circ \alpha_{\Sigma}$.
In the sequel, we define pairs of vector subbundles:

$$
\left(D_{2}, L_{2}\right) \subset\left(D_{1}, L_{1}\right) \subset\left(D_{0}, L_{0}\right) \text { and }\left(D_{2}, \hat{L}_{2}\right) \subset\left(D_{1}, \hat{L}_{1}\right) \subset\left(D_{0}, \hat{L}_{0}\right)
$$

where $D_{j} \subset T M$ and $L_{j}, \hat{L}_{j} \subset \Sigma$, with the following properties: The maps

$$
\begin{equation*}
\mathcal{L}_{j}=\left.\mathcal{L}\right|_{L_{j}}: L_{j} \rightarrow \hat{L}_{j}, \quad 0 \leq j \leq 2, \tag{12.11}
\end{equation*}
$$

are vector bundle isometries satisfying $\hat{\alpha}_{\hat{L}_{j}}=\mathcal{L}_{j} \circ \alpha_{L_{j}}$, and

$$
\begin{equation*}
D_{j}=\mathcal{N}\left(\alpha_{L_{j}^{\perp}} \oplus \hat{\alpha}_{\hat{L}_{j}^{\perp}}\right), \quad 0 \leq j \leq 2, \tag{12.12}
\end{equation*}
$$

where

$$
N_{f} M=L_{j} \oplus L_{j}^{\perp} \text { and } N_{\hat{f}} M=\hat{L}_{j} \oplus \hat{L}_{j}^{\perp} .
$$

The procedure for the construction is to go from one pair to the next by requiring an additional condition, so that the final pairs $\left(D_{2}, L_{2}\right)$ and ( $D_{2}, \hat{L}_{2}$ ) will satisfy conditions 12.9.

First, define the vector subbundles $D_{0} \subset T M$ and $L_{0} \subset \Sigma$ by

$$
D_{0}=\mathcal{N}\left(\alpha_{\Gamma} \oplus \hat{\alpha}_{\hat{\Gamma}}\right) \text { and } L_{0}=\mathcal{S}\left(\left.\alpha\right|_{D_{0} \times T M}\right)
$$

Define $\hat{L}_{0}$ in a similar way and observe that 12.11 and 12.12 hold for $j=0$. The next step is to define a vector subbundle $L_{1}$ of $L_{0}$ by requiring that

$$
\mathcal{L}_{0}\left(\nabla_{X}^{\perp} \eta\right)_{L_{0}}=\left(\hat{\nabla}_{X}^{\perp} \mathcal{L}_{0}(\eta)\right)_{\hat{L}_{0}}
$$

for all $\eta \in \Gamma\left(L_{1}\right)$. For that, consider the map $\mathcal{K}: \mathfrak{X}(M) \rightarrow \Gamma\left(\Lambda^{2}\left(L_{0}\right)\right)$ whose value at $X \in \mathfrak{X}(M)$ is the skew-symmetric tensor $\mathcal{K}(X)$ given by

$$
\mathcal{K}(X) \eta=\mathcal{L}_{0}\left(\nabla \frac{\perp}{X} \eta\right)_{L_{0}}-\left(\hat{\nabla}_{X}^{\perp} \mathcal{L}_{0}(\eta)\right)_{\hat{L}_{0}} .
$$

It is clear that $K$ gives rise to a section of $\operatorname{Hom}\left(T M ; \Lambda^{2}\left(L_{0}\right)\right)$.

Proposition 12.6. The tensor $\mathcal{K}$ has the following properties:
(i) $\mathcal{K}(Z)=0$ if $Z \in \Gamma\left(D_{0}\right)$.
(ii) $\mathcal{K}(X) \alpha(Y, Z)=\mathcal{K}(Y) \alpha(X, Z)$ if either $X, Y \in \Gamma\left(D_{0}\right)$ or $Z \in \Gamma\left(D_{0}\right)$.
(iii) $\langle\mathcal{K}(X) \alpha(Y, Z), \hat{\alpha}(T, Z)\rangle=0$ if $Z \in \Gamma\left(D_{0}\right)$.

Proof: Comparing the $\Sigma$-components of the Codazzi equation for $f$ and $\hat{f}$ gives

$$
\mathcal{L}\left(\nabla_{X}^{\perp} \alpha(Y, Z)\right)_{\Sigma}-\left(\hat{\nabla}_{X}^{\perp} \hat{\alpha}(Y, Z)\right)_{\hat{\Sigma}}=\mathcal{L}\left(\nabla_{Y}^{\perp} \alpha(X, Z)\right)_{\Sigma}-\left(\hat{\nabla}_{Y}^{\perp} \hat{\alpha}(X, Z)\right)_{\hat{\Sigma}}
$$

where we have used that $\hat{\alpha}_{\Sigma}=\mathcal{T} \circ \alpha_{\Sigma}$. If either $X, Y \in \Gamma\left(D_{0}\right)$ or $Z \in \Gamma\left(D_{0}\right)$, then

$$
\mathcal{K}(X) \alpha(Y, Z)=\mathcal{K}(Y) \alpha(X, Z),
$$

which proves part (ii). Let us denote

$$
\left\langle\mathcal{K}\left(X_{1}\right) \alpha\left(X_{2}, X_{3}\right), \hat{\alpha}\left(X_{4}, X_{5}\right)\right\rangle=\left(X_{1}, X_{2}, X_{3}, X_{4}, X_{5}\right) .
$$

Then

$$
\begin{aligned}
\left(Y, Z_{1}, Z_{2}, Z_{3}, X\right) & =-\left(Y, Z_{3}, X, Z_{1}, Z_{2}\right)=-\left(X, Z_{3}, Y, Z_{1}, Z_{2}\right)=\left(X, Z_{1}, Z_{2}, Z_{3}, Y\right) \\
& =\left(Z_{2}, Z_{1}, X, Z_{3}, Y\right)=-\left(Z_{2}, Z_{3}, Y, Z_{1}, X\right)=-\left(Z_{3}, Z_{2}, Y, Z_{1}, X\right) \\
& =\left(Z_{3}, Z_{1}, X, Z_{2}, Y\right)=\left(Z_{1}, Z_{3}, X, Z_{2}, Y\right)=-\left(Z_{1}, Z_{2}, Y, Z_{3}, X\right) \\
& =-\left(Y, Z_{1}, Z_{2}, Z_{3}, X\right) \\
& =0
\end{aligned}
$$

for all $Z_{1}, Z_{2}, Z_{3} \in \Gamma\left(D_{0}\right)$, and this proves part (i). Finally,

$$
\begin{aligned}
(X, Y, Z, T, Z) & =(Y, X, Z, T, Z)=-(Y, T, Z, X, Z)=-(T, Y, Z, X, Z) \\
& =(T, X, Z, Y, Z)=(X, T, Z, Y, Z)=-(X, Y, Z, T, Z) \\
& =0
\end{aligned}
$$

and this proves part (iii).
We can now define $L_{1}$ as the vector subbundle of $L_{0}$ whose fiber at $x \in M^{n}$ is

$$
L_{1}(x)=\cap_{X \in T_{x} M} \operatorname{ker} \mathcal{K}(X)
$$

and let $D_{1} \subset D_{0}$ be the tangent subbundle given by 12.12). To conclude, let $L_{2}$ be the vector subbundle of $L_{1}$ such that

$$
\Gamma\left(L_{2}\right)=\left\{\delta \in \Gamma\left(L_{1}\right): \nabla_{Y}^{\perp} \delta \in \Gamma\left(L_{0}\right) \text { and } \hat{\nabla}_{Y}^{\perp} \mathcal{L}_{1}(\delta) \in \Gamma\left(\hat{L}_{0}\right) \text { for all } Y \in \Gamma\left(D_{0}\right)\right\}
$$

and let $D_{2} \subset D_{1}$ be defined again by 12.12). For simplicity, set $\left(D_{2}, L_{2}\right)=(D, L)$.
Next we discuss a basic property of the tangent distributions defined above.

Proposition 12.7. The distributions $D_{0}$ and $D_{1}$ satisfy

$$
\begin{equation*}
\left[D_{1}, D_{0}\right] \subset D_{0} \tag{12.13}
\end{equation*}
$$

that is, $[Y, Z] \in \Gamma\left(D_{0}\right)$ for all $Y \in \Gamma\left(D_{1}\right)$ and $Z \in \Gamma\left(D_{0}\right)$.
Proof: Let $Z_{1}, Z_{2} \in \Gamma\left(D_{0}\right)$. Taking the inner product of both sides of the Codazzi equation

$$
\left(\nabla \frac{Z_{1}}{\perp} \alpha\right)\left(Z_{2}, U\right)=\left(\nabla \frac{1}{Z_{2}} \alpha\right)\left(Z_{1}, U\right)
$$

with $\mu \in \Gamma\left(L_{0}^{\perp}\right)$ yields

$$
\begin{equation*}
\left\langle\nabla \frac{Z_{1}}{\perp} \alpha\left(Z_{2}, U\right)-\nabla \frac{1}{Z_{2}} \alpha\left(Z_{1}, U\right), \mu\right\rangle=\left\langle\alpha_{L_{0}^{\perp}}\left(\left[Z_{1}, Z_{2}\right], U\right), \mu\right\rangle . \tag{12.14}
\end{equation*}
$$

The difference between the Codazzi equations of $f$ and $\hat{f}$ for $\delta \in \Gamma\left(L_{0}\right)$ gives

$$
\left\langle A_{\nabla \frac{1}{Z} \delta} Y-\hat{A}_{\hat{\nabla}_{\frac{1}{Z}} \mathcal{L}_{0}(\delta)} Y, X\right\rangle=\left\langle A_{\nabla \frac{1}{Y} \delta} Z-\hat{A}_{\hat{\nabla}_{\frac{Y}{Y} \mathcal{L}_{0}(\delta)}} Z, X\right\rangle .
$$

It follows using part ( $i$ ) of Proposition 12.6 that

$$
\left\langle\nabla \frac{Z_{1}}{\perp} \delta, \alpha_{L_{0}^{\perp}}(X, Y)\right\rangle-\left\langle\hat{\nabla}_{Z_{1}}^{\perp} \mathcal{L}_{0}(\delta), \hat{\alpha}_{\hat{L}_{\perp}^{\perp}}(X, Y)\right\rangle=\left\langle\hat{\alpha}\left(Z_{1}, X\right), \mathcal{K}(Y) \delta\right\rangle
$$

for all $Z_{1} \in \Gamma\left(D_{0}\right)$. Choose $\delta=\alpha\left(Z_{2}, U\right) \in \Gamma\left(L_{0}\right)$ with $Z_{2} \in \Gamma\left(D_{0}\right)$. Then

$$
\left\langle\nabla_{Z_{1}}^{\perp} \alpha\left(Z_{2}, U\right), \alpha_{L_{0}^{\perp}}(X, Y)\right\rangle-\left\langle\hat{\nabla}_{Z_{1}}^{\perp} \hat{\alpha}\left(Z_{2}, U\right), \hat{\alpha}_{\hat{L}_{\dot{\circ}}^{\perp}}(X, Y)\right\rangle=\left\langle\hat{\alpha}\left(Z_{1}, X\right), \mathcal{K}(Y) \alpha\left(Z_{2}, U\right)\right\rangle .
$$

Take $Z_{2} \in \Gamma\left(D_{1}\right)$. Then the right-hand side of the above equation vanishes, and using (12.14) for both immersions we obtain

$$
\left\langle\alpha_{L_{0}^{\perp}}\left(\left[Z_{1}, Z_{2}\right], U\right), \alpha_{L_{0}^{\perp}}(X, Y)\right\rangle-\left\langle\hat{\alpha}_{\hat{L}_{0}^{\perp}}\left(\left[Z_{1}, Z_{2}\right], U\right), \hat{\alpha}_{\hat{L}_{0}^{\perp}}(X, Y)\right\rangle=0 .
$$

Denoting $\beta=\alpha_{\Gamma} \oplus \hat{\alpha}_{\hat{\Gamma}}$, this is equivalent to

$$
\left\langle\left\langle\beta\left(\left[Z_{1}, Z_{2}\right], U\right), \mathcal{S}(\beta)\right\rangle\right\rangle=0 .
$$

By Lemma 4.21, the subspace $\mathcal{S}(\beta)$ is nondegenerate, and the proof follows easily.
Proposition 12.8. The pair $\left(\mathcal{T}=\left.\mathcal{L}\right|_{L^{\ell}}, D\right)$ defined on an open dense subset of $M^{n}$ satisfies conditions (12.9).

Proof: The isometry $\mathfrak{T}$ preserves the second fundamental forms and the normal connections since $\mathcal{L}_{1}=\left.\mathcal{L}\right|_{L_{1}}$ already has these properties. Thus it remains to show that $L$ is parallel along $D$ in the normal connection for both immersions.

We first argue that $L_{0}$ is parallel along $D$ in the normal connection, that is,

$$
\begin{equation*}
\nabla \frac{1}{Z} \mu \in \Gamma\left(L_{0}\right) \text { for all } Z \in \Gamma(D) \text { and } \mu \in \Gamma\left(L_{0}\right) \tag{12.15}
\end{equation*}
$$

Observe that

$$
\begin{equation*}
\mathcal{S}\left(\left.\alpha_{L^{\perp} \cap L_{0}}\right|_{D_{0} \times T M}\right)=L^{\perp} \cap L_{0} . \tag{12.16}
\end{equation*}
$$

The Codazzi equation and 12.13 yield

$$
\begin{equation*}
\left(\nabla_{Y}^{\perp} \alpha_{L^{\perp} \cap L_{1}}(Z, X)\right)_{L_{0}^{\perp}}=\left(\nabla_{Z}^{\perp} \alpha_{L^{\perp} \cap L_{0}}(Y, X)\right)_{L_{0}^{\perp}} \tag{12.17}
\end{equation*}
$$

for all $Z \in \Gamma\left(D_{1}\right), Y \in \Gamma\left(D_{0}\right)$ and $X \in \mathfrak{X}(M)$. The left-hand side vanishes if $Z \in \Gamma(D)$ and 12.15 ) follows from 12.16) and the definition of $L$.

We show next that

$$
\begin{equation*}
\nabla \frac{\perp}{Z} \delta \in \Gamma\left(L_{1}\right) \text { for all } Z \in \Gamma(D) \text { and } \delta \in \Gamma(L) \tag{12.18}
\end{equation*}
$$

Define $R \subset L_{0}$ by the orthogonal splitting

$$
\begin{equation*}
L_{0}=L_{1} \oplus R \tag{12.19}
\end{equation*}
$$

The skew-symmetry of $\mathcal{K}(X)$ gives

$$
\begin{equation*}
R=\operatorname{span}\{\mathcal{K}(X) \mu: X \in \mathfrak{X}(M) \text { and } \mu \in \Gamma(R)\} \tag{12.20}
\end{equation*}
$$

By the Ricci equation,

$$
\left\langle R^{\perp}(X, Z) \delta, \mu\right\rangle=\left\langle\hat{R}^{\perp}(X, Z) \mathcal{L}_{0}(\delta), \mathcal{L}_{0}(\mu)\right\rangle
$$

for all $\delta, \mu \in \Gamma\left(L_{0}\right)$. It follows that

$$
\left\langle\nabla \frac{\perp}{Z} \delta, \nabla_{X}^{\perp} \mu\right\rangle-\left\langle\hat{\nabla}_{\frac{1}{Z}}^{\perp} \mathcal{L}_{0}(\delta), \hat{\nabla}_{X}^{\perp} \mathcal{L}_{0}(\mu)\right\rangle=\left\langle\nabla \frac{\perp}{X} \delta, \nabla \frac{\perp}{Z} \mu\right\rangle-\left\langle\hat{\nabla}_{X}^{\perp} \mathcal{L}_{0}(\delta), \hat{\nabla}_{Z}^{\perp} \mathcal{L}_{0}(\mu)\right\rangle
$$

for all $\delta \in \Gamma\left(L_{1}\right)$ and $\mu \in \Gamma(R)$. Part (i) of Proposition 12.6 and 12.15 give

$$
\left\langle\mathcal{L}_{0}\left(\nabla_{Z}^{\perp} \delta\right), \mathcal{K}(X) \mu\right\rangle=0
$$

for all $Z \in \Gamma(D), \delta \in \Gamma(L), \mu \in \Gamma(R)$ and $X \in \mathfrak{X}(M)$, and 12.18) follows from 12.20).
The Ricci equation yields

$$
\left\langle R^{\perp}(Y, Z) \delta, \xi\right\rangle=\left\langle\left[A_{\xi}, A_{\delta}\right] Y, Z\right\rangle=0
$$

for all $Y, Z \in \Gamma\left(D_{0}\right)$ and $\xi \in \Gamma\left(L_{0}^{\perp}\right)$. Then 12.13), 12.15) and 12.18) give

$$
\left\langle\nabla_{Y}^{\perp} \nabla \frac{\perp}{Z} \delta, \xi\right\rangle=0
$$

for all $Y \in \Gamma\left(D_{0}\right), Z \in \Gamma(D), \delta \in \Gamma(L)$ and $\xi \in \Gamma\left(L_{0}^{\perp}\right)$. By (12.18) and the definition of $L$, it follows that $L$ and $\hat{L}=\mathcal{T}(L)$ are parallel along $D$ in the normal connections.

### 12.3 Genuine isometric deformations

The main result of this section describes the geometric structure of a genuine pair of isometric immersions. First we prove a lemma that will also be useful in the study of genuine pairs of isometric immersions in the singular sense in Section 12.8 .

Given vector bundles $E$ and $F$ over $M$, with $F$ endowed with an indefinite metric $\langle\langle\rangle$,$\rangle , and \phi \in \operatorname{Hom}^{2}(E, T M ; F)$, we say that a vector subbundle $H$ of $E$ is isotropic with respect to $\phi$ if

$$
\langle\langle\phi(\gamma, Y), \phi(\gamma, Y))\rangle\rangle=0
$$

for all $\gamma \in \Gamma(H)$ and $Y \in \mathfrak{X}(M)$.

Lemma 12.9. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ be isometric immersions such that there exists a vector bundle isometry $\mathcal{T}: L_{0} \rightarrow \hat{L}_{0}$ between vector subbundles $L_{0} \subset N_{f} M$ and $\hat{L}_{0} \subset N_{\hat{f}} M$ satisfying $\mathcal{T} \circ \alpha_{L_{0}}=\hat{\alpha}_{\hat{L}_{0}}$. Let $L_{1}$ be a vector subbundle of $L_{0}$ such that

$$
\left(\hat{\nabla}_{X}^{\perp} \mathcal{T}(\xi)\right)_{\hat{L}_{0}}=\mathcal{T}\left(\nabla_{X}^{\perp} \xi\right)_{L_{0}}
$$

for all $X \in \mathfrak{X}(M)$ and $\xi \in \Gamma\left(L_{1}\right)$. Then the following assertions hold:
(i) The map $\phi: \Gamma\left(f_{*} T M \oplus L_{1}\right) \times \mathfrak{X}(M) \rightarrow \Gamma\left(L_{0}^{\perp} \oplus \hat{L}_{0}^{\perp}\right)$, defined by

$$
\begin{equation*}
\phi\left(f_{*} X+\xi, Y\right)=\left(\left(\tilde{\nabla}_{Y}\left(f_{*} X+\xi\right)\right)_{L_{0}^{\perp}},\left(\tilde{\nabla}_{Y}\left(\hat{f}_{*} X+\mathcal{T}(\xi)\right)_{\hat{L}_{0}^{\perp}}\right)\right. \tag{12.21}
\end{equation*}
$$

gives rise to a section of the vector bundle $\operatorname{Hom}^{2}\left(f_{*} T M \oplus L_{1}, T M ; L_{0}^{\perp} \oplus \hat{L}_{0}^{\perp}\right)$ such that the bilinear form $\phi(x)$ is flat for all $x \in M^{n}$ with respect to the indefinite inner product on $L_{0}^{\perp}(x) \oplus \hat{L}_{0}^{\perp}(x)$ given by

$$
\langle\langle(\xi, \hat{\xi}),(\eta, \hat{\eta})\rangle\rangle_{L_{0}^{\perp}(x) \oplus \hat{L}_{0}^{\perp}(x)}=\langle\xi, \eta\rangle_{L_{0}^{\perp}(x)}-\langle\hat{\xi}, \hat{\eta}\rangle_{\hat{L}_{0}^{\perp}(x)} .
$$

(ii) If $S$ is an isotropic vector subbundle of $f_{*} T M \oplus L_{1}$ with respect to $\phi$ such that $S \cap f_{*} T M=\{0\}$, then the maps $F: S \rightarrow \mathbb{R}^{n+p}$ and $\hat{F}: S \rightarrow \mathbb{R}^{n+q}$, defined by

$$
F(\delta)=f(\pi(\delta))+\delta \text { and } \hat{F}(\delta)=\hat{f}(\pi(\delta))+\hat{\mathfrak{T}}(\delta)
$$

are isometric immersions on a neighborhood of the 0 -section.
Proof: (i) It is easily checked that $\phi$ is $C^{\infty}$-bilinear; hence it gives rise to a section of $\operatorname{Hom}^{2}\left(f_{*} T M \oplus L_{1}, T M ; L_{0}^{\perp} \oplus \hat{L}_{0}^{\perp}\right)$. Let $\hat{\mathfrak{T}}: f_{*} T M \oplus L_{1} \rightarrow \hat{f}_{*} T M \oplus \hat{L}_{0}$ be defined by

$$
\hat{\mathfrak{T}}\left(f_{*} X+\xi\right)=\hat{f}_{*} X+\mathcal{T}(\xi)
$$

By the assumption on $\mathfrak{T}$, we have

$$
\begin{equation*}
\left(\tilde{\nabla}_{X} \hat{\mathcal{T}}(\delta)\right)_{\hat{f}_{*} T M \oplus \hat{L}_{0}}=\hat{\mathcal{T}}\left(\tilde{\nabla}_{X} \delta\right)_{f_{*} T M \oplus L_{0}} \tag{12.22}
\end{equation*}
$$

for all $\delta \in \Gamma\left(f_{*} T M \oplus L_{1}\right)$. From

$$
\langle\tilde{R}(Y, Z) \delta, \zeta\rangle=0=\langle\tilde{R}(Y, Z) \hat{\mathcal{T}}(\delta), \hat{\mathcal{T}}(\zeta)\rangle
$$

we obtain

$$
\left\langle\tilde{\nabla}_{Y} \delta, \tilde{\nabla}_{Z} \zeta\right\rangle-\left\langle\tilde{\nabla}_{Y} \zeta, \tilde{\nabla}_{Z} \eta\right\rangle=Z\left\langle\tilde{\nabla}_{Y} \delta, \zeta\right\rangle-Y\left\langle\tilde{\nabla}_{Z} \delta, \zeta\right\rangle+\left\langle\tilde{\nabla}_{[Y, Z]} \delta, \zeta\right\rangle
$$

and

$$
\begin{aligned}
\left\langle\tilde{\nabla}_{Y} \hat{\mathcal{T}}(\delta), \tilde{\nabla}_{Z} \hat{\mathcal{T}}(\zeta)\right\rangle- & \left\langle\tilde{\nabla}_{Y} \hat{\mathcal{T}}(\zeta), \tilde{\nabla}_{Z} \hat{\mathcal{T}}(\eta)\right\rangle=\left\langle\tilde{\nabla}_{Y} \hat{\mathcal{T}}(\delta), \hat{\mathcal{T}}(\zeta)\right\rangle-Y\left\langle\tilde{\nabla}_{Z} \hat{\mathcal{T}}(\delta), \hat{\mathcal{T}}(\zeta)\right\rangle \\
& +\left\langle\tilde{\nabla}_{[Y, Z]} \hat{\mathfrak{T}}(\delta), \hat{\mathfrak{T}}(\zeta)\right\rangle
\end{aligned}
$$

for all $Y, Z \in \mathfrak{X}(M)$ and $\delta, \zeta \in \Gamma\left(f_{*} T M \oplus L_{1}\right)$. By (12.22), the right-hand sides of the preceding equations coincide. Hence

$$
\left\langle\tilde{\nabla}_{Y} \delta, \tilde{\nabla}_{Z} \zeta\right\rangle-\left\langle\tilde{\nabla}_{Y} \zeta, \tilde{\nabla}_{Z} \eta\right\rangle=\left\langle\tilde{\nabla}_{Y} \hat{\mathcal{T}}(\delta), \tilde{\nabla}_{Z} \hat{\mathcal{T}}(\zeta)\right\rangle-\left\langle\tilde{\nabla}_{Y} \hat{\mathcal{T}}(\zeta), \tilde{\nabla}_{Z} \hat{\mathcal{T}}(\eta)\right\rangle
$$

Since, again by 12.22),

$$
\begin{aligned}
& \left\langle\left(\tilde{\nabla}_{Y} \delta\right)_{L_{0}},\left(\tilde{\nabla}_{Z} \zeta\right)_{L_{0}}\right\rangle-\left\langle\left(\tilde{\nabla}_{Y} \zeta\right)_{L_{0}},\left(\tilde{\nabla}_{Z} \eta\right)_{L_{0}}\right\rangle= \\
& \quad\left\langle\left(\tilde{\nabla}_{Y} \hat{\mathcal{T}}(\delta)\right)_{\hat{L}_{0}},\left(\tilde{\nabla}_{Z} \hat{\mathcal{T}}(\zeta)\right)_{\hat{L}_{0}}\right\rangle-\left\langle\left(\tilde{\nabla}_{Y} \hat{\mathcal{T}}(\zeta)\right)_{\hat{L}_{0}},\left(\tilde{\nabla}_{Z} \hat{\mathcal{T}}(\eta)\right)_{\hat{L}_{0}}\right\rangle
\end{aligned}
$$

it follows that

$$
\langle\langle\phi(\delta, Y), \phi(\zeta, Z)\rangle\rangle-\langle\langle\phi(\zeta, Y), \phi(\delta, Z)\rangle\rangle=0 .
$$

Thus $\phi(x)$ is flat with respect to $\langle\langle\rangle$,$\rangle for all x \in M^{n}$.
(ii) As in the proof of Proposition 12.5, let $X$ denote both a tangent vector at $\pi(\delta) \in$ $M^{n}$ and its horizontal lift to $T_{\delta} S$. Choose a section of $S$ through $\delta$ along a curve in $M^{n}$ through $\pi(\delta)$ and tangent to $X$, also denoted by $\delta$. Then

$$
\begin{aligned}
\left\|F_{*}(\delta) X\right\|^{2} & =\left\|f_{*} X+\left(\tilde{\nabla}_{X} \delta\right)_{f_{*} T M \oplus L}\right\|^{2}+\left\|\left(\tilde{\nabla}_{X} \delta\right)_{L^{\perp}}\right\|^{2} \\
& =\left\|\hat{f}_{*} X+\left(\tilde{\nabla}_{X} \hat{\mathcal{T}}(\delta)\right)_{\hat{f}_{*} T M \oplus \hat{L}}\right\|^{2}+\left\|\left(\tilde{\nabla}_{X} \hat{\mathcal{T}}(\delta)\right)_{\hat{L}^{\perp}}\right\|^{2} \\
& =\left\|\hat{F}_{*}(\delta) X\right\|^{2}
\end{aligned}
$$

where in the second equality we have used 12.22 and the flatness of $\phi$. If $V$ is a vertical vector at $\delta \in S$, then the equality

$$
\left\|F_{*}(\delta) V\right\|=\left\|\hat{F}_{*}(\delta) V\right\|
$$

holds trivially. Since $S \cap f_{*} T M=\{0\}$, then $F$ and $\hat{F}$ induce the same metric on a neighborhood of the 0 -section.

Theorem 12.10. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ form a genuine pair of isometric immersions. Along each connected component of an open dense subset of $M^{n}$, let $\left(\mathcal{T}, D^{d}\right)$ be the pair given by Proposition 12.8. If $\min \{p, q\} \leq 5$ and $p+q<n$, then

$$
\begin{equation*}
d \geq n-p-q+3 \ell \tag{12.23}
\end{equation*}
$$

and the immersions $f$ and $\hat{f}$ are mutually $D^{d}$-ruled.
Proof: With notations as in the previous section, we successively obtain estimates of the ranks of $D_{0}, D_{1}$ and $D$.
Step 1. If $d_{0}=\operatorname{rank} D_{0}$ and $\ell_{j}=\operatorname{rank} L_{j}, j=0,1$, then

$$
\begin{equation*}
d_{0} \geq n-p-q+2 \ell_{0}+\ell_{1} . \tag{12.24}
\end{equation*}
$$

Define $\sigma: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \Gamma\left(L_{0}^{\perp} \oplus \hat{L}_{0}^{\perp}\right)$ by

$$
\sigma(X, Y)=\left(\alpha_{L_{0}^{\perp}}(X, Y), \hat{\alpha}_{\hat{L}_{0}^{\perp}}(X, Y)\right),
$$

also seen as a section of $\operatorname{Hom}^{2}\left(T M, T M ; L_{0}^{\perp} \oplus \hat{L}_{0}^{\perp}\right)$. Given $Y \in \mathfrak{X}(M)$, denote

$$
C_{Y}=\phi(, Y): \Gamma\left(f_{*} T M \oplus L_{1}\right) \rightarrow \Gamma\left(L_{0}^{\perp} \oplus \hat{L}_{0}^{\perp}\right)
$$

where $\phi$ is given by (12.21), and

$$
G_{Y}=\sigma(, Y): \mathfrak{X}(M) \rightarrow \Gamma\left(L_{0}^{\perp} \oplus \hat{L}_{0}^{\perp}\right) .
$$

Take $Y \in R E(\phi)$, that is, $Y(x) \in R E(\phi(x))$ for all $x \in M^{n}$, and define a vector subbundle of $f_{*} T M \oplus L_{1}$ by $H=\operatorname{ker} C_{Y}$. By Proposition 4.6,

$$
\phi(x)(V, Z) \subset \operatorname{Im} \phi_{Y}(x) \cap\left(\operatorname{Im} \phi_{Y}(x)\right)^{\perp}
$$

for all $V \in H(x)$ and $Z \in T_{x} M$, that is, the subbundle $H$ is isotropic with respect to $\phi$. Since $f$ and $\hat{f}$ form a genuine pair, it follows from part (ii) of Lemma 12.9 that $H$ must be a tangent subbundle. Therefore

$$
\begin{equation*}
\operatorname{ker} C_{Y}=\operatorname{ker} G_{Y}, \tag{12.25}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\operatorname{rank} \operatorname{Im} C_{Y}=\operatorname{rank} \operatorname{Im} G_{Y}+\ell_{1} . \tag{12.26}
\end{equation*}
$$

Since $\phi(x)$ is a flat bilinear form for all $x \in M^{n}$ by part ( $i$ ) of Lemma 12.9 , the fibers of the subbundles $\operatorname{Im} C_{Y}$ and $\mathcal{S}\left(\left.\sigma\right|_{H \times T M}\right)$ of $L_{0}^{\perp} \oplus \hat{L}_{0}^{\perp}$ are orthogonal subspaces. Thus

$$
\begin{equation*}
\operatorname{rank} \operatorname{Im} C_{Y}+\operatorname{rank} \mathcal{S}\left(\left.\sigma\right|_{H \times T M}\right) \leq p+q-2 \ell_{0} . \tag{12.27}
\end{equation*}
$$

We have seen that $\beta=\alpha_{\Gamma} \oplus \hat{\alpha}_{\hat{\Gamma}}$ is flat and that $\mathcal{S}(\beta)$ is nondegenerate. Denote

$$
B_{Y}=\beta(, Y): T M \rightarrow \Gamma \oplus \hat{\Gamma}
$$

Take $Y \in R E(\phi) \cap R E(\beta)$ and set $K=\operatorname{ker} B_{Y}$. From Lemma 4.25 we obtain

$$
\begin{equation*}
d_{0} \geq n-\operatorname{rank} \operatorname{Im} B_{Y}-\operatorname{rank} \mathcal{S}\left(\left.\beta\right|_{K \times T M}\right) \tag{12.28}
\end{equation*}
$$

It follows using (12.25) that

$$
\begin{equation*}
\operatorname{rank} \operatorname{Im} G_{Y}+\operatorname{rank} H=\operatorname{rank} \operatorname{Im} B_{Y}+\operatorname{rank} K \tag{12.29}
\end{equation*}
$$

From (12.26) to 12.29 we obtain
$d_{0} \geq n-p-q+2 \ell_{0}+\ell_{1}+\operatorname{rank} \mathcal{S}\left(\left.\sigma\right|_{H \times T M}\right)-\operatorname{rank} \mathcal{S}\left(\left.\beta\right|_{K \times T M}\right)+\operatorname{rank} K-\operatorname{rank} H$.
Clearly $H \subset K$. Thus (12.24) has been proved unless $H \neq K$ and

$$
\begin{equation*}
\operatorname{rank} \mathcal{S}\left(\left.\sigma\right|_{H \times T M}\right)-\operatorname{rank} \mathcal{S}\left(\left.\beta\right|_{K \times T M}\right)+\operatorname{rank} K-\operatorname{rank} H<0 . \tag{12.30}
\end{equation*}
$$

Assume the latter situation. We may also assume that $\left.\sigma\right|_{H \times T M} \neq 0$. Otherwise $H=$ $\mathcal{N}(\sigma)=D_{0}$, and we obtain (12.24) from (12.26) and 12.27). Hence 12.30) gives

$$
\begin{equation*}
\operatorname{rank} \mathcal{S}\left(\left.\beta\right|_{K \times T M}\right)>\operatorname{rank} \mathcal{S}\left(\left.\sigma\right|_{H \times T M}\right)+\operatorname{rank} K-\operatorname{rank} H \geq 2 . \tag{12.31}
\end{equation*}
$$

Set $\sigma=\beta \oplus \beta_{0}$, where necessarily $\beta_{0} \neq 0$. Thus $\ell_{0}<\operatorname{rank} \Sigma$ where $\Sigma$ is given by (12.10). Hence

$$
\operatorname{rank} \Gamma<p-\ell_{0} \leq p-\ell_{1} \text { and } \operatorname{rank} \hat{\Gamma}<q-\ell_{0} \leq q-\ell_{1} .
$$

By Proposition 4.6, $K$ is an isotropic subbundle of $T M$ with respect to $\beta$. Therefore

$$
\begin{equation*}
\operatorname{rank} \mathcal{S}\left(\left.\beta\right|_{K \times T M}\right)<\min \{p, q\}-\ell_{1} \tag{12.32}
\end{equation*}
$$

From (12.31), 12.32) and $p \leq 5$ we see that $\ell_{1} \leq 1$. Lemma 4.20 now yields

$$
\begin{aligned}
d_{0} & \geq n-\operatorname{rank} \mathcal{S}(\beta) \\
& \geq n-p-q+2 \ell_{0}+2 \\
& \geq n-p-q+2 \ell_{0}+\ell_{1},
\end{aligned}
$$

and that is (12.24).
Remark 12.11. Lemma 4.20 gives the estimate $d_{0} \geq n-p-q+2 \ell_{0}$. The stronger estimate (12.24) required the use of the more elaborate flat bilinear form $\phi$ and the assumption that $f$ and $\hat{f}$ form a genuine pair.

Step 2. If $d_{1}=\operatorname{rank} D_{1}$, then

$$
\begin{equation*}
d_{1} \geq d_{0}-\ell_{0}+\ell_{1} . \tag{12.33}
\end{equation*}
$$

For $R$ defined by (12.19), set rank $R=r=\ell_{0}-\ell_{1}$. We need to argue for $1 \leq r \leq 5$. Notice that $R=\mathcal{S}(\gamma)$, where

$$
\gamma=\left.\alpha_{R}\right|_{T M \times D_{0}} .
$$

Take $Z \in R E(\gamma) \subset D_{0}$ and set $m=\operatorname{rank} V_{Z}$, where $V_{Z}=\gamma(T M, Z)$. We show that

$$
\begin{equation*}
1 \leq m \leq[r / 2], \tag{12.34}
\end{equation*}
$$

where [ ] denotes the entire part function. Let $k_{0}$ be the minimal number of elements $Z_{1}, \ldots, Z_{k_{0}} \in R E(\gamma)$ such that

$$
\begin{equation*}
R=\mathcal{S}\left(\left.\gamma\right|_{T M \times \operatorname{Span}\left\{Z_{1}, \ldots, Z_{k_{0}}\right\}}\right)=\sum_{j=1}^{k_{0}} V_{Z_{j}} . \tag{12.35}
\end{equation*}
$$

Clearly, $r \geq m+k_{0}-1$. Suppose $m>[r / 2]$. Then

$$
k_{0} \leq[(r+1) / 2] .
$$

Since $r \leq 5$, it is easy to see that

$$
U=\cap_{j=1}^{k_{0}} V_{Z_{j}} \neq 0 .
$$

Part (iii) of Proposition 12.6 yields

$$
\mathcal{K}(X) V_{Z} \subset V_{Z}^{\perp} \subset R
$$

It follows using 12.35 that

$$
\mathcal{K}(X) U \subset \cap_{j=1}^{k_{0}} V_{Z_{j}}^{\perp}=0 .
$$

Thus $U \subset R \cap L_{1}=0$, and this is a contradiction that proves 12.34.
Set

$$
T(X)=\gamma(X,): D_{0} \rightarrow R
$$

That $D_{1}=\mathcal{N}(\gamma)$ is equivalent to

$$
\begin{equation*}
D_{1}=\cap_{X \in T M} \operatorname{ker} T(X) \tag{12.36}
\end{equation*}
$$

Fix $Z \in R E(\gamma)$. Then (4.8) gives

$$
R=\sum_{i=1}^{m} \operatorname{Im} T\left(X_{i}\right)
$$

where $X_{1}, \ldots, X_{m} \in T M$ are such that

$$
V_{Z}=\operatorname{span}\left\{\gamma\left(X_{i}, Z\right), 1 \leq i \leq m\right\}
$$

Let $\left\{Y_{1}, \ldots, Y_{m_{0}}\right\} \subset\left\{X_{1}, \ldots, X_{m}\right\}$ be a subset with the minimum number of elements satisfying

$$
\begin{equation*}
R=\sum_{j=1}^{m_{0}} \operatorname{Im} T\left(Y_{j}\right) \tag{12.37}
\end{equation*}
$$

By part (ii) of Proposition 12.6.

$$
\begin{equation*}
\mathcal{K}(X) \gamma\left(Y_{j}, Z\right)=\mathcal{K}\left(Y_{j}\right) \gamma(X, Z) \tag{12.38}
\end{equation*}
$$

for any $Z \in D_{0}$. From (12.20), 12.37) and 12.38) we obtain

$$
R=\sum_{j=1}^{m_{0}} \operatorname{Im} \mathcal{K}\left(Y_{j}\right)
$$

which is equivalent to

$$
\begin{equation*}
\cap_{1 \leq j \leq m_{0}} \operatorname{ker} \mathcal{K}\left(Y_{j}\right)=0 \tag{12.39}
\end{equation*}
$$

Using (12.38) we see that

$$
Z \in \cap_{j=1}^{m_{0}} \operatorname{ker} T\left(Y_{j}\right) \text { if and only if } \mathcal{K}\left(Y_{j}\right) \gamma(X, Z)=0
$$

for $1 \leq j \leq m_{0}$. It now follows from (12.39) that 12.36 can be replaced by

$$
\begin{equation*}
D_{1}=\cap_{1 \leq j \leq m_{0}} \operatorname{ker} T\left(Y_{j}\right) \tag{12.40}
\end{equation*}
$$

Suppose a nonsingular $\mathcal{K}(Y)$ exists. By (12.39), this happens if $m_{0}=1$. We have

$$
\begin{equation*}
\mathcal{K}(Y) \gamma(X, \operatorname{ker} T(Y))=\mathcal{K}(X) \gamma(Y, \operatorname{ker} T(Y))=0 . \tag{12.41}
\end{equation*}
$$

Thus $\operatorname{ker} T(Y) \subset D_{1}$. Hence

$$
d_{1} \geq \operatorname{rank} \operatorname{ker} T(Y) \geq d_{0}-r
$$

and 12.33 holds.
Since $[r / 2] \geq m \geq m_{0}$, to conclude the proof it suffices to argue for the case in which $m_{0}=2, r \geq 4$ and all the $\mathcal{K}(Y)$ are singular. After taking linear combinations, if necessary, we see from (12.39) that there are $Y_{1}, Y_{2}$ such that rank $\mathcal{K}\left(Y_{j}\right)=4$. Hence $r=5$. Moreover,

$$
\begin{equation*}
\operatorname{rank} \operatorname{ker} T\left(Y_{1}\right) \geq d_{0}-r+1 \tag{12.42}
\end{equation*}
$$

since $\operatorname{Im} T\left(Y_{1}\right) \neq R$ by (12.37). From (12.41) we have

$$
\mathcal{K}\left(Y_{1}\right) \gamma\left(Y_{2}, \operatorname{ker} T\left(Y_{1}\right)\right)=0,
$$

and 12.33 follows from (12.40) and 12.42 .
Step 3. We first give a characterization of $D^{d}$ and then prove the estimate 12.23 .
If $L^{\ell}=L_{0}$ then 12.16 gives $D^{d}=D_{1}=D_{0}$, and $\sqrt{12.23}$ ) follows from $(12.24)$. Thus we may assume that $\left.\alpha_{L^{\perp} \cap L_{0}}\right|_{D_{0} \times T M}$ does not vanish. By Proposition 4.8, there exists a subbundle of minimal rank $V_{0}^{k} \subset T M$, with $1 \leq \rho \leq \ell_{0}-\ell-k-1$, such that

$$
\begin{equation*}
\mathcal{S}\left(\left.\alpha_{L^{\perp} \cap L_{0}}\right|_{D_{0} \times V_{0}}\right)=L^{\perp} \cap L_{0} \tag{12.43}
\end{equation*}
$$

and such that the subbundle $G$ defined by

$$
G=\mathcal{N}\left(\left.\alpha_{L^{\perp} \cap L_{0}}\right|_{D_{0} \times V_{0}}\right)
$$

satisfies

$$
\begin{equation*}
\operatorname{rank} G \geq d_{0}-k(\rho-1)-1 \tag{12.44}
\end{equation*}
$$

Hence

$$
\begin{equation*}
1 \leq k \leq \ell_{0}-\ell-\rho+1 \tag{12.45}
\end{equation*}
$$

We now prove that

$$
\begin{equation*}
D^{d}=G \cap D_{1} \tag{12.46}
\end{equation*}
$$

The inclusion $D^{d} \subset G$ is clear. For the opposite one, first observe that the left-hand side of (12.17) vanishes if $Z \in G$ and $X \in V_{0}$. It follows from (12.43) and the definition of $L^{\ell}$ that $L_{0}$ is parallel along $G$ in the normal connections for both immersions. Now (12.17) yields

$$
\nabla_{Y}^{\perp} \alpha_{L^{\perp} \cap L_{0}}(Z, X) \in \Gamma\left(L_{0}\right)
$$

for all $Y \in \Gamma\left(D_{0}\right), Z \in \Gamma(G)$ and $X \in \mathfrak{X}(M)$, and similarly for $\hat{f}$. But

$$
\alpha_{L^{\perp} \cap L_{0}}(Z, X) \in \Gamma\left(L_{1}\right)
$$

since $Z \in \Gamma\left(D_{1}\right)$. Thus, by the definition of $L^{\ell}$ we have

$$
\alpha_{L^{\perp} \cap L_{0}}(Z, X)=0
$$

for all $Z \in \Gamma(G)$ and $X \in \mathfrak{X}(M)$, that is, $G \subset D^{d}$, and 12.46 has been proved.
We show next that

$$
\begin{equation*}
d \geq n-p-q+2 \ell+\ell_{1} . \tag{12.47}
\end{equation*}
$$

From (12.46) we have

$$
d_{0} \geq \operatorname{rank} G+d_{1}-d
$$

hence (12.44) gives

$$
\begin{equation*}
d \geq d_{1}-k(\rho-1)-1 \tag{12.48}
\end{equation*}
$$

It follows from the definition of $D_{1}$ that

$$
D^{d}=\mathcal{N}\left(\left.\alpha_{L^{\perp} \cap L_{1}}\right|_{D_{1} \times V_{0}}\right) .
$$

Proposition 4.8 yields

$$
d \geq d_{1}-k_{1}\left(\rho_{1}-1\right)-1
$$

with $1 \leq k_{1} \leq k$ and $\rho_{1} \leq \ell_{1}-\ell$. Thus

$$
\begin{equation*}
d \geq d_{1}-k\left(\ell_{1}-\ell-1\right)-1 . \tag{12.49}
\end{equation*}
$$

From (12.48) and (12.49) we obtain

$$
d \geq d_{1}-\min \left\{k(\rho-1)+1, k\left(\ell_{1}-\ell\right)\right\} .
$$

It follows from (12.24) and (12.33) that

$$
d \geq n-p-q+\ell_{0}+2 \ell_{1}-\min \left\{k(\rho-1)+1, k\left(\ell_{1}-\ell\right)\right\} .
$$

To prove (12.47), we have to verify that

$$
\begin{equation*}
\min \left\{k(\rho-1)+1, k\left(\ell_{1}-\ell\right)\right\} \leq \ell_{0}+\ell_{1}-2 \ell, \tag{12.50}
\end{equation*}
$$

where $1 \leq k \leq \ell_{0}-\ell-\rho+1$ from (12.45) and $1 \leq \rho \leq \ell_{0}-\ell \leq 5$.
Observe that 12.50) holds unless

$$
\begin{equation*}
k(\rho-1) \geq \ell_{0}+\ell_{1}-2 \ell \text { and }(k-2)\left(\ell_{1}-\ell\right) \geq \ell_{0}-\ell_{1}+1 \tag{12.51}
\end{equation*}
$$

Thus, for 12.50) to fail me must have $3 \leq k \leq 5, \rho \geq 2$ and $\ell_{1}-\ell \geq 1$. Hence

$$
2 \leq \rho \leq \ell_{0}-\ell-k+1 \leq 6-k .
$$

Therefore, it remains to analyze the cases in which $(k, \rho)$ is either $(4,2)$ or $(3,3)$ or $(3,2)$. In the first two cases $\ell_{0}=5$ and $\ell=0$, whereas in the third case $\ell_{0}-\ell \geq 4$. But this is in contradiction with 12.51. The estimate on $d$ now follows from 12.47, since $\ell_{1} \geq \ell$.

To conclude the proof observe that, if $f$ and $\hat{f}$ were not $D^{d}$-ruled, by Proposition 12.5 they would admit nontrivial simultaneous $R$-ruled isometric extensions, in contradiction with the fact that they form a genuine pair.

Remarks 12.12. (i) That $\ell=0$ just means that the foliation $D^{d}$ is of relative nullity for both immersions.
(ii) Notice that it is not assumed that the second fundamental form spans the full normal space as it is usually required for rigidity results. See Exercise 12.11.

Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ be $D^{d}$-ruled. At $x \in M^{n}$, consider the orthogonal splitting

$$
N_{f} M(x)=L_{D}(x) \oplus L_{D}^{\perp}(x)
$$

where

$$
L_{D}(x)=\operatorname{span}\left\{\alpha(Z, X): Z \in D^{d}(x) \text { and } X \in T_{x} M\right\}
$$

Assume that $\ell_{D}=$ rank $L_{D}$ is constant, and hence that $L_{D}$ is a smooth normal subbundle. If the isometric immersion $\hat{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ is also $D^{d}$-ruled, it follows from the Gauss equation that there is a unique vector bundle isometry $\mathcal{T}_{D}: L_{D} \rightarrow \hat{L}_{D}$ such that

$$
\left.\hat{\alpha}\right|_{D \times T M}=\left.\mathcal{T}_{D} \circ \alpha\right|_{D \times T M} .
$$

By Proposition 12.1, the pair ( $\mathcal{T}, D)$ satisfies condition $\left(\mathcal{C}_{2}\right)$. However, in general condition $\left(\mathcal{C}_{1}\right)$ does not have to be satisfied. The next result implies that this is indeed the case if the pair $\{f, \hat{f}\}$ is genuine, $p \leq 5$ and $p+q<n$.

Corollary 12.13. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+q}$ form a genuine pair of isometric immersions with $p \leq 5$ and $p+q<n$. Then, along each connected component of an open dense subset of $M^{n}$, the isometric immersions $f$ and $\hat{f}$ are mutually $D^{d}$ ruled with $d \geq n-p-q+3 \ell_{D}$ and the pair ( $\left.\mathcal{T}, D\right)$ satisfies conditions (12.9).

Proof: The proof follows from Theorem 12.10 since $L_{D} \subset L^{\ell}$ by the definition of $L_{D}$.

Remark 12.14. For some additional information in regard to the above result, see Exercises 12.8 and 12.9.

The estimate given by Theorem 12.10 is sharp, as shown by the following example.
Example 12.15. As discussed in Chapter 5, there are plenty of genuine deformations

$$
g_{n}: U \subset \mathbb{S}^{n} \rightarrow \mathbb{R}^{2 n-1}
$$

of the standard inclusion $i: U \rightarrow \mathbb{R}^{n+1}$ of an open subset $U \subset \mathbb{S}^{n}$. If

$$
g_{n_{j}}: U_{j} \subset \mathbb{S}^{n_{j}} \rightarrow \mathbb{R}^{2 n_{j}-1}
$$

is such an isometric immersion for $1 \leq j \leq r$, then the product immersion

$$
g=g_{n_{1}} \times \cdots \times g_{n_{r}} \times \mathrm{id}: U_{1} \times \cdots \times U_{r} \times \mathbb{R}^{k} \rightarrow \mathbb{R}^{n+q}
$$

is a genuine deformation of the product $i_{n_{1}} \times \cdots \times i_{n_{r}} \times$ id of inclusions with the identity map for which the equality holds in 12.23 .

The next example is for codimensions $p=q=2$. The estimate 12.23 is also sharp, but the rulings no longer come from a factor. The example also shows that the notion of isometric extensions is indeed a matter for pairs. In fact, one may have different extensions $F$ for different isometric deformations $\hat{f}$ of $f$. Moreover, it may happen that a given pair $\{f, \hat{f}\}$ extends isometrically but there exists another isometric deformation of $f$ which is genuine.

Example 12.16. Let

$$
i: M^{n}=N_{1}^{n+1} \cap N_{2}^{n+1} \hookrightarrow \mathbb{R}^{n+2}
$$

be the transversal intersection of distinct Sbrana-Cartan hypersurfaces

$$
f_{j}: N_{j}^{n+1} \rightarrow \mathbb{R}^{n+2}, \quad 1 \leq j \leq 2 .
$$

Assume that the relative nullity leaves of the hypersurfaces are transversal along every point of $M^{n}$. Hence, the index of relative nullity of $i$ is $n-4$ everywhere. Now consider two additional isometric immersions $g_{j}: M^{n} \rightarrow \mathbb{R}^{n+2}$ determined by isometric deformations $\hat{f}_{j}$ of $f_{j}, j=1,2$. It is not difficult to prove that the isometric extension of the pairs of immersions $i$ and $g_{j}$ of $M^{n}$ recreates $f_{j}$ and $\hat{f}_{j}$. Thus the extension of $i$ depends on $g_{j}$. Moreover, the immersion $g_{2}$ must be a genuine deformation of $g_{1}$ since, otherwise, their second fundamental forms would have to coincide on a normal subbundle, and that is not the case.

Example 12.17. The complete complex ruled real Kaehler submanifolds in codimension two and rank four in Euclidean space given in [114] have locally an associated one-parameter family of genuine deformations. These submanifolds are ruled and the rulings have codimension two.

### 12.4 Genuine rigidity

Theorem 12.10 has several strong consequences of both local and global nature in terms of the concept of genuine rigidity. First, one has the following basic sufficient condition for genuine rigidity.

Theorem 12.18. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ be an isometric immersion and let $q$ be $a$ positive integer such that $p+q<n$. If $\min \{p, q\}<5$ and $f$ is not $(n-p-q)$-ruled on any open subset of $M^{n}$, then $f$ is genuinely rigid in $\mathbb{Q}_{c}^{n+q}$.

Proof: Immediate from Theorem 12.10 .
The next simple applications of Theorem 12.10 provide further sufficient conditions for genuine rigidity.

Theorem 12.19. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ be an isometric immersion of a compact manifold and let $q$ be a positive integer such that $p+q<n$. If $\min \{p, q\}<5$, then there exists an open subset $U \subset M^{n}$ such that $\left.f\right|_{U}$ is genuinely rigid in $\mathbb{R}^{n+q}$.

Proof: By Corollary 1.6, there exist an open subset $U \subset M^{n}$ and a smooth unit normal vector field $\xi$ along $U$ such that $A_{\xi}$ is definite. Therefore $f$ cannot be ruled on any open subset of $U$.

Theorem 12.20. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ be an isometric immersion and let $q$ be a positive integer such that $p+q<n$. If $\min \{p, q\}<5$ and Ric $_{M}>c$ then $f$ is genuinely rigid in $\mathbb{R}^{n+q}$.

Proof: By Exercise 3.14, the isometric immersion $f$ cannot be ruled on any open subset by the assumption on the Ricci curvature.

Theorem 12.21. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p+1}$ be an isometric immersion and let $q$ be $a$ positive integer such that $p+q+1<n$ and $\min \{p+1, q\}<5$. If $f(M)$ is contained in an umbilical hypersurface $\mathbb{Q}_{\tilde{c}}^{n+p}$ of $\mathbb{Q}_{c}^{n+p+1}, \tilde{c}>c$, then $f$ is genuinely rigid in $\mathbb{Q}_{c}^{n+q}$.

Proof: The isometric immersion $f$ cannot be ruled on any open subset.
Either of the two previous theorems has the following immediate consequence.
Corollary 12.22. The umbilical inclusion $i: U \rightarrow \mathbb{Q}_{c}^{n+1}$ of an open subset $U \subset \mathbb{Q}_{\tilde{c}}^{n}$, $\tilde{c}>c$, is genuinely rigid in $\mathbb{Q}_{c}^{n+p}$ if $p \leq n-2$.

The preceding result was shown to be false for $p=n-1$ in Chapter 5. In particular, this implies that the bound for $q$ in Theorems 12.20 and 12.21 is sharp.

In the following and final result of this section, the integer $\ell$ stands for the rank of the subbundle $L$ constructed in Proposition 12.8 .

Theorem 12.23. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+q}$ form a genuine pair of isometric immersions with $p \leq 5$ and $p+q<n$. If the extrinsic curvature of $M^{n}$ is nonpositive at any point, then $f$ and $\hat{f}$ have common relative nullity subspaces of dimension $\nu \geq n-p-q+2 \ell$.

Proof: It was shown in Proposition 6.14 that if $M^{n}$ has nonpositive extrinsic curvature and $f$ has an asymptotic subspace $D^{d}$, then $\nu \geq d-s$, where $s=\operatorname{rank} \gamma(D, Y)$ for $\gamma=\left.\alpha\right|_{D \times D^{\perp}}$ and $Y \in R E(\gamma)$. By Theorem 12.10 we have $\mathcal{S}(\gamma) \subset L^{\ell}$, and we obtain the estimate for the relative nullity from the one for $d$. Observe that the relative nullity subspace contained in $D^{d}$ must be shared by $\hat{f}$ since it coincides with the set of vectors in $D^{d}$ that belong to the nullity subspace of the curvature tensor of $M^{n}$.

### 12.5 Compositions

The main result of this section gives sufficient conditions on an isometric immersion $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ which assure that any isometric immersion $\hat{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+q}$ with $q \geq p$ must be, locally on an open dense subset of $M^{n}$, a composition of $f$ with an
isometric immersion $h: U \rightarrow \mathbb{Q}_{c}^{n+q}$ of an open subset $U \subset \mathbb{Q}_{c}^{n+p}$ containing $f(M)$. In particular, for $p=q$ it recovers the rigidity Theorem 4.23.

The following general result combines Theorem 12.23 with an assumption on the $s$-nullities for large $s$ including the relative nullity index.

Proposition 12.24. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}, p \leq 5$, be an isometric immersion and let $q \geq p$ be a positive integer. For an integer $1 \leq k \leq p$, assume that

$$
\begin{equation*}
\nu_{s}^{f} \leq n+p-q-2 s-1 \text { for all } k \leq s \leq p \tag{12.52}
\end{equation*}
$$

at any point of $M^{n}$. If $\hat{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+q}$ is an isometric immersion, then there exists an open dense subset of $M^{n}$ along each connected component of which the immersions $f$ and $\hat{f}$ have (possibly trivial) isometric $R$-ruled extensions $F: N^{n+r} \rightarrow \mathbb{Q}_{c}^{n+p}$ and $\hat{F}: N^{n+r} \rightarrow \mathbb{Q}_{c}^{n+q}$ satisfying

$$
\operatorname{rank} R \geq n+2 p-q-3(k-1) \geq 4+p-k
$$

and the conclusions of Proposition 12.5.
Proof: Theorem 12.10 applies and gives (possibly trivial) isometric $R$-ruled local extensions $F$ and $F$ of maximal dimension defined on a manifold $N^{n+r}$ with $0 \leq r \leq p$. By Theorem 12.23 , there exists a vector bundle isometry $\mathcal{T}: L^{\ell} \rightarrow \hat{L}^{\ell}$ between vector subbundles $L^{\ell} \subset N_{F} N$ and $\hat{L}^{\ell} \subset N_{\hat{F}} N$ of rank $0 \leq \ell \leq p-r$ such that

$$
\alpha_{\hat{L}^{\ell}}^{\hat{F}}=\mathcal{T} \circ \alpha_{L^{\ell}}^{F}
$$

and

$$
\operatorname{rank} R \geq n-p-q+3(r+\ell)
$$

Thus $D=R \cap T M$ satisfies

$$
\operatorname{rank} D \geq \operatorname{rank} R-r \geq n-p-q+2 r+3 \ell .
$$

Therefore

$$
\begin{equation*}
\nu_{p-r-\ell}^{f} \geq n-p-q+2 r+3 \ell \tag{12.53}
\end{equation*}
$$

if $p-r-\ell \geq 1$. By 12.52, this is not possible if $k \leq p-r-\ell$. Thus $r+\ell \geq p-k+1$, and we obtain

$$
\operatorname{rank} R \geq n+2 p-q-3(k-1)
$$

To conclude the proof we use that $n+p-q-2 k-1 \geq 0$ from (12.52) for $s=k$.
Remark 12.25. In the above result, as well as in the following one, one must have $n \geq p+q+1$ for the inequality (12.52) to hold for $s=p$.

We can now state and prove the result on compositions referred to in the beginning of this section.

Theorem 12.26. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ with $p \leq 5$ be an isometric immersion and let $q \geq p$ be a positive integer. At any point of $M^{n}$, assume that

$$
\begin{equation*}
\nu_{s}^{f} \leq n+p-q-2 s-1 \text { for all } 1 \leq s \leq p \tag{12.54}
\end{equation*}
$$

For $q-p \geq 5$, replace the assumption for $s=1$ by $\nu_{1}^{f} \leq n-2(q-p)+1$. Then any isometric immersion $\hat{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+q}$ is a composition along each connected component of an open dense subset of $M^{n}$.

Proof: All we have to prove is that $r=p$. As in the proof of the last result, we see that (12.53) holds if $\ell<p-r$. Since this is in contradiction with 12.54, we conclude that $\ell=p-r$. Hence $\hat{\alpha}^{F}=\mathcal{T} \circ \alpha^{F}$ and rank $R \geq n+2 p-q$.

Suppose that $r<p$. For any normal vector field $\eta \neq 0$ to $F$, the subspace $D$ is asymptotic for $A_{\eta}$ along $M^{n}$, and thus

$$
\operatorname{rank} A_{\eta} \geq 2(n-\operatorname{rank} D)
$$

From rank $D \geq n-(q-p)+1$ we obtain $\nu_{1}^{f} \geq n-2(q-p)+2$, a contradiction.
The following special case of the preceding result extends Corollary 12.22 .
Corollary 12.27. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ and $g: M^{n} \rightarrow \mathbb{Q}_{c}^{n+q}, q \geq 2$, be isometric immersions. Assume that the rank of $f$ satisfies $\rho \geq q+2$ at any point of $M^{n}$. If $q \geq 6$, assume further that $M^{n}$ does not contain an open $(n-q+2)$-ruled subset for both immersions. Then $g$ is a composition along each connected component of an open dense subset of $M^{n}$.

### 12.6 Submanifolds of two space forms

The concept of a genuine pair of isometric immersions $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+q}$ can be extended to the case in which $f$ and $\hat{f}$ take values in space forms with distinct curvatures.

First, the pair of isometric immersions $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+q}$ is said to extend isometrically if there exist an isometric embedding $j: M^{n} \hookrightarrow N^{n+\ell}$ into a Riemannian manifold $N^{n+\ell}, 0<\ell \leq \min \{p, q\}$, and a pair of isometric immersions $F: N^{n+\ell} \rightarrow \mathbb{Q}_{c}^{n+p}$ and $\hat{F}: N^{n+\ell} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+q}$ such that

$$
f=F \circ j \text { and } \hat{f}=\hat{F} \circ j
$$

The pair $\{f, \hat{f}\}$ is then said to be genuine if there exists no open subset $U \subset M^{n}$ along which the restrictions $\left.f\right|_{U}$ and $\left.\hat{f}\right|_{U}$ extend isometrically.

The following result states that, under some assumptions on the codimensions, genuine pairs of isometric immersions into space forms with distinct curvatures are always produced by means of genuine pairs of isometric immersions into space forms with the same constant sectional curvature.

Theorem 12.28. Let $f: M^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+p}$ and $g: M^{n} \rightarrow \mathbb{Q}_{c}^{n+q}$ form a genuine pair of isometric immersions with $\tilde{c}>c$. Assume that $p+q<n-1$ and $\min \{p+1, q\} \leq 5$. Then there exist, locally on an open and dense subset of $M^{n}$, a Riemannian manifold $N^{n+1}$ that admits a genuine pair of isometric embeddings $F: N^{n+1} \rightarrow \mathbb{Q}_{c}^{n+p+1}$ and $G: N^{n+1} \rightarrow \mathbb{Q}_{c}^{n+q}$, with $F$ transversal to the inclusion $j: \mathbb{Q}_{\tilde{c}}^{n+p} \rightarrow \mathbb{Q}_{c}^{n+p+1}$, and an isometry $\psi: M^{n} \rightarrow L^{n}=F\left(N^{n+1}\right) \cap j\left(\mathbb{Q}_{\tilde{c}}^{n+p}\right)$ such that

$$
f=j^{-1} \circ \psi \text { and } g=G \circ F^{-1} \circ \psi,
$$

where $j^{-1}$ and $F^{-1}$ stand for the inverses of $j$ and $F$, respectively, regarded as maps onto their images.

Proof: Set $\bar{f}=j \circ f$. By Theorem 12.21, the isometric immersion $\bar{f}$ is genuinely rigid in $\mathbb{Q}_{c}^{n+q}$. Therefore, for each connected component $U$ of an open and dense subset of $M^{n}$, there exist a Riemannian manifold $N^{n+\ell}$ and isometric embeddings $F: N^{n+\ell} \rightarrow \mathbb{Q}_{c}^{n+p+1}$, $G: N^{n+\ell} \rightarrow \mathbb{Q}_{c}^{n+q}$ and $k: U \rightarrow N^{n+\ell}$ such that $\left.\bar{f}\right|_{U}=F \circ k$ and $\left.g\right|_{U}=G \circ k$.

If $F$ was not transversal to $j$ along an open subset $W \subset N^{n+\ell}$, there would exist $\bar{F}: W \rightarrow \mathbb{Q}_{\tilde{c}}^{n+p}$ such that $\left.F\right|_{W}=j \circ \bar{F}$. But then on $\tilde{W}=k^{-1}(W)$ we would have $\left.f\right|_{\tilde{W}}=\left.\bar{F} \circ k\right|_{\tilde{W}}$ and $\left.g\right|_{\tilde{W}}=\left.G \circ k\right|_{\tilde{W}}$, contradicting the fact the pair $\{f, g\}$ is genuine.

Hence, we can assume that $F$ is transversal to $j$, so that

$$
L^{n+\ell-1}=F\left(N^{n+\ell}\right) \cap j\left(\mathbb{Q}_{\tilde{c}}^{n+p}\right)
$$

is a manifold. If $\ell>1$, defining $\bar{L}^{n+\ell-1}=F^{-1}\left(L^{n+\ell-1}\right)$ and $\bar{F}=\left.F\right|_{\bar{L}}$, we would have $\bar{F}=j \circ \tilde{F}$ for some isometric embedding $\tilde{F}: \bar{L}^{n+\ell-1} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+p}$. Then $\left.f\right|_{U}=\tilde{F} \circ k$ and $\left.g\right|_{U}=G \circ k$, contradicting again the fact that the pair $\{f, g\}$ is genuine.

Thus $\ell=1$. Moreover, the pair $\{F, G\}$ is genuine. Otherwise, there would exist a Riemannian manifold $N^{n+s}, s \geq 2$, and isometric embeddings $\hat{F}: N^{n+s} \rightarrow \mathbb{Q}_{c}^{n+p+1}$, $\hat{G}: N^{n+s} \rightarrow \mathbb{Q}_{c}^{n+q}$ and $h: N^{n+1} \rightarrow N^{n+s}$ such that $F=\hat{F} \circ h$ and $G=\hat{G} \circ h$. Arguing as in the preceding paragraph, with $F$ and $G$ replaced by $\hat{F}$ and $\hat{G}$, respectively, and with $k$ replaced by $k \circ h$, we would reach again a contradiction with the fact that the pair $\{f, g\}$ is genuine.

Therefore

$$
\psi=\left.j \circ f\right|_{U}: U \rightarrow L^{n}=F\left(N^{n+1}\right) \cap j\left(\mathbb{Q}_{\tilde{c}}^{n+p}\right)
$$

is an isometry, and hence $\left.f\right|_{U}=j^{-1} \circ \psi$ as stated. Moreover, from $\psi=\left.j \circ f\right|_{U}=F \circ k$ it follows that $k=F^{-1} \circ \psi$, hence $\left.g\right|_{U}=G \circ k=G \circ F^{-1} \circ \psi$.

Corollary 12.29. Let $f: M^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{n+p}$ and $g: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ be isometric immersions with $\tilde{c}>c$. Assume that $n \geq 4$ and $p<n-2$. Then there exist, locally on an open and dense subset of $M^{n}$, an isometric embedding $H: U \rightarrow \mathbb{Q}_{c}^{n+p+1}$ of an open subset $U \subset \mathbb{Q}_{c}^{n+1}$, transversal to the inclusion $j: \mathbb{Q}_{\tilde{c}}^{n+p} \rightarrow \mathbb{Q}_{c}^{n+p+1}$, and an isometry $\psi: M^{n} \rightarrow L^{n}=H(U) \cap j\left(\mathbb{Q}_{\tilde{c}}^{n+p}\right)$ such that

$$
f=j^{-1} \circ \psi \text { and } g=H^{-1} \circ \psi,
$$

where $j^{-1}$ and $H^{-1}$ stand for the inverses of $j$ and $H$, respectively, regarded as maps onto their images.

Corollary 12.29 explains why there must exist a principal curvature $\lambda$ of $g$ with multiplicity greater than or equal to $n-p$ and a principal normal $\eta \in N_{f} M(x)$ such that $E_{\eta}=E_{\lambda}$ (see Exercise 4.3). The common eigenspaces arise by the intersections of the relative nullity leaves of $H$, which have dimension at least $n-p+1$, with $j\left(\mathbb{Q}_{\tilde{c}}^{n+p}\right)$.

The next lemma shows what else may happen if $p=n-2$.
Lemma 12.30. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ and $\hat{f}: M^{n} \rightarrow \mathbb{Q}_{\tilde{c}}^{2 n-2}$ be isometric immersions with $\tilde{c}>c$. Then, at each point $x \in M^{n}$, one of the following possibilities holds:
(i) There exist a principal curvature $\lambda$ of $f$ with multiplicity greater than or equal to two and a principal normal $\eta \in N_{\hat{f}} M(x)$ such that $E_{\eta}=E_{\lambda}$.
(ii) There exists an orthonormal basis of $T_{x} M$ that simultaneously diagonalizes the second fundamental forms of $f$ and $\hat{f}$.

Proof: Let $\hat{F}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p+1}$ be given by $\hat{F}=i \circ \hat{f}$, where $i: \mathbb{Q}_{\tilde{c}}^{n+p} \rightarrow \mathbb{Q}_{c}^{n+p+1}$ is an umbilical inclusion. Note that the second fundamental form of $\hat{F}$ at $x \in M^{n}$ is

$$
\begin{equation*}
\alpha^{\hat{F}}(X, Y)=i_{*} \alpha^{\hat{f}}(X, Y)+\sqrt{\tilde{c}-c}\langle X, Y\rangle \zeta \tag{12.55}
\end{equation*}
$$

for all $X, Y \in T_{x} M$, where $\zeta$ is one of the unit vectors normal to $i$ at $\hat{f}(x)$. Define

$$
W=N_{f} M(x) \oplus N_{\hat{F}} M(x)
$$

and endow $W$ with the Lorentzian inner product $\langle\langle\rangle$,$\rangle defined by$

$$
\langle\langle\xi+\nu, \tilde{\xi}+\tilde{\nu}\rangle\rangle=-\langle\xi, \tilde{\xi}\rangle+\langle\nu, \tilde{\nu}\rangle
$$

for all $\xi, \tilde{\xi} \in N_{f} M(x)$ and $\nu, \tilde{\nu} \in N_{\hat{F}} M(x)$. Now define $\beta: T_{x} M \times T_{x} M \rightarrow W$ by

$$
\beta(X, Y)=\alpha^{f}(X, Y) \oplus \alpha^{\hat{F}}(X, Y)
$$

Then $\beta$ is a flat bilinear form and $\mathcal{N}(\beta)=\{0\}$ by 12.55). Assume first that $\mathcal{S}(\beta)$ is degenerate, that is, that there is a nonzero light-like vector $e \in \mathcal{S}(\beta) \cap \mathcal{S}(\beta)^{\perp}$. Thus one can write $\mathcal{S}(\beta)=V \oplus \operatorname{span}\{e\}$ with $V$ space-like, and if $\bar{e} \in V^{\perp}$ is a light-like vector such that $\langle e, \bar{e}\rangle=1$, then the bilinear form

$$
\tilde{\beta}=\beta-\langle\beta, \bar{e}\rangle e
$$

takes values in $V$ and is also flat. From Lemma 4.10 we conclude that

$$
\operatorname{dim} \mathcal{N}(\tilde{\beta}) \geq n-\operatorname{dim} V=2
$$

Decompose $e$ as

$$
e=N+\cos \varphi \zeta+\sin \varphi i_{*} \delta
$$

where $N \in N_{f} M(x)$ and $\delta \in N_{\hat{f}} M(x)$ are unit vectors. Then $\lambda=\sqrt{\tilde{c}-c} / \cos \varphi$ is a principal curvature of $f$ and $\eta=\sqrt{\tilde{c}-c} \tan \varphi \delta$ is a principal normal of $\hat{f}$ at $x$ such that $E_{\lambda}=\mathcal{N}(\tilde{\beta})=E_{\eta}$.

Now suppose that $\mathcal{S}(\beta)$ is nondegenerate. Since $\mathcal{N}(\beta)=\{0\}$, it follows from Lemmas 4.10 and 4.14 that $\mathcal{S}(\beta)=W$. Moreover,

$$
\begin{equation*}
\langle\langle\beta(,), \zeta\rangle\rangle=\sqrt{\tilde{c}-c}\langle,\rangle \tag{12.56}
\end{equation*}
$$

by (12.55); hence $\beta$ satisfies the assumptions of Theorem 5.2. By Theorem 5.2 there exists a basis of $T_{x} M$ that diagonalizes $\beta$, and hence both $\alpha^{f}(x)$ and $\alpha^{\hat{f}}(x)$. By 12.56) such basis is orthogonal.

By Lemma 12.30, if $f: M^{3} \rightarrow \mathbb{Q}_{c}^{4}$ and $\tilde{f}: M^{3} \rightarrow \mathbb{Q}_{\tilde{c}}^{4}$ are isometric immersions with $\tilde{c} \neq c$, then at each point $x \in M^{3}$ either $f$ and $\tilde{f}$ have principal curvatures with multiplicity two with common eigenspaces or there exists an orthonormal basis of $T_{x} M$ that simultaneously diagonalizes the second fundamental forms of $f$ and $\tilde{f}$. The reader is asked to verify that $f$ and $\tilde{f}$ are given by the construction of Corollary 12.29 if the first possibility holds everywhere. The next result characterizes the pairs $\{f, f\}$ in the second and most interesting case.

Theorem 12.31. Let $f: M^{3} \rightarrow \mathbb{Q}_{c}^{4}$ be a simply connected holonomic hypersurface whose associated pair $(v, V)$ satisfies

$$
\begin{equation*}
\langle v, v\rangle=1, \quad\langle v, V\rangle=0 \text { and }\langle V, V\rangle=c-\tilde{c}, \tag{12.57}
\end{equation*}
$$

where $\langle$,$\rangle is an inner product with Lorentzian signature. Then M^{3}$ admits an isometric immersion into $\mathbb{Q}_{\tilde{c}}^{4}$, which is unique up to congruence.

Conversely, if $f: M^{3} \rightarrow \mathbb{Q}_{c}^{4}$ is a hypersurface with three distinct principal curvatures for which there exists an isometric immersion $\tilde{f}: M^{3} \rightarrow \mathbb{Q}_{\tilde{c}}^{4}$ with $\tilde{c} \neq c$, then $f$ is locally a holonomic hypersurface whose associated pair ( $v, V$ ) satisfies 12.57).

Proof: Write

$$
\begin{equation*}
\left\langle\left(x_{1}, x_{2}, x_{3}\right),\left(y_{1}, y_{2}, y_{3}\right)\right\rangle=\sum_{i=1}^{3} \delta_{i} x_{i} y_{i} \tag{12.58}
\end{equation*}
$$

where $\left(\delta_{1}, \delta_{2}, \delta_{3}\right)$ is the signature of $\langle$,$\rangle , and define$

$$
\begin{equation*}
\tilde{V}_{j}=(-1)^{j+1} \delta_{j}\left(v_{i} V_{k}-v_{k} V_{i}\right), \quad 1 \leq i \neq j \neq k \leq 3, \quad i<k \tag{12.59}
\end{equation*}
$$

Thus $\tilde{V}=\left(\tilde{V}_{1}, \tilde{V}_{2}, \tilde{V}_{3}\right)$ is the unique vector in $\mathbb{R}^{3}$, up to sign, such that the vectors $v,|C|^{-1 / 2} V$ and $|C|^{-1 / 2} \tilde{V}$ with $C=c-\tilde{c}$, form an orthonormal basis of $\mathbb{R}^{3}$ with respect to $\langle$,$\rangle . Therefore the matrix$

$$
D=\left(v,|C|^{-1 / 2} V,|C|^{-1 / 2} \tilde{V}\right)
$$

satisfies $D \delta D^{t}=\delta$, where

$$
\delta=\operatorname{diag}(1, C /|C|,-C /|C|)
$$

It follows that

$$
v_{i} v_{j}+C|C|^{-2} V_{i} V_{j}-C|C|^{-2} \tilde{V}_{i} \tilde{V}_{j}=0, \quad 1 \leq i \neq j \leq 3
$$

or equivalently, that

$$
c v_{i} v_{j}+V_{i} V_{j}=\tilde{c} v_{i} v_{j}+\tilde{V}_{i} \tilde{V}_{j}
$$

Substituting the preceding equation into equation (iii) of (1.26) yields

$$
\frac{\partial h_{i j}}{\partial u_{i}}+\frac{\partial h_{j i}}{\partial u_{j}}+h_{k i} h_{k j}+\tilde{V}_{i} \tilde{V}_{j}+\tilde{c} v_{i} v_{j}=0
$$

On the other hand, differentiating

$$
\sum_{i=1}^{3} \delta_{i} v_{i}^{2}=1 \text { and } \sum_{i=1}^{3} \delta_{i} V_{i}^{2}=c-\tilde{c}
$$

and using equations $(i)$ and (iv) of (1.26), respectively, give

$$
\begin{equation*}
\delta_{i} \frac{\partial v_{i}}{\partial u_{i}}+\delta_{j} h_{i j} v_{j}+\delta_{k} h_{i k} v_{k}=0 \tag{12.60}
\end{equation*}
$$

and

$$
\begin{equation*}
\delta_{i} \frac{\partial V_{i}}{\partial u_{i}}+\delta_{j} h_{i j} V_{j}+\delta_{k} h_{i k} V_{k}=0, \quad 1 \leq i \neq j \neq k \neq i \leq 3 . \tag{12.61}
\end{equation*}
$$

Now, differentiating (12.59) and using (12.60) and (12.61) together with equations (i) and (iv) of (1.26), we obtain

$$
\frac{\partial \tilde{V}_{j}}{\partial u_{i}}=h_{i j} \tilde{V}_{i}, \quad 1 \leq i \neq j \leq 3
$$

By Proposition 1.13, there exists an immersion $\tilde{f}: M^{3} \rightarrow \mathbb{Q}_{\tilde{c}}^{4}$ with induced metric

$$
d s^{2}=\sum_{i=1}^{3} v_{i}^{2} d u_{i}^{2}
$$

and second fundamental form

$$
\tilde{\alpha}=\sum_{i=1}^{3} \tilde{V}_{i} v_{i} d u_{i}^{2} .
$$

Thus $M^{3}$ admits an isometric immersion into $\mathbb{Q}_{\tilde{c}}^{4}$.
Conversely, let $f: M^{3} \rightarrow \mathbb{Q}_{c}^{4}$ be a hypersurface for which there exists an isometric immersion $\tilde{f}: M^{3} \rightarrow \mathbb{Q}_{\tilde{c}}^{4}$. By Lemma 12.30 , there exists an orthonormal frame $e_{1}, e_{2}, e_{3}$ of $M^{3}$ of principal directions of both $f$ and $\tilde{f}$. Let $\lambda_{1}, \lambda_{2}, \lambda_{3}$ and $\mu_{1}, \mu_{2}, \mu_{3}$ be the principal curvatures of $f$ and $\tilde{f}$ correspondent to $e_{1}, e_{2}$ and $e_{3}$, respectively. Assume that $\lambda_{1}<\lambda_{2}<\lambda_{3}$, and that the unit normal vector field to $f$ has been chosen so that $\lambda_{1}<0$. The Gauss equations for $f$ and $\tilde{f}$ yield

$$
c+\lambda_{i} \lambda_{j}=\tilde{c}+\mu_{i} \mu_{j}, \quad 1 \leq i \neq j \leq 3
$$

Thus

$$
\begin{equation*}
\mu_{i} \mu_{j}=C+\lambda_{i} \lambda_{j}, \quad 1 \leq i \neq j \leq 3 \tag{12.62}
\end{equation*}
$$

where $C=c-\tilde{c}$. It follows that

$$
\begin{equation*}
\mu_{j}^{2}=\frac{\left(C+\lambda_{j} \lambda_{i}\right)\left(C+\lambda_{j} \lambda_{k}\right)}{C+\lambda_{i} \lambda_{k}}, 1 \leq j \neq i \neq k \neq j \leq 3 \tag{12.63}
\end{equation*}
$$

By (1.19) and $\sqrt{1.20}$, the Codazzi equations for $f$ and $\tilde{f}$ are, respectively,

$$
\begin{align*}
e_{i}\left(\lambda_{j}\right) & =\left(\lambda_{i}-\lambda_{j}\right)\left\langle\nabla_{e_{j}} e_{i}, e_{j}\right\rangle, \quad i \neq j,  \tag{12.64}\\
\left(\lambda_{j}-\lambda_{k}\right)\left\langle\nabla_{e_{i}} e_{j}, e_{k}\right\rangle & =\left(\lambda_{i}-\lambda_{k}\right)\left\langle\nabla_{e_{j}} e_{i}, e_{k}\right\rangle, \quad i \neq j \neq k, \tag{12.65}
\end{align*}
$$

and

$$
\begin{align*}
e_{i}\left(\mu_{j}\right) & =\left(\mu_{i}-\mu_{j}\right)\left\langle\nabla_{e_{j}} e_{i}, e_{j}\right\rangle, \quad i \neq j,  \tag{12.66}\\
\left(\mu_{j}-\mu_{k}\right)\left\langle\nabla_{e_{i}} e_{j}, e_{k}\right\rangle & =\left(\mu_{i}-\mu_{k}\right)\left\langle\nabla_{e_{j}} e_{i}, e_{k}\right\rangle, \quad i \neq j \neq k . \tag{12.67}
\end{align*}
$$

Multiplying (12.67) by $\mu_{j}$ and using (12.63) and (12.65) give

$$
\frac{\left(\lambda_{i}-\lambda_{j}\right)\left(\lambda_{j}-\lambda_{k}\right)}{C+\lambda_{i} \lambda_{k}}\left\langle\nabla_{e_{i}} e_{j}, e_{k}\right\rangle=0, \quad i \neq j \neq k .
$$

Since the principal curvatures $\lambda_{1}, \lambda_{2}$ and $\lambda_{3}$ are distinct, it follows that

$$
\begin{equation*}
\left\langle\nabla_{e_{i}} e_{j}, e_{k}\right\rangle=0, \quad 1 \leq i \neq j \neq k \neq i \leq 3 . \tag{12.68}
\end{equation*}
$$

Computing $2 \mu_{j} e_{i}\left(\mu_{j}\right)$, first by differentiating 12.63) and then by multiplying 12.66 by $2 \mu_{j}$, and using (12.62), 12.63) and (12.64) give

$$
\begin{align*}
\left(C+\lambda_{j} \lambda_{k}\right)\left(\lambda_{k}-\lambda_{j}\right) e_{i}\left(\lambda_{i}\right) & +\left(C+\lambda_{i} \lambda_{k}\right)\left(\lambda_{k}-\lambda_{i}\right) e_{i}\left(\lambda_{j}\right) \\
& +\left(C+\lambda_{i} \lambda_{j}\right)\left(\lambda_{i}-\lambda_{j}\right) e_{i}\left(\lambda_{k}\right)=0 . \tag{12.69}
\end{align*}
$$

Now let $\left\{\omega_{1}, \omega_{2}, \omega_{3}\right\}$ be the dual frame of $\left\{e_{1}, e_{2}, e_{3}\right\}$, and define one-forms $\gamma_{j}$, $1 \leq j \leq 3$, by

$$
\gamma_{j}=\sqrt{\delta_{j} \frac{\left(\lambda_{j}-\lambda_{i}\right)\left(\lambda_{j}-\lambda_{k}\right)}{C+\lambda_{i} \lambda_{k}}} \omega_{j}, \quad 1 \leq j \neq i \neq k \neq j \leq 3,
$$

where $\delta_{j}=y_{j} /\left|y_{j}\right|$ for $y_{j}=\left(\lambda_{j}-\lambda_{i}\right)\left(\lambda_{j}-\lambda_{k}\right) /\left(C+\lambda_{i} \lambda_{k}\right)$. By 12.63), either all the three numbers $C+\lambda_{j} \lambda_{i}, C+\lambda_{j} \lambda_{k}$ and $C+\lambda_{i} \lambda_{k}$ are positive or two of them are negative and the remaining one is positive. Since $\lambda_{1}, \lambda_{2}, \lambda_{3}$ have been chosen so that $\lambda_{1}<\lambda_{2}<\lambda_{3}$ and $\lambda_{1}<0$, we must have $C+\lambda_{1} \lambda_{3}<0$ if $C+\lambda_{1} \lambda_{2}<0$. Hence, there are three possible cases:
(i) $C+\lambda_{i} \lambda_{j}>0,1 \leq i \neq j \leq 3$.
(ii) $C+\lambda_{1} \lambda_{2}<0, C+\lambda_{1} \lambda_{3}<0$ and $C+\lambda_{2} \lambda_{3}>0$.
(iii) $C+\lambda_{1} \lambda_{2}>0, C+\lambda_{1} \lambda_{3}<0$ and $C+\lambda_{2} \lambda_{3}<0$.

Notice that $\left(\delta_{1}, \delta_{2}, \delta_{3}\right)$ equals $(1,-1,1)$ in case $(i),(1,1,-1)$ in case (ii) and $(-1,1,1)$ in case (iii).

We claim that (12.69) are precisely the conditions for the one-forms $\gamma_{j}$ to be closed. To prove this, set

$$
x_{j}=\sqrt{\delta_{j} y_{j}}, \quad 1 \leq j \leq 3
$$

so that $\gamma_{j}=x_{j} \omega_{j}$. It follows from (12.68) that

$$
\begin{aligned}
d \gamma_{j}\left(e_{i}, e_{k}\right) & =e_{i} \gamma_{j}\left(e_{k}\right)-e_{k} \gamma_{j}\left(e_{i}\right)-\gamma_{j}\left(\left[e_{i}, e_{k}\right]\right) \\
& =0
\end{aligned}
$$

On the other hand, using (12.64) we obtain

$$
\begin{aligned}
d \gamma_{j}\left(e_{i}, e_{j}\right) & =e_{i} \gamma_{j}\left(e_{j}\right)-e_{j} \gamma_{j}\left(e_{i}\right)-\gamma_{j}\left(\left[e_{i}, e_{j}\right]\right) \\
& =e_{i}\left(x_{j}\right)+x_{j}\left\langle\nabla_{e_{j}} e_{i}, e_{j}\right\rangle \\
& =e_{i}\left(x_{j}\right)+x_{j} \frac{e_{i}\left(\lambda_{j}\right)}{\lambda_{i}-\lambda_{j}} .
\end{aligned}
$$

Hence, closedness of $\gamma_{j}$ is equivalent to

$$
e_{i}\left(x_{j}\right)=\frac{x_{j}}{\lambda_{j}-\lambda_{i}} e_{i}\left(\lambda_{j}\right), \quad 1 \leq i \neq j \leq 3,
$$

which can also be written as

$$
2\left(\lambda_{j}-\lambda_{k}\right) e_{i}\left(\lambda_{j}\right)=e_{i}\left(y_{j}\right)\left(C+\lambda_{i} \lambda_{k}\right) .
$$

The preceding equation is in turn equivalent to

$$
\begin{aligned}
2\left(\lambda_{j}-\lambda_{k}\right)\left(C+\lambda_{i} \lambda_{k}\right) e_{i}\left(\lambda_{j}\right)= & \left(e_{i}\left(\lambda_{j}\right)-e_{i}\left(\lambda_{i}\right)\left(\lambda_{j}-\lambda_{k}\right)\left(C+\lambda_{i} \lambda_{k}\right)\right. \\
& +\left(\lambda_{j}-\lambda_{i}\right)\left(e_{i}\left(\lambda_{j}\right)-e_{i}\left(\lambda_{k}\right)\right)\left(C+\lambda_{i} \lambda_{k}\right) \\
& -\left(\lambda_{j}-\lambda_{i}\right)\left(\lambda_{j}-\lambda_{k}\right)\left(\left(e_{i}\left(\lambda_{i}\right) \lambda_{k}+\lambda_{i} e_{i}\left(\lambda_{k}\right)\right),\right.
\end{aligned}
$$

which is the same as (12.69).
Therefore each point $x \in M^{3}$ has an open neighborhood $V$ where there exist functions $u_{j} \in C^{\infty}(V), 1 \leq j \leq 3$, such that $d u_{j}=\gamma_{j}$, and $V$ can be chosen small enough so that $\Phi=\left(u_{1}, u_{2}, u_{3}\right)$ is a diffeomorphism of $V$ onto an open subset $U \subset \mathbb{R}^{3}$, that is, $\left(u_{1}, u_{2}, u_{3}\right)$ are local coordinates on $V$. From

$$
\delta_{i j}=d u_{j}\left(\partial / \partial u_{i}\right)=x_{j} \omega_{j}\left(\partial / \partial u_{i}\right)
$$

it follows that $\partial / \partial u_{i}=v_{i} e_{i}$, with $v_{i}=1 / x_{i}$.
Now notice that

$$
\sum_{j=1}^{3} \delta_{j} v_{j}^{2}=\sum_{i, k \neq j=1}^{3} \frac{C+\lambda_{i} \lambda_{k}}{\left(\lambda_{j}-\lambda_{i}\right)\left(\lambda_{j}-\lambda_{k}\right)}=1
$$

$$
\sum_{j=1}^{3} \delta_{j} v_{j} V_{j}=\sum_{j=1}^{3} \delta_{j} \lambda_{j} v_{j}^{2}=\sum_{i, k \neq j=1}^{3} \frac{\lambda_{j}\left(C+\lambda_{i} \lambda_{k}\right)}{\left(\lambda_{j}-\lambda_{i}\right)\left(\lambda_{j}-\lambda_{k}\right)}=0
$$

and

$$
\sum_{j=1}^{3} \delta_{j} V_{j}^{2}=\sum_{j=1}^{3} \delta_{j} \lambda_{j}^{2} v_{j}^{2}=\sum_{i, k \neq j=1}^{3} \frac{\lambda_{j}^{2}\left(C+\lambda_{i} \lambda_{k}\right)}{\left(\lambda_{j}-\lambda_{i}\right)\left(\lambda_{j}-\lambda_{k}\right)}=C .
$$

It follows that the pair $(v, V)$ satisfies (12.57) with respect to the Lorentzian inner product (12.58).

### 12.7 Genuine conformal deformations

In this section we discuss an extension of the notion of genuine deformation to the conformal realm, and provide a similar description as in the isometric case of the geometric nature of a submanifold that admits conformal genuine deformations.

### 12.7.1 Reduction to the isometric case

To describe the geometric structure of a genuine pair of conformal immersions $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ and $\bar{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$, the strategy is to reduce the problem to the case of genuine pairs of isometric immersions, by endowing $M^{n}$ with the metric induced by $f$ and considering the isometric light cone representative

$$
\hat{f}=\mathcal{J}(\bar{f}): M^{n} \rightarrow \mathbb{V}^{n+q+1} \subset \mathbb{L}^{n+q+2}
$$

of $\bar{f}$. However, when trying to apply to the pair $\{f, \hat{f}\}$ the procedure developed in Section 12.2 to construct simultaneous ruled isometric extensions of $f$ and $\hat{f}$, some technical difficulties arise. These are due to the fact that, because $\hat{f}$ takes values in Lorentzian space, it may happen that, in the notations of Section 12.2.1, the projections of

$$
\Omega=\Omega(f, \hat{f})=\mathcal{S}(\alpha \oplus \hat{\alpha}) \cap \mathcal{S}(\alpha \oplus \hat{\alpha})^{\perp} \subset \mathcal{S}(\alpha) \oplus \mathcal{S}(\hat{\alpha})
$$

onto $N_{f} M$ and $N_{\hat{f}} M$ be not injective. This accounts for the two possibilities in Proposition 12.32 below, for the proof of which the reader is referred to the article [186], where he can find the details on how one can adapt the procedure of Section 12.2 to the case in which such projections are not injective.

In the next statement, that an isometric immersion $F: N^{n} \rightarrow \mathbb{L}^{m}$ is conical means that $t F(x) \in F(N)$ for any $x \in N^{n}$ if $t$ is close to 1 .

Proposition 12.32. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{V}^{n+q+1} \subset \mathbb{L}^{n+q+2}$ be isometric immersions. Assume that $p+q \leq n-1$ and $\min \{p, q\} \leq 5$. Then, along each connected component of an open dense subset of $M^{n}$, one of the following possibilities holds:
(i) There exists a vector bundle isometry $\mathfrak{T}: L^{\ell} \rightarrow \hat{L}^{\ell}$ between vector subbundles $L^{\ell}$ and $\hat{L}^{\ell}$ of $N_{f} M$ and $N_{\hat{f}} M$, respectively, such that the tangent subspaces

$$
D(x)=\mathcal{N}\left(\alpha_{L^{\perp}}(x)\right) \cap \mathcal{N}\left(\hat{\alpha}_{\hat{L}^{\perp}}(x)\right)
$$

form a tangent subbundle of rank d, the pair ( $\mathcal{T}, D^{d}$ ) satisfies conditions (12.9), and the immersions $f$ and $\hat{f}$ have (possibly trivial) maximal mutually $\Delta_{0}^{s_{0}}$-ruled isometric extensions $F^{\prime}: N_{0}^{n+r_{0}} \rightarrow \mathbb{R}^{n+p}$ and $\hat{F}: N_{0}^{n+r_{0}} \rightarrow \mathbb{L}^{n+q+2}, 1 \leq r_{0} \leq \ell$, with

$$
s_{0} \geq n-p-q-2+3 \ell
$$

(ii) Setting $f^{\prime}=\mathcal{J}(f): M^{n} \rightarrow \mathbb{V}^{n+p+1} \subset \mathbb{L}^{n+p+2}$, there exists a vector bundle isometry $\mathcal{T}: L^{\ell} \rightarrow \hat{L}^{\ell}$ between vector subbundles $L^{\ell}$ and $\hat{L}^{\ell}$ of $N_{f^{\prime}} M$ and $N_{\hat{f}} M$, respectively, such that the tangent subspaces

$$
D(x)=\mathcal{N}\left(\alpha_{L^{\perp}}^{\prime}(x)\right) \cap \mathcal{N}\left(\hat{\alpha}_{\hat{L}^{\perp}}(x)\right)
$$

form a tangent subbundle of rank $d$, the pair $\left(\mathcal{T}, D^{d}\right)$ satisfies conditions (12.9), and the immersions $f^{\prime}$ and $\hat{f}$ have mutually $\Delta_{0}^{s_{0}}$-ruled isometric Lorentzian conical extensions $F^{\prime}: N_{0}^{n+r_{0}} \rightarrow \mathbb{L}^{n+p+2}$ and $\hat{F}: N_{0}^{n+r_{0}} \rightarrow \mathbb{L}^{n+q+2}$ such that $\left\langle F^{\prime}, F^{\prime}\right\rangle=\langle\hat{F}, \hat{F}\rangle$, with

$$
s_{0} \geq n-p-q+3 \ell-4, \quad 2 \leq r_{0} \leq \ell .
$$

Moreover, there are smooth orthogonal splittings

$$
N_{F^{\prime}} N_{0}=\mathcal{L}_{0}^{\ell-r_{0}} \oplus L^{\perp} \text { and } N_{\hat{F}} N_{0}=\hat{\mathcal{L}}_{0}^{\ell-r_{0}} \oplus \hat{L}^{\perp}
$$

and a vector bundle isometry $T_{0}: \mathcal{L}_{0} \rightarrow \hat{\mathcal{L}}_{0}$ such that

$$
\Delta_{0}=\mathcal{N}\left(\alpha_{\mathcal{L}_{\frac{1}{0}}}^{F^{\prime}}\right) \cap \mathcal{N}\left(\hat{\alpha}_{\hat{\mathcal{L}}_{\dot{1}}^{\prime}}^{\hat{F}}\right)
$$

and the pair $\left(T_{0}, \Delta_{0}\right)$ satisfies conditions 12.9).

### 12.7.2 Conformally ruled extensions of conformal pairs

A conformal immersion $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ is said to be $D^{d}$-conformally ruled if $M^{n}$ carries an integrable $d$-dimensional distribution $D^{d} \subset T M$ such that the restriction $\left.f\right|_{\sigma}: \sigma \rightarrow \mathbb{R}^{n+p}$ of $f$ to each leaf $\sigma$ of $D^{d}$ is umbilical. The same definition applies for a conformal immersion into a semi-Riemannian flat space.

Given a $D^{d}$-conformally ruled conformal immersion $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ with conformal factor $\varphi \in C^{\infty}(M)$ of a Riemannian manifold, for each $x \in M^{n}$ let $\eta(x)$ be the component in $N_{f} M(x)$ of the mean curvature vector of the restriction of $f$ to the leaf of $D$ through $x \in M^{n}$. Let $\theta^{f}=\theta^{f}(x): T_{x} M \times T_{x} M \rightarrow N_{f} M(x)$ be the symmetric bilinear form defined by

$$
\theta^{f}(Z, X)=\frac{1}{\varphi}\left(\alpha^{f}(Z, X)-\langle Z, X\rangle \eta(x)\right)
$$

and consider the subspace of the normal space $N_{f} M(x)$ of $f$ at $x$ given by

$$
L_{D}^{c}(x)=L_{D}^{c}(f)(x)=\operatorname{span}\left\{\theta^{f}(Z, X): Z \in D^{d}(x) \text { and } X \in T_{x} M\right\} .
$$

We always work on open subsets where the dimension of $L_{D}^{c}(x)$ is constant, in which case such subspaces form a smooth subbundle of $N_{f} M$ that we denote by $L_{D}^{c}(f)$, or simply $L_{D}^{c}$ when it is clear to which immersion it refers to.

In the following result, for a conformal immersion $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ of a Riemannian manifold and its isometric light cone representative $F=\mathcal{J}(f): M^{n} \rightarrow \mathbb{V}^{n+p+1} \subset$ $\mathbb{L}^{n+p+2}$, the map $\phi: N_{f} M \rightarrow V$ stands for the vector bundle isometry given by part (i) of Proposition 9.17 onto the vector subbundle $V$ of $N_{F} M$, and $\zeta \in \Gamma\left(N_{F} M\right)$, $\psi \in \Gamma\left(T^{*} M \otimes T^{*} M\right)$ for the vector field and symmetric bilinear form given by (9.30) and (9.34), respectively.

Lemma 12.33. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ be a conformal immersion with conformal factor $\varphi \in C^{\infty}(M)$ of a Riemannian manifold, and let $F=\mathcal{J}(f): M^{n} \rightarrow \mathbb{V}^{n+p+1} \subset \mathbb{L}^{n+p+2}$ be its isometric light cone representative. If $F$ is $\Delta$-conformally ruled, then so is $f$. Moreover, the following assertions hold:
(i) The components $\eta^{f}$ and $\eta^{F}$ in $N_{f} M$ and $N_{F} M$, respectively, of the mean curvature vector fields of the leaves of $\Delta$ for $f$ and $F$ are related by

$$
\begin{equation*}
\eta^{f}=\frac{1}{\varphi} \rho^{F}+\mathcal{H}^{f} \tag{12.70}
\end{equation*}
$$

where $\phi\left(\rho^{F}\right)=\eta_{V}^{F}$.
(ii) The symmetric bilinear forms

$$
\theta^{f}=\frac{1}{\varphi}\left(\alpha^{f}-\langle,\rangle_{f} \eta^{f}\right) \text { and } \theta^{F}=\alpha^{F}-\langle,\rangle_{F} \eta^{F}
$$

are related by

$$
\begin{equation*}
\theta^{F}(Z, X)=\phi\left(\theta^{f}(Z, X)\right)-\left(\psi(Z, X)+\left\langle\eta^{F}, \zeta\right\rangle\langle Z, X\rangle\right) F \tag{12.71}
\end{equation*}
$$

(iii) The bundle map $\Upsilon=\left.\phi^{-1} \circ \pi\right|_{\hat{\mathcal{L}}}: \hat{\mathcal{L}} \rightarrow N_{f} M$, where $\pi$ : $N_{F} M \rightarrow V$ is the orthogonal projection, is a vector bundle isometry of $\hat{\mathcal{L}}=L_{\Delta}^{c}(F)$ onto $\mathcal{L}=L_{\Delta}^{c}(f)$ that is parallel with respect to the induced connections on $\hat{\mathcal{L}}$ and $\mathcal{L}$ and satisfies

$$
\Upsilon \circ \theta_{\hat{\mathcal{L}}}^{F}=\theta_{\mathcal{L}}^{f}
$$

Proof: (i) If $F$ is $\Delta$-conformally ruled, then

$$
\begin{equation*}
\alpha^{F}(Z, W)=\langle Z, W\rangle \eta^{F} \tag{12.72}
\end{equation*}
$$

for all $Z, W \in \Gamma(\Delta)$. Decompose

$$
\begin{equation*}
\eta^{F}=\phi\left(\rho^{F}\right)+\left\langle\eta^{F}, \zeta\right\rangle F+\left\langle\eta^{F}, F\right\rangle \zeta \tag{12.73}
\end{equation*}
$$

according to the orthogonal decomposition $N_{F} M=V \oplus V^{\perp}$. Using 9.32, (12.72) and (12.73) we obtain

$$
\begin{aligned}
\phi\left(\beta^{f}(Z, W)\right)-\psi(Z, W) F-\langle Z, W\rangle \zeta & =\alpha^{F}(Z, W) \\
& =\langle Z, W\rangle \eta^{F} \\
& =\langle Z, W\rangle\left(\phi\left(\rho^{F}\right)+\left\langle\eta^{F}, \zeta\right\rangle F+\left\langle\eta^{F}, F\right\rangle \zeta\right) .
\end{aligned}
$$

It follows that

$$
\begin{equation*}
\beta^{f}(Z, W)=\langle Z, W\rangle \rho^{F} \tag{12.74}
\end{equation*}
$$

for all $Z, W \in \Gamma(\Delta)$, and that

$$
\begin{equation*}
\left\langle\eta^{F}, F\right\rangle=-1 . \tag{12.75}
\end{equation*}
$$

Equation (12.74) yields

$$
\alpha^{f}(Z, W)=\langle Z, W\rangle_{f}\left(\varphi^{-1} \rho^{F}+\mathcal{H}^{f}\right)
$$

for all $Z, W \in \Gamma(\Delta)$, which implies that $f$ is $\Delta$-conformally ruled and that 12.70 holds.
(ii) Using (9.32), 12.73) and (12.75) we obtain

$$
\begin{aligned}
\theta^{F}(Z, X) & =\alpha^{F}(Z, X)-\langle Z, X\rangle \eta^{F} \\
& =\phi\left(\beta^{f}(Z, X)\right)-\psi(Z, X) F-\langle Z, X\rangle \zeta-\langle Z, X\rangle\left(\phi\left(\rho^{F}\right)+\left\langle\eta^{F}, \zeta\right\rangle F-\zeta\right) \\
& =\phi\left(\beta^{f}(Z, X)-\langle Z, X\rangle \rho^{F}\right)-\left(\psi(Z, X)+\left\langle\eta^{F}, \zeta\right\rangle\langle Z, X\rangle\right) F,
\end{aligned}
$$

and (12.71) is a consequence of

$$
\begin{aligned}
\beta^{f}(Z, X)-\langle Z, X\rangle \rho^{F} & =\varphi^{-1}\left(\alpha^{f}(Z, X)-\langle Z, X\rangle_{f} \mathcal{H}^{f}\right)-\varphi^{-2}\langle Z, X\rangle_{f} \rho^{F} \\
& =\varphi^{-1}\left(\alpha^{f}(Z, X)-\langle Z, X\rangle_{f}\left(\mathcal{H}^{f}+\varphi^{-1} \rho^{F}\right)\right) \\
& =\theta^{f}(Z, X)
\end{aligned}
$$

for all $Z, X \in \mathfrak{X}(M)$.
(iii) It follows from (12.71) that

$$
\Upsilon\left(\theta^{F}(Z, X)\right)=\theta^{f}(Z, X)
$$

for all $Z \in \Gamma(\Delta)$ and $X \in \mathfrak{X}(M)$. On the other hand, since $\phi: N_{f} M \rightarrow V$ is a vector bundle isometry and the position vector field $F$ is light-like and orthogonal to $V$, then

$$
\left\langle\theta^{F}(Z, X), \theta^{F}(W, Y)\right\rangle=\left\langle\theta^{f}(Z, X), \theta^{f}(W, Y)\right\rangle
$$

for all $Z, W \in \Gamma(\Delta)$ and $X, Y \in \mathfrak{X}(M)$. Thus $\Upsilon$ is a vector bundle isometry. Now,

$$
\begin{aligned}
\left\langle\Upsilon\left(\theta_{\hat{\mathcal{L}}}^{F}(Z, X)\right), \theta^{f}(W, Y)\right\rangle & =\left\langle\Upsilon\left(\theta_{\hat{\hat{\mathcal{L}}}}^{F}(Z, X)\right), \Upsilon\left(\theta^{F}(W, Y)\right)\right\rangle \\
& =\left\langle\theta_{\hat{\mathcal{L}}}^{F}(Z, X), \theta^{F}(W, Y)\right\rangle \\
& =\left\langle\theta^{F}(Z, X), \theta^{F}(W, Y)\right\rangle \\
& =\left\langle\theta^{f}(Z, X), \theta^{f}(W, Y)\right\rangle \\
& =\left\langle\theta_{\mathcal{L}}^{f}(Z, X), \theta^{f}(W, Y)\right\rangle
\end{aligned}
$$

for all $Z, X, Y \in \mathfrak{X}(M)$ and $W \in \Gamma(\Delta)$. Hence $\Upsilon \circ \theta_{\hat{\mathcal{L}}}^{F}=\theta_{\mathcal{L}}^{f}$. Finally, using that $\phi$ is a vector bundle isometry that is parallel with respect to the induced connection on $V$ by part ( $i$ ) of Proposition (9.17), we have

$$
\begin{aligned}
\left\langle{ }^{f} \nabla_{X}^{\perp} \Upsilon(\xi), \Upsilon(\eta)\right\rangle & =\left\langle\phi^{f} \nabla_{X}^{\perp} \Upsilon(\xi), \phi \Upsilon(\eta)\right\rangle \\
& =\left\langle{ }^{F} \nabla^{\perp} \phi \Upsilon(\xi), \phi \Upsilon(\eta)\right\rangle \\
& =\left\langle{ }^{F} \nabla^{\perp} \pi(\xi), \pi(\eta)\right\rangle \\
& =\left\langle{ }^{F} \nabla^{\perp} \xi, \eta\right\rangle \\
& =\left\langle\left({ }^{F} \nabla_{X}^{\perp} \xi\right)_{\hat{\mathcal{L}}}^{\perp}, \eta\right\rangle \\
& =\left\langle\Upsilon\left(\left({ }^{F} \nabla_{X}^{\perp} \xi\right)_{\hat{\mathcal{L}}}\right), \Upsilon(\eta)\right\rangle
\end{aligned}
$$

for all $\xi, \eta \in \hat{\mathcal{L}}$, where the fourth equality is due to the fact that $\xi$ differs from $\pi(\xi)$ by a multiple of $F$ by (12.71) and $F$ is a parallel light-like normal vector field. We conclude that

$$
\left({ }^{f} \nabla_{X}^{\perp} \Upsilon(\xi)\right)_{\mathcal{L}}=\Upsilon\left(\left({ }^{F} \nabla_{X}^{\perp} \xi\right)_{\hat{\mathcal{L}}}\right),
$$

that is, $\Upsilon$ is parallel with respect to the induced connections on $\hat{\mathcal{L}}$ and $\mathcal{L}$.
Theorem 12.34. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ and $\bar{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ be conformal immersions with $p+q \leq n-3$ and $\min \{p, q\} \leq 5$. Then, locally on an open dense subset of $M^{n}$, the pair $\{f, \bar{f}\}$ extends conformally (possibly trivially) to a mutually $\Delta^{s}$-conformally ruled pair of immersions $F: N^{n+r} \rightarrow \mathbb{R}^{n+p}$ and $\bar{F}: N^{n+r} \rightarrow \mathbb{R}^{n+q}$ such that

$$
\Delta^{s}=\mathcal{N}\left(\theta_{\mathcal{L} \perp}^{F}\right) \cap \mathcal{N}\left(\theta_{\overline{\mathcal{L}} \perp}^{\overline{V_{A}}}\right)
$$

where $\mathcal{L}=L_{\Delta}^{c}(F)$ and $\overline{\mathcal{L}}=L_{\Delta}^{c}(\bar{F})$. Moreover, there exists a parallel vector bundle isometry $\mathcal{T}: \mathcal{L} \rightarrow \overline{\mathcal{L}}$ such that $\mathcal{T} \circ \theta_{\mathcal{L}}^{F}=\theta_{\overline{\mathcal{L}}}^{\bar{F}}$ and

$$
s \geq n-p-q+3\left(\ell^{c}+r\right)
$$

where $\ell^{c}=\operatorname{rank} \mathcal{L}=\operatorname{rank} \overline{\mathcal{L}}$.
Proof: Endow $M^{n}$ with the metric induced by $f$ and apply Proposition 12.32 to the pair of isometric immersions $f$ and $\hat{f}=\mathcal{J}(\bar{f}): M^{n} \rightarrow \mathbb{V}^{n+q+1} \subset \mathbb{L}^{n+q+2}$. Assume first that assertion $(i)$ in Proposition 12.32 holds on a certain connected component $\mathcal{V}$ of an open and dense subset of $M^{n}$, and write $\mathcal{V}=M^{n}$ for simplicity. Thus there exists a vector bundle isometry $\mathfrak{T}: L^{\ell} \rightarrow \hat{L}^{\ell}$ between vector subbundles $L^{\ell}$ and $\hat{L}^{\ell}$ of $N_{f} M$ and $N_{\hat{f}} M$, respectively, such that the tangent subspaces

$$
D(x)=\mathcal{N}\left(\alpha_{L^{\perp}}(x)\right) \cap \mathcal{N}\left(\hat{\alpha}_{\hat{L}^{\perp}}(x)\right)
$$

form a subbundle of rank $d$, the pair ( $\mathcal{T}, D^{d}$ ) satisfies conditions 12.9 , and the immersions $f$ and $\hat{f}$ have (possibly trivial) maximal mutually $\Delta_{0}^{s_{0}}$-ruled isometric extensions $F^{\prime}: N_{0}^{n+r_{0}} \rightarrow \mathbb{R}^{n+p}$ and $\hat{F}: N_{0}^{n+r_{0}} \rightarrow \mathbb{L}^{n+q+2}, 1 \leq r_{0} \leq \ell$, with

$$
s_{0} \geq n-p-q-2+3 \ell
$$

Moreover, there are smooth orthogonal splittings

$$
N_{F^{\prime}} N_{0}=\mathcal{L}_{0}^{\ell_{0}} \oplus L^{\perp} \text { and } N_{\hat{F}} N_{0}=\hat{\mathcal{L}}_{0}^{\ell_{0}} \oplus \hat{L}^{\perp}
$$

with $\ell_{0}=\ell-r_{0}$, and a vector bundle isometry $T_{0}: \mathcal{L}_{0} \rightarrow \hat{\mathcal{L}}_{0}$ such that

$$
\Delta_{0}=\mathcal{N}\left(\alpha_{\mathcal{L}_{\frac{1}{0}}^{\prime}}^{F^{\prime}}\right) \cap \mathcal{N}\left(\hat{\alpha}_{\hat{\mathcal{L}}_{\frac{1}{0}}^{\hat{F}}}^{\hat{F}^{\prime}}\right),
$$

and the pair $\left(T_{0}, \Delta_{0}\right)$ satisfies conditions (12.9).
Since $N_{0}^{n+r_{0}}$ is Riemannian and $\hat{F}$ is ruled, the immersion $\hat{F}$ must be transversal to the light cone. By restricting to an open subset if necessary, we may assume that $\hat{F}$ is an embedding, and hence that

$$
N=\hat{F}^{-1}\left(\hat{F}\left(N_{0}\right) \cap \mathbb{V}^{n+q+1}\right) \supseteq M^{n}
$$

is an $\left(n+r_{0}-1\right)$-dimensional manifold.
Set $F=F^{\prime} \circ i$ and $\bar{F}=\mathcal{C}(\hat{F} \circ i): N \rightarrow \mathbb{R}^{n+q}$, where $i: N \rightarrow N_{0}$ is the inclusion map. Then $\{F, \bar{F}\}$ is a conformal pair, $F \circ j=f$ and $\bar{F} \circ j=\bar{f}$, where $j$ is the inclusion of $M$ into $N$, and hence $\{F, \bar{F}\}$ is a conformal extension of $\{f, \bar{f}\}$. Moreover, $F$ and $\bar{F}$ are mutually $\Delta^{s}$-conformally ruled, where $s=s_{0}-1$ and $\Delta^{s}$ is the distribution on $N$ defined by $\hat{F}_{*}\left(\Delta^{s}\right)=\hat{F}_{*}\left(\Delta_{0}\right) \cap T \mathbb{V}^{n+q+1}$. Therefore

$$
s \geq n-p-q+3\left(\ell_{0}+r\right)
$$

with $r=r_{0}-1$, and hence the estimate on $s$ will follow once we prove that $\ell_{0} \geq \ell^{c}$.
First observe that, from

$$
\alpha^{\hat{F} \circ i}(Z, X)=\alpha^{\hat{F}}\left(i_{*} Z, i_{*} X\right)+\hat{F}_{*} \alpha^{i}(Z, X)
$$

for all $Z, X \in \mathfrak{X}(N)$, we obtain

$$
\hat{\mathcal{L}}=L_{\Delta}^{c}(\hat{F} \circ i) \subset \hat{\mathcal{L}}_{0} \oplus \operatorname{span}\left\{\hat{F}_{*} \eta=\eta^{\hat{F} \circ i}\right\}
$$

where $\eta(x)$ is the component in $T_{x} N_{0} \cap N_{i} N(x)$ of the mean curvature vector at $x \in N$ of the leaf of $\Delta^{s}$ through $x$. On the other hand, since $\hat{F} \circ i=\mathcal{J}(\bar{F})$, then Lemma 12.33 can be applied to $\bar{F}$ and $\hat{F} \circ i$. In particular, from 12.71 it follows that $\phi(\overline{\mathcal{L}}) \subset$ $\hat{\mathcal{L}} \oplus \operatorname{span}\{\hat{F} \circ i\}$; hence

$$
\begin{equation*}
\phi(\overline{\mathcal{L}}) \subset \hat{\mathcal{L}}_{0} \oplus \operatorname{span}\left\{\eta^{\hat{F} \circ i}\right\} \oplus \operatorname{span}\{\hat{F} \circ i\} \tag{12.76}
\end{equation*}
$$

Now, by (12.75) we have

$$
\left\langle\eta^{\hat{F} \circ i}, \hat{F} \circ i\right\rangle=-1 .
$$

Thus the subspace $W$ on the right-hand side of $\sqrt{12.76}$ ) is Lorentzian. Therefore the subspace $\phi(\overline{\mathcal{L}})$ is a Riemannian subspace of $W$ that is orthogonal to the null vector $\hat{F} \circ i \in W$; hence $\phi(\overline{\mathcal{L}})$ has codimension at least two in $W$. We conclude that

$$
\ell^{c}=\operatorname{rank} \overline{\mathcal{L}} \leq \ell_{0}
$$

as we wished.
We now show that there exists a vector bundle isometry $\mathcal{T}: \mathcal{L} \rightarrow \overline{\mathcal{L}}$ that is parallel with respect to the induced connections on $\mathcal{L}$ and $\overline{\mathcal{L}}$ and satisfies

$$
\mathcal{T} \circ \theta_{\mathcal{L}}^{F}=\theta_{\overline{\mathcal{L}}}^{\bar{F}} .
$$

First extend $T_{0}$ to a vector bundle map $T_{1}$ between the vector subbundles

$$
\mathcal{L}_{1}=\mathcal{L}_{0} \oplus \operatorname{span}\left\{\eta^{F}=F_{*}^{\prime} \eta\right\} \text { and } \hat{\mathcal{L}}_{1}=\hat{\mathcal{L}}_{0} \oplus \operatorname{span}\left\{\eta^{\hat{F} \circ i}=\hat{F}_{*} \eta\right\}
$$

of $N_{F} N$ and $N_{\hat{F} \circ i} N$, respectively, by setting

$$
\left.T_{1}\right|_{\mathcal{L}_{0}}=T_{0} \text { and } T_{1}\left(\eta^{F}\right)=\eta^{\hat{F} \circ i} .
$$

Since $F^{\prime}$ and $\hat{F}$ are isometric, the vector bundle map $T_{1}$ is an isometry, which is easily seen to be parallel and satisfy

$$
\Delta=\mathcal{N}\left(\alpha_{\mathcal{L}_{\frac{1}{1}}}^{F}\right) \cap \mathcal{N}\left(\alpha_{\hat{\mathcal{L}}_{\frac{1}{1}}^{+}}^{\left.\hat{\hat{F}_{\circ}}\right)} \text { and } \alpha_{\hat{\mathcal{L}}_{1}}^{\hat{F} \circ i}=T_{1} \circ \alpha_{\mathcal{L}_{1}}^{F} .\right.
$$

Moreover, since $\mathcal{L} \subset \mathcal{L}_{1}$, the restriction $T=\left.T_{1}\right|_{\mathcal{L}}: \mathcal{L} \rightarrow \hat{\mathcal{L}}$ defines a parallel vector bundle isometry such that $\theta_{\hat{\mathcal{L}}}^{\hat{\mathrm{F}} \circ i}=T \circ \theta_{\mathcal{L}}^{F}$.

Composing $T$ with the vector bundle isometry $\Upsilon: \hat{\mathcal{L}} \rightarrow \overline{\mathcal{L}}$ given by part (iii) of Lemma 12.33 yields the desired vector bundle isometry $\mathcal{T}: \mathcal{L} \rightarrow \overline{\mathcal{L}}$.

Now suppose that assertion (ii) in Proposition 12.32 holds on some connected component of an open and dense subset of $M^{n}$, which we also denote by $M^{n}$ for simplicity. In this case, setting $f^{\prime}=\mathcal{J}(f): M^{n} \rightarrow \mathbb{V}^{n+p+1} \subset \mathbb{L}^{n+p+2}$, there exists a vector bundle isometry $\mathcal{T}: L^{\ell} \rightarrow \hat{L}^{\ell}$ between vector subbundles $L^{\ell}$ and $\hat{L}^{\ell}$ of $N_{f^{\prime}} M$ and $N_{\hat{f}} M$, respectively, such that the tangent subspaces

$$
D(x)=\mathcal{N}\left(\alpha_{L^{\perp}}^{\prime}(x)\right) \cap \mathcal{N}\left(\hat{\alpha}_{\hat{L}^{\perp}}(x)\right)
$$

form a tangent subbundle of rank $d$, the pair $\left(\mathcal{T}, D^{d}\right)$ satisfies conditions 12.9 , and the immersions $f^{\prime}$ and $\hat{f}$ have mutually $\Delta_{0}^{s_{0}}$-ruled isometric Lorentzian conical extensions $F_{0}^{\prime}: N_{0}^{n+r_{0}} \rightarrow \mathbb{L}^{n+p+2}$ and $\hat{F}_{0}: N_{0}^{n+r_{0}} \rightarrow \mathbb{L}^{n+q+2}$ such that $\left\langle F_{0}^{\prime}, F_{0}^{\prime}\right\rangle=\left\langle\hat{F}_{0}, \hat{F}_{0}\right\rangle$ with

$$
\begin{equation*}
s_{0} \geq n-p-q+3 \ell-4, \quad 2 \leq r_{0} \leq \ell . \tag{12.77}
\end{equation*}
$$

Moreover, there are smooth orthogonal splittings

$$
N_{F_{0}^{\prime}} N_{0}=\mathcal{L}_{0}^{\ell-r_{0}} \oplus L^{\perp} \text { and } N_{\hat{F}_{0}} N_{0}=\hat{\mathcal{L}}_{0}^{\ell-r_{0}} \oplus \hat{L}^{\perp}
$$

and a vector bundle isometry $T_{0}: \mathcal{L}_{0} \rightarrow \hat{\mathcal{L}}_{0}$ such that

$$
\Delta_{0}=\mathcal{N}\left(\alpha_{\mathcal{L}_{0}^{\perp}}^{F_{0}^{\prime}}\right) \cap \mathcal{N}\left(\hat{\alpha}_{\hat{\mathcal{L}}_{0}^{\perp}}^{\hat{F}_{0}^{\prime}}\right)
$$

and the pair $\left(T_{0}, \Delta_{0}\right)$ satisfies conditions 12.9$)$.

Since $f^{\prime}$ is tangent to $F_{0}^{\prime}$, the light-like vector $w$ is nowhere normal to $F_{0}^{\prime}$. Thus $F_{0}^{\prime}$ is transversal to the degenerate hyperplane

$$
\mathcal{H}=\mathcal{H}^{n+p+1}=\left\{v \in \mathbb{L}^{n+p+2}:\langle v, w\rangle=1\right\}
$$

and we locally define

$$
N_{1}^{n+r_{0}-1}=F_{0}^{\prime-1}\left(F_{0}^{\prime}\left(N_{0}\right) \cap \mathcal{H}\right) \subset N_{0}^{n+r_{0}}, \quad \Delta_{1}^{s_{0}-1}=\Delta_{0}^{s_{0}} \cap T N_{1},
$$

and $F_{1}^{\prime}=\left.F_{0}^{\prime}\right|_{N_{1}}, \hat{F}_{1}=\left.\hat{F}_{0}\right|_{N_{1}}$. Now, $F_{1}^{\prime}$ is transversal to $\mathbb{V}^{n+p+1}$; hence we may locally define

$$
N^{n+r}=F_{1}^{\prime-1}\left(F_{1}^{\prime}\left(N_{1}\right) \cap \mathbb{V}^{n+p+1}\right) \subset N_{1}^{n+r_{0}-1}, \quad \Delta^{s}=\Delta_{1}^{s_{0}-1} \cap T N
$$

and $F^{\prime}=\left.F_{1}^{\prime}\right|_{N}, \hat{F}=\left.\hat{F}_{1}\right|_{N}$ with $s=s_{0}-2, r=r_{0}-2$.
Since $F^{\prime}(N) \subset \mathcal{H} \cap \mathbb{V}^{n+p+1}=\mathbb{E}^{n+p}$, there is $F: N \rightarrow \mathbb{R}^{n+p}$ such that $F^{\prime}=\Psi \circ F$. On the other hand, using that $\left\langle F_{0}^{\prime}, F_{0}^{\prime}\right\rangle=\left\langle\hat{F}_{0}, \hat{F}_{0}\right\rangle$, it follows that $\hat{F}$ takes values in $\mathbb{V}^{n+q+1}$, and we may define $\bar{F}=\mathcal{C}(\hat{F}): N \rightarrow \mathbb{R}^{n+q}$. Then, as in the nondegenerate case, we see that $\{F, \bar{F}\}$ is a conformal pair, $F \circ j=f$ and $\bar{F} \circ j=\bar{f}$, where $j$ is the inclusion of $M$ into $N$; hence $\{F, \bar{F}\}$ is a conformal extension of $\{f, \bar{f}\}$. Moreover, $F$ and $\bar{F}$ are mutually $\Delta^{s}$-conformally ruled.

The estimate on $s$ follows as in the previous case. From (12.77) we have

$$
s=s_{0}-2 \geq n-p-q+3\left(\ell_{0}+r\right)
$$

so it suffices to show that $\ell_{0} \geq \ell^{c}$. As before,

$$
\mathcal{L}^{\prime}=L_{\Delta}^{c}\left(F^{\prime}\right) \subset \mathcal{L}_{0} \oplus \operatorname{span}\left\{F_{1 *}^{\prime} \eta\right\}
$$

where $\eta(x)$ is the component in $T_{x} N_{1} \cap N_{i} N(x)$, with $i: N \rightarrow N_{1}$ the inclusion, of the mean curvature vector at $x \in N$ of the leaf of $\Delta^{s}$ through $x$. On the other hand, $\phi(\mathcal{L}) \subset \mathcal{L}^{\prime} \oplus \operatorname{span}\left\{F^{\prime}\right\}$, hence

$$
\begin{equation*}
\phi(\mathcal{L}) \subset \mathcal{L}_{0} \oplus \operatorname{span}\left\{F_{1 *}^{\prime}(\eta)\right\} \oplus \operatorname{span}\left\{F^{\prime}\right\} . \tag{12.78}
\end{equation*}
$$

Arguing as before, we see that the subspace $W$ on the right-hand side of 12.78 is Lorentzian. Therefore $\phi(\mathcal{L})$ is a Riemannian subspace of $W$ that is orthogonal to the null vector $F^{\prime} \in W$, hence it has codimension at least two in $W$.

It remains to prove the existence of a parallel vector bundle isometry $\mathcal{T}: \mathcal{L} \rightarrow \overline{\mathcal{L}}$ such that

$$
\mathcal{T} \circ \theta_{\mathcal{L}}^{F}=\theta_{\overline{\mathcal{L}}}^{\bar{F}}
$$

Let $\xi^{\prime}$ and $\hat{\xi}$ be unit vector fields spanning $F_{0 *}^{\prime} T N_{0} \cap N_{F_{1}^{\prime}} N_{1}$ and $\hat{F}_{0 *} T N_{0} \cap N_{\hat{F}_{1}} N_{1}$, respectively, and set

$$
\mathcal{L}_{1}=\mathcal{L}_{0} \oplus \operatorname{span}\left\{\xi^{\prime}\right\} \subset N_{F_{1}^{\prime}} N_{1} \text { and } \hat{\mathcal{L}}_{1}=\hat{\mathcal{L}}_{0} \oplus \operatorname{span}\{\hat{\xi}\} \subset N_{\hat{F}_{1}} N_{1} .
$$

Extend the parallel vector bundle isometry $T_{0}: \mathcal{L}_{0} \rightarrow \hat{\mathcal{L}}_{0}$ to $T_{1}: \mathcal{L}_{1} \rightarrow \hat{\mathcal{L}}_{1}$ by defining

$$
\left.T_{1}\right|_{\mathcal{L}_{0}}=T_{0} \text { and } T_{1}\left(\xi^{\prime}\right)=\hat{\xi}
$$

Now set

$$
\mathcal{L}_{2}=\mathcal{L}_{1} \oplus \operatorname{span}\left\{\eta^{F^{\prime}}\right\} \subset N_{F^{\prime}} N \text { and } \hat{\mathcal{L}}_{2}=\hat{\mathcal{L}}_{1} \oplus \operatorname{span}\left\{\eta^{\hat{F}}\right\} \subset N_{\hat{F}} N,
$$

and extend $T_{1}$ to $T_{2}: \mathcal{L}_{2} \rightarrow \hat{\mathcal{L}}_{2}$ by setting

$$
\left.T_{2}\right|_{\mathcal{L}_{1}}=T_{1} \text { and } T_{2}\left(\eta_{F^{\prime}}\right)=\eta_{\hat{F}} .
$$

Since $\eta_{F^{\prime}}=F_{1 *}^{\prime} \eta$ and $\eta_{\hat{F}}=\hat{F}_{1 *} \eta$ belong to $T N_{1}$, it is easily seen that $T_{2}$ is also a parallel vector bundle isometry with

$$
\Delta^{s}=\mathcal{N}\left(\alpha_{\mathcal{L} \frac{1}{2}}^{F^{\prime}}\right) \cap \mathcal{N}\left(\alpha_{\hat{\mathcal{L}}_{\frac{1}{2}}^{\hat{F}}}^{\hat{F}}\right)
$$

and $\alpha_{\hat{\mathcal{L}}_{2}}^{\hat{F}}=T_{2} \circ \alpha_{\mathcal{L}_{2}}^{F^{\prime}}$. Moreover, since $\mathcal{L} \subset \mathcal{L}_{2}$, the restriction $T=\left.T_{2}\right|_{\mathcal{L}}: \mathcal{L} \rightarrow \hat{\mathcal{L}}$ defines a parallel vector bundle isometry such that $\theta_{\hat{\hat{L}}}^{\hat{F}}=T \circ \theta_{\mathcal{L}^{\prime}}^{F^{\prime}}$.

To obtain the desired vector bundle isometry $\mathcal{T}: \mathcal{L} \rightarrow \overline{\mathcal{L}}$, it suffices to compose $T$ with the vector bundle isometry $\Upsilon: \hat{\mathcal{L}} \rightarrow \overline{\mathcal{L}}$ obtained by applying part (iii) of Lemma 12.33 to $\bar{F}$ and $\hat{F}=\mathcal{J}(\bar{F})$.

### 12.7.3 Geometric structure of a genuine conformal pair

Given conformal immersions $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ and $\bar{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$, the pair $\{f, \bar{f}\}$ is said to extend conformally when there exist a conformal embedding $j: M^{n} \rightarrow N^{n+r}$, with $r \geq 1$, and conformal immersions $F: N^{n+r} \rightarrow \mathbb{R}^{n+p}$ and $\bar{F}: N^{n+r} \rightarrow \mathbb{R}^{n+q}$ such that $f=F \circ j$ and $\bar{f}=\bar{F} \circ j$.

A conformal immersion $\bar{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ is called a genuine conformal deformation of a given conformal immersion $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ if there exists no open subset $U \subset M^{n}$ along which the restrictions $\left.f\right|_{U}$ and $\left.\bar{f}\right|_{U}$ extend conformally. Since, in this case, the immersion $f$ is also a genuine conformal deformation of $\bar{f}$, we refer to $\{f, \hat{f}\}$ simply as a genuine conformal pair. The next result follows immediately from Theorem 12.34 ,
Corollary 12.35. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ and $\bar{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ form a genuine conformal pair with $p+q \leq n-3$ and $\min \{p, q\} \leq 5$. Then, along each connected component of an open dense subset of $M^{n}$, the immersions $f$ and $\bar{f}$ are mutually conformally $D^{d}$-ruled with

$$
D^{d}=\mathcal{N}\left(\beta_{L^{\perp}}^{f}\right) \cap \mathcal{N}\left(\beta_{\bar{L}^{\perp}}^{\bar{f}}\right)
$$

where $L=L_{D}^{c}(f)$ and $\bar{L}=L_{D}^{c}(\bar{f})$. Moreover, there exists a parallel vector bundle isometry $\mathcal{T}: L \rightarrow \bar{L}$ such that $\theta_{\bar{L}}^{f}=\mathcal{T} \circ \theta_{L}^{f}$, and

$$
d \geq n-p-q+3 \ell_{D}^{c},
$$

where $\ell_{D}^{c}=\operatorname{rank} L=\operatorname{rank} \bar{L}$.
A conformal immersion $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ is genuinely conformally rigid in $\mathbb{R}^{n+q}$ for a fixed integer $q>0$ if, for any given conformal immersion $\bar{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$, there is an open dense subset $U \subset M^{n}$ such that the pair $\left\{\left.f\right|_{U},\left.\bar{f}\right|_{U}\right\}$ extends conformally. Theorem 12.35 implies the following criterion for genuine conformal rigidity.

Corollary 12.36. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ be a conformal immersion and let $q$ be a positive integer with $p+q \leq n-3$ and $\min \{p, q\} \leq 5$. If $f$ is not $(n-p-q)$-conformally ruled on any open subset of $M^{n}$, then $f$ is genuinely conformally rigid in $\mathbb{R}^{n+q}$.

### 12.7.4 Compositions of conformal immersions

Theorem 12.34 yields the following conformal version of Theorem 12.26 in terms of the conformal $s$-nullities introduced in Section 9.9 .

Corollary 12.37. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ be a conformal immersion and let $q$ be a positive integer with $p \leq q \leq n-p-3$. Suppose that $p \leq 5$ and that $f$ satisfies

$$
\nu_{s}^{c} \leq n+p-q-2 s-1 \text { for all } 1 \leq s \leq p
$$

For $q \geq p+5$, assume further that $\nu_{1}^{c} \leq n-2(q-p)+1$. Then, for any conformal immersion $\bar{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$, there exists an open dense subset $V \subseteq M^{n}$ such that the restriction of $\bar{f}$ to any connected component $U$ of $V$ is a composition $\left.\bar{f}\right|_{U}=\left.h \circ f\right|_{U}$ of $\left.f\right|_{U}$ with a conformal immersion $h: W \subset \mathbb{R}^{n+p} \rightarrow \mathbb{R}^{n+q}$ of an open subset $W \supset f(U)$.

Proof: Given a conformal immersion $\bar{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$, Theorem 12.34 applies and yields, locally on an open dense subset of $M^{n}$, a (possibly trivial) conformal extension $\{F, \bar{F}\}$ of $\{f, \bar{f}\}$ to a mutually $\Delta^{s}$-conformally ruled pair of immersions $F: N^{n+r} \rightarrow \mathbb{R}^{n+p}$ and $\bar{F}: N^{n+r} \rightarrow \mathbb{R}^{n+q}, 0 \leq r \leq p$, with

$$
s \geq n-p-q+3\left(\ell^{c}+r\right)
$$

where $\ell^{c}=\operatorname{rank} L_{\Delta}^{c}(F)=\operatorname{rank} L_{\Delta}^{c}(\bar{F})$. It suffices to prove that $r=p$.
First we show that $L_{\Delta}^{c}(F)^{\perp}=\{0\}$. Assume otherwise that $s^{\prime}=\operatorname{rank} L_{\Delta}^{c}(f)^{\perp}>0$. If $D=\Delta \cap T M$, then

$$
\operatorname{rank} D=\operatorname{rank} \Delta-r \geq n-p-q+2 r+3 \ell^{c}
$$

Since $D \subset \mathcal{N}\left(\theta_{L_{\Delta}^{c}(F)^{\perp}}^{f}\right)$, using that $\ell^{c}=p-r-s^{\prime}$ we would have

$$
\begin{aligned}
\nu_{s^{\prime}}^{c} & \geq n-p-q+2 r+3 \ell^{c} \\
& =n-p-q+2 r+2\left(p-r-s^{\prime}\right)+\ell^{c} \\
& =n+p-q-2 s^{\prime}+\ell^{c},
\end{aligned}
$$

contradicting the assumption on $\nu_{s}^{c}$ for $1 \leq s \leq p$. Therefore

$$
s \geq n+2 p-q .
$$

Now assume that $r<p$. Since $F$ is $\Delta$-conformally ruled, there exists $\eta \in \Gamma\left(N_{F} N\right)$ such that

$$
\alpha_{F}(Z, W)=\langle Z, W\rangle \eta_{F}
$$

for all $Z, W \in \Gamma(\Delta)$. In particular, for any unit normal vector field $\xi \in \Gamma\left(N_{F} N\right)$ we obtain

$$
\left\langle\left(A_{\xi}^{f}-\left\langle\xi, \eta_{F}\right\rangle I\right) T, S\right\rangle=0
$$

for all $T, S \in \Gamma(D)$. Since rank $D=s-r \geq n-(q-p)+1$, then $\nu_{1}^{c} \geq n-2(q-p)+2$, and this is a contradiction with the assumption on $\nu_{1}^{c}$.

Notice that for $p=q$ the preceding corollary reduces to Theorem4.23. For $p=1$, it yields the following conformal version of Corollary 12.27 .

Corollary 12.38. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be an immersion and let $q$ be a positive integer such that $1 \leq q \leq n-4$. If $q \leq 5$, suppose that $f$ has no principal curvature of multiplicity greater than $n-q-2$. If $q \geq 6$, assume further that $f$ is not $(n-q+2)$ conformally ruled on any open subset. Then any immersion $\bar{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ conformal to $f$ is locally a composition, that is, there exists an open dense subset $V \subseteq M^{n}$ such that the restriction of $\bar{f}$ to any connected component $U$ of $V$ satisfies

$$
\left.\bar{f}\right|_{U}=\left.h \circ f\right|_{U}
$$

where $h: W \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+q}$ is a conformal immersion of an open subset $W \supset f(U)$.

### 12.7.5 Genuine conformal pairs from isometric ones

The main result of this section gives a geometric way to construct conformal genuine pairs by means of isometric ones, explaining the similitude between Theorem 12.35 and its isometric counterpart, namely, Theorem 12.10.

Let $N^{n+1}$ be a Riemannian manifold. Assume that there exist an isometric immersion $F^{\prime}: N^{n+1} \rightarrow \mathbb{R}^{n+p}$ and an isometric embedding $\hat{F}: N^{n+1} \rightarrow \mathbb{L}^{n+q+2}$ transversal to the light cone $\mathbb{V}^{n+q+1}$. Set

$$
M^{n}=\hat{F}^{-1}\left(\hat{F}\left(N^{n+1}\right) \cap \mathbb{V}^{n+q+2}\right)
$$

and let $i: M^{n} \rightarrow N^{n+1}$ be the inclusion map. Then the immersions $f=F^{\prime} \circ i$ and $\bar{f}=\mathcal{C}(\hat{F} \circ i)$ induce conformal metrics on $M^{n}$. The next result states that any genuine conformal pair $\{f, \bar{f}\}$ of immersions $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ and $\bar{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ with sufficiently low codimensions $p$ and $q$ is locally produced in this way by means of a genuine isometric pair $\left\{F^{\prime}, \hat{F}\right\}$ as above.

Theorem 12.39. Assume that $f: M^{n} \rightarrow \mathbb{R}^{n+p}, p \geq 1$, and $\bar{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ form a genuine conformal pair with $p+q \leq n-3$ and $\min \{p, q\} \leq 5$. Suppose further that $\bar{f}$ is nowhere conformally congruent to an immersion that is isometric to $f$. Then, locally on an open dense subset of $M^{n}$, there exist a Riemannian manifold $N^{n+1}$ that admits an isometric immersion $F^{\prime}: N^{n+1} \rightarrow \mathbb{R}^{n+p}$ and an isometric embedding $\hat{F}: N^{n+1} \rightarrow \mathbb{L}^{n+q+2}$ transversal to the light cone $\mathbb{V}^{n+q+1}$, and a conformal diffeomorphism $i: M^{n} \rightarrow \hat{F}^{-1}\left(\hat{F}\left(N^{n+1}\right) \cap \mathbb{V}^{n+q+1}\right)$ such that $\left\{F^{\prime}, \hat{F}\right\}$ is a genuine isometric pair, $f=F^{\prime} \circ i$ and $\bar{f}=\mathcal{C}(\hat{F} \circ i)$.

Proof: Endow $M^{n}$ with the metric induced by $f$ and apply Proposition 12.32 to the pair of isometric immersions $f$ and $\hat{f}=\mathcal{J}(\bar{f}): M^{n} \rightarrow \mathbb{V}^{n+q+1} \subset \mathbb{L}^{n+q+2}$. Assume that assertion (i) in Proposition 12.32 holds on a certain connected component $\mathcal{V}$ of an open and dense subset of $M^{n}$, and write $\mathcal{V}=M^{n}$ for simplicity. Thus the immersions $f$ and $\hat{f}$ have (possibly trivial) maximal mutually $\Delta_{0}^{s_{0}}$-ruled isometric extensions $F^{\prime}: N_{0}^{n+r_{0}} \rightarrow \mathbb{R}^{n+p}$ and $\hat{F}: N_{0}^{n+r_{0}} \rightarrow \mathbb{L}^{n+q+2}, 1 \leq r_{0} \leq \ell$.

As in the proof of Theorem 12.34 , since $\hat{F}$ is transversal to the light cone, by restricting to an open subset, if necessary, we may assume that $\hat{F}$ is an embedding, so that

$$
\bar{N}=\hat{F}^{-1}\left(\hat{F}(N) \cap \mathbb{V}^{n+q+1}\right) \supset M^{n}
$$

is an $(n+r-1)$-dimensional manifold. As before, setting $F=F^{\prime} \circ i$ and $\bar{F}=$ $\mathcal{C}(\hat{F} \circ i): \bar{N} \rightarrow \mathbb{R}^{n+q}$, where $i: \bar{N} \rightarrow N$ is the inclusion map, we see that $\{F, \bar{F}\}$ is a conformal pair, $F \circ j=f$ and $\bar{F} \circ j=\bar{f}$, where $j$ is the inclusion of $M$ into $\bar{N}$. Thus $\{F, \bar{F}\}$ is a conformal extension of $\{f, \bar{f}\}$.

Since $\{f, \bar{f}\}$ is a genuine conformal pair, we must have $r=1$, hence $\bar{N}=M^{n}$, $F \circ i=f$ and $\mathcal{C}(\hat{F} \circ i)=\bar{f}$. A similar argument shows that any isometric extension of the pair $\left\{F^{\prime}, \hat{F}\right\}$ would give a conformal extension of the pair $\{f, \bar{f}\}$, hence $\left\{F^{\prime}, \hat{F}\right\}$ must be a genuine isometric pair.

We refer the reader to [186] for the details on how one can reach the same conclusion under the assumption that assertion (ii) in Proposition 12.32 holds, for in this case one needs to use some of the arguments in the proof of Proposition 12.32, which we have omitted.

Remark 12.40. The assumption that $\bar{f}$ is nowhere locally conformally congruent to an immersion that is isometric to $f$ is always satisfied if $f$ is genuinely isometrically rigid in $\mathbb{R}^{n+q}$, for instance if $M^{n}$ does not carry any ruled open subset with rulings of dimension at least $n-p-q$. In particular, this is always the case after composing $f$ with a suitable inversion of $\mathbb{R}^{n+p}$.

For $p=1$, Theorem 12.39 says that any hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ that admits a genuine conformal (but not isometric) deformation in $\mathbb{R}^{n+q}$ can be locally produced as the intersection with the light cone $\mathbb{V}^{n+q+1}$ of an $(n+1)$-dimensional flat submanifold of $\mathbb{L}^{n+q+2}$ transversal to $\mathbb{V}^{n+q+1}$.

Corollary 12.41. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ and $\bar{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ form a conformal pair, with $q \leq n-4$. Assume that there exists no open subset $M^{n}$ along which $\bar{f}$ either is a composition or is conformally congruent to an isometric deformation of $f$. Then, locally on an open dense subset of $M^{n}$, there exist an isometric embedding $\bar{F}: U \subset$ $\mathbb{R}^{n+1} \rightarrow \mathbb{L}^{n+q+2}$ transversal to the light cone $\mathbb{V}^{n+q+1}$ and a conformal diffeomorphism $\tau: M^{n} \rightarrow \bar{M}^{n}=\bar{F}^{-1}\left(\bar{F}(U) \cap \mathbb{V}^{n+q+1}\right) \subset U$ such that $f=i \circ \tau$ and $\bar{f}=\mathcal{C}(\bar{F} \circ \tau)$, where $i: \bar{M}^{n} \rightarrow U$ is the inclusion map.

In the particular case $q=1$, the preceding corollary provides a nonparametric description of Cartan's conformally deformable hypersurfaces, a parametric classification of which will be given in Chapter 17 .

Another important special case of Theorem 12.39 occurs when $q=0$. In this situation, we have a conformally flat submanifold $f: M^{n} \rightarrow \mathbb{R}^{n+p}$, which clearly forms a genuine conformal pair with any conformal diffeomorphism $\bar{f}: M^{n} \rightarrow U \subset \mathbb{R}^{n}$ onto an open subset of $\mathbb{R}^{n}$. Theorem 12.39 then provides a geometric construction of all conformally flat Euclidean submanifolds with dimension $n \geq 4$ and codimension $p \leq n-3$ free of flat points. See Theorem 16.10 for a precise statement and a direct proof of this result.

### 12.8 Singular genuine deformations

When studying the possible isometric deformations of a compact Euclidean submanifold with codimension two, one is naturally led to consider isometric extensions that may have singular points, that is, that may fail to be immersions at some points. In fact, the necessity of allowing singularities in isometric extensions arises already in the study of local isometric deformations, as discussed at the end of this section.

The isometric immersions $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ are said to admit singular isometric extensions when there exist an embedding $j: M^{n} \hookrightarrow N^{n+\ell}$ into a manifold $N^{n+\ell}, 0<\ell \leq \min \{p, q\}$, and maps $F: N^{n+\ell} \rightarrow \mathbb{R}^{n+p}$ and $\hat{F}: N^{n+\ell} \rightarrow \mathbb{R}^{n+q}$, which are isometric immersions on $N^{n+\ell} \backslash j(M)$, such that $f=F \circ j$ and $\hat{f}=\hat{F} \circ j$ as in (1). Thus the maps $F$ and $\hat{F}$ may have common singular points, but these are necessarily contained in $j(M)$.

An isometric immersion $\hat{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ is called a genuine deformation in the singular sense of a given isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ if there exists no open subset $U \subset M^{n}$ along which the restrictions $\left.f\right|_{U}$ and $\left.\hat{f}\right|_{U}$ admit singular isometric extensions. In this case, since $f$ is also a singular genuine deformation of $\hat{f}$, we refer to $\{f, \hat{f}\}$ simply as a genuine pair in the singular sense.

In order to derive necessary conditions for an isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ to admit a genuine deformation $\hat{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ in the singular sense, we first prove the next two propositions.

Proposition 12.42. Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be an isometric immersion and let $\pi: D \rightarrow M^{n}$ be a tangent vector subbundle of rank $d>0$. Assume that there does not exist an open subset $U \subset M^{n}$ and $X \in \Gamma\left(\left.D\right|_{U}\right)$ such that the map $F: U \times \mathbb{R} \rightarrow \mathbb{R}^{m}$ given by

$$
F(x, t)=f(x)+t f_{*} X(x)
$$

is an immersion on some open neighborhood of $U \times\{0\}$ except on $U \times\{0\}$ itself. Then for any $x \in M^{n}$ there is an open neighborhood $V$ of the origin in $D(x)$ such that $f_{*}(x) V \subset f(M)$. In particular, $f$ is d-ruled along each connected component of an open dense subset of $M^{n}$.

Proof: Given $X \in \Gamma(D)$, define $F: M^{n} \times \mathbb{R} \rightarrow \mathbb{R}^{m}$ by

$$
F(x, t)=f(x)+t f_{*} X(x) .
$$

For each $t \in \mathbb{R}$, define $T_{t} \in \Gamma(\operatorname{End}(T M))$ by

$$
T_{t}=I+t K
$$

where $K \in \Gamma(\operatorname{End}(T M))$ is given by $K(Z)=\nabla_{Z} X$. Then

$$
\begin{equation*}
F_{*} \partial / \partial t=f_{*} X \text { and } F_{*} Z=f_{*} T_{t}(Z)+t \alpha(X, Z) \tag{12.79}
\end{equation*}
$$

for all $Z \in \mathfrak{X}(M)$. By the assumption, for any $x \in M^{n}$ there must exist a sequence $\left(x_{j}, t_{j}\right) \rightarrow(x, 0)$, with $t_{j} \neq 0$ for all $j$, such that rank $F_{*}\left(x_{j}, t_{j}\right)=n$. Since $T_{t} \rightarrow I$ as $t \rightarrow 0$, it follows from 12.79 that, for $j$ large enough, there exists $Y_{j} \in T_{x_{j}} M$ such that

$$
F_{*}\left(x_{j}, t_{j}\right) Y_{j}=f_{*} X
$$

By the second equation in (12.79), this is equivalent to

$$
\begin{equation*}
T_{t_{j}} Y_{j}=X\left(x_{j}\right) \text { and } \alpha\left(Y_{j}, X\left(x_{j}\right)\right)=0 \tag{12.80}
\end{equation*}
$$

Let $U$ be an open neighborhood of $x$ with compact closure where $\|K\|<c$ for some $c>1$. If $t \in \mathcal{J}=\left(-1 / c^{2}, 1 / c^{2}\right)$, then at any point of $U$ we have

$$
\|t K\| \leq 1 / c<1
$$

Since

$$
T_{t} \sum_{i=0}^{N}(-t)^{i} K^{i}=I+(-t K)^{N+1}
$$

it follows that $T_{t}$ is invertible on $U$ for any $t \in \mathcal{J}$, with

$$
\begin{equation*}
T_{t}^{-1}=\sum_{i \geq 0}(-t)^{i} K^{i} \tag{12.81}
\end{equation*}
$$

By 12.80), for $j$ sufficiently large we have

$$
\begin{equation*}
\alpha\left(T_{t_{j}}^{-1} X\left(x_{j}\right), X\left(x_{j}\right)\right)=0 . \tag{12.82}
\end{equation*}
$$

We claim that $\alpha(\mathcal{S}, X)=0$, where

$$
\mathcal{S}=\operatorname{span}\left\{X, K(X), K^{2}(X), \ldots\right\}
$$

Assume otherwise, and let $k$ be the first integer such that $\alpha\left(K^{k}(X), X\right) \neq 0$. Choose $x \in M^{n}$ such that $\alpha\left(K^{k}(X(x)), X(x)\right) \neq 0$. It follows from 12.81) and 12.82) that

$$
\sum_{i \geq k}\left(-t_{j}\right)^{i} \alpha\left(K^{i} X\left(x_{j}\right), X\left(x_{j}\right)\right)=0 .
$$

Dividing all terms by $t_{j}^{k}$ and letting $j \rightarrow+\infty$ we obtain

$$
\alpha\left(K^{k} X(x), X(x)\right)=0
$$

This is a contradiction, and the claim is proved.
We see from 12.81) that $T_{t}^{-1}(X) \in \mathcal{S}$ on $U$ for any $t \in \mathcal{J}$. From the claim and the second equation in (12.79) we obtain

$$
F_{*} T_{t}^{-1}(X)=f_{*} X
$$

Thus rank $F_{*}=n$ on $U \times \mathcal{J}$, and hence $F(U \times \mathcal{J})=f(U)$, that is, $f(U)$ contains the segment $t f_{*} X$ for $t \in \mathcal{J}$.

Proposition 12.43. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ be isometric immersions, and let $\mathcal{T}: L \rightarrow \hat{L}$ be a parallel vector bundle isometry between vector subbundles $L \subset N_{f} M$ and $\hat{L} \subset N_{\hat{f}} M$ that preserves the second fundamental forms. Let

$$
\phi: \Gamma\left(f_{*} T M \oplus L\right) \times \mathfrak{X}(M) \rightarrow \Gamma\left(L^{\perp} \oplus \hat{L}^{\perp}\right)
$$

be defined by (12.21), with $L_{1}=L=L_{0}$, and set $\ell=\operatorname{rank} L$. For each $x \in M^{n}$, define $\phi_{Y}(x)=\phi(x)(\cdot, Y)$, where $Y \in R E(\phi(x))$ is a right regular element of $\phi(x)$. Then, on any open subset where

$$
\begin{equation*}
\rho(x)=\operatorname{dim} \phi_{Y}\left(f_{*} T_{x} M \oplus L(x)\right) \tag{12.83}
\end{equation*}
$$

takes a constant value $\rho$, the map $x \mapsto D(x)=k e r \phi_{Y}(x)$ defines an isotropic smooth subbundle with respect to $\phi$ of rank

$$
\begin{equation*}
d=n+\ell-\rho \geq n-p-q+3 \ell . \tag{12.84}
\end{equation*}
$$

Proof: The bilinear form $\phi(x)$ is flat for any $x \in M^{n}$ by Lemma 12.9; hence Proposition 4.6 implies that

$$
\begin{equation*}
\phi(x)\left(D(x), T_{x} M\right) \subset \operatorname{Im} \phi_{Y}(x) \cap\left(\operatorname{Im} \phi_{Y}(x)\right)^{\perp} . \square \tag{12.85}
\end{equation*}
$$

Theorem 12.44. Under the assumptions of Theorem 12.44 , suppose in addition that $f$ and $\hat{f}$ form a genuine pair in the singular sense. Then any isotropic subbundle $D$ of $f_{*} T M \oplus L$ of rank $d$ with respect to $\phi$ is a tangent subbundle, and the isometric immersions $f$ and $\hat{f}$ are mutually d-ruled along each connected component of an open dense subset of $M^{n}$, with $d=n+\ell-\rho \geq n-p-q+3 \ell$.

Proof: If $D$ was not a tangent subbundle, there would exist an open subset of $M^{n}$ along which $D$ would decompose orthogonally as

$$
D=D \cap f_{*} T M \oplus \Lambda
$$

for some nontrivial subbundle $\Lambda$ of $L$. But in this case $\left.f\right|_{U}$ and $\left.\hat{f}\right|_{U}$ would admit regular ruled isometric extensions by Lemma 12.9 applied to $L_{1}=L=L_{0}$. This contradicts the assumption that $f$ and $\hat{f}$ form a genuine pair. Applying Proposition 12.42 to the distribution $D$ constructed in Proposition 12.43 yields the last assertion.

Remarks 12.45. (i) Proposition 12.43 and Theorem 12.44 still hold if $\ell=0$, that is, if $\mathcal{T}=0$ and $\phi=\phi_{0}=\alpha^{f} \oplus \alpha^{\hat{f}}$.
(ii) Unlike is the case in Theorem 12.10, the estimate of the dimension of the rulings in Theorem 12.44 does not make use of Lemma 4.20 .

An isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ is called genuinely rigid in the singular sense in $\mathbb{R}^{n+q}$, for a given $q$, if for any isometric immersion $\hat{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ there exists an open dense subset of $M^{n}$ along each connected component of which $f$ and $\hat{f}$ admit singular isometric extensions.

Examples of Euclidean submanifolds of codimension two that are genuinely rigid but not genuinely rigid in the singular sense can be constructed as follows.

Let $f: N^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, be a Sbrana-Cartan hypersurface locally parametrized, in terms of the Gauss parametrization, by the map $\psi: \Lambda \rightarrow \mathbb{R}^{n+1}$ defined on the normal bundle $\Lambda=N_{g} L$ of a surface $g: L^{2} \rightarrow \mathbb{S}^{n}$ and given by

$$
\psi(y, w)=\gamma g+g_{*} \nabla \gamma+w .
$$

By part $(i)$ of Proposition 7.19, the map $\psi$ is regular at $(y, w) \in \Lambda$ if and only if the self-adjoint operator

$$
P_{w}(y)=\gamma(y) I+\operatorname{Hess} \gamma(y)-A_{w}
$$

on $T_{y} L$ is nonsingular. For any $y \in L^{2}$, the operator

$$
P_{0}(y)=\gamma(y) I+\operatorname{Hess} \gamma(y)
$$

is nonsingular. At $y \in L^{2}$, take $w \in N_{g} L(y)$ in the open dense subset where $A_{w} \neq 0$ and consider the line $\psi(y, t w)$ with $t \in \mathbb{R}$. Clearly, the operator $P_{t w}(y)$ is singular either for 0,1 or 2 values of $t$. Thus the submanifold parametrized by $\psi$ has the same number of singular points along that line. Whenever we have a constant number 1 or 2 of singular points along each line, the subset $\Sigma^{n-1}$ of $N^{n}$ of singular points forms a smooth hypersurface in $\Lambda$. In that case, $P$ has constant rank one along $\Sigma^{n-1}$ and $h=\left.\psi\right|_{\Sigma}$ is an immersion. In this situation, the Gauss parametrization $\psi$ provides a singular extension of $h$. Notice that the normal bundle of $h$ is

$$
N_{h} \Sigma=\operatorname{span}\{\varphi, Z\}
$$

where $\operatorname{span}\{Z\}=\operatorname{ker} P$. Since $\varphi$ is constant along the leaves of relative nullity of the hypersurface, it follows that rank $A_{\varphi}^{h}=1$. Moreover, it is easy to see that rank $A_{Z}^{h} \leq 2$ and that the index of relative nullity is $n-3$ at any point. Under the isometric deformation of $f$, the tensor $A_{Z}^{h}$ remains the same, since it depends only on the metric of $N^{n}$, whereas $A_{\varphi}^{h}$ is determined by the second fundamental form of $f$ and thus has to change, inducing an isometric deformation of $h$. Therefore $h$ is not genuinely rigid in the singular sense. On the other hand, it was shown in [184] that there are examples of submanifolds in codimension two as above that are genuinely rigid.

### 12.9 Nonparallel first normal bundle

The results of this section show that the geometric structure of an isometric immersion that carries a nonparallel first normal bundle of low rank must satisfy strong conditions.

For a 1-regular isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ such that $\operatorname{dim} N_{1}=p<m-n$, define $\phi: \Gamma\left(N_{1}^{\perp}\right) \times \mathfrak{X}(M) \rightarrow \Gamma\left(N_{1}\right)$ by

$$
\phi(\mu, X)=\left(\nabla \frac{1}{X} \mu\right)_{N_{1}} .
$$

Since $\phi$ is clearly $C^{\infty}$-bilinear, it can be regarded as a section of $\operatorname{Hom}^{2}\left(N_{1}^{\perp}, T M ; N_{1}\right)$.
For each $x \in M^{n}$, denote by $s(x)>0$ the dimension of the vector subspace $\mathcal{S}(x) \subset N_{1}(x)$ given by

$$
\mathcal{S}(x)=\operatorname{span}\left\{\phi(\mu, X): \mu \in N_{1}^{\perp}(x) \text { and } X \in T_{x} M\right\} .
$$

Notice that if $s(x)=s$ is constant, then the subspaces $\mathcal{S}(x)$ form a smooth normal vector subbundle $\mathcal{S}=\mathcal{S}_{f}$ of $N_{1}$.

Theorem 12.46. Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be a 1-regular locally substantial connected submanifold such that $p<m-n$ and $s(x)=s$ is constant with $0<s<n$. If also $s \leq 6$ then one of the following possibilities holds:
(i) $s=p$ and $f$ has index of relative nullity $\nu_{f} \geq n-p$.
(ii) $s=1<p$ and $f$ has a flat extension $F: N^{n+p-1} \rightarrow \mathbb{R}^{m}$ with index of relative nullity $\nu_{F}=n+p-2$ and such that $N_{1}^{F}$ is nonparallel of rank one.
(iii) $1<s<p$ and there is an open dense subset of $M^{n}$, the union of open subsets $U_{k, d}$ with $d \geq n-s$ and $n-d \leq k \leq q=n-d+p-s$, such that:
(a) $\left.f\right|_{U_{q, d}}$ is d-ruled and $\mathcal{S}_{f}$ is parallel in $\mathbb{R}^{m}$ along the rulings, and
(b) $\left.f\right|_{U_{k, d}}, k<q$, has a $(n+p-k-s)$-ruled extension $F: N^{n+q-k} \rightarrow \mathbb{R}^{m}$ with $N_{1}^{F}$ nonparallel of rank $p+k-q$ and $\mathfrak{S}_{F}$ parallel along the rulings. If $k=n-d$, then the rulings are the relative nullity leaves of $F$.
Moreover, if $s=2$ then $U_{k, d}=\emptyset$ for $k \geq 5$.
Proof: In the sequel we define several tangent and normal subspaces and assume, for the sake of simplicity, that they have constant dimension and thus form subbundles of the tangent and normal bundles.

First we show that $\phi$ satisfies

$$
\begin{equation*}
\mathcal{N}(\phi)=\mathcal{N}\left(\alpha_{\delta}\right) . \tag{12.86}
\end{equation*}
$$

Notice that

$$
Y \in \mathcal{N}(\phi) \text { if and only if } A_{\nabla_{\frac{1}{Y}} \mu} X=0
$$

for all $\mu \in \Gamma\left(N_{1}^{\perp}\right)$ and $X \in \mathfrak{X}(M)$. Also,

$$
Y \in \mathcal{N}\left(\alpha_{\delta}\right) \text { if and only if } A_{\nabla \frac{1}{X} \mu} Y=0
$$

for all $\mu \in \Gamma\left(N_{1}^{\perp}\right)$ and $X \in \mathfrak{X}(M)$. Thus 12.86 follows from the Codazzi equation

$$
A_{\nabla_{\frac{1}{X} \mu}} Y=A_{\nabla_{\frac{1}{Y} \mu}} X
$$

for all $\mu \in \Gamma\left(N_{1}^{\perp}\right)$ and $X, Y \in \mathfrak{X}(M)$.
Assertion: The normal subbundle $P=\mathcal{S} \oplus N_{1}^{\perp}$ is parallel along $D=\mathcal{N}\left(\alpha_{\mathcal{S}}\right)$ in $\mathbb{R}^{m}$.
Let $\mu_{1} \in \Gamma\left(N_{1}^{\perp}\right)$ be a unit vector field such that $\mu_{1} \in R E(\phi)$, and set

$$
\phi_{\mu_{1}}=\phi\left(\mu_{1},\right) .
$$

Let $s_{1}$ be the rank of the subbundle $S_{1} \subset \mathcal{S}$ defined by

$$
S_{1}(x)=\phi_{\mu_{1}}\left(T_{x} M\right)
$$

Then the tangent subspaces

$$
D_{1}(x)=\operatorname{ker} \phi_{\mu_{1}}(x)
$$

form a tangent subbundle of rank $n-s_{1}$ such that $D \subset D_{1}$.
By the Ricci equation,

$$
\nabla_{Y}^{\perp} \nabla_{X}^{\perp} \mu_{1}-\nabla_{X}^{\perp} \nabla_{Y}^{\perp} \mu_{1}-\nabla_{[Y, X]}^{\perp} \mu_{1}=0
$$

for all $X, Y \in \mathfrak{X}(M)$. If $Y \in \Gamma\left(D_{1}\right)$, it follows that

$$
\nabla_{Y}^{\perp}\left(\nabla_{X}^{\perp} \mu_{1}\right)_{S_{1}}+\nabla_{Y}^{\perp}\left(\nabla_{X}^{\perp} \mu_{1}\right)_{N_{1}^{\perp}}=\nabla_{X}^{\perp} \nabla_{Y}^{\perp} \mu_{1}+\nabla_{[Y, X]}^{\perp} \mu_{1} \in \Gamma(P)
$$

for any $X \in \mathfrak{X}(M)$. By Proposition 4.6, also the second term on the left-hand side belongs to $\Gamma(P)$. Hence $\nabla_{Y}^{\frac{1}{Y}} \delta \in \Gamma(P)$ for all $\delta \in \Gamma\left(S_{1}\right)$ and $Y \in \Gamma\left(D_{1}\right)$, and the assertion follows.

By the assertion, we are under the assumptions of Proposition 12.4. Hence there is a $R$-ruled extension $F: N^{n+r} \rightarrow \mathbb{R}^{m}$ of $f$ defined along a tubular neighborhood of the 0 -section of the affine bundle $\pi: \Lambda^{r} \rightarrow M^{n}$ given by (12.6). The rulings are

$$
R(\lambda)=D(\pi(\lambda)) \oplus \Lambda(\pi(\lambda))
$$

and satisfy $R=\mathcal{N}\left(\alpha_{\mathcal{P}}^{F}\right)$, where $\mathcal{P}(\lambda)=P(\pi(\lambda)) \subset N_{F} N$. Moreover, the subbundle $\mathcal{P}(\lambda)$ is parallel in $\mathbb{R}^{m}$ along the rulings, but clearly not parallel along $T N$. If $r=0$, then $f$ is already ruled, and the dimension of the rulings cannot be increased by the extension procedure.

Assertion: The rank of $D$ satisfies

$$
\begin{equation*}
\operatorname{rank} D \geq n-s \tag{12.87}
\end{equation*}
$$

As above, let $\mu_{1} \in \Gamma\left(N_{1}^{\perp}\right)$ be a unit vector field such that $\mu_{1} \in R E(\phi)$ and $s_{1}=\operatorname{rank} S_{1}$. Since the assertion holds if $s_{1}=s$, suppose that $s_{1}<s$ and consider the orthogonal splitting

$$
\mathcal{S}=S_{1} \oplus S_{1}^{\perp}
$$

Let $\psi: \Gamma\left(N_{1}^{\perp}\right) \times \mathfrak{X}(M) \rightarrow \Gamma\left(S_{1}^{\perp}\right)$ be the map defined by

$$
\psi(\mu, X)=\left(\nabla_{X}^{\perp} \mu\right)_{S_{1}^{\perp}},
$$

which gives rise to a section of $\operatorname{Hom}^{2}\left(N_{1}^{\perp}, T M ; S_{1}^{\perp}\right)$. Take $\mu_{2} \in R E(\phi) \cap R E(\psi)$ and set

$$
t=\operatorname{rank} \psi\left(\mu_{2}, T M\right)
$$

Then $S_{2}=\phi_{\mu_{2}}(T M)$ satisfies

$$
\operatorname{rank}\left(S_{1}+S_{2}\right)=s_{1}+t \text { and rank } S_{1} \cap S_{2}=s_{1}-t
$$

It follows using Proposition 4.6 that

$$
\begin{align*}
\operatorname{rank} D_{1} \cap D_{2} & \geq \operatorname{rank} D_{1}-\operatorname{rank} S_{1} \cap S_{2} \\
& \geq n-2 s_{1}+t \tag{12.88}
\end{align*}
$$

where $D_{2}=\operatorname{ker} \phi_{\mu_{2}}$. If $t=s_{1}$, then $S_{1} \cap S_{2}=0$. Thus $D_{1}=D_{2}$. In particular, 12.87) holds if $s_{1}=1$, since this forces $t=1$. Therefore we may assume that $s_{1} \geq 2$.

We first analyze the case $t=1$. In this case, $H=\operatorname{ker} \psi\left(\mu_{2},\right)$ has rank $n-1$. The Codazzi equation gives

$$
A_{\nabla_{\frac{1}{Z}} \mu_{2}} X=A_{\nabla \frac{1}{x} \mu_{2}} Z=0
$$

for all $Z \in \Gamma\left(D_{1}\right)$ and $X \in \Gamma(H)$. This implies that rank $\phi_{\mu_{2}}\left(D_{1}\right) \leq 1$. Otherwise, there would exist a two-dimensional plane in $S_{2}$ such that the corresponding shape operators would have the same kernel of codimension one. But then a vector in this plane would belong to $N_{1}^{\perp}$, and this is a contradiction. It follows that

$$
\operatorname{rank} D_{1} \cap D_{2} \geq n-s_{1}-1
$$

If $\mathcal{S}=S_{1}+S_{2}$, then (12.87) holds, since $s=s_{1}+1$ and $D=D_{1} \cap D_{2}$. If otherwise, we just repeat the process and obtain subspaces $S_{1}, \ldots, S_{m}$ and $D_{1}, \ldots, D_{m}$, $m=s-s_{1}+1$, such that $\mathcal{S}=S_{1}+\cdots+S_{m}$ and

$$
\begin{aligned}
\operatorname{rank} D_{1} \cap \cdots \cap D_{m} & \geq n-s_{1}-m+1 \\
& =n-s .
\end{aligned}
$$

Then $D=D_{1} \cap \cdots \cap D_{m}$, and (12.87) follows.
We may assume that $t \geq 2$. We argue for the case $s=6$, the other cases being similar and easier. If $t=s_{1}$, then $s_{1}=2$, 3 . In these cases, we have seen that $D_{1}=D_{2}$, and thus (12.87) holds. Hence, we may assume that $s_{1}>t \geq 2$. Therefore, it remains
to consider the cases $\left(s_{1}, t\right)=(3,2)$ and $\left(s_{1}, t\right)=(4,2)$. In the latter case, $\mathcal{S}=S_{1}+S_{2}$, and 12.87 ) follows from 12.88 . In the first case,

$$
\operatorname{rank}\left(S_{1}+S_{2}\right)=5, \quad \text { rank } S_{1} \cap S_{2}=1 \text { and rank } D_{1} \cap D_{2} \geq n-4
$$

We now repeat the process and obtain $S_{3}$ such that

$$
\mathcal{S}=S_{1}+S_{2}+S_{3} \text { and } \operatorname{rank} S_{i} \cap S_{j}=1 \text { if } 1 \leq i \neq j \leq 3
$$

In this case, it is clear that rank $D \geq n-5$, and this proves the assertion.
To conclude the proof of the theorem, first assume that $s=p$. Then (12.86) and (12.87) imply that $\nu_{f} \geq n-p$.

Suppose that $s<p$. For each positive integer $d$, let $U_{d}$ denote the interior of the subset of all $x \in M^{n}$ such that the subspace $D(x)$ has dimension $d$. By 12.87), $d \geq n-s$. By the lower semi-continuity of the dimension, the set $\cup_{d} U_{d}$ is open and dense in $M^{n}$. Now let $U_{k, d}$ be the interior of the subset of all $x \in U_{d}$ such that the subspace $\Gamma(x)$ given by (12.3) has dimension $k$. Then (12.4), with $\ell=p-s$, gives $n-d \leq k \leq q$. Again by the lower semi-continuity of the dimension, $\cup_{k} U_{k, d}$ is open and dense in $U_{d}$.

In view of 12.86 and the parallelism of $P$ along $D$ in $\mathbb{R}^{m}$, Proposition 12.4 applies to $\left.f\right|_{U_{k, d}}$. If $k=q$, by part $(i)$ we see that $\left.f\right|_{U_{q, d}}$ is $d$-ruled and that $P$ (hence $\mathcal{S}$ ) is constant in $\mathbb{R}^{m}$ along the rulings. If $k<q$, it follows that $f$ admits a ruled extension $F: N^{n+r} \rightarrow \mathbb{R}^{m}, r=n-d+\ell-k=q-k$, with rulings of dimension $n+\ell-k=n+p-k-s$. Moreover, there is an orthogonal splitting $N_{F} N=\mathcal{L} \oplus \mathcal{P}$, where $\mathcal{P}$ is the parallel extension (in $\mathbb{R}^{m}$ ) of $P$ along the rulings, such that rank $\mathcal{L}=p-s-r$. In particular,

$$
\operatorname{rank} N_{1}^{F}=p-r=p+k-q
$$

Finally, if $k=n-d$, by part (ii) the rulings of $F$ coincide with the leaves of its relative nullity distribution.

The global assertion in (ii) for the case $1=s<p$ is due to the fact that $s=1$ implies $d=1$, and also $k=1$, as follows from (12.2). It is also a consequence of 12.2 ) that $k \leq 4$ if $s=2$; hence in this case $U_{k, d}=\emptyset$ for $k \geq 5$.

Notice that the submanifolds in the condition of Theorem 12.46 are either ruled or extend to ruled ones with nonparallel $N_{1}$. In that respect, observe that the ruled extensions in part (ii) and item (b) of part (iii) are as the submanifolds in part (i) and item (a) of part (iii), respectively.

Remark 12.47. Without the regularity assumptions, the statement of Theorem 12.46 holds on connected components of an open dense subset of the manifold.

We say that the submanifold in Theorem 12.46 does not extend if the extension procedure is trivial, which is the case in part $(i)$ of Proposition 12.4 .

Corollary 12.48. Assume that $f$ in Theorem 12.46 does not extend along any open subset. Then $s \geq 2$ and the submanifold is $d$-ruled with $d \geq n-s$.

For a ruled Euclidean submanifold, it follows from Exercise 3.14 that the Ricci curvature satisfies $\operatorname{Ric}(X) \leq 0$ for any vector $X$ tangent to a ruling, with equality if and only if $X$ belongs to the relative nullity subspace. Hence, we have the following immediate consequence of Theorem 12.46 .

Corollary 12.49. Under the assumptions of Theorem 12.46, cases (i) and (a) of (iii) cannot occur if Ric $_{M}>0$. If Ric $_{M} \geq 0$ then $\left.f\right|_{U_{q, d}}$ in case (a) of (iii) satisfies $\nu_{f}=d$.

To illustrate Theorem 12.46 we discuss next the cases $p=1,2$ and 3 .
Example 12.50. The case $p=1$. Here the only possibility is that $s=1$, and hence $\nu_{f}=n-1$. In particular, the manifold $M^{n}$ is flat.

Submanifolds as above can be easily described parametrically. Consider the image under the normal exponential map of a parallel normal subbundle of the normal bundle of a curve with nonvanishing curvature.

Example 12.51. The case $p=2$. We only have the following two possibilities:
(i) $s=2$, and hence $\nu_{f}=n-2$.
(ii) $s=1$, in which case $f$ admits a flat extension $F: N^{n+1} \rightarrow \mathbb{R}^{m}$ such that $\nu_{F}=n$ and $N_{1}^{F}$ is nonparallel of rank one.

Example 12.52. The case $p=3$. One of the following possibilities holds:
(i) $s=3$ and $f$ satisfies $\nu_{f} \geq n-3$.
(ii) $s=1$ and $f$ has a flat extension $F: N^{n+2} \rightarrow \mathbb{R}^{m}$ such that $\nu_{F}=n+1$ and $N_{1}^{F}$ is nonparallel of rank one.
(iii) $s=2<k=3$, in which case $f$ is $(n-2)$-ruled and $\mathcal{S}$ is constant along the rulings.
(iv) $s=2=k$ and $f$ has an extension $F: N^{n+1} \rightarrow \mathbb{R}^{m}$ such that $\nu_{F}=n-1$ and $N_{1}^{F}$ has rank two.

Observe that $F$ in part ( $i i$ ) of Example 12.51 and in Example 12.52 is as $f$ given in Example 12.50. Also, the extension $F$ in (iv) of Example 12.52 is as $f$ in part $(i)$ of Example 12.51

### 12.10 Notes

The results on genuine isometric deformations and genuine isometric rigidity given in this chapter are due to Dajczer-Florit [99]. The proof of Theorem 12.10 given here considerably simplifies the original one in [99]. The consequences of that theorem related to compositions of isometric immersions were previously obtained by DajczerTojeiro [133] and Dajczer-Florit [98. The results on genuine isometric deformations in the singular sense are due to Florit-Guimarães [185], and those on genuine conformal deformations to Florit-Tojeiro [186]. Corollary 12.38 on compositions of conformal immersions and the result in Exercise 12.16 were previously obtained by Dajczer-Tojeiro [138], [140], respectively.

Classifying the genuinely deformable submanifolds of space forms and their deformations, both in the local and global cases, is a very difficult problem even in codimension two. In that codimension, only the compact case has already been solved by Dajczer-Gromoll [113] and extended to other cases by Theorem 13.21 due to FloritGuimarães [185] to be given in the next chapter. For the local problem, it follows from Theorem 4.23 that one may divide its study in three distinct cases, depending on whether the submanifold has rank two, three or four. The simplest case of submanifolds of rank two has been considered in several papers and is quite well understood; see [100], [107], [121], [122] and [184]. For submanifolds of higher rank the classification problem remains wide open. In this case, all that is known so far is the result for complete minimal Kaehler submanifolds due to Dajczer-Gromoll [114] and the two families of minimal examples of rank four in Euclidean space and sphere constructed by Dajczer-Vlachos [156], [152], respectively.

Going in another direction, it follows from Corollary 12.27 that a Euclidean hypersurface of dimension $n \geq 3$ is genuinely rigid in $\mathbb{R}^{n+2}$ unless its rank is less than or equal to three. Another viewpoint for rigidity results of this sort was considered by Moore [258], based on the notion of isometric homotopy. Euclidean hypersurfaces with rank two that admit genuine deformations in codimension two have been classified by Dajczer-Florit-Tojeiro [107], extending the Sbrana-Cartan theory of Chapter 11 to this situation. The classification of Euclidean hypersurfaces with rank three that admit genuine deformations in codimension two is still an open problem.

With respect to Euclidean submanifolds that admit genuine conformal deformations, apart from the classification by Cartan [65] of the hypersurfaces $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ of dimension $n \geq 5$ that admit genuine conformal deformations $\tilde{f}: M^{n} \rightarrow \mathbb{R}^{n+1}$, which is the subject of Chapter 17, the only other classification result up to now for such a class of submanifolds is the description due to Chion-Tojeiro [88] of the Euclidean hypersurfaces of dimension $n \geq 6$ that carry a principal curvature of multiplicity $n-2$ and admit genuine conformal deformations $\tilde{f}: M^{n} \rightarrow \mathbb{R}^{n+2}$, extending to the conformal realm the aforementioned result of [107]. Notice that, by Corollary 12.38, a Euclidean hypersurface of dimension $n \geq 6$ is genuinely conformally rigid in $\mathbb{R}^{n+2}$ if all of its principal curvatures have multiplicity less than or equal to $n-4$.

The study of Riemannian manifolds of dimension $n \geq 4$ that admit isometric immersions into space forms with distinct curvatures was initiated do Carmo-Dajczer
[55], where the result in Exercise 4.3 was obtained. The geometric classification given by Corollary 12.29 of hypersurfaces of dimension $n \geq 4$ of $\mathbb{Q}_{c}^{n+1}$ that also admit an isometric immersion into $\mathbb{Q}_{\tilde{c}}^{n+p}$ with $c<\tilde{c}$ and $p<n-2$ was obtained by Dajczer-Tojeiro [133]. The dual case $c>\tilde{c}$ was also treated by Dajczer-Tojeiro [135]. The existence of hypersurfaces of dimension three of $\mathbb{Q}_{c}^{4}$ with three distinct principal curvatures that also admit an isometric immersion into $\mathbb{Q}_{\tilde{c}}^{4}$ with $c \neq \tilde{c}$ and are not produced by the construction in Corollary 12.29 was observed in [133], where Lemma 12.30 was proved. The characterization of such hypersurfaces given by Theorem 12.31 is due to CanevariTojeiro [52], and explicit parametrized examples have been produced by means of the Ribaucour transformation by Canevari-Tojeiro [53].

The characterization of the submanifolds that carry a nonparallel first normal bundle of low rank is due to Dajczer-Tojeiro [148]. Examples of submanifolds as in part ( $i$ ) of Examples 12.51 have been studied by Dajczer-Florit 97 and Dajczer-Morais [121], where a parametric classification has been obtained in most cases.

### 12.11 Exercises

Exercise 12.1. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}, p \leq 5$, be an isometric immersion and let $q<n-p$ be a positive integer. If $\operatorname{Ric}_{M} \geq 0$ and $\nu_{f}<n-p-q$ at any point show that $f$ is genuinely rigid in $\mathbb{R}^{n+q}$.

Exercise 12.2. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}, 2 \leq p \leq 3$, be an isometric immersion. If $\nu_{f}<n-2 p$ at any point, show that $f$ is genuinely rigid in $\mathbb{R}^{n+p}$.

Exercise 12.3. Let $g: M^{n} \rightarrow \mathbb{R}^{n+1}$ be a simply connected nowhere surface-like ruled hypersurface without flat points.
(i) Prove that the family $\left\{f_{\gamma}\right\}_{\gamma \in \Gamma}$ of ruled isometric immersions $f_{\gamma}: M^{n} \rightarrow \mathbb{R}^{n+2}$ is parametrized by the set $\Gamma$ of triples of smooth arbitrary functions in an interval.
(ii) Show that if $f: M^{n} \rightarrow \mathbb{R}^{n+2}$ is a ruled simply connected submanifold without flat points then $M^{n}$ admits an isometric immersion as a ruled hypersurface.

Hint for (i): From the assumptions and the Sbrana-Cartan classification, all the $f_{\gamma}$ 's have the same rulings. Thus there are smooth orthonormal frames $\{X, Y\}$ of $\Delta^{\perp}$ and $\{\xi, \eta\}$ of $N_{f} M$ such that the second fundamental form is of the form

$$
\left.A_{\xi}\right|_{\Delta^{\perp}}=\left[\begin{array}{ll}
a & b  \tag{12.89}\\
b & 0
\end{array}\right] \quad \text { and }\left.\quad A_{\eta}\right|_{\Delta^{\perp}}=\left[\begin{array}{ll}
\lambda & 0 \\
0 & 0
\end{array}\right]
$$

where $b>0$ is given by $b^{2}=-K(X, Y) \neq 0, \lambda \geq 0$ and the subspaces $R=\operatorname{span}\{Y\} \oplus \Delta$ are tangent to the rulings. It follows from the Codazzi equation for $A_{\eta}$ that $\xi$ and $\eta$ are parallel in the normal connection along the rulings. Then any ruled isometric immersion $\tilde{f}: M^{n} \rightarrow \mathbb{R}^{n+2}$ is determined by three smooth functions

$$
\left\{a, \lambda, \psi=\left\langle\nabla_{X}^{\perp} \xi, \eta\right\rangle\right\}
$$

such that (12.89) satisfies the Codazzi and Ricci equations. In fact, these equations are

$$
\left\{\begin{array} { l } 
{ Y ( a ) = \langle \nabla _ { X } X , Y \rangle a + X ( b ) } \\
{ Y ( \lambda ) = \langle \nabla _ { X } X , Y \rangle \lambda + b \psi } \\
{ Y ( \psi ) = \langle \nabla _ { X } X , Y \rangle \psi - b }
\end{array} \quad \text { and } \quad \left\{\begin{array}{l}
Z(a)=\left\langle\nabla_{X} X, Z\right\rangle a+\left\langle\nabla_{X} Y, Z\right\rangle b \\
Z(\lambda)=\left\langle\nabla_{X} X, Z\right\rangle \lambda \\
Z(\psi)=\left\langle\nabla_{X} X, Z\right\rangle \psi
\end{array}\right.\right.
$$

where $Z \in R$. It follows that $\{a, \lambda, \psi\}$ can be arbitrarily prescribed along an integrable curve of $X$, and then they are completely determined.

Exercise 12.4. Show that any element $f \in\left\{f_{\gamma}\right\}_{\gamma \in \Gamma}$ in Exercise 12.3 with $\lambda \neq 0$ is a composition $f=F \circ g$, where $F: U \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+2}$ is a flat hypersurface. Verify that the extension of $f$ is locally defined by $F: M^{n} \times(-\epsilon, \epsilon) \rightarrow \mathbb{R}^{n+2}$ given by

$$
F(x, t)=f(x)+t\left(\psi f_{*} X+\lambda \xi\right)
$$

Exercise 12.5. Let $g: M^{n} \rightarrow \mathbb{R}^{n+1}$ be a ruled hypersurface as in Exercise 12.3 and let $f \in\left\{f_{\lambda}\right\}_{\lambda \in \Lambda}$ be such that $\lambda \neq 0$ in (12.89) and the functions $a$ for both submanifolds differ. Then let $\hat{f}=h \circ g$ be a composition, where $h: U \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+2}$ is an isometric immersion, such that $g(M) \subset U$ and the generic condition $\nu_{\tilde{f}}=n-3$ is satisfied. Show that the pair $f, \hat{f}$ is genuine.

Exercise 12.6. Show that any isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{n+2}, n \geq 5$, with index of relative nullity $\nu \leq n-5$ at any point, is genuinely rigid in $\mathbb{R}^{n+2}$.

Exercise 12.7. Check that the proof of Theorem 12.23 yields the stronger estimate

$$
d \geq n-p-q+2 \ell+\ell_{1} .
$$

Exercise 12.8. Give a direct proof that $\mathcal{T}_{D}$ in Corollary 12.13 is parallel.
Exercise 12.9. Show that in Corollary 12.13 one also has

$$
D^{d}=\mathcal{N}\left(\alpha_{L_{D}^{\perp}}\right) \cap \mathcal{N}\left(\alpha_{\hat{L}_{\frac{1}{D}}}\right)
$$

Hint: Use that $L_{D} \subset L^{\ell}$.
Exercise 12.10. Under the hypotheses of Theorem 12.10, prove the estimate

$$
d \geq n-p-q+2 \ell .
$$

Hint: Instead of (12.24) use the estimate given in Remark 12.11.
Exercise 12.11. Assume that the isometric immersions $f$ and $\hat{f}$ considered in the statement of Theorem 12.10 are 1-regular. Verify that the proof still works if we replace $p$ and $q$ in the estimate 12.23 of $d$ by the ranks of $N_{1}^{f}$ and $N_{1}^{\hat{f}}$.

Exercise 12.12. Extend Theorem 12.19 to the case of nonflat ambient space forms.
Hint: Use Exercise 6.2.
Exercise 12.13. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ be isometric immersions and let $\mathcal{T}: L^{\ell} \rightarrow \hat{L}^{\ell}$ be a parallel vector bundle isometry between vector subbundles of $N_{f} M$ and $N_{\hat{f}} M$ that preserves the second fundamental forms.
(i) Prove that the bilinear form $\phi: \Gamma\left(f_{*} T M \oplus L\right) \times \mathfrak{X}(M) \rightarrow \Gamma\left(L^{\perp} \oplus \hat{L}^{\perp}\right)$ given by (12.21) is a Codazzi tensor, that is, if we define

$$
\left(\bar{\nabla}_{X} \phi\right)(\xi, Y)=\bar{\nabla}_{X} \phi(\xi, Y)-\phi\left(\left(\tilde{\nabla}_{X} \xi\right)_{L \oplus T M}, Y\right)-\phi\left(\xi, \nabla_{X} Y\right)
$$

then

$$
\left(\bar{\nabla}_{X} \phi\right)(\xi, Y)=\left(\bar{\nabla}_{Y} \phi\right)(\xi, X)
$$

where $\bar{\nabla}$ is the Levi-Civita connection on $L^{\perp} \oplus \hat{L}^{\perp}$.
(ii) Let $U \subset M^{n}$ be an open subset where the nullity subspaces $\mathcal{N}(\phi)(x) \subset T_{x} M$ have constant dimension. Show that the distribution $x \in U \mapsto \mathcal{N}(\phi)(x)$ is smooth and integrable and that the image subspaces $\mathcal{S}(\phi)$ are parallel along the leaves.

Exercise 12.14. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ be an isometric immersion and let $q$ be a positive integer such that $p+q<n$. If $M^{n}$ has positive Ricci curvature show that $f$ is genuinely rigid in the singular sense in $\mathbb{R}^{n+q}$.

Exercise 12.15. Let $h: L^{n} \rightarrow \mathbb{R}^{m}$ be a regular isometric immersion such that $N_{2}^{h}$ is nonparallel. Let $M^{2 n}$ denote $T L$ with the zero section of $\pi: T L \rightarrow L^{n}$ excluded, and let $f: M^{2 n} \rightarrow \mathbb{R}^{m}$ be the immersion given by

$$
f(X)=h(\pi(X))+h_{*} X .
$$

Show that the following facts hold:
(i) $f_{*} T M=h_{*} T L \oplus N_{1}^{h}$.
(ii) $N_{1}^{f}=N_{2}^{h}$ is not parallel.

Prove that any such $f$ is as in part ( $i$ ) of Theorem 12.46 .
Exercise 12.16. Let $f, g: M^{n} \rightarrow \mathbb{R}^{n+2}, n \geq 7$, be conformal immersions. Suppose that $\nu_{f}^{c} \leq n-5$ everywhere. Prove that there exists an open and dense subset $\mathcal{W} \subset M^{n}$ such that, for each connected component $U$ of $\mathcal{W}$, either $\left.f\right|_{U}$ and $\left.g\right|_{U}$ are conformally congruent or there exist conformal nowhere conformally congruent hypersurfaces $\bar{f}, \bar{g}: N^{n+1} \rightarrow \mathbb{R}^{n+2}$ and a conformal immersion $i: U \rightarrow N^{n+1}$ such that $\left.f\right|_{U}=\bar{f} \circ i$ and $\left.g\right|_{U}=\bar{g} \circ i$.

## Chapter 13

## Deformations of complete submanifolds

The main theorems of this chapter are of global nature and show that complete Euclidean submanifolds with low codimension that allow isometric deformations are rather special. A first basic result in this direction is a theorem due to Sacksteder, which asserts that any compact Euclidean hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, is isometrically rigid, provided that the subset of totally geodesic points of $f$ does not disconnect $M^{n}$. Even if that subset disconnects the manifold, only discrete isometric deformations are possible. In fact, any such deformation is a reflection with respect to an affine hyperplane. The corresponding versions of that result for hypersurfaces of the sphere and the hyperbolic space are also discussed.

The conclusion in Sacksteder's theorem does not hold for complete hypersurfaces. However, in this case it turns out that continuous isometric deformations can occur only along completely ruled subsets called ruled strips, as long as no open subset of the hypersurface is a cylinder over a surface in $\mathbb{R}^{3}$ or over a hypersurface of $\mathbb{R}^{4}$ of a special type that carries a one-dimensional relative nullity distribution with complete leaves. The proof of this result is preceded by a description of the geometric structure of complete Euclidean submanifolds whose rank is at most two.

A far-reaching extension of Sacksteder's theorem is then discussed. Namely, it is shown that any isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ of a compact Riemannian manifold is genuinely rigid in the singular sense in $\mathbb{R}^{n+q}$ if $p+q \leq \min \{4, n-1\}$. This is derived by first showing the general fact that, for any other isometric immersion $\hat{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ with $p+q \leq n-1$ of a compact Riemannian manifold, there exists an open dense subset of $M^{n}$ along each connected component of which $f$ and $\hat{f}$ either admit singular isometric extensions or are mutually $D^{d}$-ruled with $d \geq n-p-q+3$.

### 13.1 The Sacksteder theorem

The main result of this section asserts that a compact (respectively, complete) hypersurface $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ with $n \geq 3$ and $c \leq 0$ (respectively, $n \geq 4$ and $c>0$ ) is
rigid, except for some trivial discrete isometric deformations that can only occur when the subset of totally geodesic points of $f$ disconnects $M^{n}$.

A key role in the proof of Sacksteder's result will be played by Theorem 7.9. We will also need the next fact for submanifolds of spheres.

Lemma 13.1. Let $f: M^{n} \rightarrow \mathbb{S}_{c}^{m}$ be an isometric immersion, and let $D$ be a smooth totally geodesic distribution on an open subset $U \subset M^{n}$ such that $D(x) \subset \Delta(x)$ for all $x \in U$. If there exists a geodesic ray $\gamma:[0, \infty) \rightarrow U$ such that $\gamma^{\prime}(0) \in D(\gamma(0))$, then the splitting tensor $C_{\gamma^{\prime}(0)}$ of $D$ cannot have real eigenvalues.

Proof: Let $\gamma:[0, \infty) \rightarrow U$ be a geodesic ray such that $T=\gamma^{\prime}(0) \in D(\gamma(0))$. Assume that

$$
C_{T} Y_{0}=\lambda Y_{0}
$$

for some $\lambda \in \mathbb{R}$ and $0 \neq Y_{0} \in D^{\perp}(\gamma(0))$. From the proof of Theorem 7.7, there exists a unique vector field $Y=Y(t)$ along $\gamma$ such that $Y(0)=Y_{0}, Y(t) \in D^{\perp}(\gamma(t))$ for all $t \in[0, \infty)$, and

$$
\begin{equation*}
\frac{D}{d t} Y+C_{\gamma^{\prime}} Y=0 \tag{13.1}
\end{equation*}
$$

Moreover, $Y$ is also a solution of the differential equation

$$
\frac{D^{2}}{d t^{2}} Y+c Y=0
$$

The solution of the preceding equation with initial conditions

$$
Y(0)=Y_{0} \text { and } \frac{D Y}{d t}(0)=-C_{\gamma^{\prime}} Y(0)=-\lambda Y_{0}
$$

is

$$
Y(t)=(\cos \sqrt{c} t-(\lambda / \sqrt{c}) \sin \sqrt{c} t) Y_{0}(t)
$$

where $Y_{0}(t)$ is the parallel transport of $Y_{0}$ along $\gamma$. But then $Y(t)$ has a zero, which contradicts the fact that a solution of (13.1) never vanishes if it is nonzero at some point.

As pointed out before the statement of Theorem 1.11, given isometric immersions $f, \hat{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ there always exists a vector bundle isometry $\phi: N_{f} M \rightarrow N_{\hat{f}} M$, regardless of $M^{n}$ being orientable or not. Moreover, $\phi$ and $-\phi$ are the only such vector bundle isometries. We assume that one of them has been fixed, and whenever we write $\hat{A}=A$ on a subset of $M^{n}$ we mean that the shape operators $A_{\xi}$ and $\hat{A}_{\phi \xi}$ of $f$ and $\hat{f}$, respectively, coincide at every point of that subset.

Theorem 13.2. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$ be an isometric immersion of a compact (respectively, complete) Riemannian manifold with $n \geq 3$ and $c \leq 0$ (respectively, $n \geq 4$ and $c>0$ ). Then, for any other isometric immersion $\hat{f}: M^{n} \rightarrow \mathbb{Q}_{c}^{n+1}$, the following assertions hold:
(i) The set $B$ of totally geodesic points of $f$ coincides with that of $\hat{f}$.
(ii) On each connected component of $M \backslash B$ either $A=\hat{A}$ or $A=-\hat{A}$.

In particular, if $B$ does not disconnect $M^{n}$, then $f$ is rigid.
Proof: We use the notations in Theorem 7.9. Consider the open subset

$$
M_{3}=\left\{x \in M^{n}: \nu^{*}(x) \leq n-3\right\} .
$$

If $c \leq 0$ then $M_{3}$ is nonempty by Corollary 1.6 and Exercise 6.2. If $c>0$, the statement is trivially satisfied if $\nu^{*}(x)=n$ for all $x \in M^{n}$, that is, if both $f$ and $\hat{f}$ are totally geodesic. If this is not the case, we claim that $M_{3}$ is also nonempty if $n \geq 4$. Assume otherwise. Then the minimum value $\nu_{0}^{*}$ of $\nu^{*}(x)=\operatorname{dim} \Delta^{*}(x)$, for $x \in M^{n}$, is either $n-1$ or $n-2$.

We first prove that, for any point $x$ in an open subset where $\nu^{*}$ is everywhere equal to either $n-1$ or $n-2$, there always exists $T \in \Delta^{*}(x)$ such that the splitting tensor $C_{T}$ of $\Delta^{*}$ has a real eigenvalue. This is clear if $n \geq 2$ and $\nu^{*}=n-1$. Suppose that $\nu^{*}=n-2$ and suppose otherwise that $C_{T}$ has no real eigenvalues for all $T \in \Delta^{*}(x)$. If $T_{1}, \ldots, T_{n-2}$ is a basis of $\Delta^{*}(x)$, then for all $Z \in \Delta^{*}(x)^{\perp}$ and $a, a_{1}, \ldots, a_{n-2} \in \mathbb{R}$ the equation

$$
0=a Z+\sum_{i=1}^{n-2} a_{i} C_{T_{i}} Z=a Z+C_{\Sigma_{i} a_{i} T_{i}} Z
$$

implies $a=a_{1}=\cdots=a_{n-2}=0$. Hence $Z, C_{T_{1}} Z, \ldots, C_{T_{n-2}} Z$ are linearly independent in $\Delta^{*}(x)^{\perp}$, which is a contradiction if $n \geq 4$.

Now let $U$ be the open subset where $\nu^{*}$ attains its minimum value $\nu_{0}^{*}$. Given any $x \in U$ and $T \in \Delta^{*}(x)$ such that $C_{T}$ has a real eigenvalue, let $\gamma:[0, \infty) \rightarrow M^{n}$ be a geodesic ray with $\gamma(0)=x$ and $\gamma^{\prime}(0)=T$. Then $\gamma(t) \in U$ for all $t \in[0, \infty)$ by Corollary 7.10, in contradiction with Lemma 13.1. This completes the proof that $M_{3}$ is nonempty if $c>0$ and $n \geq 4$.

By the Beez-Killing rigidity theorem, on each connected component of $M_{3}$ the hypersurfaces $f$ and $\tilde{f}$ differ by an isometry of $\mathbb{Q}_{c}^{n+1}$. Thus $\hat{A}=A$ or $\hat{A}=-A$ on each such component, and hence $\hat{A}(x)= \pm A(x)$ at any $x \in \bar{M}_{3}$, where we denote by $\bar{S}$ the closure of a subset $S \subset M^{n}$.

Set $M_{2}=M^{n} \backslash \bar{M}_{3}$ and consider the open subset

$$
U_{2}=\left\{x \in M_{2}: \nu^{*}(x)=n-2\right\} .
$$

Let $x \in U_{2}$ and let $L$ be the maximal leaf of $\left.\Delta^{*}\right|_{U_{2}}$ through $x$. Let $\gamma: \mathbb{R} \rightarrow M^{n}$ be a geodesic such that $\gamma(0)=x$ and $\gamma^{\prime}(0)=T \in \Delta^{*}(x)$, where $T$ is chosen so that $C_{T}$ has a real eigenvalue if $c>0$. By the compactness of $M^{n}$ for $c \leq 0$, and by Lemma 13.1 for $c>0$, there must exist $b>0$ such that $\gamma([0, b)) \subset L$ but $\gamma(b) \notin U_{2}$. Since $\gamma(b) \in M_{2}$ by Theorem 7.9 , we must have $\gamma(b) \in \bar{M}_{3}$, and therefore $\hat{A}= \pm A$ at $\gamma(b)$. But also by Theorem 7.9, the splitting tensor $C_{\gamma^{\prime}}$ of $\Delta^{*}$ extends smoothly to $t=b$, and the
differential equation (7.6) holds on $[0, b]$. By the uniqueness of solutions of (7.6) with a given initial condition, we conclude that $A= \pm \hat{A}$ at $x$.

Now set $M_{1}=M_{2} \backslash \bar{U}_{2}$ and

$$
U_{1}=\left\{x \in M_{1}: \nu^{*}(x)=n-1\right\} .
$$

By the same argument as above we see that $A= \pm \hat{A}$ at $x$, and this concludes the proof.

A description of the set of isometric deformations of a compact hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ of dimension $n \geq 3$ follows from the assertion in part (ii) of Theorem 13.2 and the next fact.

Proposition 13.3. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be any isometric immersion and let $S$ be a connected component of the subset of totally geodesic points. Then $f(S)$ is contained in an $n$-dimensional affine subspace of $\mathbb{R}^{n+1}$ tangent to $f$ along $S$.

Proof: We use the fact that if $\varphi: N^{n} \rightarrow \mathbb{R}$ is a smooth function on a differentiable manifold $N^{n}$ and $B$ is a connected subset of $N^{n}$ all of whose points are critical for $\varphi$, then $\varphi$ must be constant on $B$. This is because $\varphi(B)$ is an interval in $\mathbb{R}$, which by Sard's theorem must contain regular values of $\varphi$ unless it reduces to a point.

Let $\varphi_{a} \in C^{\infty}(M)$ be given by $\varphi_{a}=\langle N, a\rangle$, where $N$ is a unit normal vector field along $f$ and $a$ is any constant vector field in $\mathbb{R}^{n+1}$. Since $d \varphi_{a}=0$ at every point of $S$, the above fact implies that $\varphi_{a}$ is constant on $S$. Thus $N$ has a constant value $N_{0} \in \mathbb{R}^{n+1}$ along $S$. Applying the preceding fact again for $\psi \in C^{\infty}(M)$ given by $\psi=\left\langle f, N_{0}\right\rangle$ implies that also $\psi$ is constant on $S$, and the conclusion follows.

Corollary 13.4. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be an isometric immersion of a compact Riemannian manifold of dimension $n \geq 3$. Then any isometric immersion $\hat{f}: M^{n} \rightarrow \mathbb{R}^{n+1}$ is congruent to the composition of $f$ with reflections with respect to affine hyperplanes containing connected components of a separating set $S$ of totally geodesic points of $f$.

Remark 13.5. Theorem 13.2 is false for $c>0$ and $n=3$. The universal cover of the three-dimensional hypersurface in $S^{4}$ given in Exercise 7.14 is a complete minimal hypersurface with constant index of relative nullity $\nu=1$ that is not rigid as seen in Exercise 11.5 ,

### 13.2 Ruled submanifolds

Section 13.4 will provide an extension of Theorem 13.2 to the case of complete Euclidean hypersurfaces of dimension $n \geq 3$. With that goal in mind, in this section we discuss some general properties of complete ruled submanifolds with constant index of relative nullity.

Let $N^{n}$ denote a Riemannian manifold, possibly with boundary $\partial N$. An isometric immersion $f: N^{n} \rightarrow \mathbb{R}^{n+p}$ is called ruled if $N^{n}$ admits a smooth integrable distribution
$D$ of codimension one whose leaves (rulings) are tangent along $\partial N$ and such that $f$ maps each leaf onto an open subset of an affine subspace of $\mathbb{R}^{n+p}$. If all the rulings are complete manifolds, then $f$ is said to be completely ruled. A connected component of a completely ruled submanifold is called a ruled strip. Thus the rulings in each ruled strip form an affine vector bundle over a curve with or without end points.

Let $f: N^{n} \rightarrow \mathbb{R}^{n+p}$ be a ruled submanifold. Take a unit speed curve $c: I \rightarrow N^{n}$ perpendicular to the rulings, let $d c / d s=T_{0}, T_{1}, \ldots, T_{n-1}$ be an orthonormal frame field along $c$ such that $T_{1}, \ldots, T_{n-1}$ are parallel with respect to the induced connection in $c^{*} D$, and let $N_{1}, \ldots, N_{p}$ be a parallel orthonormal frame of $c^{*} N_{f} N$ with respect to the induced connection. Hence, denoting $\tilde{T}_{i}=f_{*} T_{i}$ for $0 \leq i \leq n-1$, there exist $\varphi_{i}$, $\beta_{i j}, \gamma_{j} \in C^{\infty}(I), 1 \leq i \leq n-1$ and $1 \leq j \leq p$, such that

$$
\left\{\begin{array}{l}
\tilde{\nabla}_{d / d s} \tilde{T}_{0}=-\sum_{i} \varphi_{i} \tilde{T}_{i}+\sum_{j} \gamma_{j} N_{j}  \tag{13.2}\\
\tilde{\nabla}_{d / d s} \tilde{T}_{i}=\varphi_{i} \tilde{T}_{0}+\sum_{j} \beta_{i j} N_{j} \\
\tilde{\nabla}_{d / d s} N_{j}=-\gamma_{j} \tilde{T}_{0}-\sum_{i} \beta_{i j} \tilde{T}_{i} .
\end{array}\right.
$$

The vector field

$$
\varphi=-\sum_{i} \varphi_{i} T_{i}=\nabla_{T_{0}} T_{0}
$$

along $c$ is the curvature vector of $c$ in $N^{n}$. On the other hand,

$$
\beta\left(T_{i}\right)=\sum_{j} \beta_{i j} N_{j}=\alpha\left(T_{0}, T_{i}\right),
$$

whereas

$$
\gamma=\sum_{j} \gamma_{j} N_{j}=\alpha\left(T_{0}, T_{0}\right)
$$

is the mean curvature vector field of $f$ along $c$, since

$$
\alpha\left(T_{i}, T_{k}\right)=0, \quad 1 \leq i, k \leq n-1
$$

We may parametrize $f(N)$ near $\tilde{c}=f \circ c$ by means of the map $F: I \times \mathbb{R}^{n-1} \rightarrow$ $\mathbb{R}^{n+p}$, given by

$$
\begin{equation*}
F(s, t)=\tilde{c}(s)+\sum_{i=1}^{n-1} t_{i} \tilde{T}_{i}(s) \tag{13.3}
\end{equation*}
$$

restricted to a neighborhood of $I \times\{0\}$. Conversely, if $\varphi_{i}, \beta_{i j}, \gamma_{j} \in C^{\infty}(I)$ are arbitrarily prescribed for $1 \leq i \leq n-1$ and $1 \leq j \leq p$, then (13.2) has a solution frame field, which is unique up to a fixed orthogonal transformation. Defining

$$
\tilde{c}(s)=\int_{s_{0}}^{s} \tilde{T}_{0}(r) d r
$$

the map $F$ given in (13.3) provides, at regular points (in particular near $I \times\{0\}$ ), a parametrization of a smooth ruled submanifold.

Lemma 13.6. The map 13.3) is regular at any point where the ruled submanifold $f: N^{n} \rightarrow \mathbb{R}^{n+p}$ is defined.

Proof: In order to decide where $F$ is singular, at a point $(s, t)$ we compute

$$
\begin{equation*}
F_{t_{j}}=\tilde{T}_{j}, \quad F_{s}=(1-\langle\varphi, T\rangle) \tilde{T}_{0}+\beta(T) \tag{13.4}
\end{equation*}
$$

where $T=\sum_{i} t_{i} T_{i}$. Thus $F$ has maximal rank if and only if $F_{s} \neq 0$, that is, if

$$
\begin{equation*}
\left\|F_{s}\right\|^{2}=(1-\langle\varphi, T\rangle)^{2}+\|\beta(T)\|^{2}>0 \tag{13.5}
\end{equation*}
$$

Hence $F$ has precisely one singular point on each line in a direction $T$ that is not perpendicular to $\varphi$ and lies in the kernel of $\beta$. For each $s \in I$, the set of singular points of $F$ in $\{s\} \times \mathbb{R}^{n-1}$ is either empty or an affine hyperplane in the kernel of $\beta$, that is, in the relative nullity of $f$ along $c(s)$.

Consider now any open neighborhood $W$ of $I \times\{0\}$ in $I \times \mathbb{R}^{n-1}$ such that

$$
W_{s}=W \cap\{s\} \times \mathbb{R}^{n-1}
$$

is star-shaped with respect to $s \times 0$ and $F$ maps $W_{s}$ into the ruling through $c(s)$ for all $s \in I$. Then the exponential map $\left.F\right|_{W_{s}}$ is injective in $N^{n}$ by construction, and we show next that it must have maximal rank.

Take $t \in \mathbb{R}^{n-1}$ such that $t \in W_{s}$ on some open interval $I_{0} \subset I$. The field $T$ is parallel along $c$ in the bundle of rulings, and

$$
\bar{c}(s)=c(s)+T(s)
$$

is the reparametrization $\bar{c}=c_{1} \circ \psi$ of the unit-speed trajectory $c_{1}: I_{0} \rightarrow N^{n}$ orthogonal to the rulings such that $c_{1}\left(s_{0}\right)=\bar{c}\left(s_{0}\right)$ for some $s_{0} \in I_{0}$, where $\psi$ is the $C^{1}$ arc-length function of $\bar{c}$ on $I_{0}$, measured from $s_{0}$. If $T_{1}$ is parallel in the bundle of rulings along $c_{1}$ with $T_{1}\left(s_{0}\right)=-T\left(s_{0}\right)$, then $T_{1} \circ \psi=-T$ on $I_{0}$. Since

$$
c=c_{1} \circ \psi+T_{1} \circ \psi
$$

is regular, hence $\psi^{\prime} \neq 0$. But this means that $F_{s} \neq 0$ in (13.4), and $F$ is regular.
Proposition 13.7. Let $f: N^{n} \rightarrow \mathbb{R}^{m}$ be a ruled submanifold. Suppose that $f$ has constant index of relative nullity $\nu \leq n-2$ and that the leaves of relative nullity are complete. Then every point in $N^{n}$ has an open neighborhood $W$ such that $\left.f\right|_{W}$ extends uniquely to a smoothly ruled strip with constant index of relative nullity $\nu$. If $N^{n}$ is simply connected, then $f$ extends globally to a ruled strip.

Proof: We show that the index of relative nullity of $f$ is constant along a ruling if it is at most $n-2$ somewhere. It suffices to consider the parametrization (13.3). Let $\alpha^{F}$ be the second fundamental form of $F$ at $(s, t)$ and let $U$ be a parallel vector field tangent to the rulings, that is, $U=\sum_{i} u_{i} T_{i}$, where $u=\left(u_{1}, \ldots, u_{n-1}\right)$ is constant. By 13.2),

$$
\begin{equation*}
\alpha^{F}(\partial / \partial s, U)=\left(\langle\varphi, U\rangle T_{0}+\beta(U)\right)^{\perp} \tag{13.6}
\end{equation*}
$$

where the normal component is obtained by subtracting the projection in the direction of $F_{s}$. With

$$
L(U)=\langle\varphi, U\rangle T_{0}+\beta(U),
$$

we see that $F_{s}$ given by (13.4) is in the image of $L$ if and only if $\operatorname{rank} L=\operatorname{rank} \beta+1$. We conclude that

$$
\operatorname{rank} \alpha^{F}(\partial / \partial s,)=\operatorname{rank} \beta
$$

is constant on the ruling. Now, $\operatorname{ker} \beta$ is the relative nullity of $f$ exactly where $\beta$ is not identically zero, that is, where $\nu \leq n-2$.

### 13.3 Submanifolds of rank at most two

The main result of this section is a description of the complete submanifolds of dimension $n \geq 3$ in Euclidean space such that the $\operatorname{rank} \rho=n-\nu$ of the generalized Gauss map satisfies $\rho \leq 2$ at any point.

First we establish some basic facts about Euclidean submanifolds that carry a relative nullity distribution with complete leaves.

Proposition 13.8. Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be an isometric immersion. Assume that the leaves of the relative nullity distribution $\Delta$ on an open subset $U \subset M^{n}$ are complete. Then, for all $x_{0} \in U$ and any unit vector $T_{0} \in \Delta\left(x_{0}\right)$, the only possible real eigenvalue of the splitting tensor $C_{T_{0}}$ is zero. Moreover, $\operatorname{ker} C_{\gamma^{\prime}}$ is parallel along the geodesic $\gamma$ through $x_{0}$ tangent to $T_{0}$.

Proof: Suppose that $C_{0}=C_{T_{0}}$ has nonzero real eigenvalues $\lambda_{1}, \ldots, \lambda_{k}$, and set

$$
\tau^{-1}=\max _{1 \leq j \leq k}\left|\lambda_{j}\right| .
$$

By (7.3), the splitting tensor $C_{\gamma^{\prime}}$ along $\gamma$ satisfies

$$
\begin{equation*}
\frac{D}{d t} C_{\gamma^{\prime}}=C_{\gamma^{\prime}}^{2} \tag{13.7}
\end{equation*}
$$

Notice that the operator $I-t C_{0}$ is invertible for $-\tau<t<\tau$, and that

$$
\begin{equation*}
C_{t}=\mathcal{P}_{0}^{t} C_{0}\left(I-t C_{0}\right)^{-1}\left(\mathcal{P}_{0}^{t}\right)^{-1} \tag{13.8}
\end{equation*}
$$

is a solution of (13.7) with initial condition $C_{0}$ for $t=0$, where $\mathcal{P}_{0}^{t}$ denotes parallel transport along $\gamma$ from $\gamma(0)$ to $\gamma(t)$. From the uniqueness of solutions of (13.7) with a given initial condition it follows that

$$
C_{\gamma^{\prime}(t)}=C_{t}
$$

on $(-\tau, \tau)$. Then either $(\tau-t)^{-1}$ or $-(\tau+t)^{-1}$ is an eigenvalue of $C_{\gamma^{\prime}(t)},-\tau<t<\tau$, which diverges as $t \rightarrow \tau$ or $t \rightarrow-\tau$. This is a contradiction, because $C_{\gamma^{\prime}(t)}$ is well defined for all $t \in \mathbb{R}$, in view of the completeness assumption.

Corollary 13.9. If $f: M^{n} \rightarrow \mathbb{R}^{m}$ satisfies the assumptions of Proposition 13.8 and has constant rank $\rho=2$ on the open subset $U \subset M^{n}$, then the dimension of the orthogonal complement coker $C$ of $\operatorname{ker} C$ in $\Delta$ is at most one.

Proof: Otherwise, the image of $C$ would contain a self-adjoint $C_{T} \neq 0$, for dimension reasons, and that is a contradiction.

Corollary 13.10. Suppose that $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ has constant rank $\rho=2$ and that $\operatorname{dim} \operatorname{coker} C=1$ on an open subset $U \subset M^{n}$. Let $T$ be a (local) unit vector field perpendicular to $\operatorname{ker} C$. If $C_{T}$ is invertible, then the vector subbundle $f_{*} \operatorname{ker} C$ is parallel along $U$. In addition, if the leaves in $\operatorname{ker} C$ in $U$ are complete then $\left.f\right|_{U}$ is a cylinder over a submanifold $g: L^{3} \rightarrow \mathbb{R}^{3+p}$ with complete one-dimensional leaves of relative nullity.

Proof: It follows from (7.5) that

$$
\left\langle\nabla_{X} S, T\right\rangle C_{T} Y=\left\langle\nabla_{Y} S, T\right\rangle C_{T} X
$$

for all $S \in \Gamma(\operatorname{ker} C)$ and $X, Y \in \Gamma\left(\Delta^{\perp}\right)$. Thus

$$
\left\langle\nabla_{X} S, T\right\rangle Y-\left\langle\nabla_{Y} S, T\right\rangle X \in \Gamma\left(\operatorname{ker} C_{T}\right)
$$

Since $C_{T}$ is invertible, then

$$
\left\langle\nabla_{X} S, T\right\rangle Y-\left\langle\nabla_{Y} S, T\right\rangle X=0
$$

which implies that $\nabla_{X}^{v} S \in \Gamma(\operatorname{ker} C)$ for any $X \in \Gamma\left(\Delta^{\perp}\right)$. But

$$
\left\langle\nabla_{X} S, Y\right\rangle=-\left\langle C_{S} X, Y\right\rangle=0,
$$

hence $\nabla_{X} S \in \Gamma(\operatorname{ker} C)$. On the other hand, from (7.2) we have

$$
C_{\nabla_{R} S}=\nabla_{R} C_{S}-C_{S} C_{R}=0
$$

for any $R \in \Gamma(\Delta)$, and therefore $\nabla_{R} S \in \Gamma(\operatorname{ker} C)$. Thus $\operatorname{ker} C$ is parallel along $U$. Since $\operatorname{ker} C \subset \Delta$, then $f_{*} \operatorname{ker} C$ is parallel in the ambient space, and the conclusion follows.

Theorem 13.11. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}, n \geq 3$, be an isometric immersion with rank $\rho \leq 2$ at any point of a complete Riemannian manifold. Then there is a disjoint decomposition

$$
\begin{equation*}
M^{*}=M_{0} \cup M_{1} \cup M_{2} \tag{13.9}
\end{equation*}
$$

of the open subset $M^{*}$ of all points in $M^{n}$ with $\rho=2$, where $M_{0}$ is closed in $M^{*}$ and $M_{2}$ open, such that $f$ is as follows on each connected component of the interior $M_{0}^{o}$ of $M_{0}, M_{2}$ and the closure of each connected component of $M_{1}^{o}$, respectively:
(i) A cylinder over a surface $L^{2}$ in $\mathbb{R}^{2+p}$.
(ii) A cylinder over a submanifold $L^{3}$ in $\mathbb{R}^{3+p}$ with complete one-dimensional leaves of relative nullity such that the splitting tensor has complex conjugate eigenvalues.
(iii) A ruled strip.

Moreover, if $M_{0}^{o}=\emptyset=M_{2}$, then $f$ is completely ruled everywhere, and a cylinder over a curve on each component of the complement of the closure of $M_{1}$.

Proof: In view of Corollary 13.9, according to the degeneracy of the splitting tensor $C$ of $f$ we have a disjoint decomposition

$$
M^{*}=M_{0} \cup M_{1} \cup M_{2}
$$

such that $M_{0}$ is the closed (in $M^{*}$ ) subset of points where $C=0, M_{1} \cup M_{2}$ is the open subset where coker $C$ is (locally) spanned by a unit vector field $T \in \Gamma(\Delta)$ and $M_{2}$ is the open set of points where rank $C_{T}=2$. By (13.8), these three sets are saturated, that is, they are unions of (complete) leaves of relative nullity.

On any connected component of $M_{0}^{o}$, it follows from Proposition 7.4 that $f$ is a cylinder over a surface in $\mathbb{R}^{2+p}$. Moreover, $f$ is as in part (ii) on any connected component of $M_{2}$ by Corollary 13.10. By Proposition 13.8, the eigenvalues of the splitting tensor of such submanifold are complex conjugate.

We claim that the smooth distribution $\Delta \oplus \operatorname{ker} C_{T}$ is totally geodesic on $M_{1}^{o}$, and that the restriction of $f$ to each of its leaves is totally geodesic, that is, $\left.f\right|_{M_{1}^{o}}$ is ruled. To see this, let ker $C_{T}$ be locally spanned by a unit vector field $X$, that is, $\nabla_{X} T \in \Gamma(\Delta)$. On the other hand, from Proposition 13.8 we see that $\nabla_{T} X=0$. Since $\Delta$ is totally geodesic, we conclude that $\Delta \oplus \operatorname{ker} C_{T}$ is integrable. From (11.4) we have

$$
\begin{equation*}
A_{\xi} C_{T}=C_{T}^{t} A_{\xi} . \tag{13.10}
\end{equation*}
$$

Hence $C_{T}^{t} A_{\xi} X=0$. On the other hand, by Proposition 13.8 both eigenvalues of $C_{T}$ are zero, hence $C_{T}^{t}$ is nilpotent. Therefore

$$
\begin{equation*}
\operatorname{ker} C_{T}^{t}=\operatorname{Im} C_{T}^{t} \tag{13.11}
\end{equation*}
$$

We conclude from (13.10) and (13.11) that

$$
\begin{equation*}
\left\langle A_{\xi} X, X\right\rangle=0 \tag{13.12}
\end{equation*}
$$

for any $\xi \in \Gamma\left(N_{f} M\right)$. Extend $X$ to a local orthonormal frame $X, Y \in \Gamma\left(\Delta^{\perp}\right)$. Since $C_{T}$ is nilpotent, then $C_{T} Y=\mu X$, where $\mu$ is a smooth function. On the other hand, from (7.5) we obtain

$$
\begin{equation*}
\left(\nabla_{X} C_{T}\right) Y=\left(\nabla_{Y} C_{T}\right) X \tag{13.13}
\end{equation*}
$$

It follows that $\nabla_{X} X=0$, which concludes the proof of the claim.
We claim that the rulings in $M_{1}^{o}$ are complete. It also follows from (13.13) that

$$
\begin{equation*}
X(\mu)=\left\langle\nabla_{Y} Y, X\right\rangle \mu \tag{13.14}
\end{equation*}
$$

The leaves of the relative nullity foliation are complete, and thus parallel hyperplanes in each ruling. Therefore any integral curve of $X$ is a line segment in $\mathbb{R}^{n+p}$. It suffices to show that if $\gamma:[0, b] \rightarrow M^{n}$ is the segment in $\mathbb{R}^{n+p}$ whose restriction to $[0, b)$ is an integral curve of $X$, then $\gamma(b) \in M_{1}^{o}$. It follows from Proposition 13.7 that the linear differential equation (13.14) extends smoothly to the point $\gamma(b)$. Now, $\mu \neq 0$ on $[0, b]$, and thus $\gamma(b) \in M_{1}^{o}$, which gives the claim.

The closure $\bar{N}$ in $M^{n}$ of a connected component $N$ of $M_{1}^{o}$ is a smooth submanifold, with possibly nonempty boundary, on which $f$ is a ruled strip. In fact, let $x \in \bar{N}$ and let $x_{j} \in N$ be a sequence such that $x_{j} \rightarrow x$. Now, the rulings $L_{j}$ through $x_{j}$ must converge to a complete totally geodesic Euclidean space $L \cong \mathbb{R}^{n-1}$ through $x$ in $\bar{N}$. Otherwise, we would find subsequences $L_{j}^{\prime}$ and $L_{j}^{\prime \prime}$ converging to limits $L^{\prime}$ and $L^{\prime \prime}$ in $\bar{N}$ which intersect transversally at $x$. But then almost all $L_{j}^{\prime}$ and $L_{j}^{\prime \prime}$ would intersect transversally near the point $x$, which is not possible for leaves of a foliation. It follows that $L \subset \partial \bar{N}$, and $\bar{N}$ is a continuous affine vector bundle over a connected one-dimensional manifold with or without boundary. Notice that if $N_{1}, N_{2}$ are two such completely ruled strips, closed in $M^{*}$ and $N_{1} \cap N_{2} \neq \emptyset$, then $N_{1} \cup N_{2}$ is again a ruled strip.

To prove the last assertion, first consider the subset $M^{* *}$ of all points in $M^{n}$ with $\rho=1$. We claim that all leaves of the relative nullity foliation in the interior of $M^{* *}$ are complete. Otherwise, there is a geodesic $\gamma:[0, b] \rightarrow M^{n}$ such that $\gamma[0, b)$ is contained in a leaf, but $\gamma(b)$ is not. Since $\rho=1$ at $\gamma(b)$, this point lies in the closure of $M^{*}$, which is completely ruled by assumption. But the relative nullity subspace at $\gamma(b)$ is contained in the limit ruling transversal to $\gamma^{\prime}(b)$, and this is a contradiction. It follows from Propositions 7.4 and 13.8 that $f$ is a cylinder on each connected component of the interior of $M^{* *}$. The remaining arguments are straightforward.

Remark 13.12. If $f$ in Theorem 13.11 is real analytic, then the submanifold is either completely ruled or a cylinder over a surface in $\mathbb{R}^{2+p}$ or a cylinder over a threedimensional submanifold in $\mathbb{R}^{3+p}$.

Example 13.13. We give an example of how a ruled strip can be smoothly "attached" to a cylinder over a nonruled surface in $\mathbb{R}^{3}$ as a complete hypersurface in $\mathbb{R}^{n+1}$. Suppose that the smooth unit-speed curve $c$ has a Frenet frame $c^{\prime}=e_{1}, \ldots, e_{n+1}$ with curvatures $\tau_{1}, \ldots, \tau_{n}$. Then

$$
F\left(s, t_{1}, \ldots, t_{n-1}\right)=c(s)+\sum_{j=1}^{n-1} t_{j} e_{2+j}
$$

parametrizes a ruled strip. Assume that $\tau_{1}, \tau_{2} \neq 0$ everywhere, and that $\tau_{k}=0$ exactly on some interval $[a, b]$ for $3 \leq k \leq n$. Then $F$ parametrizes a completely ruled hypersurface which splits as a cylinder over a surface $N^{2}$ in $\mathbb{R}^{3}$ over $[a, b]$. Clearly, $N^{2}$ can be replaced smoothly by a nonruled surface $L^{2}$ over $(a, b)$.

### 13.4 The hypersurface case

In this section we describe all possible isometric deformations of a complete Euclidean hypersurface of dimension $n \geq 3$, thus providing an extension of Theorem 13.2 for $c=0$.

We first discuss the isometric deformations of completely ruled hypersurfaces.
Proposition 13.14. Let $f: N^{n} \rightarrow \mathbb{R}^{n+1}$, $n \geq 3$, be a completely ruled hypersurface. If $f$ is not a cylinder over a surface in $\mathbb{R}^{3}$ on any open subset of $N^{n}$, then any other isometric immersion of $N^{n}$ into $\mathbb{R}^{n+1}$ is also completely ruled, with the same rulings in $N^{n}$.

Proof: The hypersurface $f$ has rank $\rho \leq 2$ at any point of $N^{n}$. On the open set $N^{*}$ of points where $\rho=2$, the leaves of the relative nullity foliation are contained in the rulings, and are thus complete. For any isometric immersion $\tilde{f}: N^{n} \rightarrow \mathbb{R}^{n+1}$, by Exercise 4.1 the subset of points of $N^{n}$ where $\tilde{\rho}=2$ coincides with $N^{*}$, and the decomposition (13.9) remains the same because it is determined by the structure of the splitting tensor of $N^{n}$. By the assumption, $\tilde{f}$ is also completely ruled on each connected component of an open dense saturated subset of $N^{*}$. The conclusion now follows by a continuity argument.

Theorem 13.15. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, be an isometric immersion of a complete Riemannian manifold. Assume that there exists no open subset $U \subset M^{n}$ where the hypersurface is as in parts ( $i$ ) or (ii) of Theorem 13.11. Then $f$ admits continuous isometric deformations only along ruled strips. Moreover, if $f$ is nowhere completely ruled and the subset of totally geodesic points does not disconnect $M^{n}$, then $f$ is rigid.

Proof: Let $\tilde{f}: M^{n} \rightarrow \mathbb{R}^{n+1}$ be any other isometric immersion. Let $U_{k}, \tilde{U}_{k}, k \geq 0$, be the open subsets of $M^{n}$ where the ranks $\rho, \tilde{\rho} \geq k$. By Exercise 4.1, $\tilde{U}_{k}=U_{k}$ whenever $k \geq 2$, and by the Beez-Killing rigidity theorem, $\tilde{A} \equiv \pm A$ on each connected component of $\tilde{U}_{3}=U_{3}$.

First consider the open set

$$
W=U_{2}-\bar{U}_{3}
$$

Through any point $x \in W$ we have $\Delta_{x}=\tilde{\Delta}_{x}$ for the leaves of the relative nullity foliations, again by Exercise 4.1, Let $\gamma:[0, a] \rightarrow M^{n}$ be a geodesic with $\gamma(0)=x$, $\gamma[0, a) \subset \Delta_{x}$ and $\gamma(a) \notin W$. By Theorem 7.7. $\gamma(a) \in \bar{U}_{3}$. Thus $\tilde{A}= \pm A$ at $\gamma(a)$. Now, by (7.6) we have

$$
\begin{equation*}
\nabla_{\gamma^{\prime}} A=C_{\gamma}^{\prime} A \tag{13.15}
\end{equation*}
$$

on $[0, a)$, and 13.15 ) extends smoothly to $[0, a]$ according to Theorem 7.7. Therefore $\tilde{A}(x)= \pm A(x)$. Consider the open saturated subset

$$
V=\{y \in W: \tilde{A}(y) \neq \pm A(y)\}
$$

By the above, all leaves in $V$ are complete. Since $V$ does not contain an open subset isometric to a Riemannian product $L^{3} \times \mathbb{R}^{n-3}$, the proof of Theorem 13.11 shows that $f$ and $\tilde{f}$ are ruled on $V$. We argue next that these rulings must be complete in $V$.

Suppose there is an incomplete ruling $L$ in $V$. Then there exists a geodesic $\delta:[0, b] \rightarrow M^{n}$ such that $\delta[0, b) \subset L, \delta(b) \notin V$ and $\delta^{\prime} \in \Delta^{\perp}$. From Proposition 13.7 it follows that $\delta(b) \in W$, and thus $\tilde{A}=A$ at $\delta(b)$ after possibly changing the local orientation of $\tilde{f}$. Moreover, the differential equation 13.15 with $X=\delta^{\prime}$ extends smoothly to $[0, b]$. Since $\tilde{A}=A$ at $\delta(b)$, we have $\tilde{A}=A$ at $\delta(0)$, and this is a contradiction. Now, $f$ and $\tilde{f}$ are completely ruled on $V$. The closure of a connected component $V_{t}$ of $V$ is a ruled strip. Bear in mind that

$$
\begin{equation*}
\tilde{A}= \pm A \text { on } U_{2}-\bar{V} \tag{13.16}
\end{equation*}
$$

The next step deals with the open subset

$$
W^{\prime}=U_{1} \cap \tilde{U}_{1}-\bar{U}_{2}
$$

We first claim $\Delta=\tilde{\Delta}$ on $W^{\prime}$. Otherwise, consider the open set

$$
V^{*}=\left\{y \in W^{\prime}: \Delta_{y} \neq \tilde{\Delta}_{y}\right\}
$$

and the smooth $(n-2)$-dimensional foliation $y \rightarrow \Gamma_{y}=\Delta_{y} \cap \tilde{\Delta}_{y}$. The leaves are in fact complete affine subspaces. To see this, let $\varepsilon:[0, c] \rightarrow M^{n}$ be a geodesic such that $\varepsilon(0)=y \in V^{*}, \varepsilon[0, c) \subset \Gamma_{y}$ and $\varepsilon(c) \notin W^{\prime}$. We conclude that $\varepsilon(c) \in \bar{U}_{2}$ and $\tilde{A}= \pm A$ at $\varepsilon(c)$. Otherwise, $\varepsilon(c) \in \partial V_{l_{0}}$ for some $l_{0}$, and $\rho=1$ at $\varepsilon(c)$ by Theorem 7.7. This is a contradiction since $L_{\varepsilon(c)}=\Delta_{\varepsilon(c)}$ and $\varepsilon$ is transversal to $L_{\varepsilon(c)}$. Now, in particular, $\tilde{A}= \pm A$ have the same kernel at $\varepsilon(c)$, and then at $\varepsilon(0)=y$, contradicting $y \in V^{*}$. The complete leaves of $\Gamma$ must be parallel both in the leaves of $\Delta$ and $\tilde{\Delta}$. Therefore they are parallel in $M^{n}$, and then in $\mathbb{R}^{n+1}$, along $V^{*}$. This means that $V^{*}$ contains a product $L^{2} \times \mathbb{R}^{n-2}$, which we have excluded. Thus the claim is proved. Now, it follows that $\tilde{A}= \pm A$ on $W^{\prime}$. The argument is analogous to the one applied to $W$, using also the above transversality. In particular,

$$
\begin{equation*}
\tilde{A}= \pm A \text { on } U_{1} \cap \tilde{U}_{1}-\bar{V} \tag{13.17}
\end{equation*}
$$

Finally, the open set

$$
W^{\prime \prime}=U_{1}-\overline{\tilde{U}}_{1}
$$

must be empty, and the same applies to $\tilde{U}_{1}-\bar{U}_{1}$. Otherwise, there exist a point $x \in W^{\prime \prime}$ and a geodesic $\eta:[0, d] \rightarrow M^{n}$ such that $\eta(0)=x, \eta[0, d] \subset \Delta_{x}$ but $\eta(d) \notin W^{\prime \prime}$. Here we use that the relative nullity foliation $\Delta$ cannot be complete on an open subset of $W^{\prime \prime}$ by our assumption. According to Theorem 7.7, we have $\rho=1$ and $\tilde{\rho}=0$ at $\eta(d)$. Again we apply the transversality argument to obtain first that $\eta(d) \notin \bar{V}$. If $\eta(d) \notin U_{1}$, then $\eta(d) \in \bar{U}_{2}-\bar{V}$. If $\eta(d) \in U_{1}$, then

$$
\eta(d) \in U_{1} \cap \overline{\tilde{U}}_{1}-\bar{V} \subset U_{1} \cap \overline{\tilde{U}}_{1}-\bar{U}
$$

Now, 13.16) or 13.17) implies the contradiction $\rho=\tilde{\rho}$ at $\eta(d)$. We have therefore shown that $U_{1} \cap U_{1}$ is dense both in $U_{1}$ and $\tilde{U}_{1}$, and this, together with 13.17, yield $\tilde{A}= \pm A$ on $U_{1}-\bar{V}=\tilde{U}_{1}-\bar{V}$, and thus on $M^{n}-\bar{V}$.

If $V$ is empty and the set of totally geodesic points $M^{n}-U_{1}=M^{n}-\tilde{U}_{1}$ does not disconnect $M^{n}$, then $\tilde{A}=A$ or $\tilde{A}=-A$ on $M^{n}$.

Corollary 13.16. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 4$, be a complete irreducible real analytic hypersurface. Then, unless $f$ is completely ruled, $f$ is rigid in the category of analytic isometric immersions. Moreover, $f$ is also rigid in the $C^{\infty}$-category if the set of totally geodesic points does not disconnect $M^{n}$.

Theorem 13.17. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, be an isometric immersion of a complete Riemannian manifold. Assume that $f$ is not a cylinder over a surface in $\mathbb{R}^{3}$ on any open subset of $M^{n}$. If the scalar curvature of $M^{n}$ is either positive everywhere or is bounded from above by a negative real number, then $f$ is rigid.

Proof: We argue first that $f$ is nowhere completely ruled. Assume otherwise that there exists an open subset $U \subset M^{n}$ such that $f(U)$ is a ruled strip, and let $f(U)$ be parametrized by a map $F: I \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n+1}$ as in (13.3). Then 13.5) and 13.6) yield

$$
\alpha^{F}\left(\partial_{s} /\left|\partial_{s}\right|, V\right) \rightarrow 0 \text { as } t_{j} \rightarrow \infty
$$

where $\partial_{s}=\partial / \partial s$ and $V$ is a parallel vector field tangent to the ruling and transversal to the relative nullity. Thus the scalar curvature of $M^{n}$ approaches zero from below, and that has been excluded.

Suppose that $f$ has index of relative nullity $\nu=n-2$ on some open set $U \subset M^{n}$. It follows from (7.6) and (13.8) that $A\left(I-t C_{T_{0}}\right)$ is parallel along any geodesic $\gamma$ starting at a point $x_{0} \in U$ tangent to a unit vector $T_{0} \in \Delta\left(x_{0}\right)$. In particular, the (not normalized) scalar curvature $s$ along $\gamma$ satisfies

$$
s \operatorname{det}\left(I-t C_{T_{0}}\right)=s_{0}
$$

for some constant constant $s_{0}$. If $\gamma$ can be taken complete, then $s \rightarrow 0$ as $t \rightarrow \pm \infty$, unless $C_{T_{0}}$ is nilpotent. On the other hand, suppose that $C_{T_{0}}$ has nonreal eigenvalues, that is,

$$
C_{T_{0}} Z=(a+i b) Z
$$

where $Z=X+i Y$ and $b \neq 0$. Since $A C_{T_{0}}$ is symmetric by (7.6), from

$$
\left\langle A C_{T_{0}} Z, \bar{Z}\right\rangle=\left\langle A C_{T_{0}} \bar{Z}, Z\right\rangle
$$

we obtain

$$
\langle A X, X\rangle+\langle A Y, Y\rangle=0,
$$

hence $s=\operatorname{det} A<0$. Since $\nu=n-2$ is the minimum value, we can take $U$ saturated by complete leaves of relative nullity. It now follows from Proposition 13.5 that $C_{S}$ is nilpotent for any $S \in \Delta$. Furthermore, by Proposition 7.4 we see that $C \neq 0$ on an
open dense saturated subset $U_{0}$ of $U$. From the proof of Theorem 13.11 it follows that $f$ is ruled on $U_{0}$.

Finally, by one of the crucial arguments in the proof of Theorem 13.15, the immersion $f$ can be deformed along a connected ruled subset $V$ with $\nu=n-2$ only if $V$ is contained in a ruled strip. But such strips cannot exist according to the first part of the proof, and thus $f$ is rigid.

### 13.5 The compact case

The main result of this section shows that any $n$-dimensional compact submanifold of $\mathbb{R}^{n+p}$ is genuinely rigid in the singular sense in $\mathbb{R}^{n+q}$ if $p+q \leq \min \{4, n-1\}$.

We make use of the results in Section 7.2.1. Given isometric immersions $f: M^{n} \rightarrow$ $\mathbb{R}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$, let $\beta, \Delta^{*}$ and $\nu^{*}$ be defined as in that section. For $x \in M^{n}$ and $X \in R E(\beta(x))$, denote $B_{X}=\beta(x)(X$,$) and$

$$
\rho(x)=\operatorname{dim} B_{X}\left(T_{x} M\right)
$$

Proposition 13.18. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ be isometric immersions of a compact Riemannian manifold such that $p+q \leq n-1$. Given $x \in M^{n}$, assume also that $\rho(x) \geq p+q-2$ if $\min \{p, q\} \geq 6$. Then there exist unit vectors $\xi \in N_{f} M(x)$ and $\hat{\xi} \in N_{\hat{f}} M(x)$ such that $A_{\xi}=\hat{A}_{\hat{\xi}}$.

Proof: Let $U \subset M^{n}$ be the open subset of points where unit normal vectors $\xi$ and $\hat{\xi}$ as in the statement do not exist and where $\rho \geq p+q-2$ if $\min \{p, q\} \geq 6$. We claim that $\nu^{*}>0$ on $U$. To prove the claim, first observe that, by assumption, $\mathcal{S}(\beta)$ is nondegenerate at any point of $U$. Thus, if $\min \{p, q\} \leq 5$, the claim follows from the Main Lemma 4.20. Suppose that $\min \{p, q\} \geq 6$. If $X \in R E(\beta)$, using the assumption on $\rho$ we obtain

$$
\operatorname{dim} \operatorname{Im} B_{X} \cap\left(\operatorname{Im} B_{X}\right)^{\perp} \leq \operatorname{dim}\left(\operatorname{Im} B_{X}\right)^{\perp}=p+q-\operatorname{dim} \operatorname{Im} B_{X} \leq 2
$$

It is now easy to see, making use of (4.5), that there exists $Y \in R E(\beta)$ such that $\Delta^{*}=\operatorname{ker} B_{X} \cap \operatorname{ker} B_{Y}$, and hence $\nu^{*}=\left.\operatorname{dim} \operatorname{ker} B_{Y}\right|_{\text {ker } B_{X}} \geq n-p-q>0$, which proves the claim.

Let $V \subset U$ be the open subset where $\nu^{*}=\nu_{0}^{*}$ is minimal on $U$. Fix a point $x_{0} \in V$ and consider the maximal leaf $L$ of $\Delta^{*}$ through $x_{0}$. By compactness of $M^{n}$, there is a unit-speed geodesic $\gamma:[0, \ell] \rightarrow M^{n}$ so that $\gamma(0)=x_{0}$ and $\gamma([0, \ell)) \subset L$, but $\gamma(\ell) \notin L$. Let $\mathcal{P}_{0}^{t}$ denote the parallel transport along $\gamma$ from $x_{0}$ to $\gamma(t)$. We claim that

$$
R E(\beta(\gamma(t)))=\mathcal{P}_{0}^{t}\left(R E\left(\beta\left(x_{0}\right)\right),\right.
$$

and hence $\rho(\gamma(t))=\rho\left(x_{0}\right)$. Given $X_{0} \in T_{x_{0}} M$ and $Y_{0} \in\left(\Delta^{*}\right)^{\perp}\left(x_{0}\right)$, set $X=\mathcal{P}_{0}^{t} X_{0}$ and $Y=\mathcal{P}_{0}^{t} Y_{0}$. From the proof of Theorem 7.7, we see that $\beta(X, T Y)$ is parallel along $\gamma$ for arbitrary $X_{0}$. Since $T$ is invertible, the claim follows.

From the claim we obtain $\gamma(\ell) \notin U$. Thus there exist unit vectors $\xi_{0} \in N_{f} M(\gamma(\ell))$ and $\hat{\xi}_{0} \in N_{\hat{f}} M(\gamma(\ell))$ such that $A_{\xi_{0}}=\hat{A}_{\hat{\xi}_{0}}$. We extend these vectors to parallel vector fields $\xi$ and $\hat{\xi}$ along $\gamma$. By Theorem 7.9, the differential equation

$$
\nabla_{\gamma^{\prime}} A_{\xi}=A_{\xi} C_{\gamma^{\prime}}
$$

holds on $[0, \ell]$. From the uniqueness of solutions of this equation with a given initial condition, it follows that $A_{\xi}=\hat{A}_{\hat{\xi}}$ along $\gamma$. Thus $U$ is empty, and this concludes the proof.

Proposition 13.19. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ be isometric immersions of a Riemannian manifold. Assume that at each point $x \in M^{n}$ there exist unit vectors $\xi \in N_{f} M(x)$ and $\hat{\xi} \in N_{\hat{f}} M(x)$ such that $A_{\xi}=\hat{A}_{\hat{\xi}}$. Then there is an open dense subset of $M^{n}$ where one can define smooth unit vector fields $\xi \in \Gamma\left(N_{f} M\right)$ and $\hat{\xi} \in \Gamma\left(N_{\hat{f}} M\right)$ that are parallel along the leaves of $\Delta^{*}$ and satisfy $A_{\xi}=\hat{A}_{\hat{\xi}}$.

Proof: The dimensions of $\Delta^{*}, \mathcal{S}(\beta)$ and $\mathcal{S}(\beta) \cap \mathcal{S}(\beta)^{\perp}$ are constant on each connected component of an open dense subset of $M^{n}$. By Exercise 12.13, $\mathcal{S}(\beta)$ is parallel along $\Delta^{*}$ on any such component. Since $\nabla^{*}$ is compatible with the metric $\langle\langle\rangle$,$\rangle , then the$ bundle $\mathcal{S}(\beta) \cap \mathcal{S}(\beta)^{\perp}$ is smooth and parallel along $\Delta^{*}$.

Theorem 13.20. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ and $\hat{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ be isometric immersions of a compact Riemannian manifold such that $p+q \leq n-1$. Then, along each connected component of an open dense subset of $M^{n}, f$ and $\tilde{f}$ either admit singular isometric extensions or are mutually $d$-ruled with $d \geq n-p-q+3$.

Proof: Let $V \subset M^{n}$ be the open subset where $\beta$ satisfies $\rho \geq p+q-2$. By Propositions 13.18 and 13.19 , along an open dense subset $U$ of $V$ there is a line bundle isometry $\mathcal{T}: \operatorname{span}\{\xi\} \rightarrow \operatorname{span}\{\hat{\xi}\}$ that preserves the second fundamental forms. The statement now follows from Theorem 12.44 applied to each connected component of $U$ for this $\mathcal{T}$, and to $M^{n} \backslash \bar{V}$ for $\mathcal{T}=0$, since, in either case, we have $n+\ell-\rho \geq n-p-q+3$, where $\ell$ and $\rho$ are as in that result.

Theorem 13.21. Any isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ of a compact Riemannian manifold is genuinely rigid in the singular sense in $\mathbb{R}^{n+q}$ if $p+q \leq \min \{4, n-1\}$.

Proof: If $\hat{f}: M^{n} \rightarrow \mathbb{R}^{n+q}$ is an isometric immersion, we must prove that there exists an open dense subset of $M^{n}$ along each connected component of which $f$ and $\hat{f}$ admit singular isometric extensions.

Since $p+q \leq 4$, by Propositions 13.18 and 13.19 there exist, along an open dense subset $U$ of $M^{n}$, line subbundles $L=\operatorname{span}\{\xi\}$ and $\hat{L}=\operatorname{span}\{\hat{\xi}\}$ of $N_{f} M$ and $N_{\hat{f}} M$, respectively, which are parallel along $\Delta^{*}$, and an isometry $\mathcal{T}: L \rightarrow \hat{L}$ that preserves the second fundamental forms and is trivially parallel. Consider the map $\phi: \Gamma\left(f_{*} T M \oplus L\right) \times \mathfrak{X}(M) \rightarrow \Gamma\left(L^{\perp} \oplus \hat{L}^{\perp}\right)$ and the isotropic subbundle $D$ of $f_{*} T M \oplus L$
with respect to $\phi$ given by Proposition 12.43 on any open subset $U$ where $\rho(x)$, defined by (12.83), takes a constant value $\rho$. By (12.84), the rank of $D$ on $U$ satisfies

$$
\begin{equation*}
d=n+1-\rho \geq n-1 \tag{13.18}
\end{equation*}
$$

It suffices to show that the pair $\left\{\left.f\right|_{U},\left.\hat{f}\right|_{U}\right\}$ cannot be genuine in the singular sense on any such subset $U$.

Assuming otherwise, Theorem 12.44 implies that $D$ is a tangent subbundle and, moreover, that the pair $\left\{\left.f\right|_{U},\left.\hat{f}\right|_{U}\right\}$ is mutually ruled, with the rulings determined by $D$ in the sense that, for each $x_{0} \in U$, there exists an open subset $W$ of 0 in $D\left(x_{0}\right)$ such that $f_{*} W \subset f(M)$ and $\hat{f}_{*} W \subset \hat{f}(M)$.

If $d=n$, then $\left.f\right|_{U}$ and $\left.f\right|_{U}$ are totally geodesic, hence clearly admit isometric extensions. Thus, from now on we assume that $d=n-1$. In this case, it follows from (13.18) that $\rho=2$; hence 12.85) implies that $D=\mathcal{N}(\phi)$, that is,

$$
\begin{equation*}
D=\mathcal{N}\left(\alpha_{L^{\perp}}\right) \cap \mathcal{N}\left(\alpha_{\hat{L}^{\perp}}\right) \tag{13.19}
\end{equation*}
$$

In particular, $L_{D}(x) \subset L(x)$ and $\hat{L}_{D}(x) \subset \hat{L}(x)$ for all $x \in U$, where

$$
L_{D}(x)=\operatorname{span}\left\{\alpha(Z, X): Z \in D(x) \text { and } X \in T_{x} M\right\}
$$

and $\hat{L}_{D}(x)$ is similarly defined for $\hat{f}$. We can also assume, by restricting $U$ to a smaller subset, if necessary, that $U$ is free of totally geodesic points of $f$ and $\hat{f}$.

Let $R$ be the distribution given by the rulings determined as above by $D$ on an open neighborhood $U_{0}$ of $x_{0} \in U$, so that $R\left(x_{0}\right)=D\left(x_{0}\right)$. We consider separately the two possible cases: either $D$ and $R$ coincide on $U_{0}$, or there exists an open subset $U_{0}^{*} \subset U_{0}$ such that $R(y) \neq D(y)$ for all $y \in U_{0}^{*}$. In the latter case, since $U_{0}$ is free of totally geodesic points of $f$ and $\hat{f}$, it follows from 13.19 that $L(y)$ and $\hat{L}(y)$ coincide with the first normal spaces $N_{1}(y)$ and $\hat{N}_{1}(y)$ of $f$ and $f$, respectively, at any $y \in U_{0}^{*}$. We claim that $N_{1}$ and $\hat{N}_{1}$ are parallel line bundles along $U_{0}^{*}$. Otherwise, by Exercise 2.2 the relative nullity subspaces $\Delta(y)$ and $\hat{\Delta}(y)$ would have dimensions at least $n-1$ for all $y \in U_{0}^{*}$. But if, say, $\operatorname{dim} \Delta(y)=n-1$, since $R(y)$ and $D(y)$ are asymptotic subspaces of $f$ at $y$, and at least one of them would be transversal to $\Delta(y)$, then $y$ would actually be a totally geodesic point of $f$, a contradiction. Therefore, the codimensions of $\left.f\right|_{U_{0}^{*}}$ and $\left.\hat{f}\right|_{U_{0}^{*}}$ can be reduced to one on $U_{0}^{*}$ by Corollary 2.2 , and hence $\left.f\right|_{U_{0}^{*}}$ and $\left.\hat{f}\right|_{U_{0}^{*}}$ can be regarded as isometric immersions in codimension one having the same second fundamental forms. But this implies that $f$ and $\hat{f}$ admit isometric extensions in $U_{0}^{*}$, contradicting our assumption.

It remains to consider the case in which $D$ and $R$ coincide on $U_{0}$, that is, the pair $\left\{\left.f\right|_{U_{0}},\left.\hat{f}\right|_{U_{0}}\right\}$ is mutually $D$-ruled.

On any open subset $U_{0}^{\prime} \subset U_{0}$ where $L_{D}(x)$, and hence $\hat{L}_{D}(x)$, is trivial, the subspace $D(x)$ coincides with $\Delta^{*}(x)$, and hence $L$ and $\hat{L}$ are parallel on $U_{0}^{\prime}$ along $D$ in the normal connection.

Now let $\hat{U}_{0} \subset U_{0}$ be the open subset where $L_{D}(x)$, and hence $\hat{L}_{D}(x)$, is nontrivial, that is, $L_{D}(x)=L(x)$ and $\hat{L}_{D}(x)=\hat{L}(x)$. Using 7.11 we obtain

$$
\begin{aligned}
\nabla_{X}^{*} \beta(Z, Y) & =\left(\nabla_{X}^{*} \beta\right)(Z, Y)+\beta\left(\nabla_{X} Z, Y\right)+\beta\left(Z, \nabla_{X} Y\right) \\
& =\left(\nabla_{Z}^{*} \beta\right)(X, Y)+\beta\left(\nabla_{X} Z, Y\right)+\beta\left(Z, \nabla_{X} Y\right) \\
& =\nabla_{Z}^{*} \beta(X, Y)-\beta\left(\nabla_{Z} X, Y\right)-\beta\left(X, \nabla_{Z} Y\right)+\beta\left(\nabla_{X} Z, Y\right)+\beta\left(Z, \nabla_{X} Y\right)
\end{aligned}
$$

for all $X, Y \in \Gamma(D)$ and $Z \in \mathfrak{X}\left(\hat{U}_{0}\right)$. Since the line bundles $L$ and $\hat{L}$ coincide with $L_{D}$ and $\hat{L}_{D}$, respectively, and the pair $\left\{\left.f\right|_{U_{0}},\left.\hat{f}\right|_{U_{0}}\right\}$ is mutually $D$-ruled, which implies in particular that $D$ is totally geodesic on $U_{0}$, all terms on the right-hand side of the preceding equation belong to $\Gamma(L \oplus \hat{L})$. Thus $L$ and $\hat{L}$ are also parallel along $D$ on $\hat{U}_{0}$.

In either case, the pair ( $\mathcal{T}, D)$ satisfies conditions 12.9). Proposition 12.5 then implies that both $\left\{\left.f\right|_{U_{0}^{\prime}},\left.\hat{f}\right|_{U_{0}^{\prime}}\right\}$ and $\left\{\left.f\right|_{\hat{U}_{0}},\left.\hat{f}\right|_{\hat{U}_{0}}\right\}$ admit isometric extensions, and that is a contradiction.

Remark 13.22. Notice that Theorems 13.20 and 13.21 are still valid if $M^{n}$ is only assumed to be complete, provided that one of the immersions is bounded.

In the following application of Theorem 13.21, rigidity is established in terms of intrinsic assumptions on the immersed submanifold.

Theorem 13.23. Let $f, \tilde{f}: M^{n} \rightarrow \mathbb{R}^{n+2}, n \geq 5$, be isometric immersions of a compact Riemannian manifold $M^{n}$ with first geometric Pontrjagin form $p_{1}$. Then $p_{1}^{2}=p_{1} \wedge p_{1}$ vanishes on any open subset of $M^{n}$ where $f$ and $\tilde{f}$ are not congruent in any open subset. In particular, if $f$ is real analytic and the rational Pontrjagin class $\left[p_{1}\right]$ of $M^{n}$ satisfies $\left[p_{1}\right]^{2} \neq 0$, then $f$ is rigid.

Proof: In terms of the curvature forms defined by

$$
\Omega_{i j}(X, Y)=\left\langle R\left(e_{i}, e_{j}\right) X, Y\right\rangle
$$

with respect to an orthonormal basis at a given point, we have

$$
p_{1} \sim \sum_{i<j} \Omega_{i j}^{2}
$$

up to a constant. Consider the 1 -forms

$$
\varphi_{k}=\left\langle A_{\xi} e_{k},\right\rangle \text { and } \psi_{k}=\left\langle A_{\eta} e_{k},\right\rangle
$$

where $\xi, \eta$ are such that $A_{\xi}=\tilde{A}_{\tilde{\xi}}$ and rank $A_{\eta} \leq 2$. By the Gauss equation,

$$
\Omega_{i j}=\varphi_{i} \wedge \varphi_{j}+\psi_{i} \wedge \psi_{j}
$$

Since rank $A_{\eta} \leq 2$, we can choose a basis so that $\psi_{k}=0$ for $3 \leq k \leq n$. It follows that $\Omega_{i j}=0$ unless $i=1$ and $j=2$. Therefore

$$
p_{1} \sim\left(\varphi_{1} \wedge \varphi_{2}+\psi_{1} \wedge \psi_{2}\right)^{2}=2 \varphi_{1} \wedge \varphi_{2} \wedge \psi_{1} \wedge \psi_{2}
$$

and thus $p_{1}^{2}=0$.

### 13.6 Notes

The rigidity result in Euclidean space for compact hypersurfaces was obtained by Sacksteder [308] and extended to higher codimension by do Carmo-Dajczer [58]. Ferus [176] considered the cases in which the ambient space is the sphere or the hyperbolic space. The extension of Sacksteder's theorem to complete Euclidean hypersurfaces, and the description of the structure of the complete submanifolds of rank two, are due to Dajczer-Gromoll [112]. The case of complete minimal submanifolds of rank at most two in space forms was considered by Dajczer-Kasioumis-Savas Halilaj-Vlachos [117], [118], [119].

A Euclidean hypersurface is called Weingarten if there exists a differentiable function relating the mean curvature and the scalar curvature. Dajczer-Tenenblat [132] proved that a complete Weingarten hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ is rigid, provided that $n \geq 4$ and $M^{n}$ does not contain any open subset isometric to $U \times \mathbb{R}^{n-3}$. The result relies on a classification of the ruled Weingarten hypersurfaces of the Euclidean space. Ruled Weingarten hypersurfaces in the sphere and in the hyperbolic space have been considered in [23] and [22], respectively.

The classification of complete deformable hypersurfaces in the hyperbolic space has not been done yet. Exercises 13.9 and 13.10 provide partial results in this direction, namely, the existence of complete surface-like examples and the nonexistence of complete Sbrana-Cartan hypersurfaces of real type of the continuous or the discrete class. We believe that there exists an abundance of examples of complete Sbrana-Cartan hypersurfaces of complex type in the continuous class. We point out that the examples given in [262] are surface-like.

An important open question concerns the existence of a complete nonruled and not surface-like deformable hypersurface $f: M^{3} \rightarrow \mathbb{R}^{4}$ whose rank is almost everywhere equal to two. We observe that the examples in [263] do not give an answer to this question, which was asked by Dajczer-Gromoll [109] in the particular case of minimal immersions; see also the result by Savas Halilaj 310.

The results in the last section on isometric deformations of compact Euclidean submanifolds have been taken from Dajczer-Gromoll [113] and Florit-Guimarães [185]. In the latter article also the case of compact submanifolds of the sphere was considered.

### 13.7 Exercises

Exercise 13.1. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, be a compact hypersurface. Prove that the identity component $\operatorname{Iso}^{0}(M)$ of the isometry group of $M^{n}$ admits an orthogonal representation

$$
\Phi: \operatorname{Iso}^{0}(M) \rightarrow \mathrm{SO}(n+1)
$$

such that $f \circ g=\Phi(g) \circ f$ for all $g \in \operatorname{Iso}^{0}(M)$.
Hint: Given $g \in \operatorname{Iso}^{0}(M)$, let $\alpha^{f \circ g}$ denote the second fundamental form of $f \circ g$. Show that

$$
\alpha^{f \circ g}(x)=\phi_{g}(x) \circ \alpha^{f}(x)
$$

for every $g \in \operatorname{Iso}^{0}(M)$ and $x \in M^{n}$, where $\phi_{g}$ denotes a vector bundle isometry between $N_{f} M$ and $N_{f \circ g} M$, as follows: on one hand,

$$
\alpha^{f \circ g}(x)(X, Y)=\alpha^{f}(g x)\left(g_{*} X, g_{*} Y\right)
$$

for every $g \in \operatorname{Iso}^{0}(M), x \in M^{n}$ and $X, Y \in T_{x} M$ (see Exercise 1.6). In particular, this implies that, for any fixed $x \in M^{n}$, the map

$$
\Theta_{x}: \operatorname{Iso}^{0}(M) \rightarrow \operatorname{Sym}\left(T_{x} M \times T_{x} M \rightarrow N_{f} M(x)\right)
$$

into the vector space of symmetric bilinear maps of $T_{x} M \times T_{x} M$ into $N_{f} M(x)$, given by

$$
\begin{aligned}
\Theta_{x}(g)(X, Y) & =\phi_{g}(x)^{-1}\left(\alpha^{f \circ g}(x)(X, Y)\right) \\
& =\phi_{g}(x)^{-1}\left(\alpha^{f}(g x)\left(g_{*} X, g_{*} Y\right)\right)
\end{aligned}
$$

for any $X, Y \in T_{x} M$, is continuous. On the other hand, by Theorem 13.2 either

$$
\alpha^{f \circ g}(x)=\phi_{g}(x) \circ \alpha^{f}(x)
$$

or

$$
\alpha^{f \circ g}(x)=-\phi_{g}(x) \circ \alpha^{f}(x) .
$$

Thus $\Theta_{x}$ is a continuous map taking values in $\left\{\alpha^{f}(x),-\alpha^{f}(x)\right\}$, hence it must be constant because $\operatorname{Iso}^{0}(M)$ is connected. Now use that $\Theta_{x}(\mathrm{id})=\alpha^{f}(x)$. Conclude that for each $g \in \operatorname{Iso}^{0}(M)$ there exists a rigid motion $\tilde{g} \in \operatorname{Iso}\left(\mathbb{R}^{n+1}\right)$ such that $f \circ g=\tilde{g} \circ f$. Then show that $g \mapsto \tilde{g}$ defines a Lie-group homomorphism $\Phi: \operatorname{Iso}^{0}(M) \rightarrow \operatorname{Iso}\left(\mathbb{R}^{n+1}\right)$, and argue that its image must lie in $\mathrm{SO}(n+1)$ by using that it is compact and connected.

Exercise 13.2. Show that Proposition 13.14 does not hold for a cylinder over a surface in $\mathbb{R}^{3}$.

Exercise 13.3. Let $\ell$ be a straight line in $\mathbb{R}^{n+1}$ and let $\xi$ be a nowhere parallel unit normal field along $\ell$. Show that the normal subspaces to $\ell$ orthogonal to $\xi$ are the rulings of a complete hypersurface with constant rank $\rho=2$.

Exercise 13.4. Show that the scalar curvature is constant along the relative nullity leaves of a ruled strip.

Exercise 13.5. Prove that Theorem 13.17 holds under the weaker assumption that the scalar curvature of $M^{n}$ is nonnegative, provided that the set of totally geodesic points does not disconnect the manifold. Moreover, the conditions need only to be satisfied outside a compact set.

Exercise 13.6. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}$ be an isometric immersion and let $\gamma$ be a geodesic contained in a leaf of the relative nullity foliation. By (7.3), the splitting tensor $C_{\gamma^{\prime}}$ along $\gamma$ satisfies the differential equation

$$
\frac{D}{d t} C_{\gamma^{\prime}(t)}=C_{\gamma^{\prime}(t)}^{2}+c I
$$

Show that its solution is

$$
C_{\gamma^{\prime}(t)}=\mathcal{P}_{0}^{t}\left(\sin t I+\cos t C_{\gamma^{\prime}(0)}\right)\left(\cos t I-\sin t C_{\gamma^{\prime}(0)}\right)^{-1}\left(\mathcal{P}_{0}^{t}\right)^{-1}
$$

if $c=1$, and

$$
C_{\gamma^{\prime}(t)}=\mathcal{P}_{0}^{t}\left(-\sinh t I+\cosh t C_{\gamma^{\prime}(0)}\right)\left(\cosh t I-\sinh t C_{\gamma^{\prime}(0)}\right)^{-1}\left(\mathcal{P}_{0}^{t}\right)^{-1}
$$

if $c=-1$, where $\mathcal{P}_{0}^{t}$ denotes parallel transport along $\gamma$ from $\gamma(0)$ to $\gamma(t)$.
Exercise 13.7. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be a simply connected flat hypersurface with constant index of relative nullity $\nu=n-1$. Show that the set of isometric immersions of $M^{n}$ into $\mathbb{R}^{n+1}$ that preserve the relative nullity distribution is parametrized by the set of smooth functions on an interval.

Exercise 13.8. Let $f: M^{n} \rightarrow \mathbb{R}^{n+2}$ be a simply connected flat submanifold with $\operatorname{dim} N_{1}^{f}=1$ everywhere and constant index of relative nullity $\nu=n-1$. Show that the set of isometric immersions of $M^{n}$ into $\mathbb{R}^{n+1}$ that preserve the relative nullity distribution $\Delta$ is parametrized by the set of pairs of smooth functions on an interval.
Hint: Consider smooth unit vector fields $X \in \mathfrak{X}(M)$ and $\xi \in \Gamma\left(N_{f} M\right)$ such that $X$ is orthogonal to $\Delta$ and $N_{1}^{f}=\operatorname{span}\{\xi\}$. Set $A_{\xi} X=\lambda X$ and let $\xi, \eta$ be a smooth orthonormal normal frame. Then use that the Codazzi and Ricci equations reduce to

$$
T(\lambda)=\lambda\left\langle\nabla_{X} X, T\right\rangle
$$

and

$$
T\left\langle\nabla_{X}^{\perp} \xi, \eta\right\rangle+\left\langle\nabla_{X} X, T\right\rangle\left\langle\nabla_{X}^{\perp} \xi, \eta\right\rangle=0
$$

for any $T \in \Gamma(\Delta)$.
Exercise 13.9. (i) Let $f: M^{n} \rightarrow \mathbb{H}^{m}$ be a generalized cone over a submanifold $g: L^{p} \rightarrow \mathbb{Q}_{c}^{m-n+p}, 2 \leq p \leq n-1$, in an umbilical submanifold $\mathbb{Q}_{c}^{m-n+p}$ of $\mathbb{H}^{m}$. Show that $M^{n}$ is complete if and only if $c \leq 0$ and $L^{p}$ is complete.
(ii) Make use of part ( $i$ ) to conclude that there exist complete Sbrana-Cartan surfacelike hypersurfaces in the hyperbolic space.
Hint for $(i)$ : By Corollary 10.11 and the observation right before it, $M^{n}$ is isometric to a warped product $N^{n-p} \times{ }_{\sigma} L^{p}$, where

$$
N^{n-p}=\mathbb{H}^{n-p} \text { if } c \leq 0
$$

and

$$
N^{n-p}=\mathbb{H}^{n-p} \cap\left\{x \in \mathbb{L}^{n-p+1}:\langle a, x\rangle>0\right\} \text { if } c>0
$$

for a certain unit space-like vector $a \in \mathbb{L}^{n-p+1}$. Use that a warped product $N^{n-p} \times{ }_{\sigma} L^{p}$ is complete if and only if $N^{n-p}$ is complete and $\sigma$ is nowhere vanishing.

Exercise 13.10. Let $f: M^{n} \rightarrow \mathbb{H}_{-1}^{n+1}, n \geq 4$, be a Sbrana-Cartan hypersurface of real type of the continuous or the discrete class. Show that the leaves of the relative nullity distribution cannot be complete on an open subset of $M^{n}$.
Hint: Suppose that the hypersurface has complete leaves on an open set $U \subset M^{n}$. Consider the Gauss parametrization $\psi$ of $f$ along $U$, which is defined on the normal bundle $N_{g} L$ of a surface $g: L^{2} \rightarrow \mathbb{S}_{1}^{n}$. Show that the completeness of the leaves on $U$ implies that $\psi$ must have maximal rank everywhere on $N_{g} L$, and that this is equivalent to $A_{w}$ being nonsingular for any $w \in N_{g} L$ with $\langle w, w\rangle=-1$. Prove that $N_{1}^{g}$ must have rank one by showing that at any $x \in L^{2}$ where $N_{1}^{g}$ has maximal dimension two there exists $w \in N_{g} L(x)$ with $\|w\|=-1$ such that $A_{w}$ is singular. For that, first note that the assertion is trivial if the induced metric on $N_{1}^{g}$ is Riemannian. If the metric is degenerate, one may assume that $\left\|\alpha^{g}\left(\partial_{u}, \partial_{u}\right)\right\| \neq 0$ and take $0 \neq w_{0} \in N_{1}^{g} \cap\left(N_{1}^{g}\right)^{\perp}$. If $\hat{w}_{0} \in N_{g} L(x)$ is such that

$$
\left\|\hat{w}_{0}\right\|=0, \quad\left\langle w_{0}, \hat{w}_{0}\right\rangle=-1 / 2 \text { and }\left\langle\alpha^{g}\left(\partial_{u}, \partial_{u}\right), \hat{w}_{0}\right\rangle=0
$$

choose $w=w_{0}+\hat{w}_{0}$. If the metric on $N_{1}^{g}$ is Lorentzian and either $\alpha^{g}\left(\partial_{u}, \partial_{u}\right)$ or $\alpha^{g}\left(\partial_{v}, \partial_{v}\right)$, say, the former, is space-like, simply choose $w \in N_{1}^{g}$ orthogonal to $\alpha^{g}\left(\partial_{u}, \partial_{u}\right)$. Finally, if both $\alpha^{g}\left(\partial_{u}, \partial_{u}\right)$ and $\alpha^{g}\left(\partial_{v}, \partial_{v}\right)$ are time-like, let $\xi, \delta$ be an orthonormal basis of $N_{1}^{g}$, with $\xi$ collinear with $\alpha^{g}\left(\partial_{u}, \partial_{u}\right)$, and take $w=\cosh \theta \xi+\sinh \theta \delta$, where

$$
\theta=-\frac{\left\langle\alpha^{g}\left(\partial_{v}, \partial_{v}\right), \xi\right\rangle}{\left\langle\alpha^{g}\left(\partial_{v}, \partial_{v}\right), \delta\right\rangle} .
$$

Now prove that $N_{1}^{g}$ is parallel in the normal connection: if $N_{1}^{g}=\operatorname{span}\{w\}$ and there exists $\delta$ orthogonal to $w$ and $X \in T L$ such that $\left\langle\nabla \frac{1}{X} w, \delta\right\rangle \neq 0$, use the Codazzi equation for $A_{\delta}$ to show that $A_{w} Y=0$ for all $Y \in T L$ such that $\left\langle\nabla_{⿳}^{Y} w, \delta\right\rangle=0$. Conclude that $g$ reduces codimension to one, and hence $f$ is a generalized cone over the polar surface of $g$ by Exercise 13.9.

Exercise 13.11 Verify that Theorems 13.20 and 13.21 still hold if the Euclidean ambient space is replaced by the hyperbolic space.

## Chapter 14

## Infinitesimal bendings

Around the time that Sbrana obtained the local description of the isometrically deformable hypersurfaces discussed in Chapter 11, he also considered the problem of locally describing, in terms of the Gauss parametrization, the Euclidean hypersurfaces that are infinitesimally bendable, that is, the ones that admit nontrivial infinitesimal deformations. Roughly speaking, this means that the hypersurface admits a nontrivial, smooth, one-parameter variation by hypersurfaces that are isometric only "up to the first order."

It was very natural at that time for Sbrana to consider the infinitesimal version of the deformation problem for hypersurfaces. On the one hand, because there already existed a rich theory of infinitesimal deformations of surfaces; see Bianchi [36] and Spivak [317]. On the other hand, it was known that any hypersurface whose type number is at least three at any point is locally infinitesimally rigid, that is, it does not admit nontrivial infinitesimal deformations, a result contained in the book of Cesàro [76] from 1896.

In this chapter, we first discuss the general theory of infinitesimal deformations for Euclidean submanifolds of arbitrary codimension. Then we present local rigidity results that constitute the infinitesimal counterparts of the theorems of Allendoerfer and do Carmo-Dajczer on isometric rigidity given in Chapter 4. A global result for compact hypersurfaces corresponding to Sacksteder's theorem is also proved.

The continuation of the chapter is devoted to giving, in modern terms, a complete local parametric description of the nonflat infinitesimally deformable hypersurfaces. Of course, any element in the continuous class of isometrically deformable hypersurfaces is infinitesimally bendable, but the class of infinitesimally bendable hypersurfaces turns out to be much larger.

To conclude, some of the results on infinitesimally deformable hypersurfaces are used to derive a description of the Sbrana-Cartan hypersurfaces of the continuous class as envelopes of certain two-parameter congruences of affine hyperplanes.

### 14.1 Infinitesimal bendings

This section introduces the notions of an isometric bending of a submanifold and its linearized version, namely, an infinitesimal bending.

Let $F: I \times M^{n} \rightarrow \mathbb{R}^{m}$ denote a smooth variation of a given isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$. The map $F$ is called an isometric bending of $f$ if $f_{t}=F(t):, M^{n} \rightarrow \mathbb{R}^{m}$ is an isometric immersion for all $t \in I \subset \mathbb{R}$.

An isometric bending is called trivial if it is produced by a smooth family of isometries of $\mathbb{R}^{m}$, that is, if there exist a smooth family $C: I \rightarrow O(m)$ of orthogonal transformations of $\mathbb{R}^{m}$ and a smooth map $v: I \rightarrow \mathbb{R}^{m}$ such that

$$
\begin{equation*}
F(t, x)=C(t) f(x)+v(t) \tag{14.1}
\end{equation*}
$$

for all $(t, x) \in I \times M^{n}$.
An isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ is said to be isometrically bendable or unbendable whether or not it admits nontrivial isometric bendings. Notice that a submanifold may admit isometric deformations and still be unbendable, as is the case of the Sbrana-Cartan hypersurfaces in the discrete class, that is, the ones that allow a single deformation.

By Proposition 1.1, the variational vector field

$$
\mathcal{T}=F_{*} \partial /\left.\partial t\right|_{t=0}
$$

of an isometric bending satisfies the condition

$$
\begin{equation*}
\left\langle\tilde{\nabla}_{X} \mathcal{T}, f_{*} Y\right\rangle+\left\langle f_{*} X, \tilde{\nabla}_{Y} \mathcal{T}\right\rangle=0 \tag{14.2}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}(M)$. If we decompose $\mathcal{T}=f_{*} Z+\eta$ into its tangent and normal components, then (14.2) becomes

$$
\begin{equation*}
\left\langle\nabla_{X} Z, Y\right\rangle+\left\langle X, \nabla_{Y} Z\right\rangle=2\langle\alpha(X, Y), \eta\rangle \tag{14.3}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}(M)$.
For a trivial isometric bending $F$ as in (14.1), the variational vector field is

$$
\mathcal{T}(x)=\mathcal{D} f(x)+v^{\prime}(0)
$$

where $\mathcal{D}=C^{\prime}(0)$ is a skew-symmetric linear endomorphism of $\mathbb{R}^{m}$. Conversely, given a skew-symmetric linear endomorphism $\mathcal{D}$ of $\mathbb{R}^{m}$ and a vector $v \in \mathbb{R}^{m}$, the map

$$
F(t, x)=e^{t \mathbb{D}} f(x)+t v
$$

defines a trivial isometric bending of $f$.
An infinitesimal bending of an isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ is a section $\mathcal{T}$ of $f^{*} T \mathbb{R}^{m}$ that satisfies condition (14.2). If there exists a skew-symmetric linear endomorphism $\mathcal{D}$ of $\mathbb{R}^{m}$ and $v \in \mathbb{R}^{m}$ such that

$$
\mathcal{T}=\mathcal{D} f+v
$$

then $\mathcal{T}$ is said to be a trivial infinitesimal bending.
Multiplying an infinitesimal bending by a constant and adding a trivial infinitesimal bending yields a new infinitesimal bending. Therefore, it is convenient to identify two infinitesimal bendings $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ if there exist $0 \neq c \in \mathbb{R}$ and a trivial infinitesimal bending $\mathcal{T}_{0}$ such that

$$
\mathcal{T}_{2}=\mathcal{T}_{0}+c \mathcal{T}_{1}
$$

Any infinitesimal bending $\mathcal{T}$ of a submanifold $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ gives rise to a variation $F: \mathbb{R} \times M^{n} \rightarrow \mathbb{R}^{n+1}$ having $\mathcal{T}$ as variational vector field, namely,

$$
\begin{equation*}
F(t, x)=f(x)+t \mathcal{T}(x) \tag{14.4}
\end{equation*}
$$

One usually says that the immersion $f_{t}=F(t$,$) is isometric to f$ up to first order, for

$$
\begin{equation*}
\left\|f_{t *} X\right\|^{2}=\left\|f_{*} X\right\|^{2}+t^{2}\left\|\tilde{\nabla}_{X} \mathcal{T}\right\|^{2} \tag{14.5}
\end{equation*}
$$

for all $X \in \mathfrak{X}(M)$.
Examples 14.1. (i) Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be an isometric immersion. If $Z \in \mathfrak{X}(M)$ is a Killing vector field and $\eta \in \Gamma\left(N_{1}^{\perp}\right)$ then $\mathcal{T}=f_{*} Z+\eta$ is an infinitesimal bending of $f$, for both sides of (14.3) vanish identically.
(ii) Let $f, g: M^{n} \rightarrow \mathbb{R}^{m}$ be two noncongruent isometric immersions such that the map $h=f+g: M^{n} \rightarrow \mathbb{R}^{m}$ is an immersion. Then $\mathcal{T}=f-g$ is an infinitesimal bending of $h$.

### 14.2 Infinitesimal rigidity

Associated with the concept of infinitesimal bending there is a natural notion of infinitesimal rigidity defined as follows.

An isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ is called infinitesimally rigid if it only admits trivial infinitesimal bendings. It is said to be infinitesimally bendable if it admits an infinitesimal bending that is nontrivial restricted to any open subset of $M^{n}$.

Example 14.2. A submanifold $f: M^{n} \rightarrow \mathbb{R}^{m}$ that is totally geodesic on an open subset of $M^{n}$ is infinitesimally bendable. This follows from the first of Examples 14.1.

The following is an elementary but very useful fact that is already contained in the classical literature for surfaces; see Bianchi [36].

Proposition 14.3. Given an infinitesimal bending $\mathcal{T}$ of an isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$, consider for any $t \in \mathbb{R}$ the map $G_{t}: M^{n} \rightarrow \mathbb{R}^{m}$ defined by

$$
\begin{equation*}
G_{t}(x)=f(x)+t \mathcal{T}(x) . \tag{14.6}
\end{equation*}
$$

The following assertions hold:
(i) The maps $G_{t}$ and $G_{-t}$ are immersions that induce the same metric.
(ii) If $f$ is substantial and there exists $0 \neq t_{0} \in \mathbb{R}$ such that $G_{t_{0}}$ and $G_{-t_{0}}$ are congruent then $\mathfrak{T}$ is trivial.

Proof: The assertion in part ( $i$ ) follows immediately from (14.5). By the assumption of part (ii) there exist an orthogonal transformation $S$ of $\mathbb{R}^{m}$ and a vector $w \in \mathbb{R}^{m}$ such that

$$
f+t_{0} \mathcal{T}=S\left(f-t_{0} \mathcal{T}\right)+w
$$

Thus

$$
f_{*} X+t_{0} \tilde{\nabla}_{X} \mathcal{T}=S\left(f_{*} X-t_{0} \tilde{\nabla}_{X} \mathcal{T}\right)
$$

and hence

$$
\begin{equation*}
t_{0}(S+I) \tilde{\nabla}_{X} \mathcal{T}=(S-I) f_{*} X \tag{14.7}
\end{equation*}
$$

for all $X \in \mathfrak{X}(M)$.
Suppose that $S+I$ is not invertible, that is, that there exists

$$
0 \neq \delta \in \operatorname{ker}(S+I)=\operatorname{ker}(S+I)^{t}
$$

Then $(S-I)^{t} \delta=-2 \delta$. Taking the inner product of (14.7) with $\delta$ gives

$$
\left\langle f_{*} X, \delta\right\rangle=0
$$

for all $X \in \mathfrak{X}(M)$, contradicting the fact that $f$ is substantial.
Thus $S+I$ is invertible, and hence (14.7) yields

$$
\begin{equation*}
\tilde{\nabla}_{X} \mathcal{T}=\mathcal{D} f_{*} X \tag{14.8}
\end{equation*}
$$

where

$$
\mathcal{D}=\frac{1}{t_{0}}(S+I)^{-1}(S-I) .
$$

Since $f$ is substantial, it follows from (14.2) and (14.8) that $\mathcal{D}$ is skew-symmetric. Moreover, since

$$
\mathcal{D} f_{*} X=\tilde{\nabla}_{X} \mathcal{D} f
$$

then 14.8) also yields

$$
\tilde{\nabla}_{X}(\mathcal{T}-\mathcal{D} f)=0
$$

for all $X \in \mathfrak{X}(M)$, thus showing that $\mathcal{T}$ is trivial.
Theorem 14.4. An isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$ with type number $\tau \geq 3$ at any point is infinitesimally rigid.

Proof: Let $\mathcal{T}$ be an infinitesimal bending of $f$ and let $G_{t}: M^{n} \rightarrow \mathbb{R}^{m}$ be defined by (14.6) for all $t \in \mathbb{R}$. By Proposition 14.3, the immersions $G_{t}$ and $G_{-t}$ are isometric. Moreover, any point of $M^{n}$ lies in an open neighborhood $U$ where $G_{t}$ still has type number $\tau \geq 3$ if $t$ is small enough. By Theorem 4.19, the restrictions $\left.G_{t}\right|_{U}$ and $\left.G_{-t}\right|_{U}$
are congruent, and hence $\mathfrak{T}$ is trivial on $U$ by Proposition 14.3, because the assumption on the type number implies that $\left.f\right|_{U}$ is substantial.

Therefore $\mathcal{T}$ is locally trivial, that is, each point of $M^{n}$ lies in an open subset $U$ such that $\tilde{\nabla}_{X} \mathcal{T}=\mathcal{D}_{U} f_{*} X$ along $U$. If two such open subsets $U$ and $V$ intersect, then

$$
\left.\left(\mathcal{D}_{U}-\mathcal{D}_{V}\right)\right|_{f_{*} T_{x} M}=0 \text { for all } x \in U \cap V .
$$

Since $\left.f\right|_{U \cap V}$ is substantial,

$$
\operatorname{span}\left\{f_{*} T_{x} M: x \in U \cap V\right\}=\mathbb{R}^{m}
$$

Hence $\mathcal{D}_{U}=\mathcal{D}_{V}$, and thus $\mathcal{T}$ is globally trivial.
A similar argument using Theorem 4.23 proves the following result.
Theorem 14.5. An isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{n+p}, p \leq 5$, whose $s$-nullities at any point satisfy $\nu_{s} \leq n-2 s-1$ for all $1 \leq s \leq p$ is infinitesimally rigid.

The following is an infinitesimal version of Sacksteder's Theorem 13.2. Observe that the assumption on the totally geodesic subsets is different, since they are allowed to separate the manifold as long as they have empty interior.

Theorem 14.6. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, be an isometric immersion of a compact Riemannian manifold such that there are no open subsets of $M^{n}$ where $f$ is totally geodesic. Then $f$ is infinitesimally rigid.

Proof: Let $\mathcal{T}$ be an infinitesimal bending of $f$. Then there is no open subset where any of the immersions $G_{ \pm t}=f \pm t \mathcal{T}$ is totally geodesic. Otherwise, from Theorem 13.2 both immersions would be totally geodesic on that open set and the same would be the case for $f=(1 / 2)\left(G_{t}+G_{-t}\right)$.

Fix $t \neq 0$, and let $D \subset M^{n}$ denote the closed subset of points where $G_{t}$ is totally geodesic. By Theorem 13.2, $G_{t}$ and $G_{-t}$ are congruent on any connected component of the open set $M^{n} \backslash D=\cup U_{j}$. Hence Proposition 14.3 yields

$$
\left.\mathfrak{T}\right|_{U_{j}}=\mathcal{D}_{j} f+v_{j}
$$

where $\mathcal{D}_{j}$ is a skew-symmetric linear transformation in $\mathbb{R}^{m}$ and $v_{j} \in \mathbb{R}^{m}$.
At $x \in M^{n}$, let $\mathcal{D}_{x}$ be the unique skew-symmetric matrix in $\mathbb{R}^{m}$ such that
(i) $\left\langle\mathcal{D}_{x} X, Y\right\rangle=\left\langle\tilde{\nabla}_{X} \mathcal{T}, Y\right\rangle$ for all $X, Y \in T_{x} M$.
(ii) $\left\langle\mathcal{D}_{x} X, \eta\right\rangle=\left\langle\tilde{\nabla}_{X} \mathcal{T}, \eta\right\rangle$ for all $X \in T_{x} M$ and $\eta \in N_{f} M(x)$.

Since $\tilde{\nabla}_{X} \mathcal{T}=\mathcal{D}_{j} X$ on $U_{j}$, then $\mathcal{D}_{x}$ and $\mathcal{D}_{j}$ are skew-symmetric matrices that coincide on $f_{*} T_{x} M$. But two skew-symmetric matrices that coincide on a hyperplane must be equal. And since $\mathcal{T}$ is smooth, the map $x \mapsto \mathcal{D}_{x}$ is also smooth. Thus this map is globally constant. Hence there is a constant $\mathcal{D}$ such that $\mathcal{T}=\mathcal{D} f+v_{j}$ is constant on each $U_{j}$. Therefore also $\mathcal{T}-\mathcal{D} f$ is constant on $M^{n}$.

### 14.3 Infinitesimally bendable hypersurfaces

By Theorem 14.4 , any hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, with type number at least three at any point is infinitesimally rigid. Thus, as in the situation of isometrically deformable hypersurfaces studied in Chapter 11, the interesting local case is the one of hypersurfaces with constant type number two. We will see that, even in this situation, hypersurfaces are "generically" infinitesimally rigid. Surface-like hypersurfaces will be excluded from our analysis, because in this case the infinitesimal bendings are given by infinitesimal bendings of the surface (see Exercise 14.2), and the surface case is not our object of study.

### 14.3.1 The integrability conditions for an infinitesimal bending

This section is devoted to establish the integrability conditions for the system of partial differential equations of an infinitesimal bending of a Euclidean hypersurface. Some long but straightforward computations are only indicated.

Given a hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, with an infinitesimal bending $\mathcal{T}$, we work with the associated variation $F$ given by (14.4). Let $g_{t}$ be the metric on $M^{n}$ induced by $f_{t}=F(t$,$) . Then$

$$
\partial /\left.\partial t\right|_{t=0} g_{t}(X, Y)=0
$$

for all $X, Y \in \mathfrak{X}(M)$. Consequently, the associated one-parameter family of Levi-Civita connections and the corresponding family of curvature tensors satisfy

$$
\partial /\left.\partial t\right|_{t=0} \nabla_{X}^{t} Y=0
$$

and

$$
\begin{equation*}
\partial /\left.\partial t\right|_{t=0}\left\langle R^{t}(X, Y) Z, W\right\rangle=0 \tag{14.9}
\end{equation*}
$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$.
Let $N(t)$ denote a Gauss map of $f_{t}$ so that the map $t \in \mathbb{R} \mapsto N(t)$ is smooth, and let $A(t)$ be the shape operator of $f_{t}$ with respect to $N(t)$. Thus $N=N(0)$ is the Gauss map and $A=A(0)$ is the shape operator of $f$ with respect to $N$. Moreover, let $L \in \Gamma\left(\operatorname{Hom}\left(T M ; f^{*}\left(T \mathbb{R}^{n+1}\right)\right)\right)$ be defined by

$$
L X=\tilde{\nabla}_{X} \mathcal{T}=\mathcal{T}_{*} X
$$

Then (14.2) can be written as

$$
\begin{equation*}
\left\langle L X, f_{*} Y\right\rangle+\left\langle f_{*} X, L Y\right\rangle=0 \tag{14.10}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}(M)$.

Lemma 14.7. The vector field $y \in \Gamma\left(f^{*}\left(T \mathbb{R}^{n+1}\right)\right)$ defined by

$$
y=\partial /\left.\partial t\right|_{t=0} N(t)
$$

satisfies

$$
\begin{equation*}
\langle y, N\rangle=0 \tag{14.11}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle y, f_{*} X\right\rangle+\langle L X, N\rangle=0 \tag{14.12}
\end{equation*}
$$

for all $X \in \mathfrak{X}(M)$.
Proof: The derivative with respect to $t$ of $\langle N(t), N(t)\rangle=1$ at $t=0$ gives (14.11), whereas that of $\left\langle N(t), f_{t *} X\right\rangle=0$ yields (14.12).

Lemma 14.8. The tensor $B \in \Gamma(\operatorname{End}(T M))$ defined by

$$
B=\partial /\left.\partial t\right|_{t=0} A(t)
$$

is symmetric and satisfies

$$
\begin{equation*}
\left(\tilde{\nabla}_{X} L\right) Y=\langle B X, Y\rangle N+\langle A X, Y\rangle y \tag{14.13}
\end{equation*}
$$

and

$$
\begin{equation*}
y_{*} X+f_{*} B X+L A X=0 \tag{14.14}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}(M)$.
Proof: The derivatives with respect to $t$, at $t=0$, of the Gauss formula

$$
\tilde{\nabla}_{X} f_{t *} Y=f_{t *} \nabla_{X}^{t} Y+g_{t}(A(t) X, Y) N(t)
$$

and the Weingarten formula

$$
\tilde{\nabla}_{X} N(t)=-f_{t *} A(t) X
$$

easily give 14.13 ) and (14.14), respectively.
If $\mathcal{T}=\mathcal{D} f+w$ is a trivial infinitesimal bending, then $L=\mathcal{D} \circ f_{*}$. It follows that $y=\mathcal{D} N$ and that $B=0$, since

$$
\langle B X, Y\rangle=\left\langle\left(\tilde{\nabla}_{X} L\right) Y, N\right\rangle=\left\langle\left(\tilde{\nabla}_{X} \mathcal{D}\right) f_{*} Y, N\right\rangle=0
$$

for all $X, Y \in \mathfrak{X}(M)$.
Proposition 14.9. The tensor $B$ is a Codazzi tensor, that is,

$$
\begin{equation*}
\left(\nabla_{X} B\right) Y-\left(\nabla_{Y} B\right) X=0 \tag{14.15}
\end{equation*}
$$

and satisfies

$$
\begin{equation*}
B X \wedge A Y-B Y \wedge A X=0 \tag{14.16}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}(M)$.

Proof: The derivative at $t=0$ of the Codazzi equation

$$
\left(\nabla_{X}^{t} A(t)\right) Y=\left(\nabla_{Y}^{t} A(t)\right) X
$$

gives 14.15. To obtain 14.16), we compute the derivative at $t=0$ of the Gauss equation

$$
R^{t}(X, Y) Z=g(t)(A(t) Y, Z) A(t) X-g(t)(A(t) X, Z) A(t) Y
$$

and use 14.9 .
Next we consider the case of hypersurfaces of constant rank two.
Corollary 14.10. If $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, is an infinitesimally bendable hypersurface of constant rank two, then $\Delta \subset \operatorname{ker} B$.

Proof: This follows easily from 14.16).
Thus the direct statement of the following result has been proved.
Theorem 14.11. If $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, is an infinitesimally bendable hypersurface of constant rank two, then it carries a nontrivial symmetric Codazzi tensor $B \in \Gamma(E n d(T M))$ such that $\Delta \subset \operatorname{ker} B$ and

$$
\begin{equation*}
B X \wedge A Y-B Y \wedge A X=0 \tag{14.17}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}(M)$.
Conversely, if $M^{n}$ is simply connected then any such Codazzi tensor $B$ gives rise to a nontrivial infinitesimal bending of $f$.

Proof: It suffices to prove the converse statement. For $B$ as in the statement, we first show that there exist solutions $y \in \Gamma\left(f^{*}\left(T \mathbb{R}^{n+1}\right)\right)$ and $L \in \Gamma\left(\operatorname{Hom}\left(T M ; f^{*}\left(T \mathbb{R}^{n+1}\right)\right)\right)$ satisfying the system of differential equations

$$
(S)\left\{\begin{array}{l}
y_{*} X=-L A X-f_{*} B X  \tag{14.18}\\
\left(\tilde{\nabla}_{X} L\right) Y=\langle B X, Y\rangle N+\langle A X, Y\rangle y
\end{array}\right.
$$

which is easily seen to have more unknowns than equations.
The integrability condition for the first equation is

$$
\begin{equation*}
\tilde{\nabla}_{X} y_{*} Y-\tilde{\nabla}_{Y} y_{*} X-y_{*}[X, Y]=0 \tag{14.19}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}(M)$. Since

$$
\tilde{\nabla}_{X} y_{*} Y=-\left(\tilde{\nabla}_{X} L\right) A Y-L\left(\nabla_{X} A\right) Y-L A \nabla_{X} Y-f_{*} \nabla_{X} B Y-\langle A X, B Y\rangle N
$$

then (14.19) is equivalent to
$\left(\tilde{\nabla}_{X} L\right) A Y-\left(\tilde{\nabla}_{Y} L\right) A X+f_{*}\left(\left(\nabla_{X} B\right) Y-\left(\nabla_{Y} B\right) X\right)+(\langle A X, B Y\rangle-\langle A Y, B X\rangle) N=0$.

After replacing the first two terms by the use of the second equation in $(S)$, then 14.19) follows using (14.15).

It is easy to see that the integrability condition for the second equation is

$$
\begin{equation*}
\left(\tilde{\nabla}_{X} \tilde{\nabla}_{Y} L-\tilde{\nabla}_{Y} \tilde{\nabla}_{X} L-\tilde{\nabla}_{[X, Y]} L\right) Z=-L R(X, Y) Z \tag{14.20}
\end{equation*}
$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$. A straightforward computation gives

$$
\begin{aligned}
\left(\tilde{\nabla}_{X} \tilde{\nabla}_{Y} L\right) Z= & \left\langle\left(\nabla_{X} B\right) Y, Z\right\rangle N+\left\langle B \nabla_{X} Y, Z\right\rangle N-\langle B Y, Z\rangle f_{*} A X+\left\langle\left(\nabla_{X} A\right) Y, Z\right\rangle y \\
& +\left\langle A \nabla_{X} Y, Z\right\rangle y-\langle A Y, Z\rangle L A X-\langle A Y, Z\rangle f_{*} B X .
\end{aligned}
$$

The Codazzi equation together with 14.15) yields

$$
\begin{aligned}
\left(\tilde{\nabla}_{X} \tilde{\nabla}_{Y} L-\tilde{\nabla}_{Y} \tilde{\nabla}_{X} L-\tilde{\nabla}_{[X, Y]} L\right) Z= & -\langle B Y, Z\rangle f_{*} A X-\langle A Y, Z\rangle\left(L A X+f_{*} B X\right) \\
& +\langle B X, Z\rangle f_{*} A Y+\langle A X, Z\rangle\left(L A Y+f_{*} B X\right) .
\end{aligned}
$$

On the other hand, by the Gauss equation we have

$$
L R(X, Y) Z=\langle A Y, Z\rangle L A X-\langle A X, Z\rangle L A Y
$$

and 14.20 follows using 14.17 ).
We show next that there exist solutions $y$ and $L$ of the system $(S)$ that also satisfy (14.10), 14.11) and (14.12). Define a smooth function by

$$
\tau=\langle y, N\rangle,
$$

a smooth one-form by

$$
\theta(X)=\left\langle y, f_{*} X\right\rangle+\langle L X, N\rangle
$$

and a smooth symmetric bilinear tensor by

$$
\beta(X, Y)=\left\langle L X, f_{*} Y\right\rangle+\left\langle L Y, f_{*} X\right\rangle .
$$

A straightforward calculation gives

$$
\begin{gather*}
d \tau=-\theta \circ A  \tag{14.21}\\
\nabla_{X} \theta(Y)=-\beta(A X, Y)+2 \tau\langle A X, Y\rangle \tag{14.22}
\end{gather*}
$$

and

$$
\begin{equation*}
\nabla_{Z} \beta(X, Y)=\langle A Z, X\rangle \theta(Y)+\langle A Z, Y\rangle \theta(X) \tag{14.23}
\end{equation*}
$$

for all $X, Y, Z \in \mathfrak{X}(M)$.
We claim that the system formed by the above three differential equations is integrable. For the first equation, it is easy to see using (14.22) and the Codazzi equation that

$$
X Y(\tau)-Y X(\tau)-[X, Y](\tau)=-X(\theta(A Y))+Y(\theta(A X))+\theta(A[X, Y])=0
$$

A straightforward calculation shows that

$$
\begin{equation*}
\left(\nabla_{X} \nabla_{Y} \theta-\nabla_{Y} \nabla_{X} \theta-\nabla_{[X, Y]} \theta\right) Z=-\theta(R(X, Y) Z) \tag{14.24}
\end{equation*}
$$

is the integrability condition for the second equation. Using (14.21) and (14.22) we obtain

$$
\left(\nabla_{X} \nabla_{Y} \theta\right) Z=-\left(\nabla_{X} \beta\right)(A Y, Z)-\beta\left(\nabla_{X} A Y, Z\right)-2 \theta(A X)\langle A Y, Z\rangle+2 \tau\left\langle\nabla_{X} A Y, Z\right\rangle
$$

Hence

$$
\begin{aligned}
\left(\nabla_{X} \nabla_{Y} \theta-\nabla_{Y} \nabla_{X} \theta-\nabla_{[X, Y]} \theta\right) Z= & -\left(\nabla_{X} \beta\right)(A Y, Z)+\left(\nabla_{Y} \beta\right)(A X, Z) \\
& -2 \theta(A X)\langle A Y, Z\rangle+2 \theta(A Y)\langle A X, Z\rangle .
\end{aligned}
$$

Using (14.23) we see that

$$
\left(\nabla_{X} \nabla_{Y} \theta-\nabla_{Y} \nabla_{X} \theta-\nabla_{[X, Y]} \theta\right) Z=-\theta(A X)\langle A Y, Z\rangle+\theta(A Y)\langle A X, Z\rangle
$$

On the other hand, by the Gauss equation we have

$$
\theta(R(X, Y) Z)=\langle A Y, Z\rangle \theta(A X)-\langle A X, Z\rangle \theta(A Y)
$$

and 14.24 follows.
Finally, the integrability condition for the last equation is
$\left(\nabla_{X} \nabla_{Y} \beta-\nabla_{Y} \nabla_{X} \beta-\nabla_{[X, Y]} \beta\right)(Z, W)=-\beta(R(X, Y) Z, W)-\beta(R(X, Y) W, Z)$. (14.25)
Using (14.22) and (14.23) we obtain
$\nabla_{X} \nabla_{Y} \beta=\left\langle\nabla_{X} A Y, Z\right\rangle \theta(W)+\left\langle\nabla_{X} A Y, W\right\rangle \theta(Z)+\langle A Y, Z\rangle \nabla_{X} \theta(W)+\langle A Y, W\rangle \nabla_{X} \theta(Z)$.
Making use of the Codazzi equation, it follows easily that

$$
\begin{aligned}
\left(\nabla_{X} \nabla_{Y} \beta-\nabla_{Y} \nabla_{X} \beta-\nabla_{[X, Y]} \beta\right)(Z, W)= & -\langle A Y, Z\rangle \beta(A X, W)-\langle A Y, W\rangle \beta(A X, Z) \\
& +\langle A X, Z\rangle \beta(A Y, W)+\langle A X, W\rangle \beta(A Y, Z)
\end{aligned}
$$

and (14.25) is obtained using the Gauss equation.
Start with a solution $L^{*}$ and $y^{*}$ of the integrable system $(S)$ with corresponding tensors $\theta^{*}, \beta^{*}$ and function $\tau^{*}$. Fix a point $x_{0} \in M^{n}$ and let $L_{0}$ and $y_{0}$ be a solution of the integrable system

$$
\left(S_{0}\right)\left\{\begin{array}{l}
y_{*} X=-L A X \\
\left(\tilde{\nabla}_{X} L\right) Y=\langle A X, Y\rangle y
\end{array}\right.
$$

with initial conditions $\theta_{0}\left(x_{0}\right)=\theta^{*}\left(x_{0}\right), \beta_{0}\left(x_{0}\right)=\beta^{*}\left(x_{0}\right)$ and $\tau_{0}\left(x_{0}\right)=\tau^{*}\left(x_{0}\right)$. Then $L=L^{*}-L_{0}$ and $y=y^{*}-y_{0}$ are a solution of $(S)$ such that $\theta=\theta^{*}-\theta_{0}, \beta=\beta^{*}-\beta_{0}$
and $\tau=\tau^{*}-\tau_{0}$. Clearly $\theta\left(x_{0}\right)=\beta\left(x_{0}\right)=\tau\left(x_{0}\right)=0$. Since $\theta, \beta$ and $\tau$ solve the homogeneous integrable system (14.21), (14.22) and (14.23), then $\theta=\beta=\tau=0$.

We have seen that there are $y \in \Gamma\left(f^{*}\left(T \mathbb{R}^{n+1}\right)\right)$ and $L \in \Gamma\left(\operatorname{Hom}\left(T M ; f^{*}\left(T \mathbb{R}^{n+1}\right)\right)\right)$ satisfying both equations in (14.18) as well as (14.10), (14.11) and (14.12). Regarding $L$ as a one-form on $M^{n}$ with values in $f^{*}\left(T \mathbb{R}^{n+1}\right)$, it follows from the second equation in (14.18) and the symmetry of both $A$ and $B$ that $L$ is a closed one-form. Since $M^{n}$ is simply connected and the vector-bundle $f^{*}\left(T \mathbb{R}^{n+1}\right)$ is flat, $L$ is exact as a one-form, that is, there exists $\mathcal{T} \in \Gamma\left(f^{*}\left(T \mathbb{R}^{n+1}\right)\right)$ such that $L=\mathcal{T}_{*}$. In view of 14.10), $\mathcal{T}$ is an infinitesimal bending of $f$.

Given any two pairs $y_{j}, L_{j}$ as above, let $\mathcal{T}_{j}, 1 \leq j \leq 2$, be the associated infinitesimal bendings. It remains to show that $\mathfrak{T}=\mathcal{T}_{1}-\mathcal{T}_{2}$ is a trivial infinitesimal bending.

The pair $L=L_{1}-L_{2}, y=y_{1}-y_{2}$ satisfies $\left(S_{0}\right)$ as well as (14.2), (14.11) and 14.12). Fix $x_{0} \in M^{n}$ and define a skew-symmetric linear endomorphism $\mathcal{D}$ of $\mathbb{R}^{n+1}$ by

$$
\mathcal{D} f_{*}\left(x_{0}\right) X=L\left(x_{0}\right) X \text { and } \mathcal{D} N\left(x_{0}\right)=y\left(x_{0}\right)
$$

and a vector $v \in \mathbb{R}^{n+1}$ by

$$
v=\mathcal{T}\left(x_{0}\right)-\mathcal{D} f\left(x_{0}\right) .
$$

Consider the trivial infinitesimal bending $\tilde{\mathcal{T}}=\mathcal{D} f+v$ and $\tilde{y}=\mathcal{D} N$. Then the pair $\tilde{L}$ and $\tilde{y}$ satisfies $\left(S_{0}\right)$. Thus also the pair $L^{*}=L-\tilde{L}, y^{*}=y-\tilde{y}$ solves system $\left(S_{0}\right)$. Moreover, $\mathcal{T}^{*}\left(x_{0}\right)=0, \boldsymbol{y}^{*}\left(x_{0}\right)=0$ and $L^{*}\left(x_{0}\right)=L\left(x_{0}\right)-\tilde{L}\left(x_{0}\right)=0$. Thus $\mathcal{T}^{*}=0$, and hence $\mathcal{T}=\tilde{\mathcal{T}}$.

### 14.3.2 Infinitesimal bendings of ruled hypersurfaces

By Proposition 11.2, the set of local isometric bendings of a ruled hypersurface is parametrized by the set of smooth real functions on an open interval. Therefore, ruled hypersurfaces form a nontrivial class of infinitesimally bendable hypersurfaces.

Proposition 14.12. Any simply connected ruled hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, that is free of flat points and is not surface-like on any open subset of $M^{n}$ is infinitesimally bendable, and any of its infinitesimal bendings is the variational vector field of an isometric bending.

Proof: Let $J \in \Gamma(\operatorname{End}(T M))$ be defined by $\sqrt{11.15)}$ in terms of an orthonormal frame $X, Y$ of $\Delta^{\perp}$, with $X$ orthogonal to the rulings, and let $D=\delta I+\theta J$, where $\delta \in \mathbb{R}$ and $\theta \in C^{\infty}(M)$ is arbitrarily prescribed along an integral curve of $X$ and required to satisfy conditions (11.16) and (11.17). By Lemma 11.1 and Proposition 11.2, any Codazzi tensor $\tilde{A}$ on $M^{n}$ with $\Delta \subset \operatorname{ker} \tilde{A}$ is given by $A=A D$ for such a tensor $D$. It is easily checked that $\tilde{A}$ satisfies (14.17) if and only if $\delta=0$. Thus $f$ is infinitesimally bendable by Theorem 14.11, and any infinitesimal bending of $f$ is determined by a Codazzi tensor on $M^{n}$ given by $\tilde{A}=\theta A J$, with $\theta$ and $J$ as above. Therefore the one-parameter family of Codazzi tensors

$$
A(t)=A+t \tilde{A}, \quad t \in \mathbb{R},
$$

gives rise to an isometric bending of $f$ having the infinitesimal bending determined by $\tilde{A}$ as its variational vector field.

### 14.3.3 Special hyperbolic and elliptic surfaces

To proceed with the description of the infinitesimally bendable hypersurfaces in terms of the Gauss parametrization, in this section we characterize the hyperbolic and elliptic surfaces $g: L^{2} \rightarrow \mathbb{S}^{n}$ with respect to $\bar{J} \in \Gamma(\operatorname{End}(T L))$ for which there exists $\bar{\mu} \in C^{\infty}(L)$ with $\bar{\mu} \neq 0$ everywhere such that $\bar{\mu} \bar{J}$ is a Codazzi tensor on $L^{2}$. It will be shown in the following section that these surfaces are precisely the Gauss maps of the infinitesimally bendable hypersurfaces.

We call a hyperbolic surface $g: L^{2} \rightarrow \mathbb{S}^{n}$ a special hyperbolic surface if, for any local system of real conjugate coordinates $(u, v)$ on $L^{2}$ given by Proposition 11.10, the Christoffel symbols $\Gamma^{1}, \Gamma^{2}$ defined by 11.21) satisfy

$$
\begin{equation*}
\Gamma_{u}^{1}=\Gamma_{v}^{2} \tag{14.26}
\end{equation*}
$$

Hence, if $L^{2}$ is simply connected, there exists $\mu \in C^{\infty}(L)$ such that

$$
\begin{equation*}
d \mu+2 \mu \omega=0 \tag{14.27}
\end{equation*}
$$

where $\omega=\Gamma^{2} d u+\Gamma^{1} d v$.
We say that an elliptic surface $g: L^{2} \rightarrow \mathbb{S}^{n}$ is a special elliptic surface if for any local system of complex conjugate coordinates $(u, v)$ on $L^{2}$ given by Proposition 11.10 the Christoffel symbol $\Gamma$ defined by (11.22) satisfies

$$
\begin{equation*}
\Gamma_{z}=\bar{\Gamma}_{\bar{z}}, \tag{14.28}
\end{equation*}
$$

that is, $\Gamma_{z}$ is real-valued. Thus any simply connected special elliptic surface carries a real-valued $\mu \in C^{\infty}(L)$ such that

$$
\begin{equation*}
\mu_{\bar{z}}+2 \mu \Gamma=0 \tag{14.29}
\end{equation*}
$$

Notice that surfaces of first species of real type (respectively, complex type) are special hyperbolic (respectively, special elliptic).

Lemma 14.13. For a simply connected surface $g: L \rightarrow \mathbb{S}^{n}$, the following assertions are equivalent:
(i) The surface $g$ is hyperbolic (respectively, elliptic) with respect to a tensor $\bar{J}$ on $L^{2}$ satisfying $\bar{J}^{2}=\bar{I}$ (respectively, $\bar{J}^{2}=-\bar{I}$ ) and there exists $\bar{\mu} \in C^{\infty}(L), \bar{\mu} \neq 0$ everywhere, such that $\bar{D}=\bar{\mu} \bar{J}$ is a Codazzi tensor on $L^{2}$.
(ii) The surface $g$ is special hyperbolic (respectively, special elliptic).

Proof: Assume that $g$ is hyperbolic and as in part $(i)$. Let $(u, v)$ be local real conjugate coordinates on $L^{2}$ given by Proposition 11.10 such that $\bar{D} \partial u=\bar{\mu} \partial u$. Then the equation

$$
\begin{equation*}
\left(\nabla_{\partial u}^{\prime} \bar{D}\right) \partial v-\left(\nabla_{\partial v}^{\prime} \bar{D}\right) \partial u=0 \tag{14.30}
\end{equation*}
$$

is easily seen to be equivalent to the system (14.27).
Conversely, if $g$ is special hyperbolic with real conjugate coordinates $(u, v), \bar{J}$ is the complex structure on $L^{2}$ given by $\bar{J} \partial_{u}=\partial v$ and $\bar{J} \partial_{v}=-\partial_{u}$, and $\bar{\mu} \in C^{\infty}(L)$ satisfies 14.27), then $\bar{D}=\bar{\mu} \bar{J}$ satisfies 14.30 ) in view of 14.27), and hence is a Codazzi tensor on $L^{2}$. The proof for the elliptic case is similar.

### 14.3.4 The classification

We are now ready to state and prove the classification of infinitesimally bendable hypersurfaces that are neither surface-like nor ruled on any open subset. Given a surface $g: L^{2} \rightarrow \mathbb{S}^{n}$ and $\gamma \in C^{\infty}(L)$, we call $(g, \gamma)$ a special hyperbolic pair (respectively, special elliptic pair) if $(g, \gamma)$ is a hyperbolic pair (respectively, elliptic pair) and $g$ is a special hyperbolic (respectively, special elliptic) surface.

Theorem 14.14. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, be an infinitesimally bendable hypersurface with constant type number two that is neither surface-like nor ruled on any open subset of $M^{n}$. Then, on each connected component of an open dense subset of $M^{n}$, the hypersurface is parametrized in terms of the Gauss parametrization by a special hyperbolic or a special elliptic pair.

Conversely, any hypersurface parametrized in terms of the Gauss parametrization by a special hyperbolic or special elliptic pair admits locally an infinitesimal bending. Moreover, the infinitesimal bending is unique.

Proof: If $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, is an infinitesimally bendable hypersurface with constant type number two, it follows from Theorem 14.11 that there exists a nontrivial symmetric Codazzi tensor $B \in \Gamma(\operatorname{End}(T M))$ satisfying (14.17) such that $\Delta \subset \operatorname{ker} B$. Moreover, by (14.17) and the assumption that $A$ has rank two, the tensor $B$ cannot be a constant multiple of $A$ on any open subset of $M^{n}$.

By Lemma 11.1, the hypersurface $f$ is either hyperbolic, parabolic or elliptic with respect to $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ on each connected component of an open dense subset $\mathcal{U}$ of $M^{n}$, depending on whether the tensor $D=\left.\left(\left.A\right|_{\Delta^{\perp}}\right)^{-1} B\right|_{\Delta^{\perp}} \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ has two distinct real eigenvalues, one real eigenvalue of multiplicity two or a pair of complex conjugate eigenvalues, respectively. The second case cannot occur by Proposition 11.2 and the assumption that $f$ is not ruled on any open subset of $M^{n}$. Thus $f$ is either hyperbolic or elliptic with respect to $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ on each connected component of $\mathcal{U}$. Moreover, the tensor $D$ satisfies the conditions (i) and (ii) in Lemma 11.1 as well as

$$
\begin{align*}
& \langle A Y, A Z\rangle\langle A D X, A W\rangle-\langle A D X, A Z\rangle\langle A Y, A W\rangle \\
& = \tag{14.31}
\end{align*}
$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$, which follows from 14.17).
Let $g: L^{2} \rightarrow \mathbb{S}^{n+1}$ and $\gamma \in C^{\infty}(L)$ parametrize the hypersurface $f$ in terms of the Gauss parametrization. By Proposition 11.11, the tensor $J$ is the horizontal lift of a tensor $\bar{J} \in \Gamma(\operatorname{End}(T L))$ and the pair $(g, \gamma)$ is hyperbolic (respectively, elliptic) with respect to $\bar{J}$. On the other hand, by Proposition 11.12 also $D$ is the horizontal lift of a tensor $\bar{D} \in \Gamma(\operatorname{End}(T L))$ which is a Codazzi tensor on $L^{2}$ such that $\bar{D} \in \operatorname{span}\{\bar{I}, \bar{J}\}$ and $\bar{D} \notin \operatorname{span}\{\bar{I}\}$. Moreover, in view of (7.14), Eq. (14.31) gives

$$
\begin{align*}
\langle\bar{Y}, \bar{Z}\rangle^{\prime}\langle\bar{D} \bar{X}, \bar{W}\rangle^{\prime} & -\langle\bar{D} \bar{X}, \bar{Z}\rangle^{\prime}\langle\bar{Y}, \bar{W}\rangle^{\prime} \\
& =\langle\bar{X}, \bar{Z}\rangle^{\prime}\langle\bar{D} \bar{Y}, \bar{W}\rangle^{\prime}-\langle\bar{D} \bar{Y}, \bar{Z}\rangle^{\prime}\langle\bar{X}, \bar{W}\rangle^{\prime} \tag{14.32}
\end{align*}
$$

for all $\bar{X}, \bar{Y}, \bar{Z}, \bar{W} \in \mathfrak{X}(L)$, where $\langle,\rangle^{\prime}$ is the metric on $L^{2}$ induced by $g$. In terms of an orthonormal frame $\bar{X}, \bar{Y}$ of $L^{2}$, Eq. 14.32) for $\bar{W}=\bar{X}$ and $\bar{Z}=\bar{Y}$ reduces to

$$
\langle\bar{D} \bar{X}, \bar{D} X\rangle^{\prime}+\langle\bar{D} \bar{Y}, \bar{D} Y\rangle^{\prime}=0
$$

that is, $\operatorname{tr} \bar{D}=0$. Since $\bar{D} \in \operatorname{span}\{\bar{I}, \bar{J}\}$, there exists $\bar{\mu} \in C^{\infty}(L)$ such that $\bar{D}=\bar{\mu} \bar{J}$. Thus the pair $(g, \gamma)$ is special hyperbolic (respectively, special elliptic) by Lemma 14.13 .

Conversely, let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be given in terms of the Gauss parametrization by either a special hyperbolic or a special elliptic pair $(g, \gamma)$ with respect to $\bar{J} \in$ $\Gamma(\operatorname{End}(T L))$. By Proposition 11.11, the hypersurface $f$ is either hyperbolic or elliptic with respect to the horizontal lift $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ of $\bar{J}$. On the other hand, since $(g, \gamma)$ is either a special hyperbolic or a special elliptic pair, there exists $\bar{\mu} \in C^{\infty}(L)$ such that $\bar{D}=\bar{\mu} \bar{J}$ is a Codazzi tensor on $L^{2}$. In particular, $\bar{D}$ satisfies 14.32).

By Proposition 11.12, the horizontal lift $D \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ of $\bar{D}$ satisfies conditions (i) and (ii) in Lemma 11.1, and the tensor $B \in \Gamma(\operatorname{End}(T M))$, defined by

$$
\left.B\right|_{\Delta^{\perp}}=A D \text { and } \operatorname{ker} B=\Delta
$$

satisfies (11.2). Therefore $B$ is a symmetric Codazzi tensor on $M^{n}$ by Lemma 11.1. Moreover, Eq. 14.32) for $\bar{D}$ implies 14.31, which is equivalent to 14.17). By Theorem 14.11, the Codazzi tensor $B$ satisfying (14.17) determines an infinitesimal bending of $f$.

### 14.3.5 Infinitesimally bendable hypersurfaces as envelopes

If $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ is an infinitesimally bendable hypersurface parametrized, in terms of the Gauss parametrization, by a special hyperbolic or a special elliptic pair $g: L^{2} \rightarrow \mathbb{R}^{n+1}$ and $\gamma \in C^{\infty}(L)$, then $(g, \gamma)$ determines a two-parameter congruence of affine hyperplanes that is enveloped by $f$. In this section we give a simple way of producing these congruences of affine hyperplanes, thus providing an alternative description of infinitesimally bendable hypersurfaces.

The hyperbolic and elliptic cases are considered separately in the next two results.

Proposition 14.15. Let $g: L^{2} \rightarrow \mathbb{S}^{n}$ be a special hyperbolic surface and let $(u, v)$ be real conjugate coordinates on a simply connected open subset $U \subset L^{2}$ where (14.26) is satisfied. Let $\mu \in C^{\infty}(U)$ be a positive solution of (14.27). Then $\varphi \in C^{\infty}(U)$ is a solution of

$$
\begin{equation*}
\varphi_{u v}-\Gamma^{1} \varphi_{u}-\Gamma^{2} \varphi_{v}+F \varphi=0 \tag{14.33}
\end{equation*}
$$

with $F=\left\langle\partial_{u}, \partial_{v}\right\rangle$, if and only if $\psi=\sqrt{\mu} \varphi$ is a solution of

$$
\begin{equation*}
\psi_{u v}+M \psi=0 \tag{14.34}
\end{equation*}
$$

where $M \in C^{\infty}(L)$ is given by

$$
\begin{equation*}
M=F-\frac{\mu_{u v}}{2 \mu}+\frac{\mu_{u} \mu_{v}}{4 \mu^{2}} . \tag{14.35}
\end{equation*}
$$

In particular, the map $k=\sqrt{\mu} h: L^{2} \rightarrow \mathbb{R}^{n+1}$, where $h$ is the composition $h=i \circ g$ of $g$ with the inclusion $i: \mathbb{S}^{n} \rightarrow \mathbb{R}^{n+1}$, satisfies

$$
\begin{equation*}
k_{u v}+M k=0 . \tag{14.36}
\end{equation*}
$$

Conversely, let $k: L^{2} \rightarrow \mathbb{R}^{n+1}$ be a map satisfying (14.36) for some $M \in C^{\infty}(L)$ with respect to local coordinates $(u, v)$ on $L^{2}$. If $h=(1 /\|k\|) k: L^{2} \rightarrow \mathbb{R}^{n+1}$ is an immersion, then $(u, v)$ are real conjugate coordinates for the surface $g: L^{2} \rightarrow \mathbb{S}^{n}$ given by $h=i \circ g,(14.26)$ is satisfied with respect to its induced metric and $\mu=\|k\|^{2}$ is a positive solution of (14.27).
Proof: Since $\mu \in C^{\infty}(U)$ is a solution of (14.27), it satisfies

$$
\Gamma^{1}=-\frac{\mu_{v}}{2 \mu} \text { and } \Gamma^{2}=-\frac{\mu_{u}}{2 \mu} .
$$

Hence (14.33) becomes

$$
\begin{equation*}
\varphi_{u v}+\frac{\mu_{v}}{2 \mu} \varphi_{u}+\frac{\mu_{u}}{2 \mu} \varphi_{v}+F \varphi=0 \tag{14.37}
\end{equation*}
$$

It follows easily that (14.37) takes the form (14.34) for $\psi=\sqrt{\mu} \varphi$, where $M$ is given by (14.35).

Conversely, if $k: L^{2} \rightarrow \mathbb{R}^{n+1}$ is a map satisfying 14.36) for some $M \in C^{\infty}(L)$ with respect to local coordinates $(u, v)$ on $L^{2}$, then a straightforward computation shows that $h=(1 / \sqrt{\mu}) k: L^{2} \rightarrow \mathbb{R}^{n+1}$, with $\mu=\|k\|^{2}$, satisfies

$$
\begin{equation*}
h_{u v}+\frac{\mu_{v}}{2 \mu} h_{u}+\frac{\mu_{u}}{2 \mu} h_{v}+F h=0 \tag{14.38}
\end{equation*}
$$

where

$$
F=M+\frac{\mu_{u v}}{2 \mu}-\frac{\mu_{u} \mu_{v}}{4 \mu^{2}} .
$$

If $h$ is an immersion and $g: L^{2} \rightarrow \mathbb{S}^{n}$ is defined by $h=i \circ g$, then 14.38) implies that $(u, v)$ are real conjugate coordinates for $g$ and that the Christoffel symbols of the metric induced by $g$ are

$$
\Gamma^{1}=-\frac{\mu_{v}}{2 \mu} \text { and } \Gamma^{2}=-\frac{\mu_{u}}{2 \mu} .
$$

It follows that 14.26 ) is satisfied and that $\mu$ is a positive solution of (14.27).

Proposition 14.16. Let $g: L^{2} \rightarrow \mathbb{S}^{n}$ be a special elliptic surface and let $(u, v)$ be complex conjugate coordinates on a simply connected open subset $U \subset L^{2}$ where (14.28) is satisfied. If $\mu \in C^{\infty}(L)$ is a real-valued positive solution of (14.29), then $\varphi \in C^{\infty}(U)$ is a solution of

$$
\begin{equation*}
\varphi_{z \bar{z}}-\Gamma \varphi_{z}-\bar{\Gamma} \varphi_{\bar{z}}+F \varphi=0 \tag{14.39}
\end{equation*}
$$

with $F=\left\langle\partial_{z}, \partial_{\bar{z}}\right\rangle$, if and only if $\psi=\sqrt{\mu} \varphi$ is a solution of

$$
\psi_{z \bar{z}}+M \psi=0
$$

where $M \in C^{\infty}(L)$ is given by

$$
\begin{equation*}
M=F-\frac{\mu_{z \bar{z}}}{2 \mu}+\frac{\mu_{z} \mu_{\bar{z}}}{4 \mu^{2}} . \tag{14.40}
\end{equation*}
$$

In particular, the map $k=\sqrt{\mu} h: L^{2} \rightarrow \mathbb{R}^{n+1}$, where $h$ is the composition $h=i \circ g$ of $g$ with the inclusion $i: \mathbb{S}^{n} \rightarrow \mathbb{R}^{n+1}$, satisfies

$$
\begin{equation*}
k_{z \bar{z}}+M k=0 . \tag{14.41}
\end{equation*}
$$

Conversely, let $k: L^{2} \rightarrow \mathbb{R}^{n+1}$ be a map satisfying 14.41) for some $M \in C^{\infty}(L)$ with respect to local coordinates $(u, v)$. If $g=(1 /\|k\|) k: L^{2} \rightarrow \mathbb{S}^{n}$ is an immersion, then $(u, v)$ are complex conjugate coordinates for $g$, (14.28) is satisfied with respect to the induced metric and $\mu=\|k\|$ is a real-valued positive solution of (14.29).

Proof: Since $\mu \in C^{\infty}(L)$ is a real-valued solution of (14.29), then

$$
\Gamma=\frac{\mu_{z}}{2 \mu}
$$

Hence (14.39) becomes

$$
\begin{equation*}
\varphi_{z \bar{z}}+\frac{\mu_{z}}{2 \mu} \varphi_{z}+\frac{\mu_{\bar{z}}}{2 \mu} \varphi_{\bar{z}}+F \varphi=0 \tag{14.42}
\end{equation*}
$$

It follows easily that (14.42) takes the form (14.41) for $k=\sqrt{\mu} \varphi$ where $M$ is given by (14.40).

Conversely, if $k: L^{2} \rightarrow \mathbb{R}^{n+1}$ is a map satisfying (14.41) for some $M \in C^{\infty}(L)$ with respect to local coordinates $(u, v)$ on $L^{2}$, then a straightforward computation shows that $h=(1 / \sqrt{\mu}) k: L^{2} \rightarrow \mathbb{R}^{n+1}$, with $\mu=\|k\|^{2}$, satisfies

$$
\begin{equation*}
h_{z \bar{z}}+\frac{\mu_{z}}{2 \mu} h_{z}+\frac{\mu_{\bar{z}}}{2 \mu} h_{\bar{z}}+F h=0 \tag{14.43}
\end{equation*}
$$

where

$$
F=M+\frac{\mu_{z \bar{z}}}{2 \mu}-\frac{\mu_{z} \mu_{\bar{z}}}{4 \mu^{2}}
$$

If $h$ is an immersion and $g: L^{2} \rightarrow \mathbb{S}^{n}$ is defined by $h=i \circ g$, then (14.43) implies that $(u, v)$ are complex conjugate coordinates for $g$ and that the complex Christoffel symbol of the metric induced by $g$ is

$$
\Gamma=\frac{\mu_{z}}{2 \mu} .
$$

It follows that 14.28 is satisfied and that $\mu$ is a positive solution of (14.29).
The description of the infinitesimally bendable hypersurfaces $f: M^{n} \rightarrow \mathbb{R}^{n+1}$, $n \geq 3$, as envelopes of certain two-parameter congruence of affine hyperplanes can be stated as follows.

Theorem 14.17. Consider a two-parameter congruence of affine hyperplanes in $\mathbb{R}^{n+1}$ given in terms of the standard coordinates $x_{1}, \ldots, x_{n+1}$ by

$$
\begin{equation*}
k_{1} x_{1}+\cdots+k_{n+1} x_{n+1}-k_{0}=0 \tag{14.44}
\end{equation*}
$$

where $k_{0}, \ldots, k_{n+1}: U \rightarrow \mathbb{R}$ are solutions on an open subset $U \subset \mathbb{R}^{2}$ of the partial differential equation

$$
\begin{equation*}
\psi_{z_{1} z_{2}}+M \psi=0 \tag{14.45}
\end{equation*}
$$

for some $M \in C^{\infty}(U)$, with $\left(z_{1}, z_{2}\right)$ standing either for $(u, v)$ or $(z, \bar{z})$. Then any hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ that envelops such a congruence of affine hyperplanes admits locally a unique infinitesimal bending.

Conversely, any infinitesimally bendable hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, that has constant type number two and is neither surface-like nor ruled on any open subset of $M^{n}$, envelops a two-parameter family of hyperplanes as above on each connected component of an open dense subset of $M^{n}$.

Proof: Assume that $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ envelops the congruence of affine hyperplanes in $\mathbb{R}^{n+1}$ given by (14.44), where $k_{0}, \ldots, k_{n+1}: U \rightarrow \mathbb{R}$ are solutions on the open subset $U \subset \mathbb{R}^{2}$ of the partial differential equation 14.45 . We argue for the case in which $\left(z_{1}, z_{2}\right)$ stands for $(u, v)$, the other case being similar.

Let $g: U \rightarrow \mathbb{S}^{n+1}$ be defined by $i \circ g=k /\|k\|$, where $k=\left(k_{1}, \ldots, k_{n+1}\right): U \rightarrow \mathbb{R}^{n+1}$ and $i: \mathbb{S}^{n} \rightarrow \mathbb{R}^{n+1}$ is the inclusion. Then $g$ is a Gauss map for $f$, and $\gamma=k_{0} /\|k\|$ is its support function. Since the affine hyperplanes (14.44) are assumed to form a twoparameter congruence, the map $g$ is an immersion. By the assumption that $k$ satisfies (14.45), it follows from the converse statement of Proposition 14.15 that $g$ is a special hyperbolic surface.

Moreover, since $k_{0}$ also satisfies (14.45), the function $\gamma$ satisfies 14.33) by the direct statement of Proposition 14.15. Thus $(g, \gamma)$ is a special hyperbolic pair, and hence $f$ admits a unique infinitesimal bending by Theorem 14.14 .

Conversely, let $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, be an infinitesimally bendable hypersurface that has constant type number two and is neither surface-like nor ruled on any open subset of $M^{n}$. Let the pair $(g, \gamma)$ parametrize $f$ by means of the Gauss parametrization. Then $f$ envelops the two-parameter congruence of affine hyperplanes

$$
\begin{equation*}
h_{1} x_{1}+\cdots+h_{n+1} x_{n+1}-\gamma=0 \tag{14.46}
\end{equation*}
$$

where $h=\left(h_{1}, \ldots, h_{n+1}\right)$ is the composition $h=i \circ g$ of $g: L^{2} \rightarrow \mathbb{S}^{n}$ with the inclusion $i: \mathbb{S}^{n} \rightarrow \mathbb{R}^{n+1}$. By Theorem 14.14, the pair $(g, \gamma)$ is either special hyperbolic or special elliptic. We argue for the first possibility, the argument for the second being similar.

By the direct statement of Proposition 14.15, if $\mu \in C^{\infty}(U)$ is a positive solution of (14.27), then the map $k=\sqrt{\mu} h$ satisfies (14.36). Moreover, since $\gamma$ satisfies (14.33), the function $\sqrt{\mu} \gamma$ is also a solution of (14.34). Therefore Eq. (14.46) can also be written as

$$
k_{1} x_{1}+\cdots+k_{n+1} x_{n+1}-k_{0}=0
$$

with $k_{0}, \ldots, k_{n+1}$ being solutions of (14.34).

### 14.4 Sbrana-Cartan hypersurfaces as envelopes

The results of the preceding section can be used to derive Cartan's description of the Sbrana-Cartan hypersurfaces of the continuous class as envelopes of certain two-parameter congruences of affine hyperplanes.

Theorem 14.18. Consider a two-parameter congruence of affine hyperplanes in $\mathbb{R}^{n+1}$ given in terms of the standard coordinates $x_{1}, \ldots, x_{n+1}$ by

$$
\begin{equation*}
k_{1} x_{1}+\cdots+k_{n+1} x_{n+1}-k_{0}=0 \tag{14.47}
\end{equation*}
$$

where $k_{0}, \ldots, k_{n+1}: U \rightarrow \mathbb{R}$ are solutions of (14.45) on an open subset $U \subset \mathbb{R}^{2}$. Assume further that $k_{1}, \ldots, k_{n+1}$ do not vanish simultaneously at any point of $U$ and that the function $\mu=\sum_{j=1}^{n+1} k_{j}^{2}$ either is given by

$$
\mu=U(u)+V(v)
$$

for some smooth functions $U=U(u)$ and $V=V(v)$ or satisfies

$$
\mu_{u u}+\mu_{v v}=0,
$$

depending on whether $\left(z_{1}, z_{2}\right)$ stands for either $(u, v)$ or $(z, \bar{z})$ in (14.45), respectively. Then any hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ that envelops such a congruence of affine hyperplanes is, accordingly, a Sbrana-Cartan hypersurface of real or complex type in the continuous class.

Conversely, any Sbrana-Cartan hypersurface of real or complex type in the continuous class envelops a two-parameter family of hyperplanes as above on each connected component of an open dense subset of $M^{n}$.

Proof: Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ envelop the congruence of affine hyperplanes in $\mathbb{R}^{n+1}$ given by (14.47), where $k_{0}, \ldots, k_{n+1}: U \rightarrow \mathbb{R}$ are solutions of (14.45) on the open subset $U \subset \mathbb{R}^{2}$. We argue for the case in which $(u, v)$ are real conjugate coordinates on $U$, the case in which they are complex conjugate being similar.

As in the proof of Theorem 14.17, if $g: U \rightarrow \mathbb{S}^{n+1}$ is defined by $i \circ g=k /\|k\|$, where $k=\left(k_{1}, \ldots, k_{n+1}\right): U \rightarrow \mathbb{R}^{n+1}$ and $i: \mathbb{S}^{n} \rightarrow \mathbb{R}^{n+1}$ is the inclusion, and $\gamma=k_{0} /\|k\|$, then $g$ is a Gauss map for $f, \gamma$ is its support function, and $(g, \gamma)$ is a special hyperbolic pair. It remains to show, under the assumption that $\mu=U(u)+V(v)$, that the surface
$g$ is of first species of real type. By the converse statement of Proposition 14.15, the function $\mu$ satisfies (14.27). Thus

$$
\Gamma^{1}=-\frac{\mu_{v}}{2 \mu}=-\frac{V^{\prime}(v)}{2(U(u)+V(v))} \text { and } \Gamma^{2}=-\frac{\mu_{u}}{2 \mu}=-\frac{U^{\prime}(u)}{2(U(u)+V(v))}
$$

Therefore $g$ is of first species of real type, for

$$
\Gamma_{u}^{1}=\frac{U^{\prime}(u) V^{\prime}(v)}{2(U(u)+V(v))^{2}}=2 \Gamma^{1} \Gamma^{2}=\Gamma_{v}^{2} .
$$

Conversely, let $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, be a Sbrana-Cartan hypersurface of real type in the continuous class and let the pair $(g, \gamma)$ parametrize $f$ by means of the Gauss parametrization. Then $f$ envelops the two-parameter congruence of affine hyperplanes

$$
\begin{equation*}
h_{1} x_{1}+\cdots+h_{n+1} x_{n+1}-\gamma=0 \tag{14.48}
\end{equation*}
$$

where $h=\left(h_{1}, \ldots, h_{n+1}\right)$ is the composition $h=i \circ g$ of $g: L^{2} \rightarrow \mathbb{S}^{n}$ with the inclusion $i: \mathbb{S}^{n} \rightarrow \mathbb{R}^{n+1}$. By Theorem 11.16, $(g, \gamma)$ is a hyperbolic pair and $g: L^{2} \rightarrow \mathbb{S}^{n}$ is a surface of first species of real type. In particular, $(g, \gamma)$ is a special hyperbolic pair. As in the proof of Theorem 14.17, from the direct statement of Proposition 14.15 we see that, if $\mu \in C^{\infty}(U)$ is a positive solution of (14.27), then the map $k=\sqrt{\mu} h$ satisfies (14.36) and the function $\sqrt{\mu} \gamma$ is also a solution of (14.34). Therefore Eq. (14.48) can also be written as

$$
k_{1} x_{1}+\cdots+k_{n+1} x_{n+1}-k_{0}=0,
$$

with $k_{0}, \ldots, k_{n+1}$ being solutions of 14.34 . Now, since $g$ is of first species, then its Christoffel symbols satisfy (11.43). Thus

$$
\frac{\mu_{u v} \mu-\mu_{u} \mu_{v}}{2 \mu^{2}}=-\Gamma_{u}^{1}=-2 \Gamma^{1} \Gamma^{2}=-\frac{\mu_{u} \mu_{v}}{2 \mu^{2}}
$$

which gives $\mu_{u v}=0$, that is,

$$
\mu=U(u)+V(v)
$$

for some smooth functions $U(u)$ and $V(v)$.

### 14.5 Notes

Sbrana 311] stated his result on infinitesimal bendings in terms of the Gauss parametrization. For the proof, he made use of results he had obtained in [312], a paper only published the following year. In fact, what Sbrana did was to provide a complete description of one class of infinitesimally bendable hypersurfaces, but somehow ignored others.

It is a surprise that there is no reference in the literature to Sbrana's contribution to the description of the hypersurfaces that admit infinitesimal bendings. The few
places in the literature where his paper is referred to are quite old and do not discuss his result; see, e.g., 318.

Most of the basic materials of the first two sections of this chapter, as well as all rigidity results therein, have been taken from Dajczer-Rodríguez [128]; see also Goldstein-Ryan [200], as well as Florit [182] for a result on compositions. The infinitesimal version of Sacksteder rigidity result for compact hypersurfaces in Euclidean space was obtained in [128]. The more general case of complete Euclidean hypersurfaces was treated by Jimenez [225]. The description of the infinitesimally bendable hypersurfaces as envelopes of certain two-parameter congruences of hyperplanes has been taken from Dajczer-Vlachos [154].

Note that Theorem 14.11, basically contained in Sbrana's paper [311, is a kind of Fundamental theorem for infinitesimal bendings. Genuine infinitesimal bendings of submanifolds with codimension greater than one were studied by Dajczer-Jimenez [120].

### 14.6 Exercises

Exercise 14.1. Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be an isometric immersion and let $\xi \in \Gamma\left(N_{f} M\right)$ be an umbilical unit vector field that is not totally geodesic at any point, that is, $A_{\xi}=\lambda I$, where $\lambda$ never vanishes. For instance, this is the case if $M^{n}$ admits an isometric immersion into $\mathbb{S}^{m-1}$. Assume that $M^{n}$ carries a conformal Killing field $Z \in \mathfrak{X}(M)$, that is, there exists $\varphi \in C^{\infty}(M)$ such that

$$
\left\langle\nabla_{X} Z, X\right\rangle=\varphi(x)\langle X, X\rangle
$$

for any $X \in \mathfrak{X}(M)$. Show that $f$ has a nontrivial infinitesimal bending.
Exercise 14.2. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, be an $(n-\ell)$-cylinder over a hypersurface $g: L^{\ell} \rightarrow \mathbb{R}^{\ell+1}, 2 \leq \ell \leq n-1$, with type number $\tau \geq 2$ at any point. Show that any infinitesimal bending of $f$ is given by an infinitesimal bending of $g$.

Exercise 14.3. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{m}, c \neq 0$, be an isometric immersion and let $i: \mathbb{Q}_{c}^{m} \rightarrow \mathbb{R}_{\mu}^{m+1}$ denote an umbilical inclusion, where $\mathbb{R}_{\mu}^{m+1}$ stands for either $\mathbb{R}^{m+1}$ or $\mathbb{L}^{m+1}$ depending on whether $c>0$ or $c<0$, respectively. Given an infinitesimal bending $\mathcal{T}$ of $f$, for each $t \in \mathbb{R}$ consider the map $G_{t}: M^{n} \rightarrow \mathbb{R}_{\mu}^{m+1}$ defined by

$$
G_{t}(x)=\frac{1}{\sqrt{1+c t^{2}\|\mathcal{T}(x)\|^{2}}}\left(i(f(x))+t i_{*} \mathcal{T}(x)\right)
$$

Show that the following assertions hold:
(i) The maps $G_{t}$ and $G_{-t}$ are immersions that induce the same metric.
(ii) If $f$ is substantial and there is $0 \neq t_{0} \in \mathbb{R}$ such that $G_{t_{0}}$ and $G_{-t_{0}}$ are congruent then $\mathcal{T}$ is trivial.

Exercise 14.4. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, be an infinitesimally bendable hypersurface of constant rank two. Show that any hypersurface in the variation (14.4) carries the same relative nullity foliation.

## Chapter 15

## Real Kaehler submanifolds

The purpose of this chapter is to present several results on isometric immersions of Kaehler manifolds into real space forms. In fact, most of the results are about real Kaehler submanifolds. By a real Kaehler submanifold $f: M^{2 n} \rightarrow \mathbb{R}^{m}$ we mean an isometric immersion of a Kaehler manifold $M^{2 n}$ of complex dimension $n \geq 2$ into Euclidean space.

Real Kaehler submanifolds in low codimension are shown to be generically holomorphic. That $f$ is holomorphic means that $m$ is even and that $f: M^{2 n} \rightarrow \mathbb{C}^{m / 2} \approx \mathbb{R}^{m}$ is holomorphic. More precisely, it is shown that a real Kaehler submanifold must be holomorphic if its type number is greater than or equal to three at any point. In particular, this implies that real Kaehler hypersurfaces that are free of flat points have rank two. For these, a parametric description is given in terms of the Gauss parametrization.

We are mostly interested in those real Kaehler submanifolds that are not holomorphic. In fact, for a good part of the chapter we will focus on minimal real Kaehler submanifolds and, in particular, conclude that they enjoy many of the basic properties of minimal surfaces in Euclidean space. For instance, any simply connected minimal real Kaehler submanifold belongs to a one-parameter associated family of isometric submanifolds, all with the same generalized Gauss map, and can be realized as the real part of a holomorphic isometric immersion, called its holomorphic representative. The family is shown to be trivial if and only if the submanifold is holomorphic.

A classification of all complete minimal real Kaehler submanifolds in codimension two is given. They turn out to be either ruled submanifolds or cylinders of certain special types. The proof uses results on complete minimal real Kaehler submanifolds with arbitrary codimension that have a large index of relative nullity.

It is shown that a pair of minimal real Kaehler hypersurfaces that belongs to the same associate family is the only interesting case of a pair of conformal hypersurfaces that make a constant angle in the ambient space. Finally, we see that simply connected minimal real Kaehler submanifolds admit a Weierstrass type representation. In particular, this gives an alternative representation for hypersurfaces.

### 15.1 Some basic facts

An almost complex structure on a real differentiable manifold $M$ is a tensor field $J \in \Gamma(\operatorname{End}(T M))$ satisfying $J^{2}=-I$, where $I$ is the identity tensor field. The pair $(M, J)$ is called an almost complex manifold. It is easily seen that each tangent space of $M$ has a basis of the form $X_{1}, J X_{1}, \ldots, X_{n}, J X_{n}$. In particular, an almost complex manifold must have even dimension. Moreover, any two such bases differ by an isomorphism with positive determinant; hence an almost complex manifold is necessarily orientable.

The space

$$
\mathbb{C}^{n}=\left\{\left(z_{1}, \ldots, z_{n}\right): z_{k}=x_{k}+i y_{k}, x_{k}, y_{k} \in \mathbb{R}\right\}
$$

carries a natural almost complex structure defined by

$$
J\left(\partial / \partial x_{k}\right)=\partial / \partial y_{k} \text { and } J\left(\partial / \partial y_{k}\right)=-\partial / \partial x_{k}, \quad 1 \leq k \leq n
$$

A map $f: U \subset \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}$ is holomorphic if and only if $f_{*} \circ J=J \circ f_{*}$, since this condition is equivalent to the Cauchy-Riemann equations for each coordinate function.

A complex manifold $M$ of complex dimension $n$ is a $2 n$-dimensional differentiable manifold that admits an open cover $\left\{U_{\alpha}\right\}_{\alpha \in \Lambda}$ and coordinate maps $\varphi_{\alpha}: U_{\alpha} \rightarrow \mathbb{C}^{n}$ such that $\varphi_{\alpha} \circ \varphi_{\beta}^{-1}$ is holomorphic on $\varphi_{\beta}\left(U_{\alpha} \cap U_{\beta}\right)$ for all $\alpha, \beta \in \Lambda$. In particular, the existence of local isothermic coordinates implies that any two-dimensional orientable Riemannian manifold is a complex manifold of complex dimension one.

A complex manifold $M$ can be naturally endowed with an almost complex structure $J_{M}$ via the coordinate maps, that is, on each $U_{\alpha}$ define

$$
J_{M}=\left(\varphi_{\alpha}\right)_{*}^{-1} \circ J \circ\left(\varphi_{\alpha}\right)_{*},
$$

where $J$ is the almost complex structure of $\mathbb{C}^{n}$. This definition is independent of the map $\varphi_{\alpha}$, and thus $J_{M}$ is globally defined.

An almost complex structure $J$ on a manifold $M$ is called a complex structure if $M$ is the underlying differentiable manifold of a complex manifold which induces $J$ in the way just described.

A map $f: M \rightarrow \tilde{M}$ between two complex manifolds is said to be holomorphic if its representation in local coordinates is holomorphic. This turns out to be equivalent to the condition

$$
f_{*} \circ J=\tilde{J} \circ f_{*}
$$

where $J$ and $\tilde{J}$ are the almost complex structures of $M$ and $\tilde{M}$, respectively.
A Kaehler manifold $M^{2 n}$ with real dimension $2 n$ is an almost complex manifold endowed with a Riemannian metric $\langle$,$\rangle such that the almost complex structure J$ of $M^{2 n}$ is a parallel orthogonal tensor, that is,

$$
\langle J X, J Y\rangle=\langle X, Y\rangle
$$

and

$$
\left(\nabla_{X} J\right) Y=\nabla_{X} J Y-J \nabla_{X} Y=0
$$

for all $X, Y \in \mathfrak{X}(M)$. A well-known theorem of Newlander-Nirenberg states that the almost complex structure of a Kaehler manifold is a complex structure.

Proposition 15.1. The curvature tensor of a Kaehler manifold $M^{2 n}$ satisfies:
(i) $R(X, Y) \circ J=J \circ R(X, Y)$,
(ii) $R(J X, J Y)=R(X, Y)$,
(iii) $\operatorname{Ric}(J X, J Y)=\operatorname{Ric}(X, Y)$,
(iv) $\operatorname{Ric}(X, Y)=\frac{1}{2} \operatorname{tr} Z \mapsto J R(X, J Y) Z$
for all $X, Y, Z \in \mathfrak{X}(M)$.
Proof: A straightforward computation; see [230].
Proposition 15.2. If $f: M^{2 n} \rightarrow \tilde{M}^{2 m}$ is a holomorphic isometric immersion between Kaehler manifolds, then its second fundamental form satisfies

$$
\alpha(X, J Y)=\tilde{J} \alpha(X, Y)=\alpha(J X, Y)
$$

for all $X, Y \in \mathfrak{X}(M)$.
Proof: Since $f$ is holomorphic, then

$$
\begin{aligned}
\left(\tilde{\nabla}_{X} f_{*} J Y\right)^{\perp} & =\left(\tilde{\nabla}_{X} \tilde{J}_{*} Y\right)^{\perp} \\
& =\left(\tilde{J} \tilde{\nabla}_{X} f_{*} Y\right)^{\perp} \\
& \left.=\tilde{J}^{( } \tilde{\nabla}_{X} f_{*} Y\right)^{\perp}
\end{aligned}
$$

for all $X, Y \in \mathfrak{X}(M)$, and the result follows from the Gauss formula and the symmetry of $\alpha$.

Notice that the equality

$$
\alpha(X, J Y)=\alpha(J X, Y)
$$

for all $X, Y \in \mathfrak{X}(M)$ is equivalent to

$$
A_{\xi} \circ J=-J \circ A_{\xi}
$$

for all $\xi \in \Gamma\left(N_{f} M\right)$. This easily implies that every odd symmetric function of the eigenvalues of $A_{\xi}$ is zero. In particular, $f$ is minimal.

### 15.2 Unboundedness of real Kaehler submanifolds

A compact Kaehler manifold $M^{2 n}$ cannot be isometrically immersed in $\mathbb{R}^{2 n+p}$ if $p<n$. This is a consequence of Corollary 15.5 below, whose proof relies on the following fact.

Lemma 15.3. Let $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+p}$ be a real Kaehler submanifold. Given $y \in M^{2 n}$, let

$$
\beta: T_{y} M \times T_{y} M \rightarrow W=N_{f} M(y) \oplus N_{f} M(y)
$$

be the bilinear form defined by

$$
\begin{equation*}
\beta(X, Y)=(\alpha(X, Y), \alpha(X, J Y)) . \tag{15.1}
\end{equation*}
$$

Then $\beta$ is flat with respect to the inner product of signature $(p, p)$ on $W$ given by

$$
\begin{equation*}
\left\langle\left\langle\left(\xi_{1}, \eta_{1}\right),\left(\xi_{2}, \eta_{2}\right)\right\rangle\right\rangle=\left\langle\xi_{1}, \xi_{2}\right\rangle-\left\langle\eta_{1}, \eta_{2}\right\rangle . \tag{15.2}
\end{equation*}
$$

Proof: By the Gauss equation and Proposition 15.1, we have

$$
\begin{aligned}
& \langle\langle\beta(X, Y), \beta(Z, V)\rangle\rangle=\langle\alpha(X, Y), \alpha(Z, V)\rangle-\langle\alpha(X, J Y), \alpha(Z, J V)\rangle \\
& =\langle R(X, Z) V, Y\rangle+\langle\alpha(X, V), \alpha(Z, Y)\rangle-\langle R(X, Z) J V, J Y\rangle-\langle\alpha(X, J V), \alpha(Z, J Y)\rangle \\
& =\langle\langle\beta(X, V), \beta(Z, Y)\rangle\rangle
\end{aligned}
$$

for all $X, Y, Z, V \in T_{y} M$.
Theorem 15.4. Let $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+p}$ be a real Kaehler submanifold. Assume that the weak maximum principle for the Hessian holds on $M^{2 n}$. If $p<n$, then $f(M)$ is unbounded.

Proof: Suppose that $f(M)$ is bounded and let $h \in C^{\infty}(M)$ be defined by

$$
h(x)=\frac{1}{2}\|f(x)\|^{2} .
$$

By the assumption, there exists $y \in M^{2 n}$ such that

$$
\operatorname{Hess} h(y)(X, X)<\|X\|^{2}
$$

for any nonzero $X \in T_{y} M$. On the other hand, by Corollary 1.5 we have

$$
\operatorname{Hess} h(y)(X, Y)=\langle X, Y\rangle+\langle\alpha(X, Y), f(y)\rangle
$$

for all $X, Y \in T_{y} M$. Therefore

$$
\begin{equation*}
\langle\alpha(X, X), f(y)\rangle<0 \tag{15.3}
\end{equation*}
$$

for any nonzero $X \in T_{y} M$.

Let $\beta: T_{y} M \times T_{y} M \rightarrow W$ be the bilinear form defined by (15.1). Clearly, for any $X \in T_{y} M$ the subspace

$$
L(X)=\left\{Y \in T_{y} M: \beta(X, Y)=0\right\}
$$

is invariant by the complex structure of $M^{2 n}$ and

$$
\operatorname{dim} L(X) \geq 2(n-p)>0
$$

Choose $X \in R E(\beta)$. It follows from Proposition 4.6 that

$$
\left\langle\left\langle\beta\left(Y_{1}, Z_{1}\right), \beta\left(Y_{2}, Z_{2}\right)\right\rangle\right\rangle=0
$$

for all $Y_{1}, Y_{2} \in T_{y} M$ and $Z_{1}, Z_{2} \in L(X)$, that is,

$$
\left\langle\alpha\left(Y_{1}, Z_{1}\right), \alpha\left(Y_{2}, Z_{2}\right)\right\rangle=\left\langle\alpha\left(Y_{1}, J Z_{1}\right), \alpha\left(Y_{2}, J Z_{2}\right)\right\rangle .
$$

Hence, defining

$$
S=\operatorname{span}\left\{\alpha(Y, Z): Y \in T_{y} M \text { and } Z \in L(X)\right\}
$$

there exists an isometric linear isomorphism $\tilde{J}: S \rightarrow S$ such that

$$
\tilde{J} \alpha(Y, Z)=\alpha(Y, J Z)
$$

for all $Y \in T_{y} M$ and $Z \in L(X)$. In particular,

$$
\begin{aligned}
\alpha(J Z, J Z) & =\tilde{J} \alpha(J Z, Z) \\
& =\tilde{J} \alpha(Z, J Z) \\
& =\alpha\left(Z, J^{2} Z\right) \\
& =-\alpha(Z, Z)
\end{aligned}
$$

for all $Z \in L(X)$, which contradicts (15.3).
Corollary 15.5. Let $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+p}$ be a real Kaehler submanifold. Assume that $M^{2 n}$ is complete and that its sectional curvature is bounded from below. If $p<n$, then $f(M)$ is unbounded.

Proof: According to Theorem 6.6 the Omori-Yau maximum principle for the Hessian holds on $M^{2 n}$.

The proof of Theorem 15.4 has the following immediate consequence.
Corollary 15.6. Let $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+p}, p \leq n-1$, be an isometric immersion of a Kaehler manifold. Then at each point $x \in M^{2 n}$ there is a J-invariant subspace $L^{2 m} \subset T_{x} M$ with $m \geq n-p$ such that the holomorphic sectional curvature for any complex plane $P \subset L^{2 m}$ satisfies $K_{P} \leq 0$.

### 15.3 Minimal real Kaehler submanifolds

A real Kaehler submanifold $f: M^{2 n} \rightarrow \mathbb{R}^{m}$ is called pluriharmonic if its second fundamental form satisfies

$$
\alpha(X, J Y)=\alpha(J X, Y)
$$

for all $X, Y \in \mathfrak{X}(M)$, or equivalently,

$$
A_{\xi} \circ J=-J \circ A_{\xi}
$$

for all $\xi \in \Gamma\left(N_{f} M\right)$. It follows from Proposition 15.2 that any holomorphic isometric immersion is pluriharmonic. Notice that $f$ being pluriharmonic is equivalent to the restriction of $f$ to any holomorphic curve in $M^{2 n}$ being a minimal surface in $\mathbb{R}^{m}$.

Theorem 15.7. A real Kaehler submanifold $f: M^{2 n} \rightarrow \mathbb{R}^{m}$ is minimal if and only if it is pluriharmonic.

Proof: At any $x \in M^{2 n}$, consider an orthonormal basis $X_{1}, \ldots, X_{2 n}$ of $T_{x} M$ such that $X_{2 j}=J X_{2 j-1}, 1 \leq j \leq n$. Since $f$ is minimal, it follows from (3.8) that

$$
\begin{equation*}
\operatorname{Ric}\left(X_{i}, X_{i}\right)=-\sum_{j=1}^{2 n}\left\|\alpha\left(X_{i}, X_{j}\right)\right\|^{2} \text { and } \operatorname{Ric}\left(J X_{i}, J X_{i}\right)=-\sum_{j=1}^{2 n}\left\|\alpha\left(J X_{i}, X_{j}\right)\right\|^{2} \tag{15.4}
\end{equation*}
$$

On the other hand, by the Gauss equation and Proposition 15.1 we have

$$
\begin{align*}
\operatorname{Ric}\left(X_{i}, X_{i}\right) & =\sum_{j \neq i}\left\langle R\left(X_{j}, X_{i}\right) X_{i}, X_{j}\right\rangle \\
& =\sum_{j \neq i}\left\langle R\left(X_{j}, X_{i}\right) J X_{i}, J X_{j}\right\rangle \\
& =\sum_{j \neq i}\left\langle\alpha\left(X_{j}, J X_{j}\right), \alpha\left(X_{i}, J X_{i}\right)\right\rangle-\sum_{j \neq i}\left\langle\alpha\left(X_{j}, J X_{i}\right), \alpha\left(X_{i}, J X_{j}\right)\right\rangle \\
& =-\sum_{j=1}^{2 n}\left\langle\alpha\left(X_{j}, J X_{i}\right), \alpha\left(X_{i}, J X_{j}\right)\right\rangle, \tag{15.5}
\end{align*}
$$

where the last equality is a consequence of the choice of the basis.
On $V=\oplus_{j=1}^{2 n} N_{f} M(x)$, take the inner product $\langle\langle\rangle\rangle:, V \times V \rightarrow \mathbb{R}$ defined by

$$
\langle\langle,\rangle\rangle=\sum_{j=1}^{2 n}\langle,\rangle .
$$

Let $v_{i}, w_{i} \in V, 1 \leq i \leq 2 n$, be the elements

$$
v_{i}=\left(\alpha\left(X_{i}, J X_{1}\right), \ldots, \alpha\left(X_{i}, J X_{2 n}\right)\right), \quad w_{i}=\left(\alpha\left(X_{1}, J X_{i}\right), \ldots, \alpha\left(X_{2 n}, J X_{i}\right)\right) .
$$

The statement is equivalent to the assertion that $v_{i}=w_{i}$ for $1 \leq i \leq 2 n$. It follows from (15.4), 15.5 and Proposition 15.1 that

$$
\left\langle\left\langle v_{i}, w_{i}\right\rangle\right\rangle=\left\langle\left\langle w_{i}, w_{i}\right\rangle\right\rangle=\left\langle\left\langle v_{i}, v_{i}\right\rangle\right\rangle, \quad 1 \leq i \leq 2 n .
$$

Write $v_{i}=\lambda_{i} w_{i}+u_{i}$, with $\left\langle\left\langle w_{i}, u_{i}\right\rangle\right\rangle=0$ for $1 \leq i \leq 2 n$. The first of the preceding equalities yields

$$
\begin{aligned}
\lambda_{i}\left\langle\left\langle w_{i}, w_{i}\right\rangle\right\rangle & =\left\langle\left\langle v_{i}, w_{i}\right\rangle\right\rangle \\
& =\left\langle\left\langle w_{i}, w_{i}\right\rangle\right\rangle,
\end{aligned}
$$

whereas the second gives

$$
\begin{aligned}
\left\langle\left\langle w_{i}, w_{i}\right\rangle\right\rangle & =\left\langle\left\langle v_{i}, v_{i}\right\rangle\right\rangle \\
& =\lambda_{i}^{2}\left\langle\left\langle w_{i}, w_{i}\right\rangle\right\rangle+\left\langle\left\langle u_{i}, u_{i}\right\rangle\right\rangle .
\end{aligned}
$$

Thus $\lambda_{i}=1$ and $u_{i}=0$ for $1 \leq i \leq 2 n$.

### 15.3.1 The associated family

The results of this section show that simply connected minimal real Kaehler submanifolds come in one-parameter families of minimal isometric submanifolds with the same generalized Gauss map, which are trivial precisely when they contain a holomorphic member.

Theorem 15.8. If $f: M^{2 n} \rightarrow \mathbb{R}^{m}$ is a minimal simply connected real Kaehler submanifold, then there exists a one-parameter associated family of minimal isometric immersions $f_{\theta}: M^{2 n} \rightarrow \mathbb{R}^{m}, \theta \in[0, \pi)$, such that $f_{0}=f$.

Proof: For each $\theta \in[0, \pi)$, consider the tensor field $J_{\theta} \in \Gamma(\operatorname{End}(T M))$ defined by

$$
J_{\theta}=\cos \theta I+\sin \theta J
$$

where $I$ is the identity tensor and $J$ is the almost-complex structure of $M^{2 n}$. Clearly, $J_{\theta}$ is parallel and orthogonal with $J_{\theta}^{t}=J_{-\theta}$.

Let $\alpha_{\theta}: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \Gamma\left(N_{f} M\right)$ be defined by

$$
\alpha_{\theta}(X, Y)=\alpha\left(J_{\theta} X, Y\right)
$$

Since $f$ is pluriharmonic by Theorem 15.7, then $\alpha_{\theta}$ is symmetric. For $\xi \in \Gamma\left(N_{f} M\right)$, let $A_{\xi}^{\theta}$ denote the symmetric endomorphism given by

$$
\left\langle A_{\xi}^{\theta} X, Y\right\rangle=\left\langle\alpha_{\theta}(X, Y), \xi\right\rangle .
$$

Then

$$
A_{\xi} J_{\theta}=A_{\xi}^{\theta}=\left(A_{\xi}^{\theta}\right)^{t}=J_{-\theta} A_{\xi} .
$$

Next we verify that $\alpha_{\theta}$ satisfies the Gauss, Codazzi and Ricci equations with respect to the normal connection of $f$. Using part (ii) of Proposition 15.1, we obtain

$$
\begin{aligned}
\left\langle\alpha_{\theta}(X, W), \alpha_{\theta}(Y, Z)\right\rangle & -\left\langle\alpha_{\theta}(X, Z), \alpha_{\theta}(Y, W)\right\rangle \\
& =\left\langle\alpha\left(J_{\theta} X, W\right), \alpha\left(J_{\theta} Y, Z\right)\right\rangle-\left\langle\alpha\left(J_{\theta} X, Z\right), \alpha\left(J_{\theta} Y, W\right)\right\rangle \\
& =\left\langle R\left(J_{\theta} X, J_{\theta} Y\right) Z, W\right\rangle \\
& =\langle R(X, Y) Z, W\rangle .
\end{aligned}
$$

Thus the Gauss equation is satisfied. The Codazzi equation follows easily from that of $f$ and the fact that $J_{\theta}$ is parallel. For the Ricci equation, observe that

$$
A_{\xi}^{\theta} A_{\eta}^{\theta}=A_{\xi} J_{\theta} J_{-\theta} A_{\eta}=A_{\xi} A_{\eta}
$$

for all $\xi, \eta \in \Gamma\left(N_{f} M\right)$. Hence

$$
\left[A_{\xi}^{\theta}, A_{\eta}^{\theta}\right]=\left[A_{\xi}, A_{\eta}\right],
$$

and the Ricci equation for $\alpha_{\theta}$ follows from that of $f$.
By the Fundamental theorem of submanifolds, there exists an isometric immersion $f_{\theta}: M^{2 n} \rightarrow \mathbb{R}^{m}$ whose second fundamental form is $\alpha_{\theta}$. Since

$$
A_{\xi}^{\theta} J=A_{\xi} J_{\theta} J=A_{\xi} J J_{\theta}=-J A_{\xi} J_{\theta}=-J A_{\xi}^{\theta}
$$

for any $\xi \in \Gamma\left(N_{f} M\right)$, it follows that $f_{\theta}$ is minimal.
The following result provides an alternative way of defining the associated family.
Proposition 15.9. If $f: M^{2 n} \rightarrow \mathbb{R}^{m}$ is a minimal simply connected real Kaehler submanifold, then the associated family $\left\{f_{\theta}\right\}_{\theta \in[0, \pi)}$ is given by the line integral

$$
f_{\theta}(x)=\int_{x_{0}}^{x} f_{*} \circ J_{\theta}
$$

where $x_{0}$ is any fixed point of $M^{2 n}$.
Proof: Consider the one-form $\omega=f_{*} \circ J_{\theta}$ with values in $\mathbb{R}^{m}$. We have

$$
d \omega(X, Y)=\tilde{\nabla}_{X} f_{*} J_{\theta} Y-\tilde{\nabla}_{Y} f_{*} J_{\theta} X-f_{*} J_{\theta}[X, Y]
$$

for all $X, Y \in \mathfrak{X}(M)$. Since

$$
\begin{aligned}
\tilde{\nabla}_{X} f_{*} J_{\theta} Y & =f_{*} \nabla_{X} J_{\theta} Y+\alpha\left(X, J_{\theta} Y\right) \\
& =f_{*} J_{\theta} \nabla_{X} Y+\alpha\left(X, J_{\theta} Y\right),
\end{aligned}
$$

then

$$
\begin{aligned}
d \omega(X, Y) & =\alpha\left(X, J_{\theta} Y\right)-\alpha\left(Y, J_{\theta} X\right) \\
& =0
\end{aligned}
$$

because $f$ is pluriharmonic by Theorem 15.7. Thus $\omega$ is closed, and hence $f_{\theta}$ is welldefined. Clearly, $f_{\theta *}=f_{*} \circ J_{\theta}$, which shows that $f_{\theta}$ is isometric and that the tangent (normal) spaces of $f$ and $f_{\theta}$ are parallel in $\mathbb{R}^{m}$ at any point of $M^{2 n}$, that is, all $f_{\theta}$ have the same generalized Gauss map. In particular, $f_{\theta}$ and $f$ have the same normal connection. The second fundamental form of $f_{\theta}$ is given by

$$
\begin{aligned}
\alpha_{\theta}(X, Y) & =\tilde{\nabla}_{X} f_{\theta *} Y-f_{\theta *} \nabla_{X} Y \\
& =\tilde{\nabla}_{X} f_{*} J_{\theta} Y-f_{*} J_{\theta} \nabla_{X} Y \\
& =\tilde{\nabla}_{X} f_{*} J_{\theta} Y-f_{*} \nabla_{X} J_{\theta} Y \\
& =\alpha\left(X, J_{\theta} Y\right),
\end{aligned}
$$

which proves that $\left\{f_{\theta}\right\}_{\theta \in[0, \pi)}$ is the associated family of $f$.
The next result shows that the associated family of a minimal simply connected real Kaehler submanifold is trivial precisely when the submanifold is holomorphic.

Theorem 15.10. Let $f: M^{2 n} \rightarrow \mathbb{R}^{m}$ be a simply connected minimal real Kaehler submanifold. If $f$ is holomorphic, then its associated family $\left\{f_{\theta}\right\}_{\theta \in[0, \pi)}$ satisfies $f_{\theta}=f$ for any $\theta \in[0, \pi)$.

Conversely, if there exist $\theta_{1} \neq \theta_{2} \in[0, \pi)$ such that $f_{\theta_{1}}$ and $f_{\theta_{2}}$ are congruent, then $f$ has even substantial codimension $q=2 \ell$ and is holomorphic with respect to an almost complex structure in $\mathbb{R}^{2(n+\ell)}$.

Proof: If $f$ is holomorphic with respect to an almost complex structure $\tilde{J}$ in $\mathbb{R}^{m}$, then

$$
f_{\theta_{*}}=f_{*} \circ J_{\theta}=\tilde{J}_{\theta} \circ f_{*}
$$

where $\tilde{J}_{\theta}=\cos \theta I+\sin \theta \tilde{J}$. Thus $f_{\theta}$ is congruent to $f$.
In order to prove the converse, we may suppose that $\theta_{1}=0$ and denote $\theta_{2}=\theta$. Then, by assumption, there exists an orthogonal endomorphism $\tilde{T}_{\theta}$ of $\mathbb{R}^{m}$ such that

$$
f_{*} \circ J_{\theta}=f_{\theta_{*}}=\tilde{T}_{\theta} \circ f_{*}
$$

where $J_{\theta}=\cos \theta I+\sin \theta J$. Define an endomorphism $\tilde{J}$ of $\mathbb{R}^{m}$ by

$$
\sin \theta \tilde{J}=\tilde{T}_{\theta}-\cos \theta I
$$

Then

$$
\begin{aligned}
\sin \theta \tilde{J} \circ f_{*} & =\tilde{T}_{\theta} \circ f_{*}-\cos \theta f_{*} \\
& =f_{*} \circ\left(J_{\theta}-\cos \theta I\right) \\
& =\sin \theta f_{*} \circ J,
\end{aligned}
$$

hence

$$
\begin{equation*}
\tilde{J} \circ f_{*}=f_{*} \circ J \tag{15.6}
\end{equation*}
$$

Let $\mathbb{R}^{2 n+q}$ denote the subspace of $\mathbb{R}^{m}$ spanned by the union of the subspaces $f_{*} T_{y} M$ when $y$ ranges over $M^{2 n}$. It follows from (15.6) that $\tilde{J}$ leaves $\mathbb{R}^{2 n+q}$ invariant, it is an orthogonal transformation of $\mathbb{R}^{2 n+q}$, and

$$
\tilde{J}^{2} \circ f_{*}=f_{*} \circ J^{2}=-f_{*}
$$

Thus $\tilde{J}^{2}=-I$ on $\mathbb{R}^{2 n+q}$. This implies that $q$ is even, say, $q=2 \ell$, and that $f(M)$ lies in an affine subspace parallel to $\mathbb{R}^{2(n+\ell)}$. Moreover, regarded as an isometric immersion into $\mathbb{R}^{2(n+\ell)}, f$ is holomorphic with respect to $\tilde{J}$ by (15.6).

Real Kaehler submanifolds in low codimension are generically holomorphic. This can be concluded from the following result, which relates the holomorphicity of the immersion with its type number.

Theorem 15.11. Let $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+p}$ be a real Kaehler submanifold with type number $\tau(x) \geq 3$ for all $x \in M^{2 n}$. Then $p=2 \ell$ and $f$ is holomorphic.

Proof: By the assumption on the type number, Proposition 4.18 and Lemma 15.3 imply that there exists a vector bundle isometry $T: N_{f} M \rightarrow N_{f} M$ such that

$$
T \alpha(X, Y)=\alpha(X, J Y)
$$

for all $X, Y \in \mathfrak{X}(M)$. Therefore

$$
\begin{aligned}
\alpha(X, J Y) & =T \alpha(X, Y) \\
& =T \alpha(Y, X) \\
& =\alpha(Y, J X) \\
& =\alpha(J X, Y)
\end{aligned}
$$

for all $X, Y \in \mathfrak{X}(M)$. Thus $f$ is minimal, and the result follows from Theorems 4.19 and 15.10 .

The next result shows that any minimal isometric immersion $f: M^{2 n} \rightarrow \mathbb{R}^{m}$ of a simply connected Kaehler manifold arises as (a constant multiple of) the real part of a holomorphic isometric immersion $F: M^{2 n} \rightarrow \mathbb{C}^{m}$, called its holomorphic representative.

Theorem 15.12. Let $f: M^{2 n} \rightarrow \mathbb{R}^{m}$ be a simply connected minimal real Kaehler submanifold with associated family $\left\{f_{\theta}\right\}_{\theta \in[0, \pi)}$. Then the map $F: M^{2 n} \rightarrow \mathbb{R}^{2 m}=\mathbb{R}^{m} \oplus \mathbb{R}^{m}$ given by

$$
F=\frac{1}{\sqrt{2}} f \oplus \frac{1}{\sqrt{2}} f_{\pi / 2}
$$

is a holomorphic isometric immersion with respect to the standard complex structure on $\mathbb{R}^{2 m}$ given by $\tilde{J}(X, Y)=(Y,-X)$.

Proof: Since

$$
\sqrt{2} F_{*} X=\left(f_{*} X, f_{*} J X\right)
$$

it follows that $F$ is an isometric immersion and that

$$
\begin{aligned}
\sqrt{2} F_{*} J X & =\left(f_{*} J X,-f_{*} X\right) \\
& =\sqrt{2} \tilde{J} F_{*} X
\end{aligned}
$$

Thus $F$ is holomorphic with respect to $\tilde{J}$.
A well-known theorem due to Calabi [48] states that if $F: M^{2 n} \rightarrow \mathbb{C}^{m}$ and $\tilde{F}: M^{2 n} \rightarrow \mathbb{C}^{n}$ are holomorphic isometric immersions of a Kaehler manifold, then $n=m$ and $\tilde{F}=\tilde{J} \circ F$ for some almost complex structure $\tilde{J}$ of $\mathbb{C}^{m}$. Combining this result with Theorem 15.12 yields the following.

Theorem 15.13. If $f: M^{n} \rightarrow \mathbb{R}^{N}$ is a minimal simply connected real Kaehler submanifold, then the set of all minimal isometric immersions of $M^{n}$ into a Euclidean space contains a unique holomorphic one.

### 15.4 Real Kaehler hypersurfaces

In this section, a local parametric description of the real Kaehler hypersurfaces is given in terms of the Gauss parametrization.

A spherical oriented surface $g: L^{2} \rightarrow \mathbb{S}^{2 n}$ is called pseudoholomorphic if there exists an orthogonal tensor $T$ on $N_{g} L$ that is parallel with respect to the normal connection and satisfies

$$
\begin{equation*}
A_{T \xi}=J \circ A_{\xi} \tag{15.7}
\end{equation*}
$$

for all $\xi \in \Gamma\left(N_{g} L\right)$, where $J$ is the almost complex structure on $L^{2}$. Therefore, for all $\xi \in \Gamma\left(N_{g} L\right)$ the endomorphism $J \circ A_{\xi}$ is symmetric, that is,

$$
J \circ A_{\xi}=-A_{\xi} \circ J .
$$

In particular, $g$ is minimal. Notice also that (15.7) implies that

$$
A_{\left(T^{2}+I\right) \xi}=0
$$

for all $\xi \in \Gamma\left(N_{g} L\right)$. Thus $\left(T^{2}+I\right)\left(N_{g} L\right)$ is a parallel subbundle of $N_{1}^{\perp}$, and hence Proposition 2.1 implies that $T^{2}+I=0$ if $g$ is substantial.

Theorem 15.14. If $g: L^{2} \rightarrow \mathbb{S}^{2 n}$ is a pseudoholomorphic surface and $\gamma \in C^{\infty}(L)$, then the open subset of regular points of the map $\psi: N_{g} L \rightarrow \mathbb{R}^{2 n+1}$, given by

$$
\begin{equation*}
\psi(y, w)=\gamma(y) g(y)+g_{*} \operatorname{grad} \gamma(y)+w, \tag{15.8}
\end{equation*}
$$

admits a Kaehler structure.
Conversely, any real Kaehler hypersurface without flat points $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+1}$, $n \geq 2$, can be locally parametrized in this way in terms of such a pair $(g, \gamma)$. Moreover, the hypersurface $f$ is minimal if and only if $\Delta \gamma+2 \gamma=0$.

Proof: Let $V$ denote the open subset of regular points of $\psi$. By Theorem 7.18, the map $\left.\psi\right|_{V}: V \rightarrow \mathbb{R}^{2 n+1}$ defines a hypersurface with index of relative nullity $\nu=2 n-2$, with the vertical subbundle $\Delta$ of $V$ as its relative nullity distribution. We may assume that $g$ is substantial, for otherwise $V$ splits as a product $V=V^{\prime} \times \mathbb{R}^{2 k}$ and $\left.\psi\right|_{V}$ splits accordingly as $\left.\psi\right|_{V}=\left.\psi^{\prime}\right|_{V^{\prime}} \times I$, where $\psi^{\prime}: N_{g^{\prime}} L \rightarrow \mathbb{R}^{2(n-k)+1}$ is given by 15.8) with $g$ replaced by some substantial surface $g^{\prime}: L^{2} \rightarrow \mathbb{S}^{2(n-k)}$ and $V^{\prime}$ is the open subset of regular points of $\psi^{\prime}$, and then we may argue for $g^{\prime}$ and $\psi^{\prime}$.

Let $J^{\prime}$ be a parallel almost complex structure on $L^{2}$, and define $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ by

$$
\begin{equation*}
J \circ j=j \circ J^{\prime}, \tag{15.9}
\end{equation*}
$$

where $j(y, w): T_{y} L \rightarrow \Delta^{\perp}(y, w)$ is the isometry defined by 7.25 . Now let $T$ be the parallel orthogonal tensor on $N_{g} L$ such that

$$
A_{T \xi}=J^{\prime} \circ A_{\xi}
$$

for any $\xi \in \Gamma\left(N_{g} L\right)$, and extend $J$ to a section of $\operatorname{End}(T V)$ by setting

$$
\begin{equation*}
J(y, w) \xi=T(y) \xi \tag{15.10}
\end{equation*}
$$

for all $(y, w) \in V$ and $\xi \in N_{g} L(y)=\Delta(y, w)$. Since $g$ is assumed to be substantial, the orthogonal tensor $T$ satisfies $T^{2}=-I$ by the observation before the statement, and hence $J$ is an almost complex structure on $V$.

Using (7.21) and the fact that $J^{\prime}$ is parallel, we obtain

$$
\begin{align*}
\left\langle\nabla_{j X} J j Y, j Z\right\rangle & =\left\langle\nabla_{j X} j J^{\prime} Y, j Z\right\rangle \\
& =\left\langle\nabla_{P_{w}^{-1} X}^{\prime} J^{\prime} Y, Z\right\rangle^{\prime} \\
& =\left\langle J^{\prime} \nabla_{P_{w}^{-1} X}^{\prime-1} Y, Z\right\rangle^{\prime} \\
& =-\left\langle\nabla_{P_{w}^{-1} X}^{\prime} Y, J^{\prime} Z\right\rangle^{\prime} \\
& =-\left\langle\nabla_{j X} j Y, j J^{\prime} Z\right\rangle \\
& =-\left\langle\nabla_{j X} j Y, J j Z\right\rangle \\
& =\left\langle J \nabla_{j X} j Y, j Z\right\rangle \tag{15.11}
\end{align*}
$$

for all $X, Y, Z \in \mathfrak{X}(L)$.
The fact that the right-hand side of 15.10 does not depend on $w$ implies that

$$
\begin{equation*}
\nabla_{\xi} J \eta=J \nabla_{\xi} \eta \tag{15.12}
\end{equation*}
$$

for all $\xi, \eta \in \Gamma(\Delta)$. On the other hand, regarding $\xi \in \Gamma\left(N_{g} L\right)$ as an element of $\Gamma(\Delta)$ as in part (vi) of Proposition 7.19, using (7.20) and (15.9) we obtain

$$
\begin{aligned}
C_{J \xi} j & =j A_{T \xi} P_{w}^{-1} \\
& =j J^{\prime} A_{\xi} P_{w}^{-1} \\
& =J j A_{\xi} P_{w}^{-1} \\
& =J C_{\xi} j,
\end{aligned}
$$

hence $C_{J \xi}=J C_{\xi}$ for all $\xi \in \Gamma(\Delta)$. Therefore

$$
\begin{align*}
\left\langle\nabla_{j X} J \xi, j Y\right\rangle & =-\left\langle C_{J \xi} j X, j Y\right\rangle \\
& =-\left\langle J C_{\xi} j X, j Y\right\rangle \\
& =\left\langle C_{\xi} j X, J j Y\right\rangle \\
& =-\left\langle\nabla_{j X} \xi, J j Y\right\rangle \\
& =\left\langle J \nabla_{j X} \xi, j Y\right\rangle \tag{15.13}
\end{align*}
$$

for all $\xi \in \Gamma(\Delta)$ and $X, Y \in \mathfrak{X}(L)$. Moreover, using (7.22) and the fact that $T$ is parallel with respect to the normal connection of $g$, we obtain

$$
\begin{align*}
\left\langle\nabla_{j X} J \xi, \eta\right\rangle & =\left\langle\nabla_{P_{w}^{-1} X}^{\perp} T \xi, \eta\right\rangle \\
& =\left\langle T \nabla_{P_{w}^{-1} X}^{\perp} \xi, \eta\right\rangle \\
& =-\left\langle\nabla_{P_{w}^{-1}}^{\perp} \xi, T \eta\right\rangle \\
& =-\left\langle\nabla_{j X} \xi, J \eta\right\rangle \\
& =\left\langle J \nabla_{j X} \xi, \eta\right\rangle \tag{15.14}
\end{align*}
$$

for all $\xi, \eta \in \Gamma\left(N_{g} L\right)$ and $X \in \mathfrak{X}(L)$. Also, using (15.13) we have

$$
\begin{align*}
\left\langle\nabla_{j X} J j Y, \xi\right\rangle & =-\left\langle J j Y, \nabla_{j X} \xi\right. \\
& =\left\langle j Y, J \nabla_{j X} \xi\right\rangle \\
& =\left\langle j Y, \nabla_{j X} J \xi\right\rangle \\
& =-\left\langle\nabla_{j X} j Y, J \xi\right\rangle \\
& =\left\langle J \nabla_{j X} j Y, \xi\right\rangle \tag{15.15}
\end{align*}
$$

for all $\xi \in \Gamma\left(N_{g} L\right)$ and $X, Y \in \mathfrak{X}(L)$.
It follows from (15.11), (15.13), (15.14) and 15.15) that

$$
\begin{equation*}
\nabla_{j X} J \xi=J \nabla_{j X} \xi \text { and } \nabla_{j X} J j Y=J \nabla_{j X} j Y \tag{15.16}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}(L)$ and $\xi \in \Gamma(\Delta)$. Finally, by (7.18) we have

$$
\begin{equation*}
\nabla_{\xi} J j X=\nabla_{\xi} j J^{\prime} X=0=J \nabla_{\xi} j X \tag{15.17}
\end{equation*}
$$

for all $X \in \mathfrak{X}(L)$ and $\xi \in \Gamma(\Delta)$. We conclude from (15.12), (15.16) and (15.17) that $J$ is a parallel tensor.

Conversely, let $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+1}, n \geq 2$, be a real Kaehler hypersurface without flat points. For a fixed point $y \in M^{2 n}$, let

$$
\beta: T_{y} M \times T_{y} M \rightarrow W=N_{f} M(y) \oplus N_{f} M(y)
$$

be the bilinear form defined by (15.1). By Lemma $15.3, \beta$ is flat with respect to the inner product $\langle\langle\rangle$,$\rangle on W$ given by 15.2 . We show that $\mathcal{S}(\beta)$ is nondegenerate.

Otherwise, since $W$ is a Lorentzian plane, it would be spanned by an isotropic vector, and hence

$$
\langle\langle\beta(X, Y), \beta(X, Y)\rangle\rangle=0
$$

for all $X, Y \in T_{y} M$, or equivalently,

$$
\langle\alpha(X, Y), \alpha(X, Y)\rangle=\langle\alpha(X, J Y), \alpha(X, J Y)\rangle
$$

for all $X, Y \in T_{y} M$. Thus there would exist an isometry $T: N_{f} M(y) \rightarrow N_{f} M(y)$ such that

$$
T \alpha(X, Y)=\alpha(X, J Y)
$$

for all $X, Y \in T_{y} M$. Since $y$ is not a totally geodesic point, this would imply that $T^{2}+I=0$, which is a contradiction because $N_{f} M(y)$ is one-dimensional. Thus $\mathcal{S}(\beta)$ is nondegenerate, and hence $\operatorname{dim} \mathcal{N}(\beta) \geq 2 n-2$ by Exercise 4.4. From the definition of $\beta$, it follows that the relative nullity subspace $\Delta(y)$ has dimension at least $2 n-2$ and is invariant by $J$. Since $y$ is not a flat point, the dimension of $\Delta(y)$ must be exactly $2 n-2$.

By Theorem 7.18, the hypersurface $f$ can be locally parametrized by (15.8) in terms of a surface $g: L^{2} \rightarrow \mathbb{S}^{2 n}$ and a function $\gamma \in C^{\infty}(L)$. We have to show that $g$ is pseudoholomorphic. Define an almost complex structure $J^{\prime}$ on $L^{2}$ by (15.9) and an orthogonal tensor $T$ on $N_{g} L$ by 15.10 ). The latter is well defined because $J$ satisfies (15.12). Then, reversing the computations in (15.11) and (15.14), we see that $J^{\prime}$ is parallel and that $T$ is also parallel with respect to the normal connection of $g$. Moreover, since

$$
\left\langle\nabla_{j X} J \xi, j Y\right\rangle=\left\langle J \nabla_{j X} \xi, j Y\right\rangle
$$

for all $\xi \in \Gamma(\Delta)$ and $X, Y \in \mathfrak{X}(L)$, it follows that $C_{J \xi}=J C_{\xi}$ for all $\xi \in \Gamma(\Delta)$. Hence

$$
\begin{aligned}
j A_{T \xi} P_{w}^{-1} & =C_{J \xi} j \\
& =J C_{\xi} j \\
& =J j A_{\xi} P_{w}^{-1} \\
& =j J^{\prime} A_{\xi} P_{w}^{-1}
\end{aligned}
$$

for all $\xi \in \Gamma\left(N_{g} L\right)$, regarding $\xi \in \Gamma\left(N_{g} L\right)$ as an element of $\Gamma(\Delta)$ as in part (vi) of Proposition 7.19. Thus

$$
A_{T \xi}=J^{\prime} \circ A_{\xi}
$$

for all $\xi \in \Gamma\left(N_{g} L\right)$, which completes the proof that $g$ is pseudoholomorphic. Finally, the last assertion follows from Exercise 7.8.

Remark 15.15. Notice that, since the function $\gamma \in C^{\infty}(L)$ in Theorem 15.14 is arbitrary, a real Kaehler hypersurface is not necessarily real analytic.

In spite of the fact that there exists an abundance of local examples of Kaehler hypersurfaces in $\mathbb{R}^{2 n+1}$ with $n \geq 2$, according to Theorem 15.14 , the only complete ones are cylinders over complete surfaces in $\mathbb{R}^{3}$. This is the content of the next result, which is stated without proof.

Theorem 15.16. Any isometric immersion $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+1}$ with $n \geq 2$ of a complete Kaehler manifold is a $(2 n-2)$-cylinder.

### 15.5 The codimension two case

This section is devoted to real Kaehler submanifolds that are complete and have a positive lower bound for the index of relative nullity. The goal is to provide conditions that imply that they are cylinders. This is then used to give a complete description of the structure of complete minimal real Kaehler submanifolds with codimension two.

The starting point is the following basic fact.
Lemma 15.17. Let $f: M^{2 n} \rightarrow \mathbb{R}^{m}$ be a complete minimal real Kaehler submanifold. If the index of relative nullity is a positive constant $\nu$ on an open subset $U \subset M^{2 n}$, then $\nu$ is even and the relative nullity distribution $\Delta$ is invariant by the almost complex structure $J$ of $M^{2 n}$. If, in addition, $\nu=\nu_{0}$ is the minimum index of relative nullity, then the splitting tensor $C_{T}$ at any $x \in U$, with respect to any $T \in \Delta(x)$, is a nilpotent complex linear endomorphism with respect to the complex vector space structure on $\Delta^{\perp}(x)$ induced by $J$.

Proof: Since $f$ is minimal, it is pluriharmonic by Theorem 15.7. Hence the relative nullity subspaces are invariant by the almost complex structure $J$. In particular, the index of relative nullity $\nu$ must be even. Given $x \in U$ and $\xi \in N_{f} M(x)$, since $A_{\xi} C_{T}$ is symmetric by Proposition 7.3 then

$$
\begin{equation*}
\alpha\left(C_{T} X, Y\right)=\alpha\left(X, C_{T} Y\right) \tag{15.18}
\end{equation*}
$$

for all $X, Y \in \Delta^{\perp}(x)$. Moreover, the fact that $J$ is parallel implies that

$$
\begin{aligned}
C_{J T} X & =-\left(\nabla_{X} J T\right)^{h} \\
& =-\left(J \nabla_{X} T\right)^{h} \\
& =J C_{T} X
\end{aligned}
$$

for all $X \in \Gamma\left(\Delta^{\perp}\right)$ and $T \in \Gamma(\Delta)$. Thus

$$
\begin{equation*}
C_{J T}=J C_{T} . \tag{15.19}
\end{equation*}
$$

Using (15.18) and (15.19) we obtain

$$
\begin{aligned}
\alpha\left(C_{T} J X, Y\right) & =\alpha\left(J X, C_{T} Y\right) \\
& =\alpha\left(X, J C_{T} Y\right) \\
& =\alpha\left(X, C_{J T} Y\right) \\
& =\alpha\left(C_{J T} X, Y\right) \\
& =\alpha\left(J C_{T} X, Y\right)
\end{aligned}
$$

for all $X, Y \in \Delta^{\perp}(x)$. Since $C_{T} J X-J C_{T} X \in \Delta^{\perp}(x)$, it follows that $C_{T} J=J C_{T}$. Endowing $\Delta^{\perp}(x)$ with the complex vector space structure defined by $i X=J X$ for any $X \in \Delta^{\perp}(x)$, this means that $C_{T}$ is a complex linear endomorphism of $\Delta^{\perp}(x)$.

If $\nu=\nu_{0}$ is the minimum index of relative nullity, then the leaves of the relative nullity distribution are complete on $U$ by Corollary 7.8 . From Proposition 13.8 it follows that $C_{T}$ has no nonzero real eigenvalue for any $T \in \Gamma(\Delta)$. Assume that $a+i b$ is a complex eigenvalue of $C_{T}$ with corresponding eigenvector $Y$, that is,

$$
\begin{aligned}
C_{T} Y & =(a+i b) Y \\
& =a Y+b J Y .
\end{aligned}
$$

Then, using (15.19) we obtain

$$
\begin{aligned}
C_{a T-b J T} Y & =a C_{T} Y-b C_{J T} Y \\
& =a C_{T} Y-b J C_{T} Y \\
& =\left(a^{2}+b^{2}\right) Y .
\end{aligned}
$$

Hence $a=0=b$ and, consequently, $C_{T}$ is nilpotent.
In the following cylinder-type result, that the scalar curvature of a Riemannian manifold has subquadratic growth along geodesics means that its growth along any geodesic is less than that of any quadratic polynomial in the parameter of the geodesic.

Theorem 15.18. Let $f: M^{2 n} \rightarrow \mathbb{R}^{m}$ be a complete minimal real Kaehler submanifold. Assume that the minimum index of relative nullity $\nu_{0}$ is positive and that the scalar curvature has subquadratic growth along geodesics. Then $\nu_{0}=2 \ell$ and $f$ is a $2 \ell$-cylinder.

Proof: Let $U \subset M^{2 n}$ be a connected component of the open subset where the index of relative nullity takes its minimum value $\nu_{0}=2 \ell$. Fix $x \in U$ and $T \in \Delta(x)$. By Lemma 15.17, the splitting tensor $C_{T}$ is a nilpotent complex linear endomorphism. Therefore, if $\gamma$ is a geodesic through $x$ in a leaf of $\Delta$ and $\eta$ is a parallel normal vector field along $\gamma$, using (7.3) and

$$
A_{\eta}^{\prime}=\nabla_{T} A_{\eta}=A_{\eta} C_{T},
$$

as follows from Proposition 7.3, we obtain

$$
A_{\eta}^{(k)}=k!A_{\eta} C_{T}^{k}=0
$$

for some $k$. Thus the coefficients of $A_{\eta}$ are polynomials in $t$.
Let $\xi_{1}, \ldots, \xi_{q}$ be an orthonormal basis of $N_{1}(\gamma(0))$, and parallel transport $\xi_{i}$ along $\gamma$ for $1 \leq i \leq q$. Then $\xi_{1}, \ldots, \xi_{q}$ span the first normal space $N_{1}$ at each point of $\gamma$ (see Exercise 7.1). Now, the immersion $f$ being minimal, by (3.9) the scalar curvature $s$ is given by

$$
2 n(2 n-1) s=-\sum_{i=1}^{q} \sum_{j, h=1}^{2 n}\left(\left(A_{\xi_{i}}\right)_{j h}\right)^{2} .
$$

Since $s$ has subquadratic growth along $\gamma$, each term in the above sum must also have subquadratic growth along $\gamma$. But each term is the square of a polynomial in $t$, hence must be constant. This implies that

$$
A_{\xi} C_{T}=A_{\xi}^{\prime}=0
$$

for any $\xi \in \Gamma\left(N_{1}\right)$. We easily conclude that $C_{T}=0$. Since $x \in U$ and $T \in \Delta(x)$ have been chosen arbitrarily, the conclusion follows from Proposition 7.4 and the real analyticity of minimal immersions into Euclidean space.

An isometric immersion $f: M^{2 n} \rightarrow \mathbb{R}^{m}$ is called complex ruled if $M^{2 n}$ is a Kaehler manifold that admits a continuous codimension two foliation such that any leaf is a holomorphic submanifold of $M^{2 n}$ whose image by $f$ is part of an affine subspace of the ambient space. If, in addition, all leaves are complete Euclidean spaces $\mathbb{R}^{2 n-2}$, then $f$ is said to be completely complex ruled.

Theorem 15.19. Let $f: M^{2 n} \rightarrow \mathbb{R}^{m}$ be a complete minimal real Kaehler submanifold with index of relative nullity $\nu \geq 2 n-4$ at any point. Then one of the following possibilities holds:
(i) $f$ is completely complex ruled.
(ii) $f$ is a $(2 n-4)$-cylinder.

Proof: If $\nu \geq 2 n-2$ at any point, then the splitting tensor satisfies $C_{T}=0$ for all $T \in \Gamma(\Delta)$, because it is complex linear and nilpotent by Lemma 15.17. Hence, we may assume that the open subset

$$
U=\left\{x \in M^{2 n}: \nu(x)=2 n-4\right\}
$$

is nonempty. Suppose that $U$ contains an open subset $W$ such that $C_{T}=0$ for all $T \in \Gamma(\Delta)$. As before, this implies that $W$ contains an open subset where $f$ is as in part (ii). By the real analyticity of the immersion, it follows that it is globally of this type.

Therefore we may assume that the open subset

$$
V=\left\{x \in M^{2 n}: \nu(x)=2 n-4 \text { and there exists } T \in \Delta(x) \text { with } C_{T} \neq 0\right\}
$$

is dense. Let $V_{0}$ be a connected component of $V$. Given $x \in V_{0}$ and $T \in \Delta(x)$ with $C_{T} \neq 0$, we have $\operatorname{dim} \operatorname{ker} C_{T}=2$, because $C_{T}$ is complex linear and nilpotent. We first show that ker $C_{T}$ is an asymptotic plane, that is,

$$
\alpha(X, Y)=0
$$

for all $X, Y \in \operatorname{ker} C_{T}$. Since $C_{T}$ is nilpotent, it follows easily that $\operatorname{ker} C_{T}=\operatorname{Im} C_{T}$. Thus any vector $Y \in \operatorname{ker} C_{T}$ can be written as $Y=C_{T} Z$. Hence

$$
\begin{aligned}
\alpha(X, Y) & =\alpha\left(X, C_{T} Z\right) \\
& =\alpha\left(C_{T} X, Z\right) \\
& =0 .
\end{aligned}
$$

We claim that ker $C_{T} \subset \operatorname{ker} C_{S}$ for any other $S \in \Delta(x)$. Consequently, since the dimension of any of these kernels is either 2 or 4 , we have a well defined two-dimensional distribution in $V_{0}$. Take $S \in \Delta(x)$ such that $C_{S} \neq 0$ and consider the curve

$$
R(s)=\cos s T+\sin s S
$$

At least for small $\epsilon>0$, we have $\operatorname{dim} \operatorname{ker} C_{R(s)}=2$ for $0 \leq s \leq \epsilon$. Since the kernels are invariant by $J$, we can choose a smooth $X(s)$ such that

$$
\operatorname{ker} C_{R(s)}=\operatorname{span}\{X(s), J X(s)\}
$$

Unless the planes are all equal, there exists some $s_{0}<\epsilon$ such that $X^{\prime}\left(s_{0}\right)$ is not contained in the plane span $\left\{X\left(s_{0}\right), J X\left(s_{0}\right)\right\}$, and thus the same holds for $J X^{\prime}\left(s_{0}\right)$. On the other hand, from

$$
\alpha(X(s), X(s))=\alpha(X(s), J X(s))=0
$$

it follows that

$$
\alpha\left(X(s), X^{\prime}(s)\right)=\alpha\left(X(s), J X^{\prime}(s)\right)=0 .
$$

But then we should have $X\left(s_{0}\right) \in \Delta$, which is a contradiction. Therefore the linear maps $\cos s C_{T}+\sin s C_{S}$ all have the same kernels for $s$ in some open interval, and this proves the claim.

Consider the smooth $(2 n-2)$-dimensional distribution on $V_{0}$ given by

$$
L(x)=\Delta(x) \oplus \operatorname{ker} C_{T}(x),
$$

where $T(x)$ is chosen so that dim ker $C_{T(x)}=2$. It remains to show that $L$ is a totally geodesic distribution. By (7.2) we have

$$
\begin{aligned}
C_{T}\left(\nabla_{S} X\right) & =-C_{T} C_{S} X+\nabla_{S} C_{T} X-C_{\nabla_{S} T} X \\
& =0
\end{aligned}
$$

if $S \in \Gamma(\Delta)$ and $X \in \Gamma\left(\operatorname{ker} C_{T}\right)$. Now take $X, Y \in \Gamma\left(\operatorname{ker} C_{T}\right)$ and set $Y=C_{T} Z$ where $Z \in \Gamma\left(L^{\perp}\right)$. From (7.5) we obtain

$$
\begin{aligned}
\left(\nabla_{X} Y\right)^{\Delta^{\perp}} & =\left(\nabla_{X} C_{T} Z\right)^{\Delta^{\perp}} \\
& =C_{T}\left(\nabla_{X} Z\right)^{\Delta^{\perp}}-C_{T}\left(\nabla_{Z} X\right)^{\Delta^{\perp}}+C_{\left(\nabla_{X} T\right)^{\Delta}} Z \in \Gamma(L)
\end{aligned}
$$

and this concludes the proof.
Theorem 15.20. If $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+2}$ is a complete minimal real Kaehler submanifold, then one of the following possibilities holds:
(i) $f$ is holomorphic.
(ii) $f$ is completely complex ruled.
(iii) $f$ is a $(2 n-4)$-cylinder.

Moreover, if the scalar curvature of $M^{2 n}$ has subquadratic growth along geodesics, then $f$ is either of type (i) or (iii).

Proof: We may assume that $M^{2 n}$ is simply connected, because, otherwise, we can argue for its universal cover. If $\nu(x) \geq 2 n-4$ at any point of $M^{2 n}$, the result follows from Theorems 15.18 and 15.19 ,

Now suppose that there exists a point $x_{0} \in M^{2 n}$ such that $\nu\left(x_{0}\right)<2 n-4$, and let $U$ be a simply connected neighborhood of $x_{0}$ where $\nu<2 n-4$. We claim that $\left.f\right|_{U}$ is holomorphic. At a point $x \in U$, define a flat bilinear form $\beta$ as in (15.1). It follows from Lemma 4.22 that $\beta$ splits as $\beta=\beta_{1} \oplus \beta_{2}$, where
(i) $\beta_{1}$ is nonzero and null,
(ii) $\beta_{2}$ is flat and $\operatorname{dim} \mathcal{N}\left(\beta_{2}\right) \geq 2 n-2$.

We argue that $\beta_{2}=0$. Assume otherwise, that is, that $\operatorname{dim} \mathcal{N}\left(\beta_{2}\right)=2 n-2$. Take orthonormal bases $\{\xi, \eta\}$ and $\{\tilde{\xi}, \tilde{\eta}\}$ of $N_{f} M(x)$ such that

$$
\mathcal{S}\left(\beta_{1}\right)=\operatorname{span}\{\xi+\tilde{\xi}\} .
$$

Then

$$
\begin{equation*}
\langle\alpha(X, Y), \xi\rangle=\langle\alpha(X, J Y), \tilde{\xi}\rangle \tag{15.20}
\end{equation*}
$$

and

$$
\begin{equation*}
\langle\alpha(X, Y), \eta\rangle=0=\langle\alpha(X, J Y), \tilde{\eta}\rangle \tag{15.21}
\end{equation*}
$$

for all $X \in T_{x} M$ and $Y \in \mathcal{N}\left(\beta_{2}\right)$. From (15.21) it follows that $\tilde{\eta}= \pm \eta$ because, otherwise, $\mathcal{N}\left(\beta_{2}\right) \cap J \mathcal{N}\left(\beta_{2}\right)$ would be contained in $\mathcal{N}(\beta)$ and have dimension at least $2 n-4$, which is not possible. In particular, it follows that $\tilde{\xi}= \pm \xi$, and from 15.20 we obtain

$$
\begin{aligned}
\langle\alpha(X, Y), \xi\rangle & = \pm\langle\alpha(X, J Y), \xi\rangle \\
& =\left\langle\alpha\left(X, J^{2} Y\right), \xi\right\rangle \\
& =-\langle\alpha(X, Y), \xi\rangle \\
& =0 .
\end{aligned}
$$

Thus $\mathcal{N}\left(\beta_{2}\right) \subset \mathcal{N}(\beta)$, which is not possible and proves that $\beta_{2}=0$. The final argument of the proof of Theorem 15.11 now gives the claim.

Theorem 15.12 implies that $f_{U}$ and $F_{U}$ are congruent, where $F: M^{2 n} \rightarrow \mathbb{C}^{n+1}$ is the holomorphic representative of $f$. Therefore $f$ and $F$ are globally congruent by real analyticity, and this concludes the proof.

### 15.6 The case of nonflat ambient spaces

The situation of isometric immersions of Kaehler manifolds into nonflat ambient spaces is quite different from that of Euclidean space, as shown by the results of this section.

We first consider the hypersurface case.
Theorem 15.21. Let $f: M^{2 n} \rightarrow \mathbb{Q}_{c}^{2 n+1}, n \geq 2$ and $c \neq 0$, be an isometric immersion of a Kaehler manifold. Then $f(M)$ is an open subset of the image of the following standard isometric embeddings:
(i) $\mathbb{R}^{2 n}$ in $\mathbb{H}_{c}^{2 n+1}$,
(ii) $\mathbb{S}_{c_{1}}^{2} \times \mathbb{S}_{c_{2}}^{2}$ in $\mathbb{S}_{c}^{5}$,
(iii) $\mathbb{S}_{c_{1}}^{2} \times \mathbb{H}_{c_{2}}^{2}$ in $\mathbb{H}_{c}^{5}$,
where $1 / c_{1}+1 / c_{2}=1 / c$. Moreover, if $M^{2 n}$ is complete then $f$ is one of those isometric embeddings.

Proof: At $x \in M^{2 n}$, let $X_{1}, \ldots, X_{2 n}$ be an orthonormal frame of principal directions with correspondent principal curvatures $\lambda_{1}, \ldots, \lambda_{2 n}$. The Gauss equation gives

$$
R\left(X_{i}, X_{j}\right) J X_{i}=\left(c+\lambda_{i} \lambda_{j}\right)\left\langle X_{j}, J X_{i}\right\rangle X_{i}
$$

and

$$
J R\left(X_{i}, X_{j}\right) X_{i}=-\left(c+\lambda_{i} \lambda_{j}\right) J X_{j} .
$$

It follows from part ( $i$ ) of Proposition 15.1 that

$$
\begin{equation*}
\left(c+\lambda_{i} \lambda_{j}\right)\left(\left\langle X_{j}, J X_{i}\right\rangle X_{i}+J X_{j}\right)=0 \text { for } i \neq j . \tag{15.22}
\end{equation*}
$$

By (3.7), the endomorphism $T \in \Gamma(\operatorname{End}(T M))$ associated with the Ricci tensor of $M^{2 n}$ is given by

$$
T=(2 n-1) c I+2 n H A-A^{2} .
$$

On the other hand, part (iv) of Proposition 15.1 and the Gauss equation give

$$
T=c I-(J A)^{2} .
$$

Thus

$$
2(n-1) c I+2 n H A-A^{2}=-(J A)^{2} .
$$

Applying the last equation to $X_{j}$ and then computing $J$ on both sides yield

$$
\lambda_{j} A J X_{j}=\left(2(n-1) c+2 n H \lambda_{j}-\lambda_{j}^{2}\right) J X_{j} .
$$

In particular, it follows that $\lambda_{j} \neq 0$ for $1 \leq j \leq 2 n$.

Notice that

$$
\left\langle X_{j}, J X_{i}\right\rangle X_{i}+J X_{j}=0
$$

if and only if $J X_{i}= \pm X_{j}$. If this is not the case, we see from (15.22) that

$$
c+\lambda_{i} \lambda_{j}=0 \text { for } i \neq j
$$

Therefore, if $J X_{i} \neq \pm X_{j}$ for all $1 \leq i \neq j \leq 2 n$, we easily conclude that $x$ is umbilical, that $c<0$ and from the Gauss equation that $x$ is a flat point.

Assume now that $J X_{1}=X_{2}$. Since $J X_{1} \neq \pm X_{k}$ if $k>2$, it follows from 15.22) that

$$
c+\lambda_{1} \lambda_{k}=0=c+\lambda_{1} \lambda_{m}
$$

for $3 \leq k \neq m \leq 2 n$, hence $\lambda_{k}=\lambda_{m}$. Moreover, $c+\lambda_{2} \lambda_{k}=0$, so that $\lambda_{1}=\lambda_{2}$. We conclude that

$$
\lambda_{1}=\lambda_{2} \text { and } \lambda_{3}=\cdots=\lambda_{2 n}
$$

with $c+\lambda_{1} \lambda_{3}=0$. But $J X_{3}$ can be equal to at most one $X_{k}$ with $k \geq 4$. Thus, if $n>2$, then $c+\lambda_{3} \lambda_{j}=0$ for some $j \geq 4$. However, $c+\lambda_{1} \lambda_{j}=0$. Hence $\lambda_{1}=\lambda_{3}$ and $x$ is an umbilical flat point. In other words, we must have $n=2$ if $x$ is not an umbilical flat point, and in this case

$$
\lambda_{1}=\lambda_{2}, \quad \lambda_{3}=\lambda_{4} \text { and } c+\lambda_{1} \lambda_{3}=0
$$

If all points of $M^{n}$ are umbilical flat points, then $f(M)$ is an open subset of the standard umbilical inclusion of $\mathbb{R}^{2 n}$ into $\mathbb{H}_{c}^{2 n+1}$ by Proposition 1.20 .

Now assume that there exists a nonumbilical point, and hence an open neighborhood $U$ of such points. Then $n=2$,

$$
\lambda_{1}=\lambda_{2}=\lambda \text { and } \lambda_{3}=\lambda_{4}=\mu
$$

on $U$, with $c+\lambda \mu=0$. By part (iii) of Proposition 1.22, the principal curvatures $\lambda$ and $\mu$ are constant along the corresponding eigenbundles on $U$. Since $c+\lambda \mu=0$, both $\lambda$ and $\mu$ are actually constant on $U$. In particular, this implies that the open subset of nonumbilical points is also closed, hence is the whole $M^{4}$.

It follows from the Codazzi equation that the eigenbundles of $\lambda$ and $\mu$ are totally geodesic distributions on $M^{2 n}$. By the local version of de Rham theorem, $M^{4}$ is locally isometric to $\mathbb{Q}_{c_{1}}^{2} \times \mathbb{Q}_{c_{2}}^{2}$, where $c_{1}=c+\lambda^{2}$ and $c_{2}=c+\mu^{2}$. In particular, from $c+\lambda \mu=0$ we obtain

$$
\frac{1}{c_{1}}+\frac{1}{c_{2}}=\frac{1}{c} .
$$

It now follows from Corollaries 8.6 and 8.8 that $f$ is locally an extrinsic product of the identity maps id $: \mathbb{Q}_{c_{1}}^{2} \rightarrow \mathbb{Q}_{c_{1}}^{2}$ and $\mathrm{id}_{2}: \mathbb{Q}_{c_{2}}^{2} \rightarrow \mathbb{Q}_{c_{2}}^{2}$. One can now apply Exercise 1.20 to the family of such extrinsic products to conclude that $f(M)$ is an open subset of the image of a fixed one of such immersions. The global statement can be derived by first applying the global version of de Rham theorem to the universal covering of $M^{4}$, and then either Corollary 8.6 or Corollary 8.8, depending on whether $c>0$ or $c<0$, respectively.

For arbitrary codimension we have the following result.

Theorem 15.22. Let $f: M^{2 n} \rightarrow \mathbb{Q}_{c}^{m}, c \neq 0$, be an isometric immersion of a Kaehler manifold. Assume that either
(i) $c<0$ and $f$ is minimal, or
(ii) $c>0$ and

$$
\alpha(X, J Y)=\alpha(J X, Y) \text { for all } X, Y \in \mathfrak{X}(M)
$$

Then $n=1$.
Proof: At any $x \in M^{2 n}$, consider an orthonormal basis $X_{1}, \ldots, X_{2 n}$ of $T_{x} M$ such that $X_{2 j}=J X_{2 j-1}, 1 \leq j \leq n$. By (3.8), minimality of $f$ yields

$$
\begin{equation*}
\operatorname{Ric}\left(X_{i}, X_{i}\right)=(2 n-1) c-\sum_{j=1}^{2 n}\left\|\alpha\left(X_{i}, X_{j}\right)\right\|^{2} \tag{15.23}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{Ric}\left(J X_{i}, J X_{i}\right)=(2 n-1) c-\sum_{j=1}^{2 n}\left\|\alpha\left(J X_{i}, X_{j}\right)\right\|^{2} \tag{15.24}
\end{equation*}
$$

for $1 \leq i \leq 2 n$. On the other hand, a computation similar to that in the proof of Theorem 15.7 gives

$$
\begin{equation*}
\operatorname{Ric}\left(X_{i}, X_{i}\right)=c-\sum_{j=1}^{2 n}\left\langle\alpha\left(X_{j}, J X_{i}\right), \alpha\left(X_{i}, J X_{j}\right)\right\rangle \tag{15.25}
\end{equation*}
$$

Let $V$ and $v_{i}, w_{i} \in V, 1 \leq i \leq 2 n$, be defined as in the proof of Theorem 15.7. In view of Proposition 15.1, from (15.23), (15.24) and (15.25) we obtain, respectively,

$$
\begin{align*}
\operatorname{Ric}\left(X_{i}, X_{i}\right) & =(2 n-1) c-\left\|v_{i}\right\|^{2} \\
& =(2 n-1) c-\left\|w_{i}\right\|^{2} \\
& =c-\left\langle v_{i}, w_{i}\right\rangle . \tag{15.26}
\end{align*}
$$

Under the assumption in part (ii), we have $v_{i}=w_{i}, 1 \leq i \leq 2 n$, hence the preceding equalities yield $n=1$. It follows from (15.26) and the Cauchy-Schwarz inequality that

$$
\left(\operatorname{Ric}\left(X_{i}, X_{i}\right)-c\right)^{2} \leq\left(\operatorname{Ric}\left(X_{i}, X_{i}\right)-(2 n-1) c\right)^{2},
$$

hence

$$
(2 n-2) c \operatorname{Ric}\left(X_{i}, X_{i}\right) \leq n(2 n-2) c^{2} .
$$

This implies that

$$
\operatorname{Ric}\left(X_{i}, X_{i}\right) \geq n c
$$

if $c<0$ and $n>1$, which is in contradiction with 15.23 . This proves the statement under the assumption in part $(i)$.

### 15.7 Hypersurfaces making a constant angle

Two substantial immersions $f, g: M^{n} \rightarrow \mathbb{R}^{m}$ are said to make a constant angle if there exists a nonzero function $\theta \in C^{\infty}(M)$ such that, for any $x \in M^{n}$, the vectors $f_{*}(x) X$ and $g_{*}(x) X$ form an angle $\theta(x)$ for any $X \in T_{x} M$.

Trivial examples of immersions $f, g: M^{n} \rightarrow \mathbb{R}^{m}$ that make a constant angle are given by any pair of immersions that differ by the composition of a homothety and a translation in $\mathbb{R}^{m}$.

If $m$ is even, one can construct further trivial examples as follows. Consider the orthogonal transformation $\tilde{J}_{\theta}=\cos \theta I+\sin \theta \tilde{J}$, where $\tilde{J}$ is the standard complex structure in $\mathbb{R}^{m}$ and $\theta \in(0, \pi / 2)$ is a constant. Then, for any isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{m}$, the pair of isometric immersions $f, g_{\theta}=\tilde{J}_{\theta} \circ f$ makes a constant angle.

Nontrivial examples of isometric immersions making a constant angle are given by any pair of elements in the associated family of a real minimal Kaehler submanifold. In the case of hypersurfaces one has the following converse, in which the immersions are only required to be conformal.

Theorem 15.23. Let $f, g: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, be conformal immersions making a constant angle. Then either $f$ and $g$ differ by the composition of a homothety and a translation in $\mathbb{R}^{n+1}$ or, up to such a composition, one of the following possibilities holds:
(i) $M^{n}$ is a Kaehler manifold and $f, g$ are associated minimal isometric immersions.
(ii) $n$ is odd and $f, g$ are congruent by an orthogonal transformation $\tilde{J}_{\theta}$ of $\mathbb{R}^{n+1}$.

For the proof we need the following algebraic fact.
Lemma 15.24. If $S: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ is an orthogonal transformation and $P \subset \mathbb{R}^{n+1}$ is a hyperplane such that

$$
\langle S X, X\rangle=c\|X\|^{2}, \quad|c|<1
$$

for any $X \in P$, then one of the following possibilities holds:
(i) The dimension $n$ is even, $S P=P$ and

$$
\left.S\right|_{P}=\cos \theta I+\sin \theta J
$$

where $\cos \theta=c$ and $J: P \rightarrow P$ is a complex structure.
(ii) The dimension $n$ is odd and $L=S P \cap P, \operatorname{dim} L=n-1$, satisfies $S L=L$ and

$$
\left.S\right|_{L}=\cos \theta I+\sin \theta J
$$

where $\cos \theta=c$ and $J: L \rightarrow L$ is a complex structure.

Proof: First assume that $n$ is odd and $c=0$. Take $Y \in S^{-1} L$ such that $\|Y\|=1$. Then $S Y \in P \cap S P$. But $S Y \in S P$ implies that $Y \in P$, since $S$ is injective. Hence $Y, S Y \in P$ and, by assumption,

$$
\begin{aligned}
0 & =\langle S(Y+S Y), Y+S Y\rangle \\
& =\langle S Y, S Y\rangle+\left\langle S^{2} Y, Y\right\rangle \\
& =1+\left\langle S^{2} Y, Y\right\rangle .
\end{aligned}
$$

Thus, since $S$ is orthogonal, one can take $J=\left.S\right|_{S^{-1} L}$ and thus $S L=L$, proving the claim in this case.

If $c \neq 0$, take the linear transformation

$$
\tilde{S}=\frac{1}{\sqrt{1-c^{2}}}(S-c I)
$$

It suffices to show that $\tilde{S}$ is orthogonal, for then the proof reduces to the previous case. Given $X \in P$ with $\|X\|=1$, we have

$$
\begin{aligned}
\|\tilde{S} X\|^{2} & =\frac{1}{1-c^{2}}\|S X-c X\|^{2} \\
& =\frac{1}{1-c^{2}}\left(\|S X\|^{2}-2 c\langle S X, X\rangle+c^{2}\|X\|^{2}\right) \\
& =1
\end{aligned}
$$

The case when $n$ is even is now trivial.
Proof of Theorem 15.23: Without loss of generality, we may assume that $g$ is an isometric immersion. Since $f$ and $g$ are conformal, there exist a vector bundle isometry $T: g^{*} T \mathbb{R}^{n+1} \rightarrow f^{*} T \mathbb{R}^{n+1}$ and $\phi \in C^{\infty}(M)$ such that

$$
e^{\phi} f_{*}=T \circ g_{*} \text { and } \bar{N}=T N
$$

where $\bar{N}$ and $N$ are unit normal vector fields along $f$ and $g$, respectively. We have

$$
\begin{aligned}
\tilde{\nabla}_{X} T g_{*} Y & =\left\langle\tilde{\nabla}_{X} T g_{*} Y, \bar{N}\right\rangle \bar{N}+\left(\tilde{\nabla}_{X} T g_{*} Y\right)_{f_{*} T M} \\
& =\left\langle\tilde{\nabla}_{X} T g_{*} Y, T N\right\rangle T N+\left(\tilde{\nabla}_{X} e^{\phi} f_{*} Y\right)_{f_{*} T M} \\
& =-\left\langle T g_{*} Y, \tilde{\nabla}_{X} T N\right\rangle T N+X(\phi) e^{\phi} f_{*} Y+e^{\phi}\left(\tilde{\nabla}_{X} f_{*} Y\right)_{f_{*} T M}
\end{aligned}
$$

for all $X, Y \in \mathfrak{X}(M)$. Exercise 9.1 gives

$$
\begin{equation*}
\left(\tilde{\nabla}_{X} f_{*} Y\right)_{f_{*} T M}=f_{*}\left(\nabla_{X} Y-X(\phi) Y-Y(\phi) X+\langle X, Y\rangle \nabla \phi\right) \tag{15.27}
\end{equation*}
$$

Thus

$$
\begin{equation*}
\tilde{\nabla}_{X} T g_{*} Y=-\left\langle T g_{*} Y, \tilde{\nabla}_{X} T N\right\rangle T N+T g_{*}\left(\nabla_{X} Y-Y(\phi) X+\langle X, Y\rangle \nabla \phi\right) \tag{15.28}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}(M)$.

By the Cauchy-Schwarz inequality and the assumption that $f$ and $g$ make a constant angle, there exists $\theta \in C^{\infty}(M)$ such that

$$
\frac{\left\langle T g_{*} X, g_{*} X\right\rangle}{\left\|g_{*} X\right\|^{2}}=\cos \theta
$$

for all $X \in \mathfrak{X}(M)$.
Assume first that $\theta$ is identically zero modulo $\pi$. Then, up to sign, we have

$$
\begin{equation*}
e^{\phi} f_{*} X=T g_{*} X=g_{*} X \tag{15.29}
\end{equation*}
$$

for all $X \in \mathfrak{X}(M)$. Hence

$$
\begin{align*}
\left(\tilde{\nabla}_{X} e^{\phi} f_{*} Y\right)_{f_{*} T M} & =\left(\tilde{\nabla}_{X} g_{*} Y\right)_{g_{*} T M} \\
& =g_{*} \nabla_{X} Y \tag{15.30}
\end{align*}
$$

for all $X, Y \in \mathfrak{X}(M)$. On the other hand, using 15.27) we obtain

$$
\begin{aligned}
\left(\tilde{\nabla}_{X} e^{\phi} f_{*} Y\right)_{f_{*} T M} & =X(\phi) e^{\phi} f_{*} Y+e^{\phi}\left(\tilde{\nabla}_{X} f_{*} Y\right)_{f_{*} T M} \\
& =g_{*}\left(\nabla_{X} Y-Y(\phi) X+\langle X, Y\rangle \nabla \phi\right)
\end{aligned}
$$

for all $X, Y \in \mathfrak{X}(M)$. Comparing with 15.30 yields

$$
Y(\phi) X-\langle X, Y\rangle \nabla \phi=0
$$

for all $X, Y \in \mathfrak{X}(M)$, which implies that $\phi$ is constant on $M^{n}$. By 15.29), this means that $f$ and $g$ differ by the composition of a homothety and a translation in $\mathbb{R}^{n+1}$.

Now assume that the open subset where $\theta \neq 0$ modulo $\pi$ is nonempty. Since $T$ is a vector bundle isometry, at each point $x$ of a connected component $U$ of that subset we can apply Lemma 15.24 to the orthogonal transformation $S=T(x)$ of $\mathbb{R}^{n+1}$ and to the hyperplane $P=g_{*} T_{x} M$. Let $L$ be the distribution on $U$ defined by

$$
g_{*} L(x)=g_{*} T_{x} M \cap f_{*} T_{x} M=g_{*} T_{x} M \cap T\left(g_{*} T_{x} M\right)
$$

for any $x \in U$. It follows from (15.28) that

$$
\begin{equation*}
\left\langle\tilde{\nabla}_{X} T g_{*} Y, T g_{*} Z\right\rangle=\left\langle\nabla_{X} Y, Z\right\rangle-Y(\phi)\langle X, Z\rangle+Z(\phi)\langle X, Y\rangle \tag{15.31}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}(U)$ and $Z \in \Gamma(L)$. By Lemma 15.24, either $n$ is even, $L=T U$ and there exists an orthogonal tensor field $J \in \Gamma(\operatorname{End}(T U))$ such that $J^{2}=-I$ and

$$
T \circ g_{*}=g_{*}(\cos \theta I+\sin \theta J)
$$

or $n$ is odd, $\operatorname{rank} L=n-1$ and there exists an orthogonal tensor field $J \in \Gamma(\operatorname{End}(L))$ such that $J^{2}=-I$ and

$$
\left.T \circ g_{*}\right|_{L}=g_{*}(\cos \theta I+\sin \theta J)
$$

In either case,

$$
\begin{align*}
\left\langle\tilde{\nabla}_{X} T g_{*} Y, T g_{*} Z\right\rangle= & \left\langle\tilde{\nabla}_{X}\left(\cos \theta g_{*} Y+\sin \theta g_{*} J Y\right), \cos \theta g_{*} Z+\sin \theta g_{*} J Z\right\rangle \\
= & X(\theta)\langle J Y, Z\rangle+\cos ^{2} \theta\left\langle\nabla_{X} Y, Z\right\rangle+\sin ^{2} \theta\left\langle\nabla_{X} J Y, J Z\right\rangle \\
& +\cos \theta \sin \theta\left(\left\langle\nabla_{X} Y, J Z\right\rangle+\left\langle\nabla_{X} J Y, Z\right\rangle\right) \tag{15.32}
\end{align*}
$$

for all $X \in \mathfrak{X}(U)$ and $Y, Z \in \Gamma(L)$.
Applying (15.31) and 15.32) to $Y \in \Gamma(L),\|Y\|=1$, and $Z=J Y$ we obtain $\left\langle\nabla_{X} Y, J Y\right\rangle-Y(\phi)\langle X, J Y\rangle+J Y(\phi)\langle X, Y\rangle=X(\theta)+\cos ^{2} \theta\left\langle\nabla_{X} Y, J Y\right\rangle-\sin ^{2} \theta\left\langle\nabla_{X} J Y, Y\right\rangle$, hence

$$
\begin{equation*}
X(\theta)=J Y(\phi)\langle X, Y\rangle-Y(\phi)\langle X, J Y\rangle \tag{15.33}
\end{equation*}
$$

for all $X \in \mathfrak{X}(U)$ and $Y \in \Gamma(L)$.
Assume that $n \geq 4$. Given any $X \in \mathfrak{X}(U)$, we can choose $Y \in \Gamma(L)$ orthogonal to $X$ so that $J Y$ is also orthogonal to $X$. It follows from (15.33) that $\theta$ is constant on $U$. On the other hand, applying (15.33) to $X=J Y$ yields

$$
Y(\phi)=0 \quad \text { for all } Y \in \Gamma(L)
$$

It follows from (15.31) and (15.32) that

$$
\left\langle\nabla_{X} Y, Z\right\rangle=\cos ^{2} \theta\left\langle\nabla_{X} Y, Z\right\rangle+\sin ^{2} \theta\left\langle\nabla_{X} J Y, J Z\right\rangle+\cos \theta \sin \theta\left(\left\langle\nabla_{X} Y, J Z\right\rangle+\left\langle\nabla_{X} J Y, Z\right\rangle\right)
$$

for all $Y, Z \in \Gamma(L)$. Equivalently, we have

$$
\left\langle\nabla_{X} J Y, J_{\theta} Z\right\rangle-\left\langle J \nabla_{X} Y, J_{\theta} Z\right\rangle=0
$$

for all $X \in \mathfrak{X}(U)$ and $Y, Z \in \Gamma(L)$, where $J_{\theta}=\cos \theta I+\sin \theta J$. Hence

$$
\begin{equation*}
\left(\nabla_{X} J Y\right)_{L}-J\left(\nabla_{X} Y\right)_{L}=0 \tag{15.34}
\end{equation*}
$$

for all $X \in \mathfrak{X}(U)$ and $Y \in \Gamma(L)$.
We now consider separately the cases in which $n$ is even or odd.
Case $n \geq 4$ even: In this case, we have seen that $\theta$ is constant on $U$, hence $U=M^{n}$. The preceding computations have also shown that $\phi$ is constant, thus $f$ and $g$ are homothetic. By (15.34), $M^{n}$ is Kaehler. Moreover, since

$$
e^{\phi} f_{*}=T \circ g_{*}=g_{*} \circ(\cos \theta I+\sin \theta J),
$$

it follows from Proposition 15.9 that $f$ and $g$ are, up to the composition of a homothety and a translation, associated minimal Kaehler hypersurfaces.

Case $n \geq 3$ odd: Here $T U=L \oplus \operatorname{span}\{\eta\}$, where $\eta$ is a unit vector field orthogonal to $L$, and

$$
\begin{equation*}
\left.T \circ g_{*}\right|_{L}=g_{*} \circ J_{\theta} . \tag{15.35}
\end{equation*}
$$

Hence

$$
\begin{equation*}
T g_{*} \eta=\cos \theta g_{*} \eta+\sin \theta N \tag{15.36}
\end{equation*}
$$

and

$$
\begin{equation*}
T N=-\sin \theta g_{*} \eta+\cos \theta N \tag{15.37}
\end{equation*}
$$

Since $L$ is invariant by $J$, taking $X=\eta$ and $Y \in \Gamma(L)$ in 15.33) gives

$$
\eta(\theta)=0
$$

On the other hand, taking $X=Y \in \Gamma(L)$ in 15.33) yields

$$
\begin{equation*}
Y(\theta)=J Y(\phi) \tag{15.38}
\end{equation*}
$$

for any $Y \in \Gamma(L)$. Taking the inner product of both sides of (15.28) with $T g_{*} \eta$ gives

$$
\left\langle\nabla_{X} Y, \eta\right\rangle-Y(\phi)\langle X, \eta\rangle+\eta(\phi)\langle X, Y\rangle=\left\langle\tilde{\nabla}_{X} T g_{*} Y, T g_{*} \eta\right\rangle
$$

for all $X, Y \in \mathfrak{X}(U)$. In particular, using 15.35) and 15.36) we obtain

$$
\begin{equation*}
\left\langle\nabla_{X} Y, \eta\right\rangle-Y(\phi)\langle X, \eta\rangle+\eta(\phi)\langle X, Y\rangle=\cos \theta\left\langle\nabla_{X} J_{\theta} Y, \eta\right\rangle+\sin \theta\left\langle A X, J_{\theta} Y\right\rangle \tag{15.39}
\end{equation*}
$$

for all $X \in \mathfrak{X}(U)$ and $Y \in \Gamma(L)$.
Now, using again that $L$ is invariant by $J$ we obtain

$$
\begin{aligned}
-\left\langle\nabla_{X} Y, \eta\right\rangle & =\left\langle\nabla_{X} \eta, Y\right\rangle \\
& =\left\langle J_{\theta} \nabla_{X} \eta, J_{\theta} Y\right\rangle \\
& =\cos \theta\left\langle\nabla_{X} \eta, J_{\theta} Y\right\rangle+\sin \theta\left\langle J \nabla_{X} \eta, J_{\theta} Y\right\rangle \\
& =-\cos \theta\left\langle\nabla_{X} J_{\theta} Y, \eta\right\rangle+\sin \theta\left\langle J \nabla_{X} \eta, J_{\theta} Y\right\rangle
\end{aligned}
$$

for all $X \in \mathfrak{X}(U)$ and $Y \in \Gamma(L)$. Substituting in (15.39) yields

$$
\begin{equation*}
-Y(\phi)\langle X, \eta\rangle+\eta(\phi)\langle X, Y\rangle=\sin \theta\left\langle A X+J \nabla_{X} \eta, J_{\theta} Y\right\rangle \tag{15.40}
\end{equation*}
$$

for all $X \in \mathfrak{X}(U)$ and $Y \in \Gamma(L)$. In particular, taking $X=\eta$ gives

$$
\begin{equation*}
Y(\phi)=-\sin \theta\left\langle A \eta+J \nabla_{\eta} \eta, J_{\theta} Y\right\rangle \tag{15.41}
\end{equation*}
$$

for any $Y \in \Gamma(L)$.
Since $T N=\bar{N}$, we have

$$
\begin{equation*}
\tilde{\nabla}_{X} \bar{N}=\tilde{\nabla}_{X} T N . \tag{15.42}
\end{equation*}
$$

On the one hand,

$$
\begin{aligned}
-e^{\phi} \tilde{\nabla}_{X} \bar{N} & =e^{\phi} f_{*} \bar{A} X \\
& =T g_{*} \bar{A} X \\
& \left.=\langle\bar{A} X, \eta\rangle T g_{*} \eta+T g_{*}(\bar{A} X)_{L}\right) \\
& \left.=\langle\bar{A} X, \eta\rangle\left(\cos \theta g_{*} \eta+\sin \theta N\right)+g_{*} J_{\theta}(\bar{A} X)_{L}\right) .
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
-\tilde{\nabla}_{X} T N & =\tilde{\nabla}_{X}\left(\sin \theta g_{*} \eta-\cos \theta N\right) \\
& =\cos \theta g_{*}(A X+X(\theta) \eta)+\sin \theta\left(g_{*} \nabla_{X} \eta+(\langle A X, \eta\rangle+X(\theta)) N\right)
\end{aligned}
$$

Thus, taking the inner product of both sides of (15.42) with either $g_{*} \eta$ or $N$, and using the two preceding equations, we obtain

$$
\begin{equation*}
e^{-\phi}\langle\bar{A} X, \eta\rangle=\langle A X, \eta\rangle+X(\theta) \tag{15.43}
\end{equation*}
$$

for any $X \in \mathfrak{X}(U)$. Taking the inner product with $g_{*} Y, Y \in \Gamma(L)$, yields

$$
\begin{align*}
e^{-\phi}\left\langle J_{\theta} \bar{A} X, Y\right\rangle & =\cos \theta\langle A X, Y\rangle+\sin \theta\left\langle\nabla_{X} \eta, Y\right\rangle \\
& =\cos \theta\langle A X, Y\rangle+\sin \theta\left\langle J \nabla_{X} \eta, J Y\right\rangle \\
& =\left\langle A X, J_{-\theta} Y\right\rangle+\sin \theta\left\langle A X+J \nabla_{X} \eta, J Y\right\rangle \tag{15.44}
\end{align*}
$$

for all $X \in \mathfrak{X}(U)$ and $Y \in \Gamma(L)$.
Replacing $Y$ by $J_{-\theta} J Y$ in (15.40) gives

$$
\begin{equation*}
-\left\langle J_{-\theta} J Y, \nabla \phi\right\rangle\langle X, \eta\rangle+\eta(\phi)\left\langle X, J_{-\theta} J Y\right\rangle=\sin \theta\left\langle A X+J \nabla_{X} \eta, J Y\right\rangle \tag{15.45}
\end{equation*}
$$

In particular, we obtain

$$
\eta(\phi)\left\langle X, J_{-\theta} J Y\right\rangle=\sin \theta\left\langle A X+J \nabla_{X} \eta, J Y\right\rangle
$$

if $X, Y \in \Gamma(L)$. It follows from (15.44) that

$$
e^{-\phi}\langle\bar{A} X, Y\rangle=\langle A X, Y\rangle+\eta(\phi)\langle X, J Y\rangle
$$

for all $X, Y \in \Gamma(L)$. Since $e^{-\phi} \bar{A}-A$ is symmetric and $J$ is skew-symmetric, we conclude that

$$
\begin{equation*}
e^{-\phi}\langle\bar{A} X, Y\rangle=\langle A X, Y\rangle \text { and } \eta(\phi)=0 \tag{15.46}
\end{equation*}
$$

for all $X, Y \in \Gamma(L)$.
Applying (15.44) for $X=\eta$ gives

$$
e^{-\phi}\left\langle\bar{A} \eta, J_{-\theta} Y\right\rangle=\left\langle A \eta, J_{-\theta} Y\right\rangle+\sin \theta\left\langle A \eta+J \nabla_{\eta} \eta, J Y\right\rangle .
$$

Using (15.41), it follows that

$$
e^{-\phi}\langle\bar{A} \eta, Y\rangle=\langle A \eta, Y\rangle-J Y(\phi)
$$

for any $Y \in \Gamma(L)$. Comparing with 15.43) we obtain

$$
Y(\theta)=-J Y(\phi)
$$

for any $Y \in \Gamma(L)$. This and (15.38) imply that $Y(\theta)=0=Y(\phi)$ for all $Y \in \Gamma(L)$. We conclude from (15.33) and (15.46) that $\theta$ and $\phi$ are constant. In particular, the constancy of $\theta$ implies that $U=M^{n}$.

We now prove that the vector bundle isometry $T: g^{*} T \mathbb{R}^{n+1} \rightarrow f^{*} T \mathbb{R}^{n+1}$ is induced by a constant orthogonal transformation of $\mathbb{R}^{n+1}$. To see this, we must show that

$$
\begin{equation*}
\tilde{\nabla}_{X} T g_{*} Y=T \tilde{\nabla}_{X} g_{*} Y \tag{15.47}
\end{equation*}
$$

for all $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(L)$, and that

$$
\begin{equation*}
\tilde{\nabla}_{X} T g_{*} \eta=T \tilde{\nabla}_{X} g_{*} \eta \tag{15.48}
\end{equation*}
$$

for all $X \in \mathfrak{X}(M)$. Using (15.35) we obtain

$$
\begin{align*}
\tilde{\nabla}_{X} T g_{*} Y & =\tilde{\nabla}_{X}\left(\cos \theta g_{*} Y+\sin \theta g_{*} J Y\right) \\
& =\cos \theta\left(g_{*} \nabla_{X} Y+\langle A X, Y\rangle N\right)+\sin \theta\left(g_{*} \nabla_{X} J Y+\langle A X, J Y\rangle N\right) . \tag{15.49}
\end{align*}
$$

On the other hand, it follows from (15.35), 15.36) and 15.37) that

$$
\begin{align*}
T \tilde{\nabla}_{X} g_{*} Y= & T\left(\left.g_{*}\left(\nabla_{X} Y\right)\right|_{L}+\left\langle\nabla_{X} Y, \eta\right\rangle g_{*} \eta+\langle A X, Y\rangle N\right) \\
= & \cos \theta\left(\left.g_{*}\left(\nabla_{X} Y\right)\right|_{L}+\left\langle\nabla_{X} Y, \eta\right\rangle g_{*} \eta+\langle A X, Y\rangle N\right)+\sin \theta\left(\left.g_{*} J\left(\nabla_{X} Y\right)\right|_{L}\right. \\
& +\left\langle\nabla_{X} Y, \eta\right\rangle N-\langle A X, Y\rangle g_{*} \eta . \tag{15.50}
\end{align*}
$$

The $L$-components of (15.49) and 15.50 ) coincide by virtue of (15.34), whereas the equality between the components with respect to both $\eta$ and $N$ follows from

$$
\begin{equation*}
\left\langle A X+J \nabla_{X} \eta, J Y\right\rangle=0 \tag{15.51}
\end{equation*}
$$

for all $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(L)$, which is a consequence of 15.45) and the constancy of $\phi$. Thus (15.47) is proved. It remains to show that 15.48) holds. We have

$$
\begin{align*}
\tilde{\nabla}_{X} T g_{*} \eta & =\tilde{\nabla}_{X}\left(\cos \theta g_{*} \eta+\sin \theta N\right) \\
& =\cos \theta\left(g_{*} \nabla_{X} Y+\langle A X, Y\rangle N\right)-\sin \theta g_{*} A X \tag{15.52}
\end{align*}
$$

for all $X \in \mathfrak{X}(M)$. On the other hand,

$$
\begin{align*}
T \tilde{\nabla}_{X} g_{*} \eta & =T\left(g_{*} \nabla_{X} \eta+\langle A X, \eta\rangle N\right) \\
& =\cos \theta\left(g_{*} \nabla_{X} Y+\langle A X, \eta\rangle N\right)+\sin \theta g_{*}\left(J \nabla_{X} \eta-\langle A X, \eta\rangle \eta\right) . \tag{15.53}
\end{align*}
$$

The equality between the $L$-components of 15.52 and 15.53 is again a consequence of (15.51), whereas that between the components with respect to both $\eta$ and $N$ is clear. This proves (15.48). It follows that there exists a constant orthogonal transformation $T: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ satisfying (15.35), 15.36) and 15.37) such that

$$
e^{\phi} f_{*}=T \circ g_{*} .
$$

Finally, let $\tilde{J}: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ be defined by

$$
\sin \theta \tilde{J}=T-\cos \theta I
$$

Using (15.35, 15.36) and (15.37) we obtain

$$
\tilde{J} \circ g_{*}=g_{*} \circ J, \quad \tilde{J} g_{*} \eta=N \text { and } \tilde{J} N=-g_{*} \eta .
$$

Therefore $\tilde{J}$ is orthogonal and satisfies $\tilde{J}^{2}=-I$, that is, it is a complex structure in $\mathbb{R}^{n+1}$. We conclude that $T=\cos \theta I+\sin \theta \tilde{J}=\tilde{J}_{\theta}$ and $f=\tilde{J}_{\theta} \circ g$ up to the composition of a homothety and a translation in $\mathbb{R}^{n+1}$.

### 15.8 Appendix

In this appendix we first state a local Weierstrass type representation for the minimal real Kaehler submanifolds in Euclidean space. Then we discuss a Weierstrass representation for the minimal real Kaehler hypersurfaces, which is an alternative to the parametric description presented in Section 15.4. Both results are given without proof.

In the following result, that a subspace $V \subset \mathbb{C}^{m}$ is isotropic means that $u \cdot v=0$ for all $u, v \in V$ where "." denotes the standard symmetric inner product in $\mathbb{C}^{m}$.

Proposition 15.25. Let $f: M^{2 n} \rightarrow \mathbb{R}^{m}$ be a minimal real Kaehler submanifold. Given a simply connected coordinate chart $U$ of $M^{2 n}$ with $z_{j}=x_{j}+i y_{j}$, define the maps $\varphi_{j}: U \rightarrow \mathbb{C}^{m}, 1 \leq j \leq n, b y$

$$
\varphi_{j}=\sqrt{2} f_{z_{j}}=\frac{1}{\sqrt{2}}\left(f_{x_{j}}-i f_{y_{j}}\right) .
$$

Then the $\varphi_{j}$ satisfy the following conditions:
(i) The vectors $\varphi_{1}, \ldots, \varphi_{n}$ are linearly independent at any point in $U$,
(ii) The functions $\varphi_{j}$ are holomorphic,
(iii) The subspace $\operatorname{span}\left\{\varphi_{1}, \ldots, \varphi_{n}\right\} \subset \mathbb{C}^{m}$ is isotropic,
(iv) The integrability conditions $\partial \varphi_{j} / \partial z_{k}=\partial \varphi_{k} / \partial z_{j}, 1 \leq j, k \leq n$.

Furthermore, if $F: U \rightarrow \mathbb{C}^{m}$ is the holomorphic representative of $f$, then

$$
\begin{equation*}
\varphi_{j}=F_{z_{j}}, \quad 1 \leq j \leq n . \tag{15.54}
\end{equation*}
$$

Conversely, consider maps $\varphi_{1}, \ldots, \varphi_{n}: U \rightarrow \mathbb{C}^{m}$ on a simply connected open subset $U$ of $\mathbb{C}^{m}$ that satisfy conditions (i) to (iv). Then there is a holomorphic map $F: U \rightarrow \mathbb{C}^{m}$ such that (15.54) holds. Moreover, if $f: U \rightarrow \mathbb{R}^{m}$ is defined by

$$
f=\sqrt{2} R e[F]
$$

then $M^{2 n}=\left(U, f^{*}\langle\rangle,\right)$ is a Kaehler manifold and $f$ a minimal real Kaehler submanifold whose holomorphic representative is $F$.

The above result can be used to provide a local Weierstrass representation of any minimal real Kaehler hypersurface, which goes as follows.

Given a nonzero holomorphic function $\alpha_{0}: U \rightarrow \mathbb{C}$ on a simply connected domain $U \subset \mathbb{C}$, set

$$
\phi_{0}=\int \alpha_{0}(z) d z
$$

Assume that the maps $\alpha_{r}, \phi_{r}: U \rightarrow \mathbb{C}$ have been defined for some $0 \leq r \leq n-1$. Choose a nowhere zero holomorphic function $\mu_{r+1}: U \rightarrow \mathbb{C}$ and let $\alpha_{r+1}, \phi_{r+1}: U \rightarrow \mathbb{C}^{2 r+1}$ be given by

$$
\alpha_{r+1}=\frac{\mu_{r+1}}{2}\left(1-\phi_{r}^{2}, i\left(1+\phi_{r}^{2}\right), 2 \phi_{r}\right)
$$

and

$$
\phi_{r+1}=\int \alpha_{r+1}(z) d z
$$

where $\phi_{r} \cdot \phi_{r}=0$. Denote

$$
\gamma=\alpha_{n}: U \rightarrow \mathbb{C}^{2 n+1}
$$

and let $b_{0}, b_{1}, \ldots, b_{n-1}: U \rightarrow \mathbb{C}$ be holomorphic functions such that $b_{n-1}$ is nowhere zero. Take complex coordinates $\left(w_{1}, \ldots, w_{n-1}\right)$ on an open subset $W \subset \mathbb{C}^{n-1}$ and define $F: U \times W \rightarrow \mathbb{C}^{2 n+1}$ by

$$
F\left(z, w_{1}, \ldots, w_{n-1}\right)=\sum_{j=0}^{n-1} \int b_{j}(z) \gamma^{(j)}(z) d z+\sum_{j=1}^{n} w_{j} \gamma^{(j-1)}(z)
$$

where $\gamma^{(j)}=(d / d z)^{j} \gamma$. Then $f=\sqrt{2} \operatorname{Re}[F]$ is a minimal real Kaehler hypersurface given by

$$
\begin{aligned}
& \frac{1}{\sqrt{2}} f\left(x, y, u_{1}, \ldots, u_{n-1}, v_{1}, \ldots, v_{n-1}\right) \\
& \quad=\operatorname{Re}\left[\sum_{j=0}^{n-1} \int b_{j}(z) \gamma^{(j)}(z) d z\right]+\sum_{j=1}^{n}\left(u_{j} \operatorname{Re}\left[\gamma^{(j-1)}(z)\right]-v_{j} \operatorname{Im}\left[\gamma^{(j-1)}(z)\right]\right)
\end{aligned}
$$

where $z=x+i y$ and $w_{j}=u_{j}+i v_{j}$.

### 15.9 Notes

Corollary 15.5 is due to Hasanis [208]. Its consequence for the case in which $M^{2 n}$ is compact was previously obtained by Fwu [197] as a corollary of a result on proper isometric immersions $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+p}, p<n$, of complete Kaehler manifolds; see Exercises 15.2 and 15.3 .

Theorems 15.7, 15.11 and 15.22 are due to Dajczer-Rodríguez [126]. An alternative proof of Theorem 15.7 was given by Moore-Noronha [261]. Theorem 15.11 was improved for codimension at most five in [63].

The results in this chapter concerning the associated family of a minimal real Kaehler submanifold were obtained by Dajczer-Gromoll [110]. In that paper it was also shown that the isometric deformations in the associated family of a minimal simply connected real Kaehler submanifold $f: M^{2 n} \rightarrow \mathbb{R}^{m}$ are the only ones that preserve the oriented generalized Gauss map, that is, the generalized Gauss map into the Grassmannian of oriented $2 n$-planes in $\mathbb{R}^{m}$. An alternative proof of this fact is given in 261].

The case in which the orientation of the Gauss map is reversed was also considered in [110], where it was shown that the examples are of a global nature.

Theorems 15.18, 15.19 and 15.20 are due to Dajczer-Rodríguez [129]. Additional information related to these results has been given by Dajczer-Gromoll [114], [115] and Hennes [218]. For instance, examples of (nonruled) complete minimal real Kaehler submanifolds in $\mathbb{R}^{6}$ were constructed in [115] and [218]. Moreover, a Weierstrass type representation for the complete minimal complex ruled submanifolds, that are part of Theorem 15.19, was constructed by Dajczer-Gromoll [114].

The lower bound in Exercise 15.4 for the index of relative nullity of an isometric immersion $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+p}, p<n$, of a Kaehler manifold with nonnegative sectional curvature, was obtained by Fwu [197], where also the cylinder-type result in Exercise 15.5 was derived. Both results were extended by Florit-Hui-Zheng [192] for the case in which $M^{2 n}$ is only assumed to have either nonnegative Ricci curvature or nonnegative holomorphic curvature. Moreover, they proved that if $p \leq n$ and the minimum index of relative nullity $\nu_{0}$ of $f$ takes its minimum possible value $2(n-p)$, then $f$ is a $2(n-p)$ cylinder over an extrinsic product of $p$ complete surfaces in $\mathbb{R}^{3}$ with positive Gauss curvature. In fact, without the assumption on the completeness of $M^{2 n}$, but assuming that the index of relative nullity takes at any point of $M^{2 n}$ its minimum possible value $2(n-p)$, they were able to prove that each point of an open dense subset of $M^{2 n}$ has a neighborhood that splits as a Riemannian product with $p$ factors of nowhere flat Kaehler manifolds with nonnegative Ricci curvature, and that $f$, restricted to that neighborhood, is an extrinsic product of hypersurfaces. As a consequence, if $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+p}, p \leq n$, is an isometric immersion of a Kaehler manifold with either positive Ricci curvature or positive holomorphic sectional curvature, it follows that $p=n$ and that $f$ splits locally as a product of $n$ positively curved surfaces in $\mathbb{R}^{3}$. In particular, this implies that no open subset of $\mathbb{C P}^{n}$ admits an isometric immersion into $\mathbb{R}^{2 n+p}$ with $p \leq n$. Moreover, the splitting is global if $M^{2 n}$ is complete.

Interesting extensions of the above splitting theorems by Florit-Hui-Zheng [192] for isometric immersions $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+p}, p<n$, of Kaehler manifolds having either nonnegative Ricci curvature or nonnegative holomorphic curvature were obtained by Florit-Zheng [189] for the case in which the index of relative nullity of $f$ is assumed to take the constant value $\nu=2(n-p)+1$ at any point of $M^{2 n}$. They rely on an estimate of the index of pluriharmonic nullity of a real Kaehler submanifold $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+p}$, defined at $x \in M^{2 n}$ as the complex dimension $\nu_{J}(x)$ of the $J$-invariant subspace

$$
\Delta_{J}(x)=\left\{X \in T_{x} M: \alpha(X, J Y)=\alpha(J X, Y) \text { for all } Y \in T_{x} M\right\} .
$$

The authors proved that, for any real Kaehler submanifold $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+p}$, the estimate $\nu_{J}(x) \geq n-p$ holds at each $x \in M^{2 n}$, and were able to determine all such submanifolds for which the equality $\nu_{J}(x)=n-p$ is attained at any $x \in M^{2 n}$.

In the case of complete real Kaehler submanifolds $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+2}$, it was shown by Florit-Zheng [191] that if $f$ is real analytic but not minimal then $f$ has to be a ( $2 n-4$ )-cylinder.

The local classification of real Kaehler hypersurfaces given in terms of pseudoholomorphic surfaces in the sphere is due to Dajczer-Gromoll [110]. Pseudoholomorphic
surfaces $g: M^{2} \rightarrow \mathbb{S}^{2 k}$ were first studied by Calabi [50]. By definition, they induce a holomorphic map into the Hermitian symmetric space $\wp_{k}=S O(2 k+1) / U(k)$ of all oriented hyperplanes in $\mathbb{R}^{2 k+1}$ with complex structure. Conversely, any holomorphic curve in $\wp_{k}$ projects to a pseudoholomorphic surface in $\mathbb{S}^{2 k}$.

Theorem 15.16 on complete real Kaehler hypersurfaces of $\mathbb{R}^{2 n+1}$ was proved by Florit-Zheng [190. Theorem 15.21 on the classification of hypersurfaces with a Kaehler structure in nonflat space forms is due to Ryan [306]. A generalization of Ryan's result on isometric immersions $f: M^{2 n} \rightarrow \mathbb{S}^{2 n+1}$ of Kaehler manifolds into the sphere was obtained by Florit-Hui-Zheng [192]; see also [189]. Namely, it was shown that if $f: M^{2 n} \rightarrow \mathbb{S}_{c}^{2 n+p}, p<n$, is an isometric immersion of a Kaehler manifold, then $p=n-1, M^{2 n}$ is a Riemannian product $\mathbb{S}_{c_{1}}^{2} \times \cdots \times \mathbb{S}_{c_{n}}^{2}$ with $1 / c=1 / c_{1}+\cdots+1 / c_{n}$, and $f$ is an extrinsic product of the identity maps $\mathrm{id}_{a}: \mathbb{S}_{c_{a}}^{2} \rightarrow \mathbb{S}_{c_{a}}^{2}, 1 \leq a \leq n$. Isometric immersions of Kaehler manifolds into the hyperbolic space have been considered by Dajczer-Vlachos [157].

The Weierstrass type representation for minimal real Kaehler submanifolds described in the appendix was obtained by Arezzo-Pirola-Solci [20]. The Weierstrass parametrization for the case of hypersurfaces is due to Hennes [218].

The classification of pairs of conformal hypersurfaces in Euclidean space that form a constant angle is due to Dajczer-Rodríguez [130]. The result in Exercise 15.15 was obtained by Dajczer-Rodríguez [131] with a different argument.

For other results related to the subject of this chapter, see [41], [46], [102], [116], [131], [161], [167], [170], [189], [191], [192], [196], [218], [302], [348] and [349].

### 15.10 Exercises

Exercise 15.1. Provide an example showing that the assumption $p<n$ in Theorem 15.4 is the best possible.

Exercise 15.2. Let $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+p}, p<n$, be an isometric immersion of a complete Kaehler manifold. If $f$ is proper, show that $M^{2 n}$ has the homotopy type of a $C W$-complex with no cells of dimension $k>n+p$. Conclude that for any coefficient group $G$ the homology groups of $M^{n}$ satisfy $H_{k}(M ; G)=0$ for $k>n+p$.
Hint: Use Sard's theorem to obtain $q \in \mathbb{R}^{2 n+p}$ such that $h: M^{2 n} \rightarrow \mathbb{R}$, given by

$$
h(x)=\frac{1}{2}\|f(x)-q\|^{2},
$$

is a smooth function all of whose critical points are nondegenerate. If $x$ is a critical point of $h$, first obtain from the proof of Theorem 15.4 a $J$-invariant subspace $W$ of $T_{x} M$ of dimension at least $2(n-p)$ such that

$$
\alpha(J Z, J Z)=-\alpha(Z, Z)
$$

for any $Z \in W$. Then use Corollary 1.5 to show that there exists a subspace $W_{1} \subset W$ of dimension at least $n-p$ such that Hess $h$ is positive definite on $W_{1}$, and hence the
index of $h$ at $x$ is at most $n+p$. Show that the assumption that $f$ is proper implies that the subset $M^{a}=h^{-1}((-\infty, a])$ is compact for any $a \in \mathbb{R}$. Finally, as in the proof of Theorem 1.23, conclude from Theorem 3.5 in [247] that $M^{n}$ has the homotopy type of a $C W$-complex with no cells of dimension $k>n+p$.

Exercise 15.3. Use Exercise 15.2 to give another proof of the nonexistence of an isometric immersion of a compact Kaehler manifold $M^{2 n}$ into $\mathbb{R}^{2 n+p}$ with $p<n$.

Hint: Use the fact that every immersion of a compact manifold is proper and that a Kaehler manifold is always orientable, hence its top dimensional homology group is nonzero if it is compact.

Exercise 15.4. Let $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+p}, p<n$, be an isometric immersion of a Kaehler manifold of nonnegative sectional curvature. Show that the minimum index of relative nullity $\nu_{0}$ of $f$ satisfies $\nu_{0} \geq 2(n-p)$.

Hint: For $x \in M^{2 n}$, obtain from the proof of Theorem 15.4 a $J$-invariant subspace $W$ of $T_{x} M$ with $\operatorname{dim} W \geq 2(n-p)$ and an isometric linear isomorphism $\tilde{J}: U \rightarrow U$, where

$$
U=\operatorname{span}\left\{\alpha(Y, Z): Y \in T_{x} M \text { and } Z \in W\right\},
$$

such that $\tilde{J} \alpha(Y, Z)=\alpha(Y, J Z)$ for all $Y \in T_{x} M$ and $Z \in W$. In particular,

$$
\alpha(J Z, J Z)=-\alpha(Z, Z)
$$

for any $Z \in W$. Use this fact and the assumption that the sectional curvature of $M^{2 n}$ is nonnegative to show that $W$ is contained in the relative nullity subspace $\Delta(x)$ of $f$ at $x$.

Exercise 15.5. Show that any isometric immersion $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+p}, p<n$, of a complete Kaehler manifold of nonnegative sectional curvature is a $2 \ell$-cylinder, with $\ell \geq n-p$.
Hint: Combine Exercise 15.4 with Theorem 7.15
Exercise 15.6. Let $f: M^{2 n} \rightarrow \mathbb{R}^{m}, n \geq 2$, be a minimal real Kaehler submanifold. Show that any holomorphic submanifold $N^{2 \ell}$ of $M^{2 n}$ is also a minimal real Kaehler submanifold of $\mathbb{R}^{m}$.

Exercise 15.7. If $f: M^{2 n} \rightarrow \mathbb{C}^{n+q}, n \geq 2$, is a substantial holomorphic isometric immersion and $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+p}$ is a substantial minimal immersion, show that

$$
2 n+p \leq 2(n+q) \leq 2(2 n+p)
$$

Exercise 15.8. Let $f: M^{2 n} \rightarrow \mathbb{R}^{m}, n \geq 2$, be a simply connected minimal real Kaehler submanifold. Show that $f$ is a cone if and only if it is the real part of a holomorphic isometric immersion $F: M^{2 n} \rightarrow \mathbb{C}^{m}$ obtained by lifting a holomorphic isometric immersion $F: M^{2 n} \rightarrow \mathbb{C P}^{m-1}$ by the Hopf projection $\pi: \mathbb{C}^{m} \rightarrow \mathbb{C P}^{m-1}$.

Hint for the "only if" part: Let $\left\{f_{\theta}\right\}_{\theta \in[0, \pi)}$ be the associated family of $f$, and let $F: M^{2 n} \rightarrow \mathbb{R}^{2 m}=\mathbb{R}^{m} \oplus \mathbb{R}^{m}$, given by

$$
F=\frac{1}{\sqrt{2}} f \oplus \frac{1}{\sqrt{2}} g,
$$

be its holomorphic representative, where $g=f_{\pi / 2}$. Since $f$ is a cone, there exists a unit vector field $R$ and $\gamma \in C^{\infty}(M)$ such that the map $h=f+\gamma^{-1} f_{*} R$ is constant. Show that the map $\ell=g+\gamma^{-1} g_{*} R$ is also constant and that the distribution $\mathcal{L}$ spanned by $R$ and $J R$ is totally geodesic and its leaves are mapped by $f$ and $g$ into affine planes of $\mathbb{R}^{m}$. This implies that the images by $F$ of the leaves of $\mathcal{L}$ give rise to a foliation of $F(M)$ by complex lines of $\mathbb{C}^{m}$ through a common point.

Exercise 15.9. Show that any minimal real Kaehler submanifold $f: M^{2 n} \rightarrow \mathbb{R}^{m}$, $n \geq 2$, with flat normal bundle is totally geodesic.

Exercise 15.10. Let $f: M^{2 n} \rightarrow \mathbb{R}^{m}$ be a complete minimal real Kaehler submanifold. Assume that $\nu>0$ at any point and that there is a point $x_{0} \in M^{2 n}$ where $\nu$ assumes its minimum value $2 \ell$ and all holomorphic curvatures of planes in $\Delta^{\perp}\left(x_{0}\right)$ are different from zero. Show that $f$ is a $2 \ell$-cylinder.
Hint: Suppose that $C_{T} \neq 0$ at $x_{0}$. Then there exist $X, Y \in \Delta^{\perp}\left(x_{0}\right)$ such that $C_{T} X=0$ and $C_{T} Y=X$ with $X \neq 0$. Now use the Gauss equation to conclude that

$$
\langle R(X, J X) J X, X\rangle=0
$$

which is not possible.
Exercise 15.11. Let $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+2}$ be a real Kaehler submanifold. If the index of nullity of the curvature tensor of $M^{2 n}$ satisfies $\mu<2 n-4$ at any point, prove that $f$ is holomorphic.

Exercise 15.12. Assume that a real Kaehler submanifold $M^{2 n}$ of $\mathbb{R}^{2 n+p}$ satisfies

$$
\alpha(J X, Y)=-\alpha(X, J Y)
$$

for all $X, Y \in \mathfrak{X}(M)$. Prove that its second fundamental form $\alpha$ is parallel.
Hint: Combine the fact that

$$
\nabla \stackrel{\perp}{X} \alpha(J Y, Z)=-\nabla_{X}^{\perp} \alpha(Y, J Z)
$$

and the Codazzi equation in a similar fashion as in the argument for Proposition 4.17. Note: From the classification due to Ferus [173] of the Euclidean submanifolds with parallel second fundamental form it follows that the submanifold must be an open part of a standard embedded Hermitian symmetric $R$-space, of an affine subspace, or of a product of such spaces. This exercise was taken from Ferus [175].

Exercise 15.13. Let $f: L^{2} \rightarrow \mathbb{C}^{2} \approx \mathbb{R}^{4}$ be a holomorphic curve without flat points. For each $x \in L^{2}$ and each unit normal vector $\xi \in N_{f} L(x)$, let $\lambda^{\xi}(x)$ denote the positive principal curvature of the shape operator of $f$ with respect to $\xi$. Let $F: N_{f}^{1} L \rightarrow \mathbb{R}^{4}$ be the map defined on the unit normal bundle of $f$ by

$$
F(x, \xi)=f(x)+\lambda^{\xi}(x) \xi .
$$

(i) Compute the singular points of $F$.
(ii) Verify that the scalar curvature of the metric induced by $F$ on the open subset of its regular points vanishes everywhere.

Exercise 15.14. Let $f: M^{8} \rightarrow \mathbb{C}^{5}$ be a holomorphic isometric immersion of a Kaehler manifold. When considered as a real submanifold in $\mathbb{R}^{10}$, show that the type number satisfies $\tau \geq 4$ at all points where the index of relative nullity vanishes.

Exercise 15.15. If $f: M^{2 n} \rightarrow \mathbb{Q}_{c}^{2 n+p}, n \geq 2$, is an isometric immersion of a Kaehler manifold into a space form with $c \neq 0$, show that the index of relative nullity of $f$ vanishes at any point of $M^{2 n}$.

Hint: Assume that there exists $x \in M^{2 n}$ such that the subspace of relative nullity $\Delta(x)$ is nontrivial. Take $X \in \Delta(x)$ and $Y, J Y$ orthogonal to $X$. Use that

$$
\langle R(X, Y) Y, X\rangle=\langle R(J X, J Y) Y, X\rangle
$$

to obtain a contradiction from the Gauss equation.
Exercise 15.16. Let $f: M^{2 n} \rightarrow \mathbb{R}^{2 n+1}, n \geq 2$, be an isometric immersion of a locally irreducible Kaehler manifold. Prove that $f$ is a Sbrana-Cartan hypersurface if and only if it is minimal.

Exercise 15.17. Let $\mathcal{T}$ be a nontrivial infinitesimal bending on an isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{n+1}$. If $f$ and $F=f+t \mathcal{T}$ are conformal for some $t \neq 0$, prove that $f$ must be a minimal real Kaehler hypersurface.

## Chapter 16

## Conformally flat submanifolds

This chapter brings us back to the conformal realm. Here our main interest is on geometric and topological properties of conformally flat submanifolds of Euclidean space, that is, isometric immersions into Euclidean space of Riemannian manifolds that are locally conformally diffeomorphic to an open subset of Euclidean space.

The first subject we consider is the classical characterization of conformally flat manifolds in terms of the Weyl and Schouten tensors. It will be derived as a consequence of the fact that conformally flat manifolds are precisely those Riemannian manifolds that admit locally (globally, if simply connected) an isometric immersion with codimension one into the light-cone of Lorentzian space. This basic fact is also used to prove that any simply connected compact conformally flat manifold is conformally diffeomorphic to the sphere.

The remaining results of the chapter deal with conformally flat submanifolds in Euclidean space with low codimension. First, it is shown that any such submanifold $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ of dimension $n \geq 4$ and codimension $p \leq n-3$ carries a principal normal at any point whose multiplicity is at least $n-p$. This implies that a generic conformally flat Euclidean submanifold satisfying these conditions is the envelope of a $p$-parameter family of spheres, and hence carries a $(n-p)$-dimensional foliation by round spheres.

A geometric explanation for the above fact is provided, which goes as follows: since conformally flat manifolds are locally characterized as those Riemannian manifolds that admit an isometric immersion with codimension one into the light-cone, in order to obtain examples of conformally flat submanifolds $M^{n}$ of $\mathbb{R}^{n+p}$, it suffices to consider a Riemannian manifold $N^{n+1}$ that admits isometric immersions $F: N^{n+1} \rightarrow \mathbb{R}^{n+p}$ and $G: N^{n+1} \rightarrow \mathbb{L}^{n+2}$, and then take $M^{n}$ as the intersection $G(N) \cap \mathbb{V}^{n+1}$. The leaves of the foliation of $M^{n}$ by $(n-p)$-dimensional round spheres then arise as the intersections with the light-cone of the leaves of the common relative nullity leaves of both $F$ and $G$, which has dimension at least $n-p+1$. It turns out that, if $n \geq 4$ and $p \leq n-3$, this procedure generates all simply connected examples.

We then discuss how one can obtain an explicit parametrization of all conformally flat submanifolds $M^{n}$ of $\mathbb{R}^{n+2}$ by putting together the preceding construction with a description of the hypersurfaces $F: N^{n+1} \rightarrow \mathbb{R}^{n+2}$ for which $N^{n+1}$ also admits an
isometric immersion $G: N^{n+1} \rightarrow \mathbb{L}^{n+2}$.
In the codimension one case, the preceding results say that a conformally flat Euclidean hypersurface of dimension $n \geq 4$ must carry a principal curvature whose multiplicity is at least $n-1$ at any point. In other words, a generic conformally flat Euclidean hypersurface of dimension $n \geq 4$ is the envelope of a one parameter family of spheres. It is shown how this leads to an explicit parametrization of any such hypersurface by means of the conformal version of the Gauss parametrization. On the other hand, a conformally flat Euclidean hypersurface of dimension three may not carry any principal curvature of multiplicity two. The interesting class of conformally flat Euclidean hypersurfaces of dimension three that have three distinct principal curvatures is discussed in the last section of the chapter, where they are characterized as holonomic hypersurfaces satisfying some additional conditions.

### 16.1 Hypersurfaces of the light-cone

In this section we obtain a rigidity theorem for hypersurfaces in the light-cone. This result is used in the next section to give a proof of the classical characterization of conformally flat manifolds in terms of the Weyl and Schouten tensors.

The Schouten tensor of a Riemannian manifold $M^{n}$ is defined as

$$
\begin{equation*}
L(X, Y)=\frac{1}{n-2}(\operatorname{Ric}(X, Y)-(1 / 2) n s\langle X, Y\rangle), \tag{16.1}
\end{equation*}
$$

whereas the Weyl tensor (or conformal curvature tensor) is defined by

$$
\begin{aligned}
\langle C(X, Y) Z, W\rangle= & \langle R(X, Y) Z, W\rangle-L(X, W)\langle Y, Z\rangle-L(Y, Z)\langle X, W\rangle \\
& +L(X, Z)\langle Y, W\rangle+L(Y, W)\langle X, Z\rangle
\end{aligned}
$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$.
Theorem 16.1. Let $M^{n}$ be a Riemannian manifold of dimension $n \geq 3$. If there exists an isometric immersion $f: M^{n} \rightarrow \mathbb{V}^{n+1} \subset \mathbb{L}^{n+2}$, then
(i) $C=0$.
(ii) L is a Codazzi tensor.

Moreover, any other isometric immersion $\tilde{f}: M^{n} \rightarrow \mathbb{V}^{n+1} \subset \mathbb{L}^{n+2}$ is given by $\tilde{f}=T \circ f$ for some $T \in \mathbb{O}_{1}^{n+2}$.

Conversely, any simply connected Riemannian manifold $M^{n}, n \geq 3$, satisfying conditions (i) and (ii) admits an isometric immersion into $\mathbb{V}^{n+1}$.

Proof: Let $f: M^{n} \rightarrow \mathbb{V}^{n+1} \subset \mathbb{L}^{n+2}$ be an isometric immersion. By Proposition 9.1, the position vector field $f$ is a light-like parallel normal vector field satisfying $A_{f}=-I$, where $I$ stands for the identity tensor. Since the normal bundle of $f$ is a time-like plane
bundle having the position vector field $f$ as a section, there exists a unique light-like normal vector field $\delta$ such that $\langle f, \delta\rangle=-1$. Thus the second fundamental form of $f$ can be written as

$$
\alpha(X, Y)=\langle X, Y\rangle \delta-\left\langle A_{\delta} X, Y\right\rangle f
$$

for all $X, Y \in \mathfrak{X}(M)$.
Next we prove that the shape operator $A_{\delta}$ coincides with the endomorphism

$$
\begin{equation*}
\hat{L}=\frac{1}{n-2}(T-(1 / 2) n s I) \tag{16.2}
\end{equation*}
$$

associated with the Schouten tensor of $M^{n}$, where $T$ is the endomorphism associated with its Ricci tensor. The third fundamental form of $f$ is given by

$$
\begin{aligned}
\operatorname{III}(X, Y) & =\sum_{i=1}^{n}\left\langle\alpha\left(X, X_{i}\right), \alpha\left(Y, X_{i}\right)\right\rangle \\
& =\sum_{i=1}^{n}\left(\left\langle X, X_{i}\right\rangle\left\langle A_{\delta} Y, X_{i}\right\rangle+\left\langle Y, X_{i}\right\rangle\left\langle A_{\delta} X, X_{i}\right\rangle\right) \\
& =2\left\langle A_{\delta} X, Y\right\rangle
\end{aligned}
$$

where $X_{1}, \ldots, X_{n}$ is an orthonormal tangent frame. It follows from (3.6) that

$$
T=n A_{\mathcal{H}}-2 A_{\delta}
$$

On the other hand,

$$
\begin{aligned}
n \mathcal{H} & =-n\langle\mathcal{H}, f\rangle \delta-n\langle\mathcal{H}, \delta\rangle f \\
& =n \delta-\operatorname{tr} A_{\delta} f .
\end{aligned}
$$

Hence

$$
n A_{\mathcal{H}}=n A_{\delta}+\operatorname{tr} A_{\delta} I
$$

and therefore

$$
\begin{equation*}
T=(n-2) A_{\delta}+\operatorname{tr} A_{\delta} I \tag{16.3}
\end{equation*}
$$

Also

$$
\begin{align*}
n(n-1) s & =\operatorname{tr} T \\
& =2(n-1) \operatorname{tr} A_{\delta} . \tag{16.4}
\end{align*}
$$

Substituting (16.3) and 16.4 into 16.2 yields $\hat{L}=A_{\delta}$.
Now observe that $\delta$ is a parallel normal vector field, for

$$
\begin{aligned}
\nabla_{X}^{\perp} \delta & =-\langle\nabla \stackrel{\perp}{X} \delta, f\rangle \delta-\left\langle\nabla \frac{\perp}{X} \delta, \delta\right\rangle f \\
& =0
\end{aligned}
$$

Therefore the Codazzi equation of $f$ is equivalent to $A_{f}$ and $A_{\delta}$ being Codazzi tensors. Since $A_{f}=-I$, this reduces to $\hat{L}=A_{\delta}$, or equivalently, to $L$, being a Codazzi tensor. Now, from

$$
\begin{aligned}
\langle\alpha(X, W), \alpha(Y, Z)\rangle & =-\langle\alpha(X, W), f\rangle\langle\alpha(Y, Z), \delta\rangle-\langle\alpha(X, Z), f\rangle\langle\alpha(Y, W), \delta\rangle \\
& =\langle X, W\rangle L(Y, Z)+\langle Y, Z\rangle L(X, W)
\end{aligned}
$$

it follows that

$$
\langle R(X, Y) Z, W\rangle-\langle\alpha(X, W), \alpha(Y, Z)\rangle+\langle\alpha(X, Z), \alpha(Y, W)\rangle=\langle C(X, Y) Z, W\rangle
$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$. Thus the Gauss equation of $f$ is equivalent to the vanishing of $C$.

The last assertion in the direct statement follows from the uniqueness part of the Fundamental theorem of submanifolds, taking into account that $A_{f}=-I$ and $A_{\delta}=\hat{L}$.

Conversely, assume that $M^{n}$ is simply connected and that conditions (i) and (ii) hold on $M^{n}$. Consider the trivial Lorentzian vector bundle $M^{n} \times \mathbb{L}^{2}$ over $M^{n}$. Endow this bundle with its canonical connection $\nabla^{\prime}$, and choose a parallel pseudo-orthonormal frame $e_{1}, e_{2}$ of $M^{n} \times \mathbb{L}^{2}$ with $\left\langle e_{1}, e_{2}\right\rangle=-1$. Define

$$
\gamma(X, Y)=\langle X, Y\rangle e_{1}-L(X, Y) e_{2} .
$$

Conditions (i) and (ii) imply that $\nabla^{\prime}$ and $\gamma$ satisfy the Gauss and Codazzi equations for constant curvature zero. The Ricci equations are trivially satisfied since $\nabla^{\prime}$ is flat and $\gamma$ is orthogonally diagonalizable.

By the Fundamental theorem of submanifolds, there exists an isometric immersion $G: M^{n} \rightarrow \mathbb{L}^{n+2}$ whose normal bundle, normal connection and second fundamental form are $M^{n} \times \mathbb{L}^{2}, \nabla^{\prime}$ and $\gamma$, up to a vector bundle isometry. Set $h=G-e_{2}$. From

$$
\begin{aligned}
h_{*} Z & =G_{*} Z+G_{*} A_{e_{2}} Z \\
& =0
\end{aligned}
$$

for any $Z \in \mathfrak{X}(M)$ we see that $h$ is constant. Therefore the map $f=G-h=e_{2}$ defines an isometric immersion of $M^{n}$ into $\mathbb{V}^{n+1}$.

### 16.2 Conformally flat manifolds

A Riemannian manifold $M^{n}$ is called conformally flat if each point lies in a neighborhood which is conformally diffeomorphic to an open subset of Euclidean space.

Examples 16.2. (i) Any Riemannian manifold of dimension two is conformally flat since it locally admits isothermic coordinates.
(ii) Any Riemannian manifold with constant sectional curvature is conformally flat, as follows from Examples 9.10 .
(iii) By Corollary 9.29, a Riemannian product $M_{1} \times M_{2}$ is conformally flat if and only if one of the following possibilities holds:
(a) One of the factors is one-dimensional and the other one has constant sectional curvature.
(b) Both factors have dimension greater than one and are either both flat or have opposite constant sectional curvatures.
(iv) A warped product $M_{1} \times{ }_{\rho} M_{2}$ is conformally flat if and only if one of the following possibilities holds:
(a) $M_{1}$ has dimension one and $M_{2}$ has constant sectional curvature.
(b) $M_{2}$ has dimension one and the metric $\langle,\rangle_{1} \tilde{\sim}=\left(1 / \rho^{2}\right)\langle,\rangle_{1}$ on $M_{1}$ has constant sectional curvature, where $\langle,\rangle_{i}$ is the metric of $M_{i}, 1 \leq i \leq 2$.
(c) Both $M_{1}$ and $M_{2}$ have dimension greater than one and the metrics $\langle,\rangle_{1}^{\sim}$ on $M_{1}$ and $\langle,\rangle_{2}$ on $M_{2}$ have opposite constant sectional curvatures.

The assertion in part (iv) follows from that in part (iii) by noticing that the warped product metric of $M_{1} \times \rho M_{2}$ is conformal to the product metric

$$
\pi_{1}^{*}\langle,\rangle_{1}^{\sim}+\pi_{2}^{*}\langle,\rangle_{2}
$$

on $M_{1} \times M_{2}$ with conformal factor $\rho \circ \pi_{1}$, where $\pi_{i}: M_{1} \times M_{2} \rightarrow M_{i}$ is the projection, $1 \leq i \leq 2$.

Necessary and sufficient conditions for a Riemannian manifold $M^{n}$ of dimension $n \geq 3$ to be conformally flat are given in the next result.

Theorem 16.3. A Riemannian manifold $M^{n}$ of dimension $n \geq 3$ is conformally flat if and only if the following conditions are satisfied:
(i) $C=0$.
(ii) L is a Codazzi tensor.

Moreover, ( $i$ ) implies (ii) when $n \geq 4$, and ( $i$ ) is automatically satisfied if $n=3$.
Proof: If $M^{n}$ is conformally flat, then it admits locally an isometric immersion into $\mathbb{V}^{n+1}$ by part (i) of Proposition 9.9. Thus conditions (i) and (ii) hold on $M^{n}$ by Theorem 16.1. Conversely, if these conditions are satisfied, then $M^{n}$ admits locally an isometric immersion into $\mathbb{V}^{n+1}$ by Theorem 16.1, and hence a conformal immersion into $\mathbb{R}^{n}$ by part (ii) of Proposition 9.9 .

It remains to prove the two last assertions. The proof that part (i) always holds for $n=3$ is a straightforward computation that is left to the reader. That $(i)$ implies (ii) when $n \geq 4$ will follow using the second Bianchi identity (e.g. [317])

$$
\begin{equation*}
\left(\nabla_{X} R\right)(Y, Z, V, W)+\left(\nabla_{Y} R\right)(Z, X, V, W)+\left(\nabla_{Z} R\right)(X, Y, V, W)=0 \tag{16.5}
\end{equation*}
$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$.

If the Weyl tensor vanishes identically, a direct calculation shows that

$$
\begin{aligned}
\left(\nabla_{X} R\right)(Y, Z, V, W)= & \left(\nabla_{X} L\right)(Z, V)\langle Y, W\rangle+\left(\nabla_{X} L\right)(Y, W)\langle Z, V\rangle \\
& -\left(\nabla_{X} L\right)(Y, V)\langle Z, W\rangle-\left(\nabla_{X} L\right)(Z, W)\langle Y, V\rangle .
\end{aligned}
$$

Using (16.5) and the preceding equation we obtain

$$
\begin{align*}
& {\left[\left(\nabla_{X} L\right)(Z, V)-\left(\nabla_{Z} L\right)(X, V)\right]\langle Y, W\rangle+\left[\left(\nabla_{X} L\right)(Y, W)-\left(\nabla_{Y} L\right)(X, W)\right]\langle Z, V\rangle} \\
& +\left[\left(\nabla_{Y} L\right)(X, V)-\left(\nabla_{X} L\right)(Y, V)\right]\langle Z, W\rangle+\left[\left(\nabla_{Z} L\right)(X, W)-\left(\nabla_{X} L\right)(Z, W)\right]\langle Y, V\rangle \\
& +\left[\left(\nabla_{Y} L\right)(Z, W)-\left(\nabla_{Z} L\right)(Y, W)\right]\langle X, V\rangle+\left[\left(\nabla_{Z} L\right)(Y, V)-\left(\nabla_{Y} L\right)(Z, V)\right]\langle X, W\rangle \\
& =0 . \tag{16.6}
\end{align*}
$$

Let $X_{1}, \ldots, X_{n}$ be a local orthonormal frame in $M^{n}$. If we take $Y=W=X_{i}$, $X=X_{j}, Z=X_{k}$ and $V=X_{\ell}$ in the preceding expression for pairwise distinct indices $i, j, k, \ell$, we obtain

$$
\begin{equation*}
\left(\nabla_{X_{j}} L\right)\left(X_{k}, X_{\ell}\right)-\left(\nabla_{X_{k}} L\right)\left(X_{j}, X_{\ell}\right)=0 \tag{16.7}
\end{equation*}
$$

It remains to show that (16.7) holds when $j \neq k=\ell$. Now, choosing in 16.6) successively: $X=X_{j}, Y=W=X_{i}, Z=V=X_{k} ; X=V=X_{i}, Y=X_{j}, Z=W=X_{h}$ and $X=W=X_{h}, Y=V=X_{k}, Z=X_{j}$, we obtain

$$
\begin{aligned}
& {\left[\left(\nabla_{X_{j}} L\right)\left(X_{k}, X_{k}\right)-\left(\nabla_{X_{k}} L\right)\left(X_{j}, X_{k}\right)\right]+\left[\left(\nabla_{X_{j}} L\right)\left(X_{i}, X_{i}\right)-\left(\nabla_{X_{i}} L\right)\left(X_{j}, X_{i}\right)\right]=0,} \\
& {\left[\left(\nabla_{X_{j}} L\right)\left(X_{i}, X_{i}\right)-\left(\nabla_{X_{i}} L\right)\left(X_{j}, X_{i}\right)\right]+\left[\left(\nabla_{X_{j}} L\right)\left(X_{h}, X_{h}\right)-\left(\nabla_{X_{h}} L\right)\left(X_{j}, X_{h}\right)\right]=0}
\end{aligned}
$$

and

$$
\left[\left(\nabla_{X_{j}} L\right)\left(X_{h}, X_{h}\right)-\left(\nabla_{X_{h}} L\right)\left(X_{j}, X_{h}\right)\right]+\left[\left(\nabla_{X_{j}} L\right)\left(X_{k}, X_{k}\right)-\left(\nabla_{X_{k}} L\right)\left(X_{j}, X_{k}\right)\right]=0 .
$$

Subtracting the second of the preceding equations from the sum of the other two gives

$$
\left(\nabla_{X_{j}} L\right)\left(X_{k}, X_{k}\right)-\left(\nabla_{X_{k}} L\right)\left(X_{j}, X_{k}\right)=0
$$

and this completes the proof that $L$ is a Codazzi tensor.
Theorem 16.4. Any simply connected conformally flat manifold $M^{n}, n \geq 3$, admits a conformal immersion into $\mathbb{S}^{n}$, which is unique up to a conformal transformation of $\mathbb{S}^{n}$. In particular, if compact, then $M^{n}$ is conformal to $\mathbb{S}^{n}$.
Proof: In view of Theorem 16.3, a simply connected conformally flat manifold $M^{n}$ admits an isometric immersion into $\mathbb{V}^{n+1}$ by the converse statement of Theorem 16.1 , which gives rise to a conformal immersion into $\mathbb{S}^{n}$ by part ( $i i$ ) of Exercise 9.7. Moreover, the isometric immersion of $M^{n}$ into $\mathbb{V}^{n+1} \subset \mathbb{L}^{n+2}$ is unique up to an orthogonal transformations of $\mathbb{L}^{n+2}$ by Theorem 16.1. Hence, the corresponding conformal immersion of $M^{n}$ into $\mathbb{S}^{n}$ is unique up to a conformal transformations of $\mathbb{S}^{n}$ by part (iv) of Exercise 9.7 .

The last assertion follows from a standard covering map argument. In fact, if $M^{n}$ is compact, then a conformal immersion $f: M^{n} \rightarrow \mathbb{S}^{n}$ is a covering map, hence a diffeomorphism because $\mathbb{S}^{n}$ is simply connected for $n \geq 2$.

### 16.3 Conformally flat submanifolds of low codimension

Conformally flat Euclidean submanifolds of dimension $n \geq 4$ and codimension $p \leq n-3$ are shown to carry principal normal vector fields with multiplicity at least $n-p$. On the one hand, this implies that conformally flat hypersurfaces of dimension $n \geq 4$ must have a principal curvature with multiplicity at least $n-1$ at any point, and it will be proved that the converse is also true. On the other hand, this yields a topological condition for a compact conformally flat manifold of dimension $n \geq 4$ to admit an isometric immersion with low codimension into Euclidean space.

### 16.3.1 Structure of the second fundamental form

The following result makes use of the theory of flat bilinear forms to describe the structure of the second fundamental form of a conformally flat Euclidean submanifold of low codimension.

Theorem 16.5. If $f: M^{n} \rightarrow \mathbb{R}^{n+p}, n \geq 4$, is an isometric immersion of a conformally flat manifold, then the following assertions hold:
(i) If $p \leq n-3$, then for each $x \in M^{n}$ there exists a principal normal $\eta \in N_{f} M(x)$ such that $\operatorname{dim} E_{\eta}(x) \geq n-p \geq 3$.
(ii) If $p=n-2$ and at $x \in M^{n}$ there exists no principal normal $\eta \in N_{f} M(x)$ such that $\operatorname{dim} E_{\eta}(x) \geq 2$, then $f$ has flat normal bundle at $x$.

Proof: Let $\mathbb{L}^{2}$ be a Lorentzian plane with inner product denoted by $\langle,\rangle \sim$ and let $e_{1}, e_{2}$ be a pseudo-orthonormal basis of $\mathbb{L}^{2}$ such that

$$
\left\langle e_{1}, e_{1}\right\rangle^{\sim}=0=\left\langle e_{2}, e_{2}\right\rangle^{\sim} \text { and }\left\langle e_{1}, e_{2}\right\rangle^{\sim}=-1 .
$$

Given $x \in M^{n}$, define a symmetric bilinear form $\gamma: T_{x} M \times T_{x} M \rightarrow \mathbb{L}^{2}$ by

$$
\gamma(X, Y)=\langle X, Y\rangle e_{1}-L(X, Y) e_{2},
$$

where $L$ is the Schouten tensor of $M^{n}$. Endow $W=\mathbb{L}^{2} \oplus N_{f} M(x)$ with the Lorentzian inner product

$$
\left\langle\left\langle(\xi, \zeta),\left(\xi^{\prime}, \zeta^{\prime}\right)\right\rangle\right\rangle=-\left\langle\xi, \xi^{\prime}\right\rangle^{\sim}+\left\langle\zeta, \zeta^{\prime}\right\rangle
$$

and let $\beta: T_{x} M \times T_{x} M \rightarrow W$ be the symmetric bilinear form defined by

$$
\beta=\gamma \oplus \alpha^{f}(x) .
$$

It follows from the Gauss equation of $f$ and the vanishing of the Weyl tensor of $M^{n}$ that $\beta$ is a flat bilinear form. Since $\mathcal{N}(\beta)=\{0\}$, for $\mathcal{N}(\gamma)=\{0\}$, Lemmas 4.10 and 4.14 imply that either $\mathcal{S}(\beta)$ is a degenerate subspace of $W$ or $p=n-2$ and $\mathcal{S}(\beta)=W$.

In the latter case, since $\left\langle\beta,\left(e_{2}, 0\right)\right\rangle=\langle$,$\rangle is positive-definite, and in particular$ $0=\operatorname{dim} \mathcal{N}(\beta)=\operatorname{dim} T_{x} M-\operatorname{dim} W$, by Theorem 5.2 there exists a basis $X_{1}, \ldots, X_{n}$ of $T_{x} M$ that diagonalizes $\beta$, that is,

$$
\beta\left(X_{i}, X_{j}\right)=0, \quad 1 \leq i \neq j \leq n .
$$

In particular,

$$
\left\langle X_{i}, X_{j}\right\rangle=\left\langle\left\langle\beta\left(X_{i}, X_{j}\right),\left(e_{2}, 0\right)\right\rangle\right\rangle=0, \quad 1 \leq i \neq j \leq n
$$

that is, $X_{1}, \ldots, X_{n}$ is an orthonormal basis. Since $X_{1}, \ldots, X_{n}$ also diagonalizes $\alpha^{f}(x)$, it follows that $f$ has flat normal bundle at $x$.

Suppose now that $\mathcal{S}(\beta)$ is a degenerate subspace of $W$. By Lemma 4.21, and since $\mathcal{N}(\beta)=\{0\}$, there exist a nonzero light-like vector $e \in W$ and a symmetric bilinear form $\phi: T_{x} M \times T_{x} M \rightarrow \mathbb{R}$ such that

$$
\operatorname{dim} \mathcal{N}(\beta-\phi e) \geq n-p
$$

Write $e=\xi-\zeta$, where $\xi \in \mathbb{L}^{2}$ and $\zeta \in N_{f} M(x)$. From

$$
\begin{equation*}
\beta(n, Y)=\phi(n, Y) e \tag{16.8}
\end{equation*}
$$

for all $n \in \mathcal{N}(\beta-\phi e)$ and $Y \in T_{x} M$ we obtain

$$
\phi(n, Y)\left\langle\xi, e_{2}\right\rangle^{\sim}=-\langle n, Y\rangle .
$$

Thus $\left\langle\xi, e_{2}\right\rangle \neq 0$, and hence (16.8) yields

$$
\alpha^{f}(n, Y)=\frac{1}{\left\langle\xi, e_{2}\right\rangle^{\sim}}\langle n, Y\rangle \zeta .
$$

Therefore $\eta=\frac{1}{\left\langle\xi, e_{2}\right\rangle^{\sim}} \zeta$ is a principal normal of $f$ at $x$ with $\mathcal{N}(\beta-\phi e) \subset E_{\eta}(x)$.
Corollary 16.6. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}, n \geq 4$, be a compact conformally flat submanifold. If $p \leq n / 2-1$, then $M^{n}$ has the homotopy type of a $C W$-complex with no cells of dimension $p<r<n-p$. In particular, the homology groups of $M^{n}$ satisfy

$$
H_{r}(M ; G)=0, \quad p<r<n-p,
$$

for any coefficient group $G$.
Proof: The statement is a direct consequence of Theorems 1.23 and 16.5 , for the assumptions on $p$ and $n$ imply that $p \leq n-3$ and that $n-p>n / 2$.

If an isometric immersion $f: M^{n} \rightarrow \mathbb{R}^{n+p}, n \geq 4$, of a conformally flat manifold has flat normal bundle, which is always the case by Theorem 16.5 if $p=n-2$ and $f$ does not have any principal normal with multiplicity greater than one at any point, then one has the following additional information, which we state without proof.

Theorem 16.7. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}, n \geq 4$, be an isometric immersion of a conformally flat Riemannian manifold. Assume that $f$ has flat normal bundle and a constant number of principal normal vector fields. Then $f$ is locally holonomic and at most one of the principal normal vector fields has multiplicity greater than one.

For conformally flat Euclidean hypersurfaces of dimension $n \geq 4$, Theorem 16.5 immediately implies the direct statement of the following result.

Corollary 16.8. Any conformally flat hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}$, $n \geq 4$, has a principal curvature with multiplicity at least $n-1$ at every point.

Conversely, if a hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, carries a principal curvature with multiplicity at least $n-1$ at every point, then $M^{n}$ is conformally flat.

Proof: To prove the converse statement assume first that $n \geq 4$. By Theorem 16.3 , it suffices to show that $C=0$. In terms of the endomorphism $T$ associated with the Ricci tensor, the Weyl tensor can be written as

$$
C(X, Y)=R(X, Y)-\frac{1}{n-2}(T X \wedge Y+X \wedge T Y)+\frac{\operatorname{tr} T}{(n-1)(n-2)} X \wedge Y
$$

Using (3.7) we obtain

$$
\begin{aligned}
C(X, Y)= & A X \wedge A Y+\frac{1}{n-2}\left(A^{2} X \wedge Y+X \wedge A^{2} Y\right)-\frac{\operatorname{tr} A}{(n-2)}(A X \wedge Y+X \wedge A Y) \\
& +\frac{(\operatorname{tr} A)^{2}-\operatorname{tr} A^{2}}{(n-1)(n-2)} X \wedge Y
\end{aligned}
$$

Let $X_{1}, \ldots, X_{n}$ be an orthonormal basis of $T_{x} M$ such that $A X_{j}=\lambda X_{j}, 1 \leq j \leq n-1$, and $A X_{n}=\mu X_{n}$. Then

$$
\operatorname{tr} A=(n-1) \lambda+\mu \text { and } \operatorname{tr} A^{2}=(n-1) \lambda^{2}+\mu^{2},
$$

from which we conclude after a simple calculation that $C=0$.
Now suppose that $n=3$. We have to prove that $L$ is a Codazzi tensor. Let $\hat{L}$ be the endomorphism of $T M$ associated with $L$. It follows from (3.7) and (16.2) that $\hat{L}$ has two eigenvalues $\hat{\lambda}$ and $\hat{\mu}$, with the same eigendistributions $E_{\lambda}$ and $E_{\mu}$ as those of the eigenvalues $\lambda$ and $\mu$ of $A$, respectively, which are related to $\lambda$ and $\mu$ by

$$
\hat{\lambda}=\frac{1}{2}\left(c+\lambda^{2}\right) \text { and } \hat{\mu}=\frac{1}{2}\left(c-\lambda^{2}+2 \lambda \mu\right)
$$

Since $A$ is a Codazzi tensor, it follows from Exercise 1.18 that $\lambda$ is constant along $E_{\lambda}$ and that $E_{\lambda}$ and $E_{\mu}$ are umbilical distributions with mean curvature vector fields

$$
\eta=\frac{1}{\lambda-\mu} \operatorname{grad} \lambda \text { and } \zeta=\frac{1}{\mu-\lambda}(\operatorname{grad} \mu)_{E_{\lambda}}
$$

respectively. Clearly, also $\hat{\lambda}$ is constant along $E_{\hat{\lambda}}=E_{\lambda}$. Since

$$
\hat{\lambda}-\hat{\mu}=\lambda(\lambda-\mu),
$$

$$
\operatorname{grad} \hat{\lambda}=\lambda \operatorname{grad} \lambda \text { and } \operatorname{grad} \hat{\mu}=(\mu-\lambda) \operatorname{grad} \lambda+\lambda \operatorname{grad} \mu
$$

we have

$$
\frac{1}{\hat{\lambda}-\hat{\mu}} \operatorname{grad} \hat{\lambda}=\frac{1}{\lambda-\mu} \operatorname{grad} \lambda
$$

and

$$
\frac{1}{\hat{\mu}-\hat{\lambda}} \operatorname{grad} \hat{\mu}=\frac{1}{\lambda} \operatorname{grad} \lambda+\frac{1}{\mu-\lambda} \operatorname{grad} \mu
$$

Hence

$$
\frac{1}{\hat{\mu}-\hat{\lambda}}(\operatorname{grad} \hat{\mu})_{E_{\hat{\lambda}}}=\frac{1}{\mu-\lambda}(\operatorname{grad} \mu)_{E_{\lambda}}
$$

We conclude from Exercise 1.18 that $\hat{L}$ is also a Codazzi tensor.
In particular, we obtain from Theorem 9.6 the following parametric description of the conformally flat hypersurfaces of dimension $n \geq 4$, and also of those with dimension $n=3$ that carry a principal curvature of multiplicity two.

Corollary 16.9. If $\gamma: I \rightarrow \mathbb{R}^{n+1}, n \geq 3$, is a unit-speed curve and $r \in C^{\infty}(I)$ is positive with $\left\|r^{\prime}\right\|<1$, then the map $\phi: N_{\gamma}^{1} I \rightarrow \mathbb{R}^{n+1}$, defined on the unit normal bundle of $\gamma$ by

$$
\begin{equation*}
\phi(s, u)=\gamma(s)-r(s) r^{\prime}(s) \gamma^{\prime}(s)-r(s) \sqrt{1-\left\|r^{\prime}(s)\right\|^{2}} u \tag{16.9}
\end{equation*}
$$

parametrizes, on the open subset of regular points, a conformally flat hypersurface.
Conversely, let $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, be an orientable conformally flat hypersurface. If $n=3$, assume further that $f$ carries a principal curvature of constant multiplicity 2. Then there exist a unit-speed curve $\gamma: I \rightarrow \mathbb{R}^{n+1}$, a positive $r \in C^{\infty}(I)$ with $\left\|r^{\prime}\right\|<1$ and a diffeomorphism $\theta: N_{\gamma}^{1} I \rightarrow M^{n}$ such that $f \circ \theta$ is given by 16.9.

### 16.3.2 A nonparametric description

The following result sheds light on the geometrical origin of the principal normal vector field given by Theorem 16.5 of a conformally flat submanifold $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ of dimension $n \geq 4$ and codimension $p \leq n-2$ that is free of points where the normal curvature tensor of $f$ vanishes if $p=n-2$.

Theorem 16.10. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}, n \geq 4, p \leq n-2$, be an isometric immersion of a conformally flat manifold free of flat points. If $p=n-2$, assume further that $f$ does not have flat normal bundle at any point of $M^{n}$. Then each point of an open dense subset $\mathcal{V} \subset M^{n}$ has an open neighborhood $V_{0} \subset \mathcal{V}$ for which there exist a Riemannian manifold $N^{n+1}$ that admits both an isometric immersion $F: N^{n+1} \rightarrow \mathbb{R}^{n+p}$ and an isometric embedding $\hat{F}: N^{n+1} \rightarrow \mathbb{L}^{n+2}$, and an isometry $j: V_{0} \rightarrow \hat{F}^{-1}\left(\hat{F}(N) \cap \mathbb{V}^{n+1}\right)$ such that $f=F \circ j$.

Proof: By Proposition 9.9, there exists locally an isometric immersion $\hat{f}: M^{n} \rightarrow \mathbb{V}^{n+1}$. It was shown in the proof of Theorem 16.1 that the second fundamental form of $\hat{f}$, as an isometric immersion into $\mathbb{L}^{n+2}$, is given at any $x \in M^{n}$ by

$$
\alpha^{\hat{f}}(X, Y)=\langle X, Y\rangle \zeta-L(X, Y) \hat{f}(x)
$$

where $\hat{f}(x)$ stands for the position vector at $x, L$ is the Schouten tensor of $M^{n}$ and $\zeta, \hat{f}$ is a pseudo-orthonormal frame of $N_{\hat{f}} M$ with $\langle\zeta, \zeta\rangle=0$ and $\langle\zeta, \hat{f}\rangle=-1$. Notice that the second fundamental form $\alpha^{\hat{f}}(x)$ at each $x \in M^{n}$ is precisely the bilinear form $\gamma$ defined in the proof of Theorem 16.5, with $e_{1}=\zeta$ and $e_{2}=\hat{f}(x)$ in terms of the notations therein. It follows from the proof of that result that, under the assumptions, there exist $\xi \in N_{\hat{f}} M(x)$ and $\eta \in N_{f} M(x)$, with $\|\xi\|=\|\eta\|$, and a tangent subspace $\mathcal{U}(x)=E_{\eta}^{f}(x)$ with dimension at least $n-p$ such that

$$
\begin{equation*}
\alpha^{f}(U, Y)=\langle U, Y\rangle \eta \text { and } \alpha^{\hat{f}}(U, Y)=\langle U, Y\rangle \xi \tag{16.10}
\end{equation*}
$$

for all $U \in U(x)$ and $Y \in T_{x} M$. Notice that the assumption that $M^{n}$ is free of flat points implies that $\eta$, hence $\xi$, is nonzero. In fact, if $\eta=0$, then

$$
\begin{aligned}
K(U, Y) & =\left\langle\alpha^{f}(U, U), \alpha^{f}(Y, Y)\right\rangle-\left\|\alpha^{f}(U, Y)\right\|^{2} \\
& =0
\end{aligned}
$$

for all $U \in U(x)$ and $Y \in T_{x} M$. If $X, Y \in \mathcal{U}^{\perp}(x)$ and $U, V \in U(x)$ are orthogonal vectors, from

$$
K(X, Y)+K(U, V)=K(X, V)+K(U, Y)
$$

(see Exercise 16.4) it follows that $x$ is a flat point, a contradiction.
We now show that

$$
\begin{equation*}
A_{\eta}^{f}=A_{\xi}^{\hat{f}} . \tag{16.11}
\end{equation*}
$$

That

$$
\left\langle A_{\eta}^{f} Y, U\right\rangle=\left\langle A_{\xi}^{\hat{f}} Y, U\right\rangle
$$

for all $U \in \mathcal{U}(x)$ and $Y \in T_{x} M$ follows immediately from 16.10). Now, if $X, Y \in \mathcal{U}^{\perp}(x)$ are orthogonal, choosing $U \in \mathcal{U}(x)$ the Gauss equations for $f$ and $\hat{f}$ yield

$$
\begin{aligned}
\left\langle\alpha^{f}(X, Y), \eta\right\rangle & =\left\langle\alpha^{f}(X, Y), \alpha^{f}(U, U)\right\rangle-\left\langle\alpha^{f}(X, U), \alpha^{f}(Y, U)\right\rangle \\
& =\left\langle\alpha^{\hat{f}}(X, Y), \alpha^{\hat{f}}(U, U)\right\rangle-\left\langle\alpha^{\hat{f}}(X, U), \alpha^{\hat{f}}(Y, U)\right\rangle \\
& =\left\langle\alpha^{f}(X, Y), \xi\right\rangle,
\end{aligned}
$$

and the proof of 16.11 is completed.
Now let $\mathcal{V} \subset M^{n}$ be the open dense subset where the tangent vector subspaces $\mathcal{U}(x)$ have locally constant dimension, and hence where one has smooth principal normal vector fields $\eta \in \Gamma\left(N_{f} \mathcal{V}\right)$ and $\xi \in \Gamma\left(N_{\hat{f}} \mathcal{V}\right)$ such that $E_{\eta}^{f}(x)=\mathcal{U}(x)=E_{\xi}^{\hat{f}}(x)$ for all $x \in \mathcal{V}$.

Given $x \in \mathcal{V}$, let $V_{0}$ be an open connected neighborhood of $x$ and let $L$ and $\hat{L}$ be the line subbundles of $N_{f} V_{0}$ and $N_{\hat{f}} V_{0}$ spanned by $\eta$ and $\xi$, respectively. Then condition $\left(\mathcal{C}_{1}\right)$ in (12.9) is trivially satisfied, for $L$ and $\hat{L}$ have rank one, and $\left(\mathcal{C}_{2}\right)$ holds in view of part (iii) of Proposition 1.22. It follows from Proposition 12.5 that there exist ruled isometric immersions $F: N^{n+1} \rightarrow \mathbb{R}^{n+p}$ and $\hat{F}: N^{n+1} \rightarrow \mathbb{L}^{n+2}$ and an isometric inclusion $j: V_{0} \rightarrow N^{n+1}$ such that

$$
f=F \circ j \text { and } \hat{f}=\hat{F} \circ j
$$

We can assume that $V_{0}$ and $N^{n+1}$ are chosen sufficiently small so that $\hat{F}$ is an isometric embedding, in which case, taking into account that $\hat{f}$ takes values in $\mathbb{V}^{n+1}, j$ is an isometry of $V_{0}$ onto $\hat{F}^{-1}\left(\hat{F}(N) \cap \mathbb{V}^{n+1}\right)$.

Remarks 16.11. (i) For $p=1$, the map $F$ in Theorem 16.10 becomes a local isometry. Therefore, in this case the theorem roughly says that any conformally flat hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 4$ is locally obtained as the intersection of a flat hypersurface $\hat{F}: U \subset \mathbb{R}^{n+1} \rightarrow \mathbb{L}^{n+2}$ with the light-cone $\mathbb{V}^{n+1} \subset \mathbb{L}^{n+2}$. This remains true for a conformally flat hypersurface $f: M^{3} \rightarrow \mathbb{R}^{4}$ that carries a principal curvature of constant multiplicity two.
(ii) The isometric extensions $F$ and $\hat{F}$ of $f$ and $\hat{f}$ have been constructed so as to replace the leaves of the distribution $\mathcal{U}$ by the leaves of the common relative nullity distributions of $F$ and $\hat{F}$. In Exercise 16.8 , the reader is asked to prove that, for any pair of isometric immersions $F: N^{n+1} \rightarrow \mathbb{R}^{n+p}, p \leq n$, and $\hat{F}: N^{n+1} \rightarrow \mathbb{L}^{n+2}$, there exists at any $x \in N^{n+1}$ a subspace $\Delta(x)$ of $T_{x} N$ with dimension at least $n-p+1$ that is contained in the relative nullity subspaces of both $F$ and $\hat{F}$ at $x$.

### 16.4 A parametrization for codimension two

In this section we provide an explicit parametrization of the conformally flat submanifolds $f: M^{n} \rightarrow \mathbb{R}^{n+2}$ of dimension $n \geq 5$ that are called generic. The latter assumption is needed to exclude compositions of conformally flat hypersurfaces in $\mathbb{R}^{n+1}$ with local isometric immersions of $\mathbb{R}^{n+1}$ into $\mathbb{R}^{n+2}$.

A conformally flat submanifold $f: M^{n} \rightarrow \mathbb{R}^{n+2}, n \geq 5$, is said to be generic if it carries a nowhere vanishing principal normal vector field of multiplicity $n-2$. It follows from part (iii) of Exercise 16.4 that $M^{n}$ cannot have flat points.

Proposition 16.12. Let $f: M^{n} \rightarrow \mathbb{R}^{n+2}, n \geq 5$, be a conformally flat submanifold free of flat points. Then $f$ is locally either generic or a composition along an open dense subset of $M^{n}$.

Proof: By Theorem 16.5 there is an open dense subset of $M^{n}$ along any connected component of which $f$ is either generic or carries a unit principal normal vector field $\eta$ of multiplicity $\ell \geq n-1$. In the latter case, it follows from part (iii) of Exercise 16.4
that $\operatorname{rank} A_{\xi} \leq 1$ where $\xi$ is a unit vector field orthogonal to $\eta$. On any open subset where $A_{\xi}$ has constant rank, it is easy to verify that $A_{\eta}$ satisfies the Gauss and Codazzi equations for a hypersurface in $\mathbb{R}^{n+1}$. The proof now follows from Corollary 12.27 .

Now let $g: L^{2} \rightarrow \mathbb{S}^{n+1}$ be a surface of first or second species with conjugate coordinates $(u, v)$ and set $h=i \circ g: L^{2} \rightarrow \mathbb{R}^{n+2}$. Then

$$
h_{u v}-\Gamma^{1} h_{u}-\Gamma^{2} h_{v}+F h=0,
$$

where $F=\langle\partial u, \partial v\rangle$. Let $T^{*}$ denote the adjoint of $T \in \Gamma(\operatorname{End}(T L))$ given by

$$
T \partial u=\frac{1}{\theta} \partial u \text { and } T \partial v=-\theta \partial v
$$

where $\theta=\sqrt{-\tau}$ and $\tau$ is a negative solution of system 11.36. For $\beta: L^{2} \rightarrow \mathbb{R}^{n+2}$, consider the system of first order

$$
\left\{\begin{array}{l}
\beta_{u}=\theta \rho h_{u}-\frac{\rho_{u}}{\theta} h  \tag{16.12}\\
\beta_{v}=-\frac{\rho}{\theta} h_{v}+\theta \rho_{v} h,
\end{array}\right.
$$

where $\rho \in C^{\infty}(L)$. It is easy to see that the integrability condition of the system is satisfied if and only if $\rho$ is a solution of the differential equation

$$
\rho_{u v}+\theta^{2} \Gamma^{2} \rho_{v}+\frac{1}{\theta} \Gamma^{1} \rho_{u}+\rho F=0
$$

Hence, choosing such a $\rho$, system (16.12) has a unique solution up to translations.
One has the following procedure to generate parametrically all generic conformally flat submanifolds $M^{n}$ of $\mathbb{R}^{n+2}, n \geq 5$, which we state without proof.

Theorem 16.13. The map $\psi: N_{h}^{1} L \rightarrow \mathbb{R}^{n+2}$ defined on the unit normal bundle of $h$ by

$$
\psi(y, w)=\beta(y)-T^{*} \operatorname{grad} \rho(y)+\sqrt{\rho^{2}(y)-\left\|T^{*} \operatorname{grad} \rho(y)\right\|^{2}} w
$$

parametrizes a generic conformally flat submanifold $M^{n}$ of $\mathbb{R}^{n+2}$.
Conversely, any generic conformally flat submanifold $f: M^{n} \rightarrow \mathbb{R}^{n+2}, n \geq 5$, can be locally parametrized this way.

### 16.5 Conformally flat hypersurfaces of dimension three

According to Corollary 16.8, a conformally flat hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ of dimension $n \geq 4$ always has a principal curvature with multiplicity at least $n-1$ at any point. This is no longer true for $n=3$, as shown by the following examples.

Examples 16.14. (i) Let $g: M^{2} \rightarrow \mathbb{R}^{3}$ be a umbilic-free surface with nonzero constant Gaussian curvature and let $f=g \times \mathrm{id}: M^{3}=M^{2} \times \mathbb{R} \rightarrow \mathbb{R}^{3} \times \mathbb{R}=\mathbb{R}^{4}$ be the cylinder over $g$. Then $f$ has three distinct principal curvatures and $M^{3}=M^{2} \times \mathbb{R}$ is conformally flat by part (iii) of Examples 16.2.
(ii) Let $g: M^{2} \rightarrow \mathbb{S}^{3}$ be a umbilic-free surface with constant Gaussian curvature $c \neq 1$ and let $f: \mathbb{R}_{+} \times M^{2} \rightarrow \mathbb{R}^{4}$ be the cone over $g$ (see Exercise 1.5). Again, $f$ has three distinct principal curvatures and the metric induced on $\mathbb{R}_{+} \times M^{2}$ by $f$ is a warped product metric

$$
d s^{2}=d t^{2}+t^{2} d \sigma^{2}
$$

where $d \sigma^{2}$ is the constant curvature metric of $M^{2}$. Thus $d s^{2}$ is conformal to the product metric $d \tilde{t}^{2}+d \sigma^{2}$ on $\mathbb{R}_{+} \times M^{2}, \tilde{t}=-t^{-1}$, and therefore $f$ is a conformally flat hypersurface.
(iii) Let $g: M^{2} \rightarrow \mathbb{H}^{3}$ be an umbilic-free surface with constant Gaussian curvature $c \neq-1$. Consider the half-space model $\mathbb{R}_{+}^{3}$ of $\mathbb{H}^{3}$, and regard $g$ as a surface into $\mathbb{R}_{+}^{3}$. Now let $f: \mathbb{S}^{1} \times M^{2} \rightarrow \mathbb{R}^{4}$ be the rotation hypersurface having $g$ as profile. Then $f$ has also three distinct principal curvatures and the metric induced on $\mathbb{S}^{1} \times M^{2}$ by $f$ is again a warped product metric

$$
d s^{2}=\rho^{2} d t^{2}+d \tilde{\sigma}^{2}
$$

where $d \tilde{\sigma}^{2}$ is the metric on $M^{2}$ induced by $g$ from the Euclidean metric on $\mathbb{R}_{+}^{3}$. Thus

$$
\begin{aligned}
d s^{2} & =\rho^{2}\left(d t^{2}+\frac{1}{\rho^{2}} d \tilde{\sigma}^{2}\right) \\
& =\rho^{2}\left(d t^{2}+d \sigma^{2}\right),
\end{aligned}
$$

where $d \sigma^{2}$ is the constant curvature metric on $M^{2}$ induced by $g$ from the hyperbolic metric on $\mathbb{R}_{+}^{3}$. Thus $f$ is a conformally flat hypersurface.

In order to give a characterization of conformally flat hypersurfaces $f: M^{3} \rightarrow \mathbb{R}^{4}$ with three distinct principal curvatures as holonomic hypersurfaces satisfying some additional conditions, we first prove the following result.

Proposition 16.15. Let $f: M^{3} \rightarrow \mathbb{R}^{4}$ be an isometric immersion with three distinct principal curvatures $\lambda_{1}<\lambda_{2}<\lambda_{3}$. Let $e_{1}, e_{2}, e_{3}$ be a correspondent orthonormal frame of principal directions and let $\omega_{1}, \omega_{2}, \omega_{3}$ be its dual frame. The following assertions are equivalent:
(i) $M^{3}$ is conformally flat.
(ii) The relations

$$
\begin{equation*}
\left\langle\nabla_{e_{i}} e_{j}, e_{k}\right\rangle=0 \tag{16.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\lambda_{j}-\lambda_{k}\right) e_{i}\left(\lambda_{i}\right)+\left(\lambda_{i}-\lambda_{k}\right) e_{i}\left(\lambda_{j}\right)+\left(\lambda_{j}-\lambda_{i}\right) e_{i}\left(\lambda_{k}\right)=0 \tag{16.14}
\end{equation*}
$$

hold for all $1 \leq i \neq j \neq k \neq i \leq 3$.
(iii) The one-forms $\gamma_{j}$ defined by

$$
\begin{equation*}
\gamma_{j}=\sqrt{\delta_{j}\left(\lambda_{j}-\lambda_{i}\right)\left(\lambda_{j}-\lambda_{k}\right)} \omega_{j}, \quad 1 \leq j \neq i \neq k \neq j \leq 3 \tag{16.15}
\end{equation*}
$$

where $\left(\delta_{1}, \delta_{2}, \delta_{3}\right)=(1,-1,1)$, are closed.
Proof: If $M^{3}$ is conformally flat, then the Schouten tensor $L$ is a Codazzi tensor by Theorem 16.3. Denoting by $\mu_{1}, \mu_{2}, \mu_{3}$ the eigenvalues of $L$, then the Codazzi equations for $f$ and $L$ are equivalent, respectively, to the sets of equations

$$
\begin{align*}
e_{i}\left(\lambda_{j}\right) & =\left(\lambda_{i}-\lambda_{j}\right)\left\langle\nabla_{e_{j}} e_{i}, e_{j}\right\rangle, \quad i \neq j,  \tag{16.16}\\
\left(\lambda_{j}-\lambda_{k}\right)\left\langle\nabla_{e_{i}} e_{j}, e_{k}\right\rangle & =\left(\lambda_{i}-\lambda_{k}\right)\left\langle\nabla_{e_{j}} e_{i}, e_{k}\right\rangle, \quad i \neq j \neq k, \tag{16.17}
\end{align*}
$$

and

$$
\begin{align*}
e_{i}\left(\mu_{j}\right) & =\left(\mu_{i}-\mu_{j}\right)\left\langle\nabla_{e_{j}} e_{i}, e_{j}\right\rangle, \quad i \neq j,  \tag{16.18}\\
\left(\mu_{j}-\mu_{k}\right)\left\langle\nabla_{e_{i}} e_{j}, e_{k}\right\rangle & =\left(\mu_{i}-\mu_{k}\right)\left\langle\nabla_{e_{j}} e_{i}, e_{k}\right\rangle, \quad i \neq j \neq k . \tag{16.19}
\end{align*}
$$

From (16.1) we have

$$
\begin{equation*}
2 \mu_{j}=\lambda_{i} \lambda_{j}+\lambda_{k} \lambda_{j}-\lambda_{i} \lambda_{k}, \quad 1 \leq j \leq 3 . \tag{16.20}
\end{equation*}
$$

Substituting (16.20) into (16.19) and using 16.17) yield

$$
\left(\lambda_{i}-\lambda_{j}\right)\left(\lambda_{j}-\lambda_{k}\right)\left\langle\nabla_{e_{i}} e_{j}, e_{k}\right\rangle=0, \quad i \neq j \neq k .
$$

Since $\lambda_{1}, \lambda_{2}$ and $\lambda_{3}$ are pairwise distinct, then (16.13) follows.
Differentiating 16.20) with respect to $e_{i}$ gives

$$
\begin{equation*}
2 e_{i}\left(\mu_{j}\right)=\left(\lambda_{i}+\lambda_{k}\right) e_{i}\left(\lambda_{j}\right)+\left(\lambda_{j}-\lambda_{k}\right) e_{i}\left(\lambda_{i}\right)+\left(\lambda_{j}-\lambda_{i}\right) e_{i}\left(\lambda_{k}\right) \tag{16.21}
\end{equation*}
$$

On the other hand, from (16.16), 16.18) and 16.20 we obtain

$$
\begin{equation*}
e_{i}\left(\mu_{j}\right)=\lambda_{k} e_{i}\left(\lambda_{j}\right) \tag{16.22}
\end{equation*}
$$

Hence (16.14) follows from (16.21) and $\sqrt{16.22)}$. This completes the proof that $(i)$ implies (ii).

To prove that (ii) implies (i), assume that (16.13) and (16.14) hold. In order to show that $M^{3}$ is conformally flat, again by Theorem 16.3 it suffices to prove that $L$ is a Codazzi tensor, that is, that (16.18) and (16.19) are satisfied. The latter is clear from (16.13). In view of (16.16), the former is equivalent to

$$
\begin{equation*}
e_{i}\left(\mu_{j}\right)\left(\lambda_{i}-\lambda_{j}\right)=e_{i}\left(\lambda_{j}\right)\left(\mu_{i}-\mu_{j}\right), \quad i \neq j . \tag{16.23}
\end{equation*}
$$

It follows from 16.20 that

$$
\mu_{i}-\mu_{j}=\lambda_{k}\left(\lambda_{i}-\lambda_{j}\right)
$$

Since (16.14) and (16.21) give (16.22), then (16.23) holds.

We now prove the equivalence between (ii) and (iii), that is, that (16.13) and 16.14 are precisely the conditions for the one-forms $\gamma_{j}, 1 \leq j \leq 3$, to be closed. Set

$$
x_{j}=\sqrt{\delta_{j}\left(\lambda_{j}-\lambda_{i}\right)\left(\lambda_{j}-\lambda_{k}\right)}, \quad 1 \leq j \leq 3
$$

so that $\gamma_{j}=x_{j} \omega_{j}$. We have

$$
\begin{align*}
d \gamma_{j}\left(e_{i}, e_{k}\right) & =e_{i} \gamma_{j}\left(e_{k}\right)-e_{k} \gamma_{j}\left(e_{i}\right)-\gamma_{j}\left(\left[e_{i}, e_{k}\right]\right) \\
& =x_{j}\left(\left\langle\nabla_{e_{k}} e_{i}, e_{j}\right\rangle-\left\langle\nabla_{e_{i}} e_{k}, e_{j}\right\rangle\right) \tag{16.24}
\end{align*}
$$

if $1 \leq i \neq j \neq k \neq i \leq 3$. Therefore, if (16.13) holds then

$$
\begin{equation*}
d \gamma_{j}\left(e_{i}, e_{k}\right)=0 \tag{16.25}
\end{equation*}
$$

for all $1 \leq i \neq j \neq k \neq i \leq 3$. Conversely, if (16.25) is satisfied, then

$$
\begin{aligned}
\left\langle\nabla_{e_{i}} e_{j}, e_{k}\right\rangle & =-\left\langle\nabla_{e_{i}} e_{k}, e_{j}\right\rangle=-\left\langle\nabla_{e_{k}} e_{i}, e_{j}\right\rangle=\left\langle\nabla_{e_{k}} e_{j}, e_{i}\right\rangle=\left\langle\nabla_{e_{j}} e_{k}, e_{i}\right\rangle=-\left\langle\nabla_{e_{j}} e_{i}, e_{k}\right\rangle \\
& =-\left\langle\nabla_{e_{i}} e_{j}, e_{k}\right\rangle
\end{aligned}
$$

by 16.24 , hence 16.13 holds.
On the other hand, using (16.16) we obtain

$$
\begin{aligned}
d \gamma_{j}\left(e_{i}, e_{j}\right) & =e_{i} \gamma_{j}\left(e_{j}\right)-e_{j} \gamma_{j}\left(e_{i}\right)-\gamma_{j}\left(\left[e_{i}, e_{j}\right]\right) \\
& =e_{i}\left(x_{j}\right)+x_{j}\left\langle\nabla_{e_{j}} e_{i}, e_{j}\right\rangle \\
& =e_{i}\left(x_{j}\right)+\frac{x_{j}}{\lambda_{i}-\lambda_{j}} e_{i}\left(\lambda_{j}\right) .
\end{aligned}
$$

Thus $\gamma_{j}$ is closed if and only if

$$
e_{i}\left(x_{j}\right)=\frac{x_{j}}{\lambda_{j}-\lambda_{i}} e_{i}\left(\lambda_{j}\right), \quad 1 \leq i \neq j \leq 3
$$

or equivalently, if and only if

$$
e_{i}\left(\delta_{j} x_{j}^{2}\right)\left(\lambda_{j}-\lambda_{i}\right)=2 \delta_{j} x_{j}^{2} e_{i}\left(\lambda_{j}\right), \quad 1 \leq i \neq j \leq 3,
$$

which is easily checked to be the same as (16.14).
Theorem 16.16. Let $f: M^{3} \rightarrow \mathbb{R}^{4}$ be a holonomic hypersurface whose associated pair $(v, V)$ satisfies

$$
\begin{equation*}
\langle v, v\rangle=0,\langle V, v\rangle=0 \text { and }\langle V, V\rangle=1 \tag{16.26}
\end{equation*}
$$

with respect to an inner product of Lorentzian signature. Then $M^{3}$ is conformally flat and $f$ has three distinct principal curvatures.

Conversely, any conformally flat hypersurface $f: M^{3} \rightarrow \mathbb{R}^{4}$ with three distinct principal curvatures is locally a holonomic hypersurface whose associated pair ( $v, V$ ) satisfies (16.26).

Proof: Let $f: M^{3} \rightarrow \mathbb{R}^{4}$ be a holonomic hypersurface whose associated pair $(v, V)$ satisfies (16.26). We may assume that

$$
\begin{equation*}
\sum_{i=1}^{3} \delta_{i} v_{i}^{2}=0, \quad \sum_{i=1}^{3} \delta_{i} v_{i} V_{i}=0 \text { and } \sum_{i=1}^{3} \delta_{i} V_{i}^{2}=1 \tag{16.27}
\end{equation*}
$$

where $\left(\delta_{1}, \delta_{2}, \delta_{3}\right)=(1,-1,1)$. That (16.13) holds follows from 1.22). We now prove that the principal curvatures $\lambda_{i}=V_{i} / v_{i}$ of $f, 1 \leq i \leq 3$, satisfy (16.14). We can write the left-hand side of (16.14) as

$$
\begin{align*}
\frac{1}{v_{i}^{2}}\left(\frac{V_{j}}{v_{j}}-\frac{V_{k}}{v_{k}}\right)\left(v_{i} \frac{\partial V_{i}}{\partial u_{i}}-V_{i} \frac{\partial v_{i}}{\partial u_{i}}\right) & +\frac{1}{v_{j}^{2}}\left(\frac{V_{i}}{v_{i}}-\frac{V_{k}}{v_{k}}\right)\left(v_{j} \frac{\partial V_{j}}{\partial u_{i}}-V_{j} \frac{\partial v_{j}}{\partial u_{i}}\right) \\
& +\frac{1}{v_{k}^{2}}\left(\frac{V_{j}}{v_{j}}-\frac{V_{i}}{v_{i}}\right)\left(v_{k} \frac{\partial V_{k}}{\partial u_{i}}-V_{k} \frac{\partial v_{k}}{\partial u_{i}}\right) . \tag{16.28}
\end{align*}
$$

Differentiating (16.26) we obtain

$$
\begin{equation*}
\delta_{i} \partial v_{i} / \partial u_{i}+\delta_{j} h_{i j} v_{j}+\delta_{k} h_{i k} v_{k}=0 \tag{16.29}
\end{equation*}
$$

and

$$
\begin{equation*}
\delta_{i} \partial V_{i} / \partial u_{i}+\delta_{j} h_{i j} V_{j}+\delta_{k} h_{i k} V_{k}=0, \quad 1 \leq i \neq j \neq k \neq i \leq 3 . \tag{16.30}
\end{equation*}
$$

Substituting (16.29) and (16.30) into (16.28) and using (i) and (iv) of (1.26), it follows that (16.28) vanishes if and only if

$$
\begin{aligned}
& h_{i j} v_{k}\left(v_{j} V_{i}-v_{i} V_{j}\right)\left(\delta_{j} v_{j}\left(v_{k} V_{j}-v_{j} V_{k}\right)+\right. \\
&\left.-\delta_{i} v_{i}\left(v_{k} V_{i}-v_{i} V_{k}\right)\right) \\
&-h_{i k} v_{j}\left(v_{k} V_{i}-v_{i} V_{k}\right)\left(\delta_{k} v_{k}\left(v_{j} V_{k}-v_{k} V_{j}\right)+\delta_{i} v_{i}\left(v_{j} V_{i}-v_{i} V_{j}\right)\right)=0 .
\end{aligned}
$$

Thus it suffices to observe that the first two relations in (16.26) give

$$
\begin{aligned}
\delta_{k} v_{k}\left(v_{j} V_{k}-v_{k} V_{j}\right)+\delta_{i} v_{i}\left(v_{j} V_{i}-v_{i} V_{j}\right) & =v_{j}\left(\delta_{i} v_{i} V_{i}+\delta_{j} v_{j} V_{j}+\delta_{k} v_{k} V_{k}\right) \\
& =0
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_{j} v_{j}\left(v_{k} V_{j}-v_{j} V_{k}\right)+\delta_{i} v_{i}\left(v_{k} V_{i}-v_{i} V_{k}\right) & =v_{k}\left(\delta_{i} v_{i} V_{i}+\delta_{j} v_{j} V_{j}+\delta_{k} v_{k} V_{k}\right) \\
& =0 .
\end{aligned}
$$

It remains to show that the principal curvatures of $f$ are pairwise distinct. Since $v=\left(v_{1}, v_{2}, v_{3}\right)$ is a null vector and $V=\left(V_{1}, V_{2}, V_{3}\right)$ is a unit space-like vector orthogonal to $v$, we may write

$$
V=\frac{\rho}{v_{2}} v+\frac{\lambda}{v_{2}}\left(-v_{3}, 0, v_{1}\right)
$$

where $\lambda= \pm 1$ and $\rho \in C^{\infty}(M)$. This is equivalent to

$$
V_{1}=\frac{1}{v_{2}}\left(V_{2} v_{1}-\lambda v_{3}\right) \text { and } V_{3}=\frac{1}{v_{2}}\left(V_{2} v_{3}+\lambda v_{1}\right) .
$$

In particular,

$$
V_{i} v_{j}-V_{j} v_{i}=-\lambda v_{k}, \quad 1 \leq i<j \leq 3 \text { and } k \notin\{i, j\}
$$

hence $\lambda_{i}-\lambda_{j} \neq 0$ for $1 \leq i \neq j \leq 3$.
Conversely, assume that $f: M^{3} \rightarrow \mathbb{R}^{4}$ is a conformally flat hypersurface with three distinct principal curvatures $\lambda_{1}<\lambda_{2}<\lambda_{3}$. Let $\left\{e_{1}, e_{2}, e_{3}\right\}$ be a correspondent orthonormal frame of principal directions and let $\omega_{1}, \omega_{2}, \omega_{3}$ be its dual frame. By Proposition 16.15, the one-forms $\gamma_{j}, 1 \leq j \leq 3$, given by (16.15), are closed. Thus each point $x \in M^{3}$ has an open neighborhood $V$ where one can find functions $u_{j} \in C^{\infty}(V)$, $1 \leq j \leq 3$, such that $d u_{j}=\gamma_{j}$. Choosing $V$ small enough, $\Phi=\left(u_{1}, u_{2}, u_{3}\right)$ is a diffeomorphism of $V$ onto an open subset $U \subset \mathbb{R}^{3}$, that is, $u_{1}, u_{2}, u_{3}$ define local coordinates on $V$. From

$$
\delta_{i j}=d u_{j}\left(\partial / \partial u_{i}\right)=x_{j} \omega_{j}\left(\partial / \partial u_{i}\right)
$$

it follows that $\partial / \partial u_{j}=v_{j} e_{j}, 1 \leq j \leq 3$, with

$$
v_{j}=\frac{1}{\sqrt{\delta_{j}\left(\lambda_{j}-\lambda_{i}\right)\left(\lambda_{j}-\lambda_{k}\right)}}
$$

Now notice that

$$
\begin{aligned}
\sum_{j=1}^{3} \delta_{j} v_{j}^{2} & =\sum_{i, k \neq j=1}^{3} \frac{1}{\left(\lambda_{j}-\lambda_{i}\right)\left(\lambda_{j}-\lambda_{k}\right)} \\
& =0
\end{aligned}
$$

that

$$
\begin{aligned}
\sum_{j=1}^{3} \delta_{j} v_{j} V_{j} & =\sum_{j=1}^{3} \delta_{j} \lambda_{j} v_{j}^{2} \\
& =\sum_{i, k \neq j=1}^{3} \frac{\lambda_{j}}{\left(\lambda_{j}-\lambda_{i}\right)\left(\lambda_{j}-\lambda_{k}\right)} \\
& =0
\end{aligned}
$$

and that

$$
\begin{aligned}
\sum_{j=1}^{3} \delta_{j} V_{j}^{2} & =\sum_{j=1}^{3} \delta_{j} \lambda_{j}^{2} v_{j}^{2} \\
& =\sum_{i, k \neq j=1}^{3} \frac{\lambda_{j}^{2}}{\left(\lambda_{j}-\lambda_{i}\right)\left(\lambda_{j}-\lambda_{k}\right)} \\
& =1
\end{aligned}
$$

It follows that $(v, V)$ satisfies 16.26).

### 16.6 Notes

The classical characterization of conformally flat manifolds in terms of the Weyl and Schouten tensors in Theorem 16.3 is due to Schouten [313]. The fact that conformally flat manifolds are precisely those Riemannian manifolds that admit locally (globally, if simply connected) an isometric immersion with codimension one into the light-cone of Lorentzian space, and that this yields Schouten's characterization of conformally flat manifolds, was shown by Asperti-Dajczer [21], although the first statement goes back to Brinkmann [42]. Theorem 16.4 on global conformal immersions into $\mathbb{S}^{n}$ of an $n$-dimensional simply connected conformally flat manifold is due to Kuiper [228]. A classification of conformally flat warped product manifolds with arbitrarily many factors, extending the classification in part (iv) of Examples 16.2 for the case of warped products with two factors, was independently obtained by Tojeiro [334] and Brozos-García Río-Vázquez [43].

The assertion in part ( $i$ ) of Theorem 16.5 on the existence of principal normal vectors of multiplicity at least $n-p$ at any point of a Euclidean conformally flat submanifold of dimension $n \geq 4$ and codimension $p \leq n-3$ was proved by Moore [256], where he also derived Corollary 16.6 on the topology of such submanifolds. The assertion in part (ii) of Theorem 16.5 does not seem to be part of the literature.

Corollary 16.8 on conformally flat hypersurfaces was already known to Cartan [65], who proved with his own methods that a conformally flat Euclidean hypersurface of dimension $n \geq 4$ is locally an envelope of a one-parameter family of hyperspheres. For a class of Euclidean hypersurfaces that contain the conformally flat ones, called almost conformally flat hypersurfaces, see Onti-Vlachos [281]. Restrictions on the topology of almost conformally flat Euclidean submanifolds were given by Onti-Vlachos [282]. Theorem 16.7 on conformally flat submanifolds with flat normal bundles is due to Dajczer-Onti-Vlachos [124], generalizing previous results by Donaldson-Terng [164].

The explicit parametrization in Corollary 16.9 of conformally flat Euclidean hypersurfaces of dimension $n \geq 4$, as well as of those with dimension three that have a principal curvature with multiplicity two, is due to do Carmo-Dajczer-Mercuri 57]. It was used as part of the tools needed for the description of compact conformally flat Euclidean hypersurfaces, which is roughly as follows: a compact conformally flat Euclidean hypersurface $M^{n}$ is diffeomorphic to a sphere $\mathbb{S}^{n}$ with $b_{1}(M)$ handles attached, where $b_{1}(M)$ is the first Betti number of $M^{n}$. Geometrically, it is made up by (perhaps infinitely many) non-umbilical submanifolds of $\mathbb{R}^{n+1}$ that are foliated by complete round $(n-1)$-spheres, and are joined through their boundaries to the following three types of umbilical submanifolds of $\mathbb{R}^{n+1}:(a)$ an open piece of an $n$-sphere or an $n$-plane bounded by a round $(n-1)$-sphere, $(b)$ a round $(n-1)$-sphere, $(c)$ a point.

Compact conformally flat Euclidean hypersurfaces were also studied by Pinkall [294] and Suyama [319]. Pinkall proved that every compact conformally flat hypersurface of dimension at least four of Euclidean space is conformally equivalent to a classical Schottky manifold. Classical Schottky manifolds are constructed by starting with the standard sphere $\mathbb{S}^{n}$, closed round balls $B_{1}, \ldots, B_{k}$ and $\tilde{B}_{1}, \ldots, \tilde{B}_{k}$ which are pairwise disjoint, and Moebius transformations $f_{1}, \ldots, f_{k}: \mathbb{S}^{n} \rightarrow \mathbb{S}^{n}$ such that $f_{i}\left(B_{i}^{0}\right)=\mathbb{S}^{n} \backslash \tilde{B}_{i}$,
$1 \leq i \leq k$, and then taking the quotient space obtaining from $\mathbb{S}^{n} \backslash \cup\left(B_{i}^{0} \cup \tilde{B}_{i}^{0}\right)$ by identifying $\partial B_{i}$ with $\partial \tilde{B}_{i}$ via $f_{i}, 1 \leq i \leq k$, in the canonical way. Suyama constructed explicit examples of compact conformally flat hypersurfaces in the sphere $\mathbb{S}^{n+1}$ without umbilical points and which are diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^{1}, n>3$. He then derived an explicit conformal correspondence between these hypersurfaces and some classical Schottky manifolds using Pinkall's method.

The nonparametric description in Theorem 16.10 of Euclidean conformally flat submanifolds $M^{n}$ of dimension $n \geq 4$ and codimension $p \leq n-3$ free of flat points was obtained by Dajczer-Florit [94], where also Theorem 16.13, that is, the parametrization of Euclidean conformally flat submanifolds of dimension $n \geq 5$ and codimension two, was derived.

The parametric procedure to generate the Euclidean conformally flat submanifolds $M^{n}$ of dimension $n \geq 5$ and codimension two, given by Theorem 16.13, is due to Dajczer-Florit [94]. In that paper, it is shown that these submanifolds can be divided into three classes, namely, the surface-like ones, those which admit locally a continuous one-parameter family of isometric deformations, and those which are locally isometrically rigid. In addition, explicit examples of elements in the first and second classes are provided. Examples belonging to the third class were constructed by Dajczer-Florit [96], each of which is determined by two curves and two functions in one variable. They can be obtained by a geometric procedure, namely, as intersections starting from two flat hypersurfaces. It was shown in [94] that the claims on conformally flat submanifolds in codimension two made in [272] and [273] are not correct.

The existence of conformally flat Euclidean hypersurfaces of dimension three with three distinct principal curvatures was already observed by Cartan 65], who provided a somewhat mysterious characterization of them in terms of the integrability of six umbilic complex distributions on the hypersurface (see Lafontaine [231] for details). Hertrich-Jeromin [219] proved the closedness of the "conformal fundamental forms" $\gamma_{j}$ in our Proposition 16.15. From this he derived the existence of a Guichard net on every conformally flat Euclidean hypersurface of dimension three with three distinct principal curvatures, that is, local coordinates $u_{1}, u_{2}, u_{3}$ with respect to which the metric of the hypersurface can be written as

$$
d s^{2}=\sum_{i=1}^{3} v_{i}^{2} d u_{i}^{2}
$$

with, say, $v_{2}^{2}=v_{1}^{2}+v_{3}^{2}$. Then he used the conformal invariance of this condition to associate with each such hypersurface a Guichard net in $\mathbb{R}^{3}$, that is, a conformally flat metric on an open subset of $\mathbb{R}^{3}$ satisfying the Guichard condition, which is unique up to a Moebius transformation. He also proved in [219] (see also Section 2.4.6 in [220]) that the converse holds, that is, that each conformally flat 3-metric satisfying the Guichard condition gives rise to a unique (up to a Moebius transformation) conformally flat hypersurface in $\mathbb{R}^{4}$. In this way, the classifications of conformally flat Euclidean hypersurfaces of dimension three with three distinct principal curvatures and of conformally flat 3-metrics satisfying the Guichard condition are equivalent problems.

This point of view was pursued in some subsequent papers; see, for instance, Hertrich-Jeromin-Suyama [221] (respectively, Hertrich-Jeromin-Suyama [222]), where a classification was given of conformally flat Euclidean hypersurfaces associated with cyclic (respectively, Bianchi-type) Guichard nets in $\mathbb{R}^{3}$, that is, Guichard nets in $\mathbb{R}^{3}$ for which one of the coordinate line families consists of circular arcs (respectively, the coordinate surfaces have constant sectional curvature).

The understanding of the space of conformally flat 3-metrics that satisfy the Guichard condition has had some significant advances in Burstall-Hertrich-JerominSuyama [47]. Namely, for a conformally flat 3-metric with the Guichard condition in the interior of the space, an evolution of orthogonal Riemannian 2-metrics with constant Gauss curvature -1 was determined; conversely, for a 2 -metric belonging to a certain class of orthogonal analytic 2 -metrics with constant Gauss curvature -1 , a one-parameter family of conformally flat 3 -metrics with the Guichard condition was determined as evolutions issuing from the 2-metric.

However, it is generally not an easy task to translate results on conformally flat 3-metrics satisfying the Guichard condition to corresponding ones for their associated conformally flat Euclidean hypersurfaces, making even the construction of further examples of conformally flat Euclidean hypersurfaces in $\mathbb{R}^{4}$ with three distinct principal curvatures a challenging problem.

The characterization of conformally flat Euclidean hypersurfaces of dimension three with three distinct principal curvatures in Theorem 16.16 is due to CanevariTojeiro [52]. It allowed to derive in Canevari-Tojeiro [53] a Ribaucour transformation for this class of hypersurfaces, providing a process to generate, from a given element of the class, a family of new ones, which depend on the solutions of a linear system of PDEs. In particular, explicit parametrizations of new examples of such hypersurfaces were given. On the other hand, making use of Theorem 16.16, the existence of precisely a one-parameter family of minimal conformally flat Euclidean hypersurfaces of dimension three with three distinct principal curvatures was established by do Rei Filho-Tojeiro [299], whereas conformally flat Euclidean hypersurfaces of dimension three with three distinct principal curvatures and constant scalar curvature were classified by do Rei Filho-Tojeiro [300].

The result in Exercise 16.6 was obtained by Dajczer-Florit-Tojeiro [104]. The assertion in part (ii) of Exercise 16.11 under the assumptions in (a) and (b) were taken from Moore-Morvan [260] and Chen-Verstraelen [84], respectively. Related to Exercise 16.11, it was shown by Dajczer-Onti-Vlachos [124] that if $f: M^{n} \rightarrow \mathbb{R}^{n+p}$, $n \geq 4$, is an isometric immersion with flat normal bundle of a conformally flat manifold having a constant number of principal normal vector fields, then one of them can have multiplicity greater than one only if $p \geq n-m$ and $f$ is quasiumbilical.

### 16.7 Exercises

Exercise 16.1. Prove that any conformally flat Einstein manifold has constant sectional curvature.

Exercise 16.2. Show that Theorem 16.5, and hence Corollary 16.8, remains valid if the ambient space $\mathbb{R}^{n+p}$ is replaced by an arbitrary conformally flat manifold. Show that the same is true for Theorem 16.7.
Hint: Use Exercise 9.2 .
Exercise 16.3. Let $M^{n}, n \geq 4$, be a conformally flat Riemannian manifold. Show that the sectional curvature along a plane spanned by the orthonormal vectors $X, Y \in T_{x} M$ at $x \in M^{n}$ is given in terms of the Schouten tensor $L$ by

$$
K(X, Y)=L(X, X)+L(Y, Y)
$$

Exercise 16.4. Let $M^{n}, n \geq 4$, be a Riemannian manifold. Show that the following assertions are equivalent:
(i) $M^{n}$ is conformally flat.
(ii) At any $x \in M^{n}$, and for every four-dimensional subspace $S \subset T_{x} M$, there is a constant $C(S)$ such that

$$
K\left(\sigma_{1}\right)+K\left(\sigma_{2}\right)=C(S)
$$

for any two mutually orthogonal 2-planes $\sigma_{1}, \sigma_{2}$ spanning $S$.
(iii) At any $x \in M^{n}$, the condition

$$
K\left(X_{1}, X_{2}\right)+K\left(X_{3}, X_{4}\right)=K\left(X_{1}, X_{3}\right)+K\left(X_{2}, X_{4}\right)
$$

holds for every quadruple of pairwise orthogonal vectors $X_{1}, X_{2}, X_{3}, X_{4} \in T_{x} M$.
Exercise 16.5. Let $f: M^{3} \rightarrow \mathbb{R}^{4}$ be a conformally flat hypersurface with three distinct principal curvatures. If one of the principal curvatures is everywhere vanishing, show that there exists a conformal transformation $T$ of $\mathbb{R}^{4}$ such that $T(f(M))$ is an open subset of a hypersurface as in parts $(i)$ or (ii) of Examples 16.14 .
Hint: Let $\lambda_{1}, \lambda_{2}$ and $\lambda_{3}$ be the principal curvatures of $f$ and let $e_{1}, e_{2}, e_{3}$ be a correspondent orthonormal frame of principal directions. Use the Codazzi equation and 16.13) to show that

$$
\begin{equation*}
\nabla_{e_{i}} e_{i}=\sum_{j \neq i} \frac{e_{j}\left(\lambda_{i}\right)}{\lambda_{i}-\lambda_{j}} e_{j} . \tag{16.31}
\end{equation*}
$$

If, say, $\lambda_{3}=0$, show that 16.14 yields

$$
\lambda_{2} e_{3}\left(\lambda_{1}\right)=\lambda_{1} e_{3}\left(\lambda_{2}\right) .
$$

Conclude from 16.31) that the distribution spanned by $e_{1}$ and $e_{2}$ is umbilic in $M^{3}$, and then apply Proposition 7.6.

Exercise 16.6. Let $f: M^{3} \rightarrow \mathbb{R}^{4}$ be a conformally flat hypersurface with three distinct principal curvatures. Assume that (the images by $f$ of) the lines of curvature correspondent to one of the principal curvatures are segments of circles or straight lines in $\mathbb{R}^{4}$. Show that there exists a conformal transformation $T$ of $\mathbb{R}^{4}$ such that $T(f(M))$ is an open subset of a hypersurface as in one of Examples 16.14 .

Hint: With notations as in the preceding exercise, if, say, $e_{3}\left(\lambda_{3}\right)=0$, argue as in the hint of that exercise to show that the distribution spanned by $e_{1}$ and $e_{2}$ is umbilic in $M^{3}$, and then apply Corollary 9.33 .

Exercise 16.7. Let $f: M^{n} \rightarrow \mathbb{R}^{m}$ be a conformally flat submanifold with flat normal bundle and a constant number of principal normal vector fields $\eta_{1}, \ldots, \eta_{\ell}$ such that $E_{\eta_{1}}, \ldots, E_{\eta_{\ell}}$ have constant rank. Show that $E_{\eta_{k}}^{\perp}$ is integrable if the rank of $E_{\eta_{k}}$ is at least 2.

Hint: Use part (iii) of Exercise 16.4, the assumption on the rank of $E_{\eta_{k}}$ and the Gauss equation to show that

$$
\left\langle\eta_{i}-\eta_{k}, \eta_{j}-\eta_{k}\right\rangle=0
$$

for any pair $i \neq j$ with $i \neq k$ and $j \neq k$. Then use Exercise 1.40 .
Exercise 16.8. Let $F: N^{n+1} \rightarrow \mathbb{R}^{n+p}, p \leq n$, and $\hat{F}: N^{n+1} \rightarrow \mathbb{L}^{n+2}$ be isometric immersions of a Riemannian manifold. Show that at each $x \in N^{n+1}$ there exists a subspace of $T_{x} N$ with dimension at least $n-p+1$ that is contained in the relative nullity subspaces of both $F$ and $\hat{F}$ at $x$.

Exercise 16.9. Let $\gamma: I \rightarrow \mathbb{R}^{n+1}, n \geq 3$, be a unit-speed curve and let $r \in C^{\infty}(I)$ be a positive smooth function such that $\left\|r^{\prime}\right\|<1$. Show that a point $(t, u) \in N_{\gamma}^{1} I$ is a singular point for the map $\phi: N_{\gamma}^{1} I \rightarrow \mathbb{R}^{n+1}$ defined by 16.9 if and only if the functions

$$
S=r r^{\prime} /\left\|\gamma^{\prime}\right\|^{2} \text { and } R=r \sqrt{1-\left(r^{\prime} /\left\|\gamma^{\prime}\right\|\right)^{2}}
$$

satisfy

$$
1-S^{\prime}=\frac{1}{\left\|\gamma^{\prime}\right\|^{2}}\left(R\left\langle u, \gamma^{\prime \prime}\right\rangle+S\left\langle\gamma^{\prime}, \gamma^{\prime \prime}\right\rangle\right)
$$

and

$$
R^{\prime}=S\left\langle\gamma^{\prime \prime}, u\right\rangle
$$

Moreover, show that the first equation implies the second, and that they are equivalent if $S^{\prime} \neq 0$ (that is, $r^{\prime} \neq 0$ ).

Exercise 16.10. A Riemannian manifold $M^{n}, n \geq 3$, is said to satisfy the axiom of conformally flat hypersurfaces if for every point $x \in M^{n}$ and every hyperplane $H \subset T_{x} M$ there exists a conformally flat hypersurface $S$ of $M^{n}$ passing through $x$ such that $T_{x} S=H$. Show that if $M^{n}, n \geq 4$, admits an isometric immersion as a hypersurface into the Lorentz space $\mathbb{L}^{n+1}$, then $M^{n}$ satisfies the axiom of conformally flat hypersurfaces.

Exercise 16.11. An isometric immersion $f: M^{n} \rightarrow \tilde{M}^{n+p}$ is said to be quasiumbilical at $x \in M^{n}$ if there exists an orthonormal basis $\xi_{1}, \ldots, \xi_{p}$ of $N_{f} M(x)$ such that the shape operator $A_{\xi_{j}}$ has an eigenvalue of multiplicity at least $n-1$ for all $1 \leq j \leq p$. If $f$ is quasiumbilical at any $x \in M^{n}$, then it is said to be a quasiumbilical isometric immersion.

Show that, if $f: M^{n} \rightarrow \mathbb{R}^{n+p}, n \geq 4$, is an isometric immersion, then the following assertions hold:
(i) $M^{n}$ is conformally flat if $f$ is quasiumbilical,
(ii) $f$ is quasiumbilical if $M^{n}$ is conformally flat and either of the conditions below is satisfied:
(a) $p \leq \min \{4, n-3\}$;
(b) $p \leq n-3$ and $f$ has flat normal bundle.

Hint for (ii): If $f: M^{n} \rightarrow \mathbb{R}^{n+p}$, with $n \geq 4$ and $p \leq n-3$, is a conformally flat submanifold, for each $x \in M^{n}$ let $\eta \in N_{f} M(x)$ be the principal normal at $x$ with $\operatorname{dim} E_{\eta}(x) \geq n-p$ given by Theorem 16.5. Set $\lambda=\|\eta\|$. If $X_{1}, \ldots, X_{n}$ is an orthonormal basis of $T_{x} M$, use the formula

$$
\operatorname{Ric}(Y, X)=\langle\alpha(Y, X), n H\rangle-\sum_{i=1}^{n}\left\langle\alpha\left(Y, X_{i}\right), \alpha\left(X, X_{i}\right)\right\rangle
$$

to show that

$$
\operatorname{Ric}(T, X)=(n-1) \lambda^{2}\langle T, X\rangle
$$

for all $T \in E_{\eta}(x)$ and $X \in T_{x} M$. Then use the definition of $L$ to obtain

$$
L(T, X)=\frac{1}{2} \lambda^{2}\langle T, X\rangle
$$

for all $T \in E_{\eta}(x)$ and $X \in T_{x} M$. Prove first that this implies that

$$
L(Y, X)-\lambda\langle\alpha(Y, X), \xi\rangle+\frac{1}{2} \lambda^{2}\langle Y, X\rangle=0
$$

for $Y=T \in E_{\eta}(x)$ and $X \in T_{x} M$, and then for all $Y, X \in T_{x} M$ by using that

$$
\begin{aligned}
L(T, T)+L(X, X) & =K(T, X) \\
& =\langle\alpha(T, T), \alpha(X, X)\rangle
\end{aligned}
$$

for all $T \in E_{\eta}(x)$ and $X \in T_{x} M$. Conclude that $\beta: T_{x} M \times T_{x} M \rightarrow N_{f} M(x)$ given by

$$
\beta(X, Y)=\alpha(X, Y)-\langle X, Y\rangle \eta
$$

is a flat bilinear form. If $\mathcal{N}(\beta)=n-\operatorname{dim} S(\beta)$, use Theorem 5.2. Now prove that one always has $\mathcal{N}(\beta)=n-\operatorname{dim} S(\beta)$ if $f$ has flat normal bundle, and if $\mathcal{N}(\beta)>n-\operatorname{dim} S(\beta)$ and $p \leq 4$, then the conclusion follows by applying the result due to Cartan, given in part (ii) of Remark 5.4 .

Exercise 16.12. Prove that an isometric immersion $f: M^{n} \rightarrow \tilde{M}^{n+p}$ being quasiumbilical is a conformal invariant, that is, it is preserved under a conformal change of the metric of the ambient space. Conclude that Exercise 16.11 remains true if in assertions ( $i$ ) and ( $i i$ ) the ambient space $\mathbb{R}^{n+p}$ is replaced by any conformally flat manifold.

## Chapter 17

## Conformally deformable hypersurfaces


#### Abstract

This chapter is devoted to provide a modern presentation of Cartan's classification of Euclidean hypersurfaces $M^{n}$ of dimension $n \geq 5$ that admit nontrivial conformal deformations. Besides conformally flat hypersurfaces, the simplest examples are those that are conformally congruent to cylinders and rotation hypersurfaces over surfaces in $\mathbb{R}^{3}$, and to cylinders over three-dimensional hypersurfaces of $\mathbb{R}^{4}$ that are cones over surfaces in $\mathbb{S}^{3}$. These examples are called conformally surface-like hypersurfaces.

Other examples are the conformally ruled hypersurfaces, which are foliated by round spheres of codimension one. But the most interesting examples are envelopes of some two-parameter congruences of hyperspheres, which are determined by certain space-like surfaces in the de Sitter space $\mathbb{S}_{1,1}^{n+2}$. Our approach is to determine which of those surfaces give rise to conformally deformable hypersurfaces that are neither conformally surface-like nor conformally ruled.


### 17.1 Cartan hypersurfaces

Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 5$, be an isometric immersion whose principal curvatures have multiplicity at most $n-2$ at any point. Assume that $M^{n}$ admits a conformal immersion $\tilde{f}: M^{n} \rightarrow \mathbb{R}^{n+1}$ that is not conformally congruent to $f$ on any open subset of $M^{n}$. It follows from Corollary 9.25 that $f$ has a principal curvature with constant multiplicity $n-2$. For the convenience of the reader, we provide below a direct proof of that result as well as of some additional facts needed in the classification of the conformally deformable hypersurfaces.

Proposition 17.1. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 5$, be an orientable hypersurface whose principal curvatures have multiplicity at most $n-2$ at any point. Assume that there exists a conformal immersion $\tilde{f}: M^{n} \rightarrow \mathbb{R}^{n+1}$ that is not conformally congruent to $f$ on any open subset of $M^{n}$, and let $\tilde{F}=\mathcal{J}(\tilde{f}): M^{n} \rightarrow \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$ be its isometric light-cone representative. Then the following assertions hold:
(i) At each point $y$ of a closed subset $\mathcal{V} \subset M^{n}$, there exists a pseudo-orthonormal basis $\zeta_{1}, \zeta_{2}, \tilde{F}$ of $N_{\tilde{F}} M(y)$ with

$$
\left\langle\zeta_{2}, \zeta_{2}\right\rangle=0,\left\langle\zeta_{2}, \tilde{F}\right\rangle=1 \text { and } \zeta_{1} \in L=\operatorname{span}\left\{\zeta_{2}, \tilde{F}\right\}^{\perp}
$$

such that

$$
\begin{equation*}
\alpha_{L}^{\tilde{F}}(X, Y)=-\langle X, Y\rangle \zeta_{2} \tag{17.1}
\end{equation*}
$$

for all $X, Y \in T_{y} M$. Moreover, $\operatorname{ker} A_{N}^{f} \cap \operatorname{ker} A_{\zeta_{1}}^{\tilde{F}}$ has dimension $n-2$, where $N$ is a unit normal vector field.
(ii) For each $x \in \mathcal{U}=M^{n} \backslash \mathcal{V}$ there exist $\mu \in N_{\tilde{F}} M(x)$ of unit length and a flat bilinear form $\gamma: T_{x} M \times T_{x} M \rightarrow \operatorname{span}\{\mu\}^{\perp}$ such that

$$
\alpha^{\tilde{F}}(X, Y)=\left\langle A_{N}^{f} X, Y\right\rangle \mu+\gamma(X, Y)
$$

for all $X, Y \in T_{x} M$. Moreover, $\langle\mu, \tilde{F}\rangle \neq 0$, the function $\lambda=-1 /\langle\mu, \tilde{F}\rangle$ is a principal curvature of $f$ and $\Delta=\mathcal{N}(\gamma)$ is an ( $n-2$ )-dimensional eigenspace of both $\lambda$ and a principal curvature $\tilde{\lambda}$ of $\tilde{f}$.

Proof: The normal bundle of $\tilde{F}$ splits orthogonally as

$$
N_{\tilde{F}} M=\Psi_{*} N_{\tilde{f}} M \oplus \mathbb{L}^{2},
$$

where $\Psi$ is given by (9.1) and $\mathbb{L}^{2}$ is a Lorentzian plane bundle having the position vector field $\tilde{F}$ as a section. Thus there exist unique sections $\xi$ and $\eta$ of $\mathbb{L}^{2}$ satisfying

$$
\langle\xi, \xi\rangle=-1, \quad\langle\xi, \eta\rangle=0 \text { and }\langle\eta, \eta\rangle=1
$$

such that

$$
\tilde{F}=\xi+\eta .
$$

At any $x \in M^{n}$, endow $W=N_{f} M(x) \oplus N_{\tilde{F}} M(x)$ with the inner product of signature $(2,2)$ given by

$$
\langle\langle,\rangle\rangle_{N_{f} M \oplus N_{\tilde{F}} M}=\langle,\rangle_{N_{f} M}-\langle,\rangle_{N_{\tilde{F}} M} .
$$

The bilinear form

$$
\beta=\alpha^{f} \oplus \alpha^{\tilde{F}}: T_{x} M \times T_{x} M \rightarrow W
$$

is flat by the Gauss equations of $f$ and $\tilde{F}$. Moreover, $\mathcal{N}(\beta)=\{0\}$ since

$$
\begin{equation*}
\left\langle\alpha^{\tilde{F}}(X, Y), \tilde{F}\right\rangle=-\langle X, Y\rangle \tag{17.2}
\end{equation*}
$$

for all $X, Y \in T_{x} M$. It follows from the Main lemma 4.22 for $(p, q)=(2,2)$ that the subspace $\mathcal{S}(\beta)$ is degenerate, that is, the isotropic vector subspace

$$
\Omega=\mathcal{S}(\beta) \cap \mathcal{S}(\beta)^{\perp}
$$

is nontrivial.

Since the metric $\langle\langle\rangle$,$\rangle is positive definite on W_{1}=\operatorname{span}\{N, \xi\}$ and negative definite on $W_{2}=\operatorname{span}\left\{\Psi_{*} N^{\tilde{f}}, \eta\right\}$, where $N^{\tilde{f}}$ is a unit normal vector field to $\tilde{f}$, then the orthogonal projections $P_{1}: W \rightarrow W_{1}$ and $P_{2}: W \rightarrow W_{2} \operatorname{map} \Omega$ isomorphically onto $P_{1}(\Omega)$ and $P_{2}(\Omega)$, respectively.

Assume first that $\Omega$ has dimension two on some open subset $U \subset M^{n}$, that is, the bilinear form $\beta$ is null at any point of $U$. The projections $P_{1}$ and $P_{2}$ then map $\Omega$ isomorphically onto $W_{1}$ and onto $W_{2}$, respectively, for all $x \in U$. Let $\zeta \in \Omega$ be such that $\xi=P_{1}(\zeta)$. Then $\zeta$ is an isotropic vector contained in $\mathcal{S}\left(\alpha^{\tilde{F}}\right)^{\perp}$. Since $\tilde{F} \notin \Omega$ in view of $\sqrt{17.2}$, the vectors $\zeta$ and $\tilde{F}$ are linearly independent, and hence span a Lorentzian plane in $N_{\tilde{F}} M(x)$. Choosing a unit vector $\zeta_{1} \in N_{\tilde{F}} M(x)$ spanning $\{\zeta, \tilde{F}\}^{\perp}$ and denoting $\zeta_{2}=\langle\tilde{F}, \zeta\rangle^{-1} \zeta$, we obtain a pseudo-orthonormal basis $\zeta_{1}, \zeta_{2}, \tilde{F}$ of $N_{\tilde{F}} M(x)$ such that (17.1) holds. Moreover, since $\beta$ is null so is

$$
\hat{\beta}=\alpha^{f} \oplus\left\langle\alpha^{\tilde{F}}, \zeta_{1}\right\rangle \zeta_{1}: T_{x} M \times T_{x} M \rightarrow \operatorname{span}\left\{N, \zeta_{1}\right\} .
$$

Hence $A_{\zeta_{1}}^{\tilde{F}}=A_{N}^{f}$, after changing the sign of $\zeta_{1}$ if necessary. Summarizing, we have

$$
\begin{equation*}
\alpha^{\tilde{F}}(X, Y)=\left\langle A_{N}^{f} X, Y\right\rangle \zeta_{1}-\langle X, Y\rangle \zeta_{2} \tag{17.3}
\end{equation*}
$$

for all $X, Y \in T_{x} M$.
It is easily seen from (17.3) that $\zeta_{1}$ and $\zeta_{2}$ define smooth vector fields on $U$. We now prove that they are parallel in the normal connection of $\tilde{F}$.

Comparing the Codazzi equations of $f$ and $\tilde{F}$ for $A_{N}^{f}=A_{\zeta_{1}}^{\tilde{F}}$, we obtain

$$
\begin{equation*}
A_{\nabla \frac{1}{X} \zeta_{1}}^{\tilde{F}} Y=A_{\nabla_{\frac{1}{Y} \zeta_{1}}^{\tilde{F}}}^{\tilde{T}} X \tag{17.4}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}(U)$. Since $\tilde{F}$ is parallel in the normal connection, then

$$
\nabla \stackrel{\perp}{X} \zeta_{1}=\left\langle\nabla_{X}^{\perp} \zeta_{1}, \zeta_{2}\right\rangle \tilde{F}
$$

for all $X \in \mathfrak{X}(U)$. We conclude from (17.4) that $\zeta_{1}$ is parallel, and hence $\zeta_{1}, \zeta_{2}, \tilde{F}$ is a parallel normal frame.

Now set

$$
F=\mathcal{J}(f)=\Psi \circ f: M^{n} \rightarrow \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3},
$$

regarded as a map into $\mathbb{L}^{n+3}$. Then

$$
\begin{equation*}
\alpha^{F}(X, Y)=\left\langle A_{N}^{f} X, Y\right\rangle \Psi_{*} N-\langle X, Y\rangle w \tag{17.5}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}(U)$. Define a vector bundle isometry $\tau: N_{F} U \rightarrow N_{\tilde{F}} U$ by setting

$$
\tau\left(\Psi_{*} N\right)=\zeta_{1}, \quad \tau(w)=\zeta_{2} \text { and } \tau(F)=\tilde{F} .
$$

Then $\alpha^{\tilde{F}}=\tau \circ \alpha^{F}$ from 17.3 and 17.5 . Moreover, since $\left\{\Psi_{*} N, w, F\right\}$ is also parallel in the normal connection of $F$, it follows that $\tau$ is parallel. By the Fundamental theorem of submanifolds, there exists $T \in O_{1}(n+3)$ such that $\tilde{F}=T \circ F$ and $\left.T_{*}\right|_{N_{F} U}=\tau$. By

Proposition 9.18, there is a conformal transformation $\nu$ of $\mathbb{R}^{n+1}$ such that $\left.\tilde{f}\right|_{U}=\left.\nu \circ f\right|_{U}$. This contradicts the assumption and proves that $\Omega$ must have dimension one at every point of $M^{n}$.

Assume that $P_{1}(\Omega)=\operatorname{span}\{\xi\}$ at some point $x \in M^{n}$. Arguing as before, we obtain a pseudo-orthonormal basis $\zeta_{1}, \zeta_{2}, \tilde{F}$ of $N_{\tilde{F}} M(x)$ such that 17.1) holds, but now the bilinear form

$$
\hat{\beta}=\alpha^{f} \oplus\left\langle\alpha^{\tilde{F}}, \zeta_{1}\right\rangle \zeta_{1}: T_{x} M \times T_{x} M \rightarrow \operatorname{span}\left\{N, \zeta_{1}\right\}
$$

is flat and $\mathcal{S}(\hat{\beta})$ is nondegenerate. Therefore $\mathcal{N}(\hat{\beta})$ has dimension at least $n-2$. Since

$$
\mathcal{N}(\hat{\beta})=\operatorname{ker} A_{N}^{f} \cap \operatorname{ker} A_{\zeta 1}^{\tilde{F}},
$$

we must have $\operatorname{dim} \mathcal{N}(\hat{\beta})=n-2$ and

$$
\operatorname{ker} A_{N}^{f}=\mathcal{N}(\hat{\beta})=\operatorname{ker} A_{\zeta_{1}}^{\tilde{F}}
$$

by the assumption that $f$ does not have a principal curvature with multiplicity greater than $n-2$. Therefore 0 is a principal curvature of $f$ with multiplicity at least $n-2$ and $\mathcal{N}(\hat{\beta})$ is a common eigenspace of 0 and a principal curvature $\tilde{\lambda}$ of $\tilde{f}$.

Suppose now that $P_{1}(\Omega) \neq \operatorname{span}\{\xi\}$. This is equivalent to requiring the orthogonal projection $\Pi_{1}: W \rightarrow N_{f} M$ to map $\Omega$ isomorphically onto $N_{f} M$, say, $N=\Pi_{1}(\nu)$ for some $\nu \in \Omega$. Set $\mu=\Pi_{2}(\nu)$, where $\Pi_{2}: W \rightarrow N_{\tilde{F}} M$ is the orthogonal projection onto $N_{\tilde{F}} M$. Then $A_{\mu}^{\tilde{F}}=A_{N}^{f}$, for $N+\mu=\nu \in \Omega \subset \mathcal{S}(\beta)^{\perp}$, and hence

$$
\alpha^{\tilde{F}}(X, Y)=\left\langle A_{N}^{f} X, Y\right\rangle \mu+\gamma(X, Y)
$$

where $\gamma: T_{x} M \times T_{x} M \rightarrow\{\mu\}^{\perp}$ is a flat bilinear form such that $\mathcal{S}(\gamma)$ is nondegenerate. By Lemma 4.14, the subspace $\Delta=\mathcal{N}(\gamma)$ has dimension at least $n-2$. Moreover,

$$
\begin{aligned}
-\langle T, X\rangle & =\left\langle\alpha^{\tilde{F}}(T, X), \tilde{F}\right\rangle \\
& =\langle\mu, \tilde{F}\rangle\left\langle\alpha^{\tilde{F}}(T, X), \mu\right\rangle \\
& =\langle\mu, \tilde{F}\rangle\left\langle A_{N}^{f} T, X\right\rangle
\end{aligned}
$$

for all $T \in \Delta$ and $X \in T_{x} M$. This implies that $\langle\mu, \tilde{F}\rangle \neq 0$, and that $\lambda=-1 /\langle\tilde{F}, \mu\rangle$ is a principal curvature of $f$ whose eigenspace contains $\Delta$. By the assumption that $f$ does not have a principal curvature with multiplicity greater than $n-2$, we conclude that $\operatorname{dim} \Delta=n-2$ and that $\Delta$ is the common eigenspace of $\lambda$ and a principal curvature $\tilde{\lambda}$ of $\tilde{f}$.

As a consequence of Proposition 17.1, if $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 5$, is an isometric immersion that satisfies its assumptions, then $f$ has, in particular, a principal curvature $\lambda$ with constant multiplicity $n-2$. We call $f$ a Cartan hypersurface if, in addition, $\lambda$ is nowhere vanishing. In the remaining of this chapter we determine which oriented hypersurfaces $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 5$, carrying a nowhere vanishing principal curvature with constant multiplicity $n-2$, are indeed Cartan hypersurfaces.

### 17.2 The first step

This section contains the first step towards the classification of Cartan hypersurfaces $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ that are not conformally surface-like on any open subset of $M^{n}$. It is shown that the existence of a conformal immersion $\tilde{f}: M^{n} \rightarrow \mathbb{R}^{n+1}$ that is not conformally congruent to $f$ on any open subset of $M^{n}$ is equivalent to the existence of a tensor on $M^{n}$ satisfying several properties, which come from putting together the Gauss and Codazzi equations of $f$ and the Gauss, Codazzi and Ricci equations of the isometric light-cone representative of $\tilde{f}$.

Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be a hypersurface that carries a principal curvature of constant multiplicity $n-2$ with eigenbundle $\Delta$, and let

$$
C: \Gamma(\Delta) \rightarrow \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)
$$

be its splitting tensor. As in the case in which $\Delta$ is the relative nullity distribution, the hypersurface $f$ is said to be hyperbolic (respectively, parabolic or elliptic) if there exists $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ satisfying the following conditions:
(i) $J^{2}=I$ (respectively, $J^{2}=0$, with $J \neq 0$, and $J^{2}=-I$ ),
(ii) $\nabla_{T}^{h} J=0$ for all $T \in \Gamma(\Delta)$,
(iii) $C_{T} \in \operatorname{span}\{I, J\}$ for all $T \in \Gamma(\Delta)$.

Lemma 17.2. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be an oriented hypersurface with shape operator $A$ with respect to a unit normal vector field $N$. Assume that $f$ carries a nowhere vanishing principal curvature $\lambda$ of constant multiplicity $n-2$ and let $\Delta$ denote its eigenbundle. Assume also that $f$ is not conformally surface-like on any open subset of $M^{n}$. If $f$ is a Cartan hypersurface, then it is either hyperbolic, parabolic or elliptic with respect to $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ on each connected component of an open dense subset of $M^{n}$, and there exists $D \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ such that:
(i) $D \in \operatorname{span}\{I, J\}$, and $D \neq \pm I$ everywhere,
(ii) $\nabla_{T}^{h} D=0$ for all $T \in \Gamma(\Delta)$,
(iii) $\operatorname{det} D=1$,
(iv) $\left(\nabla_{X}(A-\lambda I) D\right) Y-\left(\nabla_{Y}(A-\lambda I) D\right) X=X \wedge Y\left(D^{t} \operatorname{grad} \lambda\right)$ for all $X, Y \in \Gamma\left(\Delta^{\perp}\right)$,
(v) $\left\langle\left(\nabla_{Y} D\right) X-\left(\nabla_{X} D\right) Y, \operatorname{grad} \lambda\right\rangle+$ Hess $\lambda(D X, Y)-$ Hess $\lambda(X, D Y)$

$$
=\lambda(\langle A X,(A-\lambda I) D Y\rangle-\langle(A-\lambda I) D X, A Y\rangle)
$$

for all $X, Y \in \Gamma\left(\Delta^{\perp}\right)$.

Conversely, if $M^{n}$ is simply connected, the hypersurface is either hyperbolic, parabolic or elliptic with respect to $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ and there exists a tensor $D \in$ $\Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ satisfying conditions $(i)$ to $(v)$ above, then $f$ is a Cartan hypersurface. Moreover, two such tensors that do not coincide up to sign on any open subset of $M^{n}$ give rise to conformal immersions of $M^{n}$ into $\mathbb{R}^{n+1}$ that are not conformally congruent on any open subset of $M^{n}$.

Proof: Let $\tilde{f}: M^{n} \rightarrow \mathbb{R}^{n+1}$ be a conformal immersion that is not conformally congruent to $f$ on any open subset of $M^{n}$, and let $\tilde{F}=\mathcal{J}_{w}(\tilde{f}): M^{n} \rightarrow \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$ be its isometric light-cone representative. Since $f$ is Cartan, only part (ii) of Proposition 17.1 occurs at any point. Thus there exist $\mu \in \Gamma\left(N_{\tilde{F}} M\right)$ of unit length and a smooth flat bilinear form $\gamma: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow\{\mu\}^{\perp}$ such that

$$
\alpha^{\tilde{F}}(X, Y)=\langle A X, Y\rangle \mu+\gamma(X, Y)
$$

for all $X, Y \in \mathfrak{X}(M)$. Since $\mu$ spans the line bundle $\Pi_{2}(\Omega)$ defined in the proof of Proposition 17.1, it is indeed smooth. Moreover, $\lambda=-1 /\langle\mu, \tilde{F}\rangle$ is the principal curvature of $f$ with multiplicity $n-2$ with respect to $N$ and $\Delta=\mathcal{N}(\gamma)$ is its eigenbundle.

Defining $\zeta \in \Gamma\left(N_{\tilde{F}} M\right)$ by $\zeta=\lambda \tilde{F}+\mu$, we have $\langle\zeta, \zeta\rangle=-1$ and $\langle\mu, \zeta\rangle=0$. Using (17.2), it follows that

$$
A_{\zeta}=A-\lambda I
$$

Extend $\mu, \zeta$ to an orthonormal frame $\mu, \zeta, \bar{\zeta}$ of $N_{\tilde{F}} M$ and denote also by $A, A_{\zeta}, A_{\bar{\zeta}}$ the restrictions of these shape operators to $\Delta^{\perp}$. Define $D \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ by

$$
D=(A-\lambda I)^{-1} A_{\bar{\zeta}} .
$$

The flatness of $\gamma$ implies that

$$
\begin{aligned}
\operatorname{det} D \operatorname{det}(A-\lambda I) & =\operatorname{det}(A-\lambda I) D \\
& =\operatorname{det} A_{\bar{\zeta}} \\
& =\operatorname{det} A_{\zeta} \\
& =\operatorname{det}(A-\lambda I),
\end{aligned}
$$

and this yields part (iii).
The Codazzi equation for $A$ gives

$$
\begin{equation*}
\nabla_{T}^{h} A=(A-\lambda I) C_{T} \tag{17.6}
\end{equation*}
$$

whereas the Codazzi equation for $A_{\bar{\zeta}}$ yields

$$
\begin{equation*}
\nabla_{T}^{h} A_{\bar{\zeta}}=A_{\bar{\zeta}} C_{T} \tag{17.7}
\end{equation*}
$$

In particular, the endomorphisms on the right-hand sides of (17.6) and 17.7) are symmetric, that is,

$$
\begin{equation*}
(A-\lambda I) C_{T}=C_{T}^{t}(A-\lambda I) \text { and } A_{\bar{\zeta}} C_{T}=C_{T}^{t} A_{\bar{\zeta}} \tag{17.8}
\end{equation*}
$$

Using (17.8) we obtain

$$
\begin{aligned}
(A-\lambda I) D C_{T} & =A_{\bar{\zeta}} C_{T} \\
& =C_{T}^{t} A_{\bar{\zeta}} \\
& =C_{T}^{t}(A-\lambda I) D \\
& =(A-\lambda I) C_{T} D .
\end{aligned}
$$

Hence

$$
\begin{equation*}
\left[D, C_{T}\right]=0 \tag{17.9}
\end{equation*}
$$

for all $T \in \Gamma(\Delta)$.
Equation (17.6) yield

$$
(A-\lambda I) C_{T} D=\left(\nabla_{T}^{h} A\right) D,
$$

whereas (17.7) gives

$$
\begin{aligned}
(A-\lambda I) D C_{T} & =A_{\bar{\zeta}} C_{T} \\
& =\nabla_{T}^{h} A_{\bar{\zeta}} \\
& =\nabla_{T}^{h}(A-\lambda I) D \\
& =\nabla_{T}^{h}(A D)-\lambda \nabla_{T}^{h} D .
\end{aligned}
$$

Thus

$$
(A-\lambda I)\left[D, C_{T}\right]=(A-\lambda I) \nabla_{T}^{h} D
$$

and part (ii) follows from (17.9).
Differentiating $\zeta-\mu=\lambda F$ and taking normal components yields

$$
\begin{equation*}
\nabla_{X}^{\perp}(\zeta-\mu)=\frac{1}{\lambda} X(\lambda)(\zeta-\mu) \tag{17.10}
\end{equation*}
$$

for all $X \in \mathfrak{X}(M)$. Define $\omega \in \Gamma\left(T^{*} M\right)$ by

$$
\omega(X)=\left\langle\nabla_{X}^{\perp} \bar{\zeta}, \mu\right\rangle .
$$

Using (17.10) we obtain

$$
\begin{equation*}
\nabla_{X}^{\perp} \mu=-\frac{1}{\lambda} X(\lambda) \zeta-\omega(X) \bar{\zeta}, \quad \nabla_{X}^{\perp} \zeta=-\frac{1}{\lambda} X(\lambda) \mu-\omega(X) \bar{\zeta} \tag{17.11}
\end{equation*}
$$

and

$$
\begin{equation*}
\nabla_{X}^{\perp} \bar{\zeta}=\omega(X)(\mu-\zeta) . \tag{17.12}
\end{equation*}
$$

In view of the Codazzi equation for $A$, the Codazzi equation for $A_{\mu}=A$ reduces to

$$
\begin{equation*}
A_{\nabla \frac{+}{X} \mu} Y=A_{\nabla_{\frac{⿺}{Y} \mu}} X \tag{17.13}
\end{equation*}
$$

for all $X, Y \in \mathfrak{X}(M)$. Using (17.11) and $T(\lambda)=0$ for all $T \in \Gamma(\Delta)$, it follows that

$$
\begin{equation*}
\omega(T)=0 \tag{17.14}
\end{equation*}
$$

for all $T \in \Gamma(\Delta)$. Hence the vector fields $\mu, \zeta$ and $\bar{\zeta}$ are parallel along $\Delta$.
Equation 17.13) also yield

$$
\begin{equation*}
\lambda \omega(X) D Y+X(\lambda) Y=\lambda \omega(Y) D X+Y(\lambda) X \tag{17.15}
\end{equation*}
$$

for all $X, Y \in \Gamma\left(\Delta^{\perp}\right)$. Using that $\operatorname{det} D=1$, this can also be written as

$$
\begin{aligned}
D(\lambda \omega(X) Y-\lambda \omega(Y) X) & =Y(\lambda) X-X(\lambda) Y \\
& =\langle Y, \operatorname{grad} \lambda\rangle X-\langle X, \operatorname{grad} \lambda\rangle Y \\
& =X \wedge Y(\operatorname{grad} \lambda) \\
& =(D X \wedge D Y)(\operatorname{grad} \lambda) \\
& =\langle D Y, \operatorname{grad} \lambda\rangle D X-\langle D X, \operatorname{grad} \lambda\rangle D Y \\
& =D(\langle D Y, \operatorname{grad} \lambda\rangle X-\langle D X, \operatorname{grad} \lambda\rangle Y) .
\end{aligned}
$$

Hence

$$
\begin{equation*}
\omega(X)=-\frac{1}{\lambda}\langle D X, \operatorname{grad} \lambda\rangle \tag{17.16}
\end{equation*}
$$

for all $X \in \Gamma\left(\Delta^{\perp}\right)$. On the other hand, the Codazzi equation for $A_{\bar{\zeta}}=(A-\lambda I) D$ and (17.12) give

$$
\begin{equation*}
\left(\nabla_{X}(A-\lambda I) D\right) Y-\lambda \omega(X) Y=\left(\nabla_{Y}(A-\lambda I) D\right) X-\lambda \omega(Y) X . \tag{17.17}
\end{equation*}
$$

Notice that, here and elsewhere, wherever necessary we regard $D$ as an element of $\Gamma(\operatorname{End}(T M))$ by assuming that ker $D=\Delta$. From (17.16) we have

$$
\begin{aligned}
\lambda \omega(X) Y-\lambda \omega(Y) X & =\langle D Y, \operatorname{grad} \lambda\rangle X-\langle D X, \operatorname{grad} \lambda\rangle Y \\
& =\left\langle Y, D^{t} \operatorname{grad} \lambda\right\rangle X-\left\langle X, D^{t} \operatorname{grad} \lambda\right\rangle Y \\
& =X \wedge Y\left(D^{t} \operatorname{grad} \lambda\right)
\end{aligned}
$$

and then part (iv) follows from (17.17).
Using (17.11) and 17.12), the Ricci equations for $\mu, \bar{\zeta}$ or $\zeta, \bar{\zeta}$ yields

$$
\begin{align*}
& d \omega(X, Y)+\frac{1}{\lambda}(X(\lambda) \omega(Y)-Y(\lambda) \omega(X))  \tag{17.18}\\
&=\langle A X,(A-\lambda I) D Y\rangle-\langle(A-\lambda I) D X, A Y\rangle
\end{align*}
$$

for all $X, Y \in \Gamma\left(\Delta^{\perp}\right)$. Using (17.16) we obtain

$$
\begin{aligned}
Y \omega(X) & =\frac{1}{\lambda^{2}} Y(\lambda)\langle D X, \operatorname{grad} \lambda\rangle-\frac{1}{\lambda}\left\langle\nabla_{Y} D X, \operatorname{grad} \lambda\right\rangle-\frac{1}{\lambda} \operatorname{Hess} \lambda(D X, Y) \\
& =-\frac{1}{\lambda}\left(Y(\lambda) \omega(X)+\left\langle\nabla_{Y} D X, \operatorname{grad} \lambda\right\rangle+\operatorname{Hess} \lambda(D X, Y)\right)
\end{aligned}
$$

Hence

$$
\begin{align*}
& d \omega(X, Y)+\frac{1}{\lambda}(X(\lambda) \omega(Y)-Y(\lambda) \omega(X))  \tag{17.19}\\
& \quad=\frac{1}{\lambda}\left(\left\langle\left(\nabla_{Y} D\right) X-\left(\nabla_{X} D\right) Y, \operatorname{grad} \lambda\right\rangle-\operatorname{Hess} \lambda(D X, Y)+\operatorname{Hess} \lambda(X, D Y)\right)
\end{align*}
$$

and part $(v)$ follows from (17.18).
We now show that one cannot have $D= \pm I$ on any open subset of $M^{n}$. Assume otherwise that $D=\delta I$ on the open subset $U \subset M^{n}$, where $\delta= \pm 1$. Then the second fundamental form of $\tilde{F}$ on $U$ is given by

$$
\begin{align*}
\alpha^{\tilde{F}}(X, Y) & =\langle A X, Y\rangle \mu-\langle(A-\lambda I) X, Y\rangle \zeta+\langle(A-\lambda I) \delta X, Y\rangle \bar{\zeta} \\
& =\langle A X, Y\rangle(\mu-\zeta+\delta \bar{\zeta})+\lambda\langle X, Y\rangle(\zeta-\delta \bar{\zeta}) \tag{17.20}
\end{align*}
$$

for all $X, Y \in \mathfrak{X}(U)$. By (17.16), the one-form $\omega$ is given on $U$ by

$$
\omega(X)=-\frac{\delta}{\lambda} X(\lambda)
$$

for all $X \in \mathfrak{X}(U)$. In particular, the vector field $\gamma=\delta \bar{\zeta}-\zeta$ satisfies

$$
\begin{aligned}
\nabla \frac{\perp}{X} \gamma & =\delta \nabla_{X}^{\perp} \bar{\zeta}-\nabla_{X}^{\perp} \zeta \\
& =\delta \omega(X)(\mu-\zeta))+\frac{1}{\lambda} X(\lambda) \mu+\omega(X) \bar{\zeta} \\
& =-\frac{1}{\lambda} X(\lambda) \gamma
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\nabla_{X}^{\perp} \mu & =-\frac{1}{\lambda} X(\lambda) \zeta-\omega(X) \bar{\zeta} \\
& =-\frac{1}{\lambda} X(\lambda) \gamma
\end{aligned}
$$

Thus the vector fields $\rho_{1}=\mu-\gamma$ and $\rho_{2}=\lambda \gamma$ are parallel in the normal connection.
Let

$$
F=\mathcal{J}(f)=\Psi \circ f: M^{n} \rightarrow \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}
$$

be the isometric light-cone representative of $f$, regarded as a map into $\mathbb{L}^{n+3}$. Define a vector bundle isometry $\tau: N_{F} U \rightarrow N_{\tilde{F}} U$ by setting

$$
\tau\left(\Psi_{*} N\right)=\rho_{1}, \quad \tau(w)=\rho_{2} \text { and } \tau(F)=\tilde{F} .
$$

Then $\alpha^{\tilde{F}}=\tau \circ \alpha^{F}$ from (17.5) and 17.20 . Moreover, since $\left\{\Psi_{*} N, w, F\right\}$ and $\left\{\rho_{1}, \rho_{2}, \tilde{F}\right\}$ are parallel in the normal connections of $F$ and $\tilde{F}$, respectively, it follows that $\tau$ is parallel. By the Fundamental theorem of submanifolds there exists $T \in O_{1}(n+3)$ such that $\tilde{F}=T \circ F$ and $\left.T_{*}\right|_{N_{F} U}=\tau$. By Proposition 9.18, there is a conformal transformation $\nu$ of $\mathbb{R}^{n+1}$ such that $\left.\tilde{f}\right|_{U}=\left.\nu \circ f\right|_{U}$. This contradicts the assumption
that $\tilde{f}$ is not conformally congruent to $f$ on any open subset and proves that one cannot have $D=\delta I$ on $U$.

Let $\mathcal{U}$ be the open dense subset of $M^{n}$ where $D \neq \delta I$. Then let $U \subset \mathcal{U}$ be an open subset where $D$ has either two smooth distinct real eigenvalues, a single real eigenvalue of multiplicity two or a pair of smooth complex conjugate eigenvalues. Looking at the Jordan form of $D$, there exists $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ satisfying $J^{2}=\epsilon I$, with $\epsilon=1,0$ or -1 , respectively, such that

$$
D=a I+b J,
$$

where $a, b \in C^{\infty}(U)$, with $b$ nowhere vanishing and $b=1$ if $\epsilon=0$.
If $S \subset \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ is the subspace of all elements that commute with $D$, or equivalently, with $J$, then $S=\operatorname{span}\{I, J\}$, and $C(\Gamma(\Delta)) \subset S$ by (17.9). To complete the proof that $f$ is either hyperbolic, parabolic or elliptic with respect to $J$, corresponding to $\epsilon=1,0$ or -1 , respectively, it remains to show that $\nabla_{T}^{h} J=0$ for any $T \in \Gamma(\Delta)$.

Proceeding as in the proof of Lemma 11.1, from part (ii) we obtain

$$
T(a) I+T(b) J+b \nabla_{T}^{h} J=0
$$

for any $T \in \Gamma(\Delta)$. Hence

$$
T(a) J+\epsilon T(b) I+b\left(\nabla_{T}^{h} J\right) J=0 \text { and } T(a) J+\epsilon T(b) I+b J\left(\nabla_{T}^{h} J\right)=0
$$

Adding the two equations yields $T(a)=T(b)=0$, and hence $\nabla_{T}^{h} J=0$.
Conversely, consider the trivial vector bundle $\mathcal{E}=M^{n} \times \mathbb{L}^{3}$, where $\mathbb{L}^{3}$ is the three-dimensional Lorentz space. Let $\mu, \zeta, \bar{\zeta}$ be an orthonormal frame of $\mathcal{E}$ such that $\langle\zeta, \zeta\rangle=-1$. Define on $\mathcal{E}$ the connection $\nabla^{\prime}$ determined by (17.11), (17.12) and (17.14), with $\omega \in \Gamma\left(T^{*} M\right)$ given by (17.16). Let $\alpha \in C^{\infty}(\operatorname{Hom}(T M \times T M, \mathcal{E}))$ be given by

$$
\alpha=\langle A,\rangle \mu-\left\langle A_{\zeta},\right\rangle \zeta+\left\langle A_{\bar{\zeta}},\right\rangle \bar{\zeta}
$$

where $A_{\zeta}=A-\lambda I$ and $A_{\bar{\zeta}}=A_{\zeta} \circ D$.
Notice that one cannot have $C\left(\Gamma(\Delta) \subset \operatorname{span}\{I\}\right.$ on any open subset $U \subset M^{n}$. Otherwise, the distribution $\Delta^{\perp}$ would be umbilical on $U$, and hence $\left.f\right|_{U}$ would be conformally surface-like by Corollary 9.33 .

Since $C(\Gamma(\Delta) \subset \operatorname{span}\{I, J\}$ and $C(\Gamma(\Delta) \not \subset \operatorname{span}\{I\}$ on any open subset, it follows from the first equation of 17.8 ) that

$$
\begin{equation*}
(A-\lambda I) J=J^{t}(A-\lambda I) . \tag{17.21}
\end{equation*}
$$

Thus $A_{\bar{\zeta}}$, and hence $\alpha$, is symmetric. We claim that $\left(\mathcal{E}, \nabla^{\prime}, \alpha\right)$ satisfies the Gauss, Codazzi and Ricci equations for an isometric immersion into $\mathbb{L}^{n+3}$.

The Gauss equation can be written as

$$
R(X, Y)=A X \wedge A Y-A_{\zeta} X \wedge A_{\zeta} Y+A_{\bar{\zeta}} X \wedge A_{\bar{\zeta}} Y
$$

for all $X, Y \in \mathfrak{X}(M)$. Since $\operatorname{ker} A_{\zeta}=\Delta=\operatorname{ker} A_{\bar{\zeta}}$, this follows immediately from the Gauss equation of $f$ if either $X$ or $Y$ belongs to $\Delta$. If $\{X, Y\}$ spans $\Delta^{\perp}$, it is a consequence of the Gauss equation of $f$ and the fact that

$$
\begin{aligned}
A_{\bar{\zeta}} X \wedge A_{\bar{\zeta}} Y & =\operatorname{det} A_{\bar{\zeta}}(X \wedge Y) \\
& =\operatorname{det} A_{\zeta}(X \wedge Y) \\
& =A_{\zeta} X \wedge A_{\zeta} Y,
\end{aligned}
$$

where we used that $\operatorname{det} D=1$ in the second equality.
Using the Codazzi equation for $A$, the Codazzi equation for $A_{\mu}=A$ reduces to

$$
A_{\nabla_{X}^{\prime} \mu} Y=A_{\nabla_{Y}^{\prime} \mu} X
$$

for all $X, Y \in \mathfrak{X}(M)$. The preceding equation is trivially satisfied if either $X$ or $Y$ belongs to $\Delta$, because $\zeta$ is parallel along $\Delta$ with respect to $\nabla^{\prime}$ and $\operatorname{ker} A_{\zeta}=\Delta=\operatorname{ker} A_{\bar{\zeta}}$. If $\{X, Y\}$ spans $\Delta^{\perp}$, it is equivalent to 17.15, which follows from

$$
\begin{aligned}
\lambda \omega(X) D Y-\lambda \omega(Y) D X & =\langle D Y, \operatorname{grad} \lambda\rangle D X-\langle D X, \operatorname{grad} \lambda\rangle D Y \\
& =D X \wedge D Y(\operatorname{grad} \lambda) \\
& =X \wedge Y(\operatorname{grad} \lambda) \\
& =Y(\lambda) X-X(\lambda) Y
\end{aligned}
$$

where we used again that $\operatorname{det} D=1$.
To verify the Codazzi equation for $A_{\zeta}$, it suffices to do the same for $A_{\mu-\zeta}=\lambda I$, that is, it suffices to verify that

$$
X(\lambda) Y-A_{\nabla_{X}^{\prime}(\mu-\zeta)} Y=Y(\lambda) X-A_{\nabla_{Y}^{\prime}(\mu-\zeta)} X
$$

for all $X, Y \in \mathfrak{X}(M)$. Again, this is trivial if either $X$ or $Y$ belongs to $\Delta$, because $\lambda$ is constant along $\Delta$ and $\mu-\zeta$ is parallel along $\Delta$ with respect to $\nabla^{\prime}$. For $X, Y \in \Gamma\left(\Delta^{\perp}\right)$, it follows from

$$
\nabla_{X}^{\prime}(\mu-\zeta)=\frac{1}{\lambda} X(\lambda)(\mu-\zeta)
$$

The Codazzi equation for $A_{\bar{\zeta}}$ is

$$
\begin{align*}
\left(\nabla_{X}(A-\lambda I) D\right) Y-\left(\nabla_{Y}(A-\lambda I) D\right) X & =A_{\nabla_{X}^{\prime} \bar{\zeta}} Y-A_{\nabla_{Y}^{\prime} \bar{\zeta}} X \\
& =\lambda \omega(X) Y-\lambda \omega(Y) X \tag{17.22}
\end{align*}
$$

for all $X, Y \in \mathfrak{X}(M)$. For $X \in \Gamma\left(\Delta^{\perp}\right)$ and $Y=T \in \Gamma(\Delta)$, the horizontal component of 17.22 is

$$
\begin{equation*}
(A-\lambda I) D C_{T}=\nabla_{T}^{h}(A-\lambda I) D, \tag{17.23}
\end{equation*}
$$

whereas the vertical component can be written as

$$
\begin{equation*}
\left\langle(A-\lambda I) \nabla_{T} S, D X\right\rangle=\lambda \omega(X)\langle T, S\rangle \tag{17.24}
\end{equation*}
$$

for all $T, S \in \Gamma(\Delta)$.
To prove (17.23), first notice that condition $(i)$ and $C(\Delta) \subset \operatorname{span}\{I, J\}$ imply that

$$
\left[D, C_{T}\right]=0
$$

for all $T \in \Gamma(\Delta)$. On the other hand,

$$
\nabla_{T}^{h} A=(A-\lambda I) C_{T}
$$

by the Codazzi equation of $f$. Therefore

$$
\begin{aligned}
\nabla_{T}^{h}(A-\lambda I) D & =\nabla_{T}^{h} A D-\lambda \nabla_{T}^{h} D \\
& =\left(\nabla_{T}^{h} A\right) D+(A-\lambda I) \nabla_{T}^{h} D \\
& =(A-\lambda I) C_{T} D \\
& =(A-\lambda I) D C_{T},
\end{aligned}
$$

where we have also used part (ii) in the third equality. By (1.28) we have

$$
(A-\lambda I) \nabla_{T} S=-\langle T, S\rangle \operatorname{grad} \lambda
$$

for all $T, S \in \Gamma(\Delta)$. Hence (17.24) follows from (17.16). Finally, that (17.22) holds for $X, Y \in \Gamma\left(\Delta^{\perp}\right)$ follows from part (iv) and 17.16 ).

Let $R^{\prime}$ denote the curvature tensor of $\left(\varepsilon, \nabla^{\prime}\right)$. It follows easily from (17.11) that the left-hand side of the Ricci equation

$$
\left\langle R^{\prime}(X, Y) \mu, \zeta\right\rangle=\left\langle\left[A_{\mu}, A_{\zeta}\right] X, Y\right\rangle
$$

vanishes, and the same holds for the right-hand side since $A_{\mu}=A$ and $A_{\zeta}=(A-\lambda I)$ commute. From (17.10) and (17.12) we obtain

$$
R^{\prime}(X, Y) \bar{\zeta}=(d \omega(X, Y)-(1 / \lambda) Y(\lambda) \omega(X)+(1 / \lambda) X(\lambda) \omega(Y))(\mu-\zeta)
$$

On the other hand,

$$
\begin{aligned}
\left\langle\left[A_{\bar{\zeta}}, A_{\mu}\right] X, Y\right\rangle & =\langle A X,(A-\lambda I) D Y\rangle-\langle(A-\lambda I) D X, A Y\rangle \\
& =\left\langle\left[A_{\bar{\zeta}}, A_{\zeta}\right] X, Y\right\rangle,
\end{aligned}
$$

where in the second equality we have used that $(A-\lambda I) D$ is symmetric. Thus the Ricci equations

$$
\left\langle R^{\prime}(X, Y) \bar{\zeta}, \mu\right\rangle=\left\langle\left[A_{\bar{\zeta}}, A_{\mu}\right] X, Y\right\rangle
$$

and

$$
\left\langle R^{\prime}(X, Y) \bar{\zeta}, \zeta\right\rangle=\left\langle\left[A_{\bar{\zeta}}, A_{\zeta}\right] X, Y\right\rangle
$$

are both equivalent to

$$
\begin{align*}
& d \omega(X, Y)-\frac{1}{\lambda}(Y(\lambda) \omega(X)-X(\lambda) \omega(Y))  \tag{17.25}\\
& \quad=\langle A X,(A-\lambda I) D Y\rangle-\langle(A-\lambda I) D X, A Y\rangle
\end{align*}
$$

for all $X, Y \in \mathfrak{X}(M)$. In view of (17.19), Eq. 17.25) follows from part ( $v$ ), and this completes the proof of the claim.

By Theorem 1.25, there exist an isometric immersion $\tilde{F}: M^{n} \rightarrow \mathbb{L}^{n+3}$ and a vector bundle isometry $\Phi: \mathcal{E} \rightarrow N_{\tilde{F}} M$ such that $\Phi \alpha=\alpha^{\tilde{F}}$ and $\Phi \nabla^{\prime}=\nabla^{\perp} \Phi$. Moreover, the vector field $\rho=(1 / \lambda) \Phi(\zeta-\mu)$ satisfies

$$
\begin{aligned}
\lambda \tilde{\nabla}_{X} \rho & =-X(\lambda) \rho+\tilde{\nabla}_{X} \Phi(\zeta-\mu) \\
& =-X(\lambda) \rho-\tilde{F}_{*} A_{\Phi(\zeta-\mu)}^{\tilde{F}} X+\nabla_{X}^{\perp} \Phi(\zeta-\mu) \\
& =-X(\lambda) \rho-\tilde{F}_{*}\left(A_{\zeta}-A\right) X+\Phi \nabla_{X}^{\prime}(\zeta-\mu) \\
& =\lambda \tilde{F}_{*} X
\end{aligned}
$$

for all $X \in \mathfrak{X}(M)$. Therefore

$$
\tilde{\nabla}_{X}(\tilde{F}-\rho)=0
$$

for all $X \in \mathfrak{X}(M)$. Hence $\tilde{F}-\rho$ is a constant vector $P_{0} \in \mathbb{L}^{n+3}$. It follows that

$$
\left\langle\tilde{F}-P_{0}, \tilde{F}-P_{0}\right\rangle=\langle\rho, \rho\rangle=0,
$$

that is, $\tilde{F}$ takes values in $P_{0}+\mathbb{V}^{n+2}$. Thus $\tilde{F}$ gives rise to a conformal immersion $\tilde{f}=\mathcal{C}(\tilde{F}): M^{n} \rightarrow \mathbb{R}^{n+1}$ by Proposition 9.9.

We now show that $\tilde{f}$ is not conformally congruent to $f$ on any open subset of $M^{n}$. Assume otherwise that $U \subset M^{n}$ is an open subset such that $\left.\tilde{f}\right|_{U}$ is conformally congruent to $\left.f\right|_{U}$. Then $\left.\tilde{F}\right|_{U}$ is congruent to $\left.F\right|_{U}$ by Proposition 9.18 , where $F=$ $\mathcal{J}(f)=\Psi \circ f$, that is, there exists $T \in \mathbb{O}_{1}(n+3)$ such that $T \circ F=F$. In particular, $\alpha^{\mathscr{F}}=T \circ \alpha^{F}$.

Write

$$
\left\{\begin{array}{l}
T \Psi_{*} N=a_{11} \mu+a_{12} \zeta+a_{13} \bar{\zeta} \\
T w=a_{21} \mu+a_{22} \zeta+a_{23} \bar{\zeta}
\end{array}\right.
$$

From

$$
\begin{aligned}
\langle A X, Y\rangle \mu-\langle(A-\lambda I) X, Y\rangle \zeta+\langle(A-\lambda I) D X, Y\rangle \bar{\zeta} & =\alpha^{\tilde{F}}(X, Y) \\
& =T \circ \alpha^{F}(X, Y) \\
& =\langle A X, Y\rangle T \Psi_{*} N-\langle X, Y\rangle T w
\end{aligned}
$$

for all $X, Y \in \Gamma\left(\left.\Delta^{\perp}\right|_{U}\right)$ we obtain

$$
\left\{\begin{array}{l}
a_{11} A-a_{21} I=A  \tag{17.26}\\
a_{12} A-a_{22} I=-(A-\lambda I) \\
a_{13} A-a_{23} I=(A-\lambda I) D
\end{array}\right.
$$

Set $a=a_{11}$. Since

$$
\begin{gathered}
T F=\tilde{F}=\frac{1}{\lambda}(\zeta-\mu), \\
\left\langle T \Psi_{*} N, T \Psi_{*} N\right\rangle=1=\langle T w, T F\rangle
\end{gathered}
$$

and

$$
0=\langle T w, T w\rangle=\left\langle T \Psi_{*} N, T F\right\rangle=\left\langle T \Psi_{*} N, T w\right\rangle
$$

then

$$
a_{12}=-a, \quad a_{13}=\delta= \pm 1, \quad a_{21}=\frac{\lambda}{2}\left(a^{2}-1\right) \text { and } a_{22}=-\frac{\lambda}{2}\left(a^{2}+1\right) .
$$

If $a=1$, then the last equation in 17.26 reduces to

$$
(A-\lambda I) D=(A-\lambda I) \delta I .
$$

Hence $D=\delta I$. Otherwise, either of the first two equations in (17.26) imply that

$$
A=\frac{\lambda}{2}(a+1) I .
$$

Then the last equation gives

$$
\lambda \delta(a-1) \delta I=\lambda \delta(a-1) D
$$

and we conclude as before that $D=\delta I$. This is a contradiction with part $(i)$ and shows that $\left.\tilde{f}\right|_{U}$ is not conformally congruent to $\left.f\right|_{U}$.

It remains to prove that if two tensors $D_{1}, D_{2} \in \Gamma\left(\Delta^{\perp}\right)$ satisfying conditions (i) to $(v)$ do not coincide up to sign on any open subset of $M^{n}$, then they give rise to conformal immersions $\tilde{f}_{1}: M^{n} \rightarrow \mathbb{R}^{n+1}$ and $\tilde{f}_{2}: M^{n} \rightarrow \mathbb{R}^{n+1}$ that are not conformally congruent on any open subset of $M^{n}$. In other words, we must show that if $\left.\tilde{f}_{1}\right|_{U}$ and $\left.\tilde{f}_{2}\right|_{U}$ are conformally congruent for some open subset $U \subset M^{n}$, then $D_{2}= \pm D_{1}$ on $U$. For simplicity, we may assume that $U=M^{n}$.

By Proposition 9.18 , if $\tilde{f}_{1}$ and $\tilde{f}_{2}$ are conformally congruent, then their isometric light-cone representatives $\tilde{F}_{1}=\mathcal{J}\left(\tilde{f}_{1}\right)$ and $\tilde{F}_{2}=\mathcal{J}\left(\tilde{f}_{2}\right)$ are congruent, that is, there exists $T \in \mathbb{O}_{1}(n+3)$ such that $T \circ \tilde{F}_{1}=\tilde{F}_{2}$. In particular, $\alpha^{\tilde{F}_{2}}=T \circ \alpha^{\tilde{F_{1}}}$. We have

$$
\alpha^{\tilde{F}_{i}}(X, Y)=\langle A X, Y\rangle \mu_{i}-\langle(A-\lambda I) X, Y\rangle \zeta_{i}+\left\langle(A-\lambda I) D_{i} X, Y\right\rangle \bar{\zeta}_{i}
$$

for all $X, Y \in \Gamma\left(\Delta^{\perp}\right)$, and

$$
\tilde{F}_{i}=\frac{1}{\lambda}\left(\zeta_{i}-\mu_{i}\right),
$$

where $\mu_{i}, \zeta_{i}, \bar{\zeta}$ is an orthonormal frame of $N_{\tilde{F}_{i}} M$ such that $\left\langle\mu_{i}, \mu_{i}\right\rangle=1=\left\langle\bar{\zeta}_{i}, \bar{\zeta}_{i}\right\rangle$ and $\left\langle\zeta_{i}, \zeta_{i}\right\rangle=-1,1 \leq i \leq 2$. Write

$$
\left\{\begin{array}{l}
T \mu_{1}=b_{11} \mu_{2}+b_{12} \zeta_{2}+b_{13} \bar{\zeta}_{2} \\
T \zeta_{1}=b_{21} \mu_{2}+b_{22} \zeta_{2}+b_{23} \bar{\zeta}_{2} \\
T \bar{\zeta}_{1}=b_{31} \mu_{2}+b_{32} \zeta_{2}+b_{33} \bar{\zeta}_{2} .
\end{array}\right.
$$

From $\alpha^{\tilde{F_{2}}}=T \circ \alpha^{\tilde{F_{1}}}$ we obtain

$$
\left\{\begin{array}{l}
b_{11} A-b_{21}(A-\lambda I)+b_{31}(A-\lambda I) D_{1}=A \\
b_{12} A-b_{22}(A-\lambda I)+b_{32}(A-\lambda I) D_{1}=-(A-\lambda I) \\
b_{13} A-b_{23}(A-\lambda I)+b_{33}(A-\lambda I) D_{1}=(A-\lambda I) D_{2},
\end{array}\right.
$$

whereas $T \tilde{F}_{1}=\tilde{F}_{2}$ gives

$$
\left\{\begin{array}{l}
b_{21}-b_{11}=-1 \\
b_{22}-b_{12}=1 \\
b_{23}-b_{13}=0
\end{array}\right.
$$

Therefore

$$
\left\{\begin{array}{l}
\lambda b_{21} I+b_{31}(A-\lambda I) D_{1}=0 \\
\lambda b_{12} I+b_{32}(A-\lambda I) D_{1}=0 \\
\lambda b_{23} I+b_{33}(A-\lambda I) D_{1}=(A-\lambda I) D_{2}
\end{array}\right.
$$

In particular, the first two of the preceding equations imply that $b_{21} b_{32}=b_{12} b_{31}$. Set $b_{31}=b$. Using that the matrix $B=\left(b_{i j}\right)$ satisfies $B \theta B^{t}=\theta$, where $\theta=\operatorname{diag}(1,-1,1)$, it follows that

$$
B=\left(\begin{array}{ccc}
1-\frac{1}{2} b^{2} & \frac{1}{2} b^{2} & -\delta b \\
-\frac{1}{2} b^{2} & 1+\frac{1}{2} b^{2} & -\delta b \\
b & -b & \delta
\end{array}\right)
$$

If $b=0$, then the last of the preceding equations yields

$$
(A-\lambda I)\left(D_{2}-\delta D_{1}\right)=0 .
$$

Hence $D_{2}=\delta D_{1}$. Otherwise, either of the remaining ones gives

$$
(A-\lambda I) D_{1}=\frac{1}{2} \lambda b I,
$$

and then the last one yields

$$
(A-\lambda I) D_{2}=\frac{\delta}{2} \lambda b I .
$$

Thus, also in this case, we conclude that $D_{2}=\delta D_{1}$.
A hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, is said to be conformally ruled if it carries an umbilical distribution $L$ of rank $n-1$ such that the restriction of $f$ to each leaf of $L$ is also umbilical.

The next result shows that parabolic Cartan hypersurfaces are precisely the conformally ruled ones.

Proposition 17.3. Any parabolic Cartan hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ is conformally ruled.

Conversely, if $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 3$, is a simply connected conformally ruled hypersurface free of points with a principal curvature of multiplicity at least $n-1$ which is not conformally surface-like on any open subset of $M^{n}$, then $f$ is a parabolic Cartan hypersurface. Moreover, all conformal immersions of $M^{n}$ into $\mathbb{R}^{n+1}$ that are not conformally congruent to $f$ on any open subset are conformally ruled with the same rulings, and their congruence classes are in one-to-one correspondence with the smooth functions on an open interval.

Proof: Let $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ be such that $J^{2}=0, J \neq 0, \nabla_{T}^{h} J=0$ for all $T \in \Gamma(\Delta)$ and $C(\Gamma(\Delta)) \subset \operatorname{span}\{I, J\}$. Choose $Y \in \Gamma(\operatorname{ker} J)$ of unit length and $X \in \Gamma\left(\Delta^{\perp}\right)$ orthogonal to $Y$ such that $J X=Y$. Arguing as in the beginning of the proof of Proposition 11.2, we may assume that also $X$ has unit length.

We prove next that the distribution $L=\Delta \oplus \operatorname{span}\{Y\}$ is umbilical. Since $C(\Gamma(\Delta)) \subset \operatorname{span}\{I, J\}$ and $J Y=0$, it follows that $\left\langle C_{T} Y, X\right\rangle=0$ for all $T \in \Gamma(\Delta)$. Hence

$$
\begin{equation*}
\left\langle\nabla_{Y} T, X\right\rangle=-\left\langle C_{T} Y, X\right\rangle=0 \tag{17.27}
\end{equation*}
$$

On the other hand,

$$
0=\left(\nabla_{T}^{h} J\right) Y=-J \nabla_{T}^{h} Y
$$

for all $T \in \Gamma(\Delta)$. Hence $\nabla_{T}^{h} Y=0$, or equivalently,

$$
\begin{equation*}
\left\langle\nabla_{T} Y, X\right\rangle=0 \tag{17.28}
\end{equation*}
$$

Let $D \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ satisfy conditions $(i)$ to $(v)$ in Lemma 17.2. Since $D \in$ $\operatorname{span}\{I, J\}$ and $\operatorname{det} D=1$, there exists $\theta \in C^{\infty}(M)$ such that

$$
D=\delta I+\theta J
$$

where $\delta= \pm 1$. Now observe that

$$
(A-\lambda I) D=\delta(A-\lambda I)+\theta(A-\lambda I) J
$$

The tensor

$$
\Phi=\theta(A-\lambda I) J
$$

is symmetric by 17.21 . By part (iv) and the Codazzi equation for $A$ it satisfies

$$
\begin{align*}
\left(\nabla_{X} \Phi\right) Y-\left(\nabla_{Y} \Phi\right) X & =\langle(D-\delta I) Y, \operatorname{grad} \lambda\rangle X-\langle(D-\delta I) X, \operatorname{grad} \lambda\rangle Y \\
& =\langle\theta J Y, \operatorname{grad} \lambda\rangle X-\langle\theta J X, \operatorname{grad} \lambda\rangle Y \\
& =-\theta Y(\lambda) Y \tag{17.29}
\end{align*}
$$

Writing

$$
\begin{aligned}
\mu & =\langle A Y, X\rangle \\
& =\langle(A-\lambda I) Y, X\rangle,
\end{aligned}
$$

it follows that $\Phi X=\theta \mu X$ and $\Phi Y=0$. Substituting in (17.29) and taking the inner product of both sides with $Y$ yield

$$
\begin{equation*}
\mu\left\langle\nabla_{Y} Y, X\right\rangle=-Y(\lambda) \tag{17.30}
\end{equation*}
$$

Similarly, the inner product of both sides with $X$ gives

$$
\begin{equation*}
\theta \mu\left\langle\nabla_{X} X, Y\right\rangle=Y(\theta \mu) \tag{17.31}
\end{equation*}
$$

On the other hand, taking the inner product with $T \in \Gamma(\Delta)$ of unit length of both sides of the Codazzi equation

$$
\nabla_{T} A Y-A \nabla_{T} Y=\nabla_{Y} A T-A \nabla_{Y} T
$$

we obtain

$$
\begin{equation*}
\left\langle\nabla_{T} T,(A-\lambda I) Y\right\rangle=-Y(\lambda) \tag{17.32}
\end{equation*}
$$

Now, from (17.21) and $J Y=0$ we have

$$
\begin{align*}
\langle(A-\lambda I) Y, Y\rangle & =\langle(A-\lambda I) J X, Y\rangle \\
& =\langle X,(A-\lambda I) J Y\rangle \\
& =0 . \tag{17.33}
\end{align*}
$$

Using 17.32 we obtain

$$
\begin{equation*}
\mu\left\langle\nabla_{T} T, X\right\rangle=-Y(\lambda) \tag{17.34}
\end{equation*}
$$

for any $T \in \Gamma(\Delta)$ of unit length. It follows from (17.27), 17.28$),(17.30)$ and (17.34) that the distribution $L$ is umbilical. Finally, Eq. (17.33) and $\Delta \subset \operatorname{ker}(A-\lambda I)$ imply that the restriction of $f$ to each leaf of $L$ is umbilical. Thus $f$ is conformally ruled.

We now prove the converse. Let $L$ be an umbilical distribution of rank $n-1$ on $M^{n}$ such that the restriction of $f$ to each leaf of $L$ is also umbilical. Then, at each point $x \in M^{n}$, the subspace $(A-\lambda I) L(x)$ is contained in the one-dimensional subspace $L^{\perp}(x)$. Thus the kernel $\Delta(x)$ of $A-\lambda I$ at $x$ has dimension at least $n-2$. Since $\Delta(x)$ cannot have dimension greater than $n-2$ by assumption, it follows that $\lambda$ is a principal curvature of $f$ with constant multiplicity $n-2$. Since $M^{n}$ is simply connected, there is a global orthonormal frame $X, Y$ of $\Delta^{\perp}$ such that $X$ is orthogonal to $L$. In particular,

$$
\begin{equation*}
\langle(A-\lambda I) Y, Y\rangle=0 \tag{17.35}
\end{equation*}
$$

Define $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ by setting

$$
J X=Y \text { and } J Y=0
$$

We prove that $f$ is parabolic with respect to $J$. First notice that

$$
J^{t} X=0 \text { and } J^{t} Y=X,
$$

hence (17.35) implies that

$$
(A-\lambda I) J=J^{t}(A-\lambda I) .
$$

Now, since $L$ is umbilical, then

$$
\begin{equation*}
\nabla_{T}^{h} Y=0, \tag{17.36}
\end{equation*}
$$

which is equivalent to $\nabla_{T}^{h} J=0$. To show that $C(\Gamma(\Delta)) \subset \operatorname{span}\{I, J\}$ it suffices to prove that

$$
C_{T} \circ J=J \circ C_{T}
$$

for all $T \in \Gamma(\Delta)$, which is easily seen to be equivalent to

$$
\left\langle\nabla_{Y} T, X\right\rangle=0
$$

and

$$
\left\langle\nabla_{X} X, T\right\rangle=\left\langle\nabla_{Y} Y, T\right\rangle
$$

for all $T \in \Gamma(\Delta)$. The first equation follows from the fact that $L$ is umbilical. To prove the latter, set

$$
\begin{aligned}
\mu & =\langle A X, Y\rangle \\
& =\langle(A-\lambda I) X, Y\rangle
\end{aligned}
$$

so that $(A-\lambda I) Y=\mu X$. Now, taking the $Y$-component of the Codazzi equation

$$
\begin{aligned}
\nabla_{T}^{h}(A-\lambda I) & =\nabla_{T}^{h} A \\
& =(A-\lambda I) C_{T}
\end{aligned}
$$

applied to $X$, the $X$-component applied to $Y$ and using (17.36) give

$$
\left\langle\nabla_{X} X, T\right\rangle=T(\log \mu)=\left\langle\nabla_{Y} Y, T\right\rangle .
$$

It remains to prove the last assertion. According to Lemma 17.2, each tensor $\underset{\sim}{D} \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ satisfying conditions $(i)$ to $(v)$ gives rise to a conformal immersion $\tilde{f}: M^{n} \rightarrow \mathbb{R}^{n+1}$ that is not conformally congruent to $f$ on any open subset of $M^{n}$, and two such tensors that do not coincide up to sign on any open subset of $M^{n}$ give rise to conformal immersions of $M^{n}$ into $\mathbb{R}^{n+1}$ that are not conformally congruent on any open subset of $M^{n}$.

Any $D \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ satisfying conditions $(i)$ to (iii) is given by

$$
\begin{equation*}
D=\delta I+\theta J \tag{17.37}
\end{equation*}
$$

where $\delta= \pm 1, \theta \in C^{\infty}(M)$ is nowhere vanishing and $T(\theta)=0$ for all $T \in \Gamma(\Delta)$. Now, part $(v)$ holds for $D$ if and only if it is satisfied for $\theta J$ in the place of $D$. We have

$$
\begin{aligned}
\left\langle\left(\nabla_{Y} \theta J\right) X-\left(\nabla_{X} \theta J\right) Y, \operatorname{grad} \lambda\right\rangle & =\left\langle Y(\theta) Y+\theta \nabla_{Y} Y+\theta J \nabla_{X} Y, \operatorname{grad} \lambda\right\rangle \\
& =\left(Y(\theta)-\theta\left\langle\nabla_{X} X, Y\right\rangle\right) Y(\lambda)+\theta\left\langle\nabla_{Y} Y, X\right\rangle X(\lambda) \\
& =-\frac{\theta}{\mu} Y(\lambda)(Y(\mu)+X(\lambda))
\end{aligned}
$$

where for the last equality we have used (17.30) and (17.31). On the other hand,

$$
\text { Hess } \begin{aligned}
\lambda(\theta J X, Y)-\text { Hess } \lambda(X, \theta J Y) & =\theta\left(Y Y(\lambda)-\left\langle\nabla_{Y} Y, X\right\rangle X(\lambda)\right) \\
& =\theta(Y Y(\lambda)+(1 / \mu) Y(\lambda) X(\lambda))
\end{aligned}
$$

and

$$
\lambda(\langle(A-\lambda I) \theta J X, A Y\rangle-\langle A X,(A-\lambda I) \theta J Y\rangle)=\lambda \theta \mu^{2}
$$

Therefore part $(v)$ for $D$ is equivalent to

$$
Y Y(\lambda)-\frac{1}{\mu} Y(\lambda) Y(\mu)=-\lambda \mu^{2}
$$

which can also be written as

$$
\begin{equation*}
Y((1 / \mu) Y(\lambda))=-\lambda \mu . \tag{17.38}
\end{equation*}
$$

We claim that 17.38 is a consequence of the Gauss equation

$$
\begin{align*}
\langle R(Y, T) T, X\rangle & =\langle A T, T\rangle\langle A Y, X\rangle-\langle A Y, T\rangle\langle A T, X\rangle \\
& =\lambda \mu . \tag{17.39}
\end{align*}
$$

Indeed, on one hand,

$$
\begin{aligned}
\left\langle\nabla_{Y} \nabla_{T} T, X\right\rangle & =Y\left\langle\nabla_{T} T, X\right\rangle-\left\langle\nabla_{T} T, \nabla_{Y} X\right\rangle \\
& =-Y((1 / \mu) Y(\lambda))+\left\langle\nabla_{T} T, Y\right\rangle\left\langle\nabla_{Y} Y, X\right\rangle .
\end{aligned}
$$

On the other hand,

$$
\left\langle\nabla_{T} \nabla_{Y} T, X\right\rangle=-\left\langle\nabla_{Y} T, \nabla_{T} X\right\rangle=0
$$

and

$$
\begin{aligned}
\left\langle\nabla_{[Y, T]} T, X\right\rangle & =-\left\langle\nabla_{\nabla_{T} Y} T, X\right\rangle \\
& =\left\langle\nabla_{T} T, Y\right\rangle\left\langle\nabla_{T} T, X\right\rangle .
\end{aligned}
$$

Therefore

$$
\langle R(Y, T) T, X\rangle=-Y((1 / \mu) Y(\lambda)),
$$

and hence 17.39 is equivalent to 17.38 .
Finally, in view of the Codazzi equation of $f$, the tensor $D$ given by 17.37) satisfies part (iv) if and only if (17.29) holds for the tensor $\Phi=\theta(A-\lambda I) J$. As shown in the proof of the direct statement, the $Y$-component of that equation is equivalent to

$$
\mu\left\langle\nabla_{Y} Y, X\right\rangle=-Y(\lambda)
$$

which is satisfied because the distribution $L=\Delta \oplus \operatorname{span}\{Y\}$ is umbilical, and

$$
\mu\left\langle\nabla_{T} T, X\right\rangle=-Y(\lambda)
$$

is Eq. (17.34) for any $T \in \Gamma(\Delta)$ of unit length. On the other hand, the $X$-component of Eq. (17.29) is equivalent to the equation

$$
Y(\log \theta \mu)=\left\langle\nabla_{X} X, Y\right\rangle .
$$

Choosing an arbitrary smooth function as initial condition along one maximal integral curve of $X$, there exists a unique function $\theta$ such that $T(\theta)=0$ for all $T \in \Gamma(\Delta)$ and $\theta \mu$ is a solution of the preceding equation.

### 17.3 The reduction

Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 5$, be an oriented hypersurface with a nowhere vanishing principal curvature $\lambda$ of constant multiplicity $n-2$ with respect to a unit normal vector field $N$. By Proposition 9.4, the map $h: M^{n} \rightarrow \mathbb{R}^{n+1}$ given by

$$
h(x)=f(x)+\frac{1}{\lambda(x)} N(x)
$$

and the function $r=1 / \lambda$ determine a two-parameter congruence of hyperspheres $S: M^{n} \rightarrow \mathbb{S}_{1,1}^{n+2}$ that is enveloped by $f$, with $\operatorname{ker} S_{*}=E_{\lambda}$. By 9.7 , the map $S$ is given by

$$
\begin{equation*}
S(x)=\lambda(x) \Psi(f(x))+\Psi_{*}(f(x)) N(x) \tag{17.40}
\end{equation*}
$$

and gives rise to a map $s: M^{n} \rightarrow \mathbb{S}_{1,1}^{n+2}$ such that $S \circ \pi=s$, where $\pi: M^{n} \rightarrow L^{2}$ is the canonical projection onto the quotient space of leaves of $E_{\lambda}$.

The next result shows that if $f$ is either hyperbolic or elliptic, then the problem of whether there exists a tensor $D$ satisfying all the conditions in Lemma 17.2 can be reduced to a similar but easier one for the surface $s$.

Lemma 17.4. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be a hypersurface that envelops a two-parameter congruence of hyperspheres $s: L^{2} \rightarrow \mathbb{S}_{1,1}^{n+2} \subset \mathbb{L}^{n+3}$ and is not conformally surface-like on any open subset of $M^{n}$. Let $\Delta$ be the eigenbundle of $f$ correspondent to its principal curvature $\lambda$ of multiplicity $n-2$. If $f$ is hyperbolic (respectively, elliptic) with respect to $J \in \Gamma\left(E n d\left(\Delta^{\perp}\right)\right)$ and there exists $D \in \Gamma\left(E n d\left(\Delta^{\perp}\right)\right)$ satisfying conditions (i) to (v) in Lemma 17.2 , then $J$ and $D$ are the horizontal lifts of tensors $\bar{J}$ and $\bar{D} \in \operatorname{span}\{\bar{I}, \bar{J}\}$ on $L^{2}$, with $J^{2}=\bar{I}$ (respectively, $\bar{J}^{2}=-\bar{I}$ ), the surface $s$ is hyperbolic (respectively, elliptic) with respect to $\bar{J}$ and the tensor $\bar{D}$ satisfies:
(i) $\operatorname{det} \bar{D}=1$,
(ii) $\left(\nabla_{\bar{X}}^{\prime} \bar{D}\right) \bar{Y}-\left(\nabla_{\bar{Y}}^{\prime} \bar{D}\right) \bar{X}=0$ for all $\bar{X}, \bar{Y} \in \mathfrak{X}(L)$, where $\nabla^{\prime}$ is the Levi-Civita connection of the metric induced by $s$.

Conversely, if $s: L^{2} \rightarrow \mathbb{S}_{1,1}^{n+2} \subset \mathbb{L}^{n+3}$ is hyperbolic (respectively, elliptic) with respect to a tensor $\bar{J}$ on $L^{2}$ satisfying $\bar{J}^{2}=\bar{I}$ (respectively, $\bar{J}^{2}=-\bar{I}$ ), then the hypersurface $f$ is hyperbolic (respectively, elliptic) with respect to the horizontal lift $J$ of $\bar{J}$, and the horizontal lift $D$ of a tensor $\bar{D} \in \operatorname{span}\{\bar{I}, \bar{J}\}$ satisfying parts (i) and (ii) has all the properties $(i)$ to $(v)$ in Lemma 17.2.

Proof: Since $D \in \operatorname{span}\{I, J\}$ and $C_{T} \in \operatorname{span}\{I, J\}$ for all $T \in \Gamma(\Delta)$, then 17.9 holds. By Corollary 11.7, this and condition (ii) in Lemma 17.2 imply that $D$ is projectable, that is, there exists a tensor $\bar{D}$ on $L^{2}$ such that

$$
\bar{D} \circ \pi_{*}=\pi_{*} \circ D
$$

where $\pi: M^{n} \rightarrow L^{2}$ is the canonical projection. In particular, $\operatorname{det} \bar{D}=1$ by part (iii).

Since $\left[J, C_{T}\right]=0$ and $\nabla_{T}^{h} J=0$ for all $T \in \Gamma(\Delta)$, also $J$ is projectable, thus there exists a tensor $\bar{J}$ on $L^{2}$ such that

$$
\bar{J} \circ \pi_{*}=\pi_{*} \circ J
$$

Clearly, the tensor $\bar{J}$ satisfies $\bar{J}^{2}=I$ (respectively, $\bar{J}^{2}=-I$ ) if $f$ is hyperbolic (respectively, elliptic), and $\bar{D} \in \operatorname{span}\{\bar{I}, \bar{J}\}$. In the following, we prove that $\bar{D}$ satisfies condition (ii) in the statement and that $s$ is hyperbolic (respectively, elliptic) with respect to $\bar{J}$.

Differentiating 17.40) with respect to $Y \in \Gamma\left(\Delta^{\perp}\right)$ gives

$$
S_{*} Y=-\Psi_{*} f_{*}(A-\lambda I) Y+Y(\lambda) \Psi \circ f .
$$

Hence

$$
\Psi_{*} f_{*}(A-\lambda I) D Y=\langle D Y, \operatorname{grad} \lambda\rangle \Psi \circ f-S_{*} D Y .
$$

Differentiating with respect to $X \in \Gamma\left(\Delta^{\perp}\right)$ yields

$$
\begin{aligned}
\tilde{\nabla}_{X} \Psi_{*} f_{*}(A-\lambda I) D Y= & \left(\left\langle\nabla_{X} D Y, \operatorname{grad} \lambda\right\rangle+\operatorname{Hess} \lambda(X, D Y)\right) \Psi \circ f \\
& +\langle D Y, \operatorname{grad} \lambda\rangle \Psi_{*} f_{*} X-\tilde{\nabla}_{X} S_{*} D Y
\end{aligned}
$$

Given $\bar{X}, \bar{Y} \in \mathfrak{X}(L)$, let $X, Y \in \Gamma\left(\Delta^{\perp}\right)$ be their horizontal lifts to $M^{n}$. We have

$$
\begin{aligned}
\tilde{\nabla}_{X} S_{*} D Y & =\tilde{\nabla}_{\pi_{*} X} s_{*} \pi_{*} D Y \\
& =\tilde{\nabla}_{\bar{X}} s_{*} \bar{D} \bar{Y} \\
& =s_{*} \nabla_{\bar{X}}^{\prime} \bar{D} \bar{Y}+\alpha^{s}(\bar{X}, \bar{D} \bar{Y})-\langle\bar{X}, \bar{D} \bar{Y}\rangle^{\prime} s \circ \pi
\end{aligned}
$$

and

$$
\begin{aligned}
\langle\bar{X}, \bar{D} \bar{Y}\rangle^{\prime} & =\left\langle s_{*} \pi_{*} X, s_{*} \bar{D} \pi_{*} Y\right\rangle \\
& =\left\langle S_{*} X, S_{*} D Y\right\rangle \\
& =\left\langle f_{*}(A-\lambda I) X, f_{*}(A-\lambda I) D Y\right\rangle \\
& =\langle(A-\lambda I) X,(A-\lambda I) D Y\rangle .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\tilde{\nabla}_{X} \Psi_{*} f_{*}(A-\lambda I) D Y= & \left(\left\langle\nabla_{X} D Y, \operatorname{grad} \lambda\right\rangle+\operatorname{Hess} \lambda(X, D Y)\right) \Psi \circ f \\
& +\langle D Y, \operatorname{grad} \lambda\rangle \Psi_{*} f_{*} X-s_{*} \nabla_{\bar{X}} \bar{D} \bar{Y}-\alpha^{s}(\bar{X}, \bar{D} \bar{Y}) \\
& +(\langle(A-\lambda I) X,(A-\lambda I) D Y\rangle)\left(\lambda \Psi \circ f+\Psi_{*} N\right) .
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\tilde{\nabla}_{X} \Psi_{*} & f_{*}(A-\lambda I) D Y=\Psi_{*} \bar{\nabla}_{X} f_{*}(A-\lambda I) D Y+\alpha^{\Psi}\left(f_{*} X, f_{*}(A-\lambda I) D Y\right) \\
= & \Psi_{*} f_{*} \nabla_{X}(A-\lambda I) D Y+\Psi_{*}\langle A X,(A-\lambda I) D Y\rangle N-\langle X,(A-\lambda I) D Y\rangle w \\
= & \Psi_{*} f_{*}\left(\nabla_{X}(A-\lambda I) D\right) Y+\Psi_{*} f_{*}(A-\lambda I) D \nabla_{X} Y+\Psi_{*}\langle A X,(A-\lambda I) D Y\rangle N \\
& -\langle X,(A-\lambda I) D Y\rangle w \\
= & \Psi_{*} f_{*}\left(\nabla_{X}(A-\lambda I) D\right) Y+\left\langle D \nabla_{X} Y, \operatorname{grad} \lambda\right\rangle \Psi \circ f-S_{*} D \nabla_{X} Y \\
& +\langle A X,(A-\lambda I) D Y\rangle \Psi_{*} N-\langle X,(A-\lambda I) D Y\rangle w \\
= & \Psi_{*} f_{*}\left(\nabla_{X}(A-\lambda I) D\right) Y+\left\langle D \nabla_{X} Y, \operatorname{grad} \lambda\right\rangle \Psi \circ f-s_{*} \bar{D} \pi_{*} \nabla_{X} Y \\
& +\langle A X,(A-\lambda I) D Y\rangle \Psi_{*} N-\langle X,(A-\lambda I) D Y\rangle w .
\end{aligned}
$$

Comparing the two expressions for

$$
\tilde{\nabla}_{X} \Psi_{*} f_{*}(A-\lambda I) D Y-\tilde{\nabla}_{Y} \Psi_{*} f_{*}(A-\lambda I) D X
$$

that follow from the above and using that

$$
\begin{aligned}
\pi_{*} \nabla_{X} Y-\pi_{*} \nabla_{Y} X & =\pi_{*}[X, Y] \\
& =\left[\pi_{*} X, \pi_{*} Y\right] \\
& =[\bar{X}, \bar{Y}],
\end{aligned}
$$

we obtain

$$
\begin{align*}
\Psi_{*} f_{*} B(X, Y) & -\lambda \psi(X, Y) \Psi_{*} N+\varphi(X, Y) \Psi \circ f+\psi(X, Y) w \\
& =s_{*}\left(\left(\bar{\nabla}_{\bar{Y}} \bar{D}\right) \bar{X}-\left(\bar{\nabla}_{\bar{X}} \bar{D}\right) \bar{Y}\right)+\alpha^{s}(\bar{Y}, \bar{D} \bar{X})-\alpha^{s}(\bar{D} \bar{X}, \bar{Y}) \tag{17.41}
\end{align*}
$$

where

$$
\begin{aligned}
B(X, Y)= & \left(\nabla_{X}(A-\lambda I) D\right) Y-\left(\nabla_{Y}(A-\lambda I) D\right) X-X \wedge Y\left(D^{t} \operatorname{grad} \lambda\right) \\
\psi(X, Y)= & \langle Y,(A-\lambda I) D X\rangle-\langle X,(A-\lambda I) D Y\rangle \\
\varphi(X, Y)= & \left\langle\left(\nabla_{Y} D\right) X-\left(\nabla_{X} D\right) Y, \operatorname{grad} \lambda\right\rangle+\operatorname{Hess} \lambda(D X, Y)-\operatorname{Hess} \lambda(X, D Y) \\
& -\lambda(\langle(A-\lambda I) X,(A-\lambda I) D Y\rangle-\langle(A-\lambda I) D X,(A-\lambda I) Y\rangle)
\end{aligned}
$$

for all $X, Y \in \Gamma\left(\Delta^{\perp}\right)$.
It follows from parts $(i v)$ and $(v)$ that $B$ and $\varphi$ vanish. On the other hand, arguing as in the proof of Lemma 17.2 , the endomorphism $(A-\lambda I) D$ is symmetric, hence $\psi$ also vanishes identically.

Therefore (17.41) implies that condition (ii) in the statement holds, as well as

$$
\begin{equation*}
\alpha^{s}(\bar{D} \bar{X}, \bar{Y})=\alpha^{s}(\bar{X}, \bar{D} \bar{Y}) . \tag{17.42}
\end{equation*}
$$

Since $\bar{D} \in \operatorname{span}\{I, \bar{J}\}$ and $\bar{D} \notin \operatorname{span}\{I\}$, the preceding equation is equivalent to

$$
\begin{equation*}
\alpha^{s}(\bar{J} \bar{X}, \bar{Y})=\alpha^{s}(\bar{X}, \bar{J} \bar{Y}) . \tag{17.43}
\end{equation*}
$$

Thus $s$ is hyperbolic (respectively, elliptic) with respect to $\bar{J}$.
Conversely, suppose that $s: L^{2} \rightarrow \mathbb{S}_{1,1}^{n+2} \subset \mathbb{L}^{n+3}$ is hyperbolic (respectively, elliptic) with respect to a tensor $\bar{J}$ on $L^{2}$ satisfying $\bar{J}^{2}=I$ (respectively, $\bar{J}^{2}=-I$ ), and that $\bar{D} \in \operatorname{span}\{\bar{I}, \bar{J}\}$ satisfies parts $(i)$ and $(i i)$. Let $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ (respectively, $D \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ be the horizontal lift of $\bar{J}$ (respectively, $\left.\bar{D}\right)$. Let us prove that $D$ satisfies conditions $(i)$ to $(v)$ in Lemma 17.2 and that $f$ is hyperbolic (respectively, elliptic) with respect to $J$.

Conditions (i) and (iii) are clear. Since $s$ is hyperbolic (respectively, elliptic) with respect to $\bar{J}$, then 17.43$)$ holds for all $X, Y \in \Gamma\left(\Delta^{\perp}\right)$. This and the fact that $\bar{D} \in \operatorname{span}\{\bar{I}, \bar{J}\}$ imply that $(17.42)$ holds for all $\bar{X}, \bar{Y} \in \mathfrak{X}(L)$. Using that $\bar{D}$ also satisfies part (ii), it follows from (17.41) that $(A-\lambda I) D$ is a symmetric tensor and that parts (iv) and (v) hold.

To prove part (ii), recall from Corollary 11.7 that

$$
\begin{equation*}
\nabla_{T}^{h} D=\left[D, C_{T}\right] \tag{17.44}
\end{equation*}
$$

for all $T \in \Gamma(\Delta)$. Therefore

$$
\begin{aligned}
\nabla_{T}^{h}(A-\lambda I) D-(A-\lambda I) D C_{T} & =\left(\nabla_{T}^{h} A\right) D+(A-\lambda I) \nabla_{T}^{h} D-(A-\lambda I) D C_{T} \\
& =(A-\lambda I)\left(\nabla_{T}^{h} D-\left[D, C_{T}\right]\right. \\
& =0
\end{aligned}
$$

where we have used the Codazzi equation

$$
\nabla_{T}^{h}(A-\lambda I)=(A-\lambda I) C_{T}
$$

in the second equality. In particular, this implies that $(A-\lambda I) D C_{T}$ is symmetric. Hence

$$
\begin{aligned}
(A-\lambda I) D C_{T} & =C_{T}^{t} D^{t}(A-\lambda I) \\
& =C_{T}^{t}(A-\lambda I) D \\
& =(A-\lambda I) C_{T} D
\end{aligned}
$$

where we have used that $(A-\lambda I) D$ is symmetric in the second equality and that $(A-\lambda I) C_{T}=\nabla_{T}^{h} A$ is symmetric in the third equality. In view of 17.44), this proves part (ii). Moreover, from $\left[D, C_{T}\right]=0$ for all $T \in \Gamma(\Delta)$ and $J \in \operatorname{span}\{I, D\}$ it also follows that $C(\Gamma(\Delta)) \subset \operatorname{span}\{I, J\}$. Finally, arguing as in the proof of Lemma 17.2 we see that $\nabla_{T}^{h} J=0$. Thus $f$ is hyperbolic (respectively, elliptic) with respect to $J$.

### 17.4 The classification

We are now ready to state and prove the classification of Cartan hypersurfaces.
Theorem 17.5. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ be a Cartan hypersurface that is neither conformally surface-like nor conformally ruled on any open subset of $M^{n}$. Then, on each connected component of an open dense subset of $M^{n}$, the hypersurface $f$ is the envelope of a two-parameter congruence of hyperspheres $s: L^{2} \rightarrow \mathbb{S}_{1,1}^{n+2}$, which is a surface of first or second species of real or complex type.

Conversely, a simply connected hypersurface that envelops such a two-parameter congruence of hyperspheres $s: L^{2} \rightarrow \mathbb{S}_{1,1}^{n+2}$ is a Cartan hypersurface that admits either a one-parameter family of conformal deformations (continuous class) or a single one (discrete class), according to whether $s$ is of first or second species, respectively.

Proof: By Proposition 17.1, the hypersurface $f$ carries a nowhere vanishing principal curvature of constant multiplicity $n-2$, and hence it envelops a two-parameter congruence of hyperspheres $s: L^{2} \rightarrow \mathbb{S}_{1,1}^{n+2} \subset \mathbb{L}^{n+3}$ by Proposition 9.4. By Lemma 17.2 , on each connected component of an open dense subset of $M^{n}$, the hypersurface $f$ is
either hyperbolic, parabolic or elliptic with respect to $J \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$, and there exists $D \in \Gamma\left(\operatorname{End}\left(\Delta^{\perp}\right)\right)$ satisfying conditions $(i)$ to $(v)$. Proposition 17.3 rules out the possibility that $f$ be parabolic on an open subset of $M^{n}$.

By Lemma 17.4 , if $f$ is hyperbolic (respectively, elliptic) with respect to $J$, then $J$ and $D$ can be projected down to tensors $\bar{J}$ and $\bar{D} \in \operatorname{span}\{I, \bar{J}\}$ on $L^{2}$, with $\bar{J}^{2}=I$ (respectively, $\bar{J}^{2}=-I$ ). Moreover, the surface $s$ is hyperbolic (respectively, elliptic) with respect to $\bar{J}$, and the tensor $\bar{D}$ satisfies parts (i) and (ii). It follows from Proposition 11.13 that $s$ is a surface of first or second species of real or complex type.

Conversely, suppose that $f: M^{n} \rightarrow \mathbb{R}^{n+1}$ is a simply connected hypersurface that envelops a two-parameter congruence of hyperspheres given by a surface $s: L^{2} \rightarrow \mathbb{S}_{1,1}^{n+2}$ of first or second species of real or complex type. If the surface $s$ is of real (respectively, complex) type, by Proposition 11.13 it is hyperbolic (respectively, elliptic) with respect to a tensor $\bar{J}$ on $L^{2}$ satisfying $J^{2}=I$ (respectively, $\bar{J}^{2}=-\bar{I}$ ), and there exists a tensor $\bar{D} \in \operatorname{span}\{\bar{I}, \bar{J}\}$ satisfying parts $(i)$ and (ii). It now follows from Lemma 17.4 that the hypersurface $f$ is hyperbolic (respectively, elliptic) with respect to the horizontal lift $J$ of $\bar{J}$, and the horizontal lift $D$ of a tensor $\bar{D} \in \operatorname{span}\{\bar{I}, \bar{J}\}$ satisfying parts (i) and (ii) satisfies conditions $(i)$ to $(v)$ in Lemma 17.2 . We conclude from Lemma 17.2 that $f$ is a Cartan hypersurface.

The only thing left to prove is the last assertion. By (11.49) and (11.50) in Exercise 11.6 if the surface $s: L^{2} \rightarrow \mathbb{S}_{1,1}^{n+2}$ is of first (respectively, second) species, then there exists a one-parameter family of nontrivial positive solutions (respectively, a single nontrivial positive solution) of either system (11.36) or Eq. 11.37), according to whether the surface $s$ is of real or complex type. By Propositon 11.13, each such solution gives rise to a tensor $\bar{D}$ on $L^{2}$ satisfying parts $(i)$ and (ii) in Lemma 17.4, with distinct solutions yielding tensors that do not coincide up to sign on any open subset of $L^{2}$. The horizontal lift $D$ of such a tensor $\bar{D}$, in turn, satisfies conditions $(i)$ to $(v)$ in Lemma 17.2, as follows from Lemma 17.4. Finally, Lemma 17.2 implies that each such tensor $D$ gives rise a conformal immersion $\tilde{f}: M^{n} \rightarrow \mathbb{R}^{n+1}$ that is not congruent to $f$ on any open subset of $M^{n}$, with tensors $D_{1}$ and $D_{2}$ that do not coincide up to sign on any open subset of $M^{n}$ yielding conformal immersions $\tilde{f}_{1}: M^{n} \rightarrow \mathbb{R}^{n+1}$ and $\tilde{f}_{2}: M^{n} \rightarrow \mathbb{R}^{n+1}$ that are not conformally congruent on any open subset of $M^{n}$.

### 17.5 Notes

Starting in 1916, E. Cartan devoted five years to the study of isometric, conformal and projective deformations of Euclidean hypersurfaces by using the method of moving frames. Shortly after his paper on isometric deformations [64], he released a long and much more difficult paper [65], where he classified conformally deformable Euclidean hypersurfaces of dimension $n \geq 5$. The special cases $n=4$ and $n=3$ were subsequently treated by Cartan in [66, although in these cases a classification is far from being complete. For reasons we can only guess (maybe uncertainty about the very existence of examples), Cartan's statement in the introduction of 64] completely ignores the discrete class, although the possibility of existence of the latter arises in his proof.

A version of Cartan's parametric result, closer in spirit to the one in this chapter, was provided by Dajczer-Tojeiro [139]. In the same paper, the nonparametric description of all conformally deformable Euclidean hypersurfaces of dimension $n \geq 5$ given by Corollary 12.41 for $q=1$ was derived. Roughly speaking, it was shown that a hypersurface $M^{n}$ in $\mathbb{R}^{n+1}, n \geq 5$, which admits a conformal deformation can be locally characterized as the intersection $M^{n}=N^{n+1} \cap \mathbb{V}^{n+2}$ of a flat ( $n+1$ )-dimensional Riemannian submanifold $N^{n+1}$ of $\mathbb{L}^{n+3}$ with the light cone.

It was shown in [139] that the classification due to Sbrana and Cartan of the isometrically deformable hypersurfaces, namely Theorem 11.16, can be obtained from the results in this chapter, but only for dimension at least five. The result given by Exercise 17.2 has been taken from [139].

A classification of the Euclidean hypersurfaces that admit conformal deformations preserving the Gauss map was given by Dajczer-Vergasta [150]. The corresponding problem for surfaces in $\mathbb{R}^{3}$, namely, finding all surfaces $f: M^{2} \rightarrow \mathbb{R}^{3}$ that are not determined, up to homothety and translation, by its conformal structure and its Gauss map, was studied by Christoffel [89] and became known as Christoffel's problem. For Euclidean surfaces of arbitrary codimension, the problem was studied by the eminent algebraic geometer P. Samuel [309] in 1947 in his very first publication. He showed that exceptions are, as in the case of surfaces in $\mathbb{R}^{3}$, minimal surfaces and isothermic surfaces, depending on whether the deformation preserves or reverses orientation, respectively. Samuel's result was totally or partially rediscovered by several authors later on.

The general problem of looking for all Euclidean submanifolds $f: M^{n} \rightarrow \mathbb{R}^{m}$ that admit nontrivial conformal deformations preserving the Gauss map was also addressed by Samuel [309], who obtained partial results on their classification. The classification was completed by Dajczer-Tojeiro [149.

Surprisingly, there exits few examples of submanifolds that admit conformal nonisometric deformations preserving the Gauss map. A trivial example is obtained by taking the cone over a submanifold contained in a sphere and then considering its image under an inversion with respect to that sphere. Since the Gauss map is constant along the rulings and these are preserved by the inversion, the deformation is conformal and preserves the Gauss map. Notice that in this example the submanifold is left invariant under the deformation.

The preceding construction can be combined with a special type of isometric deformation preserving the Gauss map to produce examples of conformal deformations of a submanifold that preserve the Gauss map but do not leave the submanifold invariant. Namely, start with a minimal real Kaehler cone $f: M^{n} \rightarrow \mathbb{R}^{m}$. Any such submanifold arises as the real part of a holomorphic isometric immersion of $M^{n}$ into $\mathbb{C}^{m}$ obtained by lifting a holomorphic isometric immersion of $M^{n}$ into complex projective space $\mathbb{C P}^{m-1}$ (see Exercise 15.8). Any member of the associated family $\left\{f_{\theta}\right\}_{\theta \in[0, \pi)}$ of $f$ is also a (minimal real Kaehler) cone. Hence the composition of any such $f_{\theta}$ with an inversion with respect to a sphere centered at the vertex of $f_{\theta}$ is conformal to $f$ and has the same Gauss map as $f$. It was shown in [149] that, apart from these examples with somewhat trivial deformations in the conformal realm, all remaining ones of dimension $n \geq 3$ are extrinsic warped products of either curves or minimal surfaces with spherical
submanifolds. An interesting example of this type that cannot occur as a hypersurface is a triply warped product submanifold having as profile a degenerate minimal surface, a minimal surface that has a pair of nonconstant harmonic functions as coordinate functions. The related problem of classifying the Euclidean hypersurfaces that admit conformal deformations preserving the third fundamental form was solved by Vlachos [340].

Prescribing the Gauss map of a surface $f: M^{2} \rightarrow \mathbb{R}^{3}$ can be thought of as giving a two-parameter congruence of affine subspaces of $\mathbb{R}^{3}$ to be enveloped by $f$. Christoffel's problem can thus be rephrased as finding which surfaces are not determined by their conformal structure and a prescribed plane congruence which they are to envelop. A similar problem in the realm of Moebius geometry was studied by Blaschke [38] (see also [220]) and is known as Blaschke's problem. It consists of finding the surfaces that are not determined, up to Moebius transformations, by their conformal structure and a given two-parameter congruence of spheres enveloped by them. Blaschke's problem for hypersurfaces, namely, finding all hypersurfaces $f, \tilde{f}: M^{n} \rightarrow \mathbb{R}^{n+1}$ that envelop a common sphere congruence and induce conformal metrics on $M^{n}$, was solved by Dajczer-Tojeiro (145).

Umbilic-free Euclidean hypersurfaces with principal curvatures of constant multiplicities that are not conformally congruent and admit deformations that preserve the Moebius metric, which are special types of conformal deformations, were classified by Li-Ma-Wang [236].

### 17.6 Exercises

Exercise 17.1. Let $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 5$, be a hypersurface that carries a relative nullity distribution of rank $n-2$ everywhere. Prove that any conformal immersion $\tilde{f}: M^{n} \rightarrow \mathbb{R}^{n+1}$ is given by $\tilde{f}=\nu \circ \bar{f}$, where $\bar{f}: M^{n} \rightarrow \mathbb{R}^{n+1}$ is an isometric immersion and $\nu$ is a conformal transformation of $\mathbb{R}^{n+1}$.

Exercise 17.2. Show that a hypersurface $f: M^{n} \rightarrow \mathbb{R}^{n+1}, n \geq 5$, is conformally but not isometrically congruent to a Sbrana-Cartan hypersurface if and only if it is a Cartan hypersurface such that the spheres in $\mathbb{R}^{n+1}$ containing $n$ - 2 -dimensional spherical leaves have a common point. Moreover, any conformal (nowhere conformally congruent) deformation of the hypersurface is conformally congruent to an isometric (nowhere congruent) deformation of the Sbrana-Cartan hypersurface. Conclude that any hypersurface conformally congruent to a Sbrana-Cartan hypersurfaces in the discrete class of Sbrana-Cartan's classification belongs to the discrete class in the classification of Cartan hypersurfaces.

## Appendix A

## Vector bundles

In this appendix we recall some basic definitions and results on vector bundles that are used throughout the book.

Let $E$ and $M$ be differentiable manifolds. A differentiable map $\pi: E \rightarrow M$ is called a differentiable vector bundle of rank $k$, or simply a vector bundle, if for each point $x \in M$,
(i) $\pi^{-1}(x)$ is a real vector space of dimension $k$,
(ii) there is an open neighborhood $U$ of $x$ in $M$ and a diffeomorphism

$$
\varphi: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^{k}
$$

whose restriction to $\pi^{-1}(y)$ is an isomorphism onto $\{y\} \times \mathbb{R}^{k}$ for each $y \in U$.
The manifolds $E$ and $M$ are called the total space and the base, respectively, and the map $\pi$ the projection. It is a common abuse of language to refer to the "vector bundle $E$ ". For each $x \in M$, the vector space $E_{x}=\pi^{-1}(x)$ is called the fiber of $\pi$ over $x$.

The simplest examples of vector bundles of rank $k$ are the product vector bundles, which consist of the projection

$$
\pi: M \times V \rightarrow M
$$

onto the first factor of a product of a differentiable manifold $M$ with a vector space $V$ of dimension $k$. These are also called trivial vector bundles. For this reason, the map $\varphi: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^{k}$ in the definition of a vector bundle $\pi: E \rightarrow M$ is said to be a local trivialization of $\pi$. Note that, by condition (ii), the diffeomorphism $\varphi$ has the form

$$
\varphi(e)=\left(\pi(e), \varphi_{\pi(e)}(e)\right),
$$

where $\varphi_{\pi(e)}: \pi^{-1}(\pi(e)) \rightarrow \mathbb{R}^{k}$ is an isomorphism for every $e \in \pi^{-1}(U)$. A family of local trivializations $\left\{\left(U_{\alpha}, \varphi_{\alpha}\right)\right\}$ such that $\left\{U_{\alpha}\right\}$ is an open cover of $M$ is called a vector bundle atlas for $\pi: E \rightarrow M$.

Given local trivializations $(U, \varphi)$ and $(V, \psi)$ such that $U \cap V \neq \emptyset$, for each $x \in U \cap V$ the map $g(x)=\varphi_{x} \circ \psi_{x}^{-1}$ is an automorphism of $\mathbb{R}^{k}$. For any $w \in \mathbb{R}^{k}$ and $x \in V$ we have

$$
\varphi \circ \psi^{-1}(x, w)=(x, g(x) w),
$$

hence $g: U \cap V \rightarrow \operatorname{Aut}\left(\mathbb{R}^{k}\right)$ is a differentiable mapping, called the transition function from $\psi$ to $\varphi$.

In order to construct vector bundles, the following Gluing Principle is useful.
Theorem A.1. Assume that $E$ and $L$ are sets, $M$ is a differentiable manifold and $\pi: E \rightarrow M$ is a surjective map. Set $E_{x}=\pi^{-1}(x)$ for each $x \in M$. Assume that $V$ is a real vector space of dimension $k$ and that for each $\lambda \in L$ there is a map $\psi_{\lambda}: \pi^{-1}\left(U_{\lambda}\right) \rightarrow U_{\lambda} \times V$ of the form

$$
e \mapsto\left(\pi(e), \psi_{\lambda, \pi(e)} e\right)
$$

such that the family $\left\{\left(U_{\lambda}, \psi_{\lambda}\right): \lambda \in L\right\}$ satisfies the following conditions:
(i) $\cup_{\lambda \in L} U_{\lambda}=M$,
(ii) $\psi_{\lambda, x}: E_{x} \rightarrow V$ is a bijection for every $x \in U_{\lambda}$ and $\lambda \in L$,
(iii) $\psi_{\lambda} \circ \psi_{\mu}^{-1}$ is a diffeomorphism from $\left(U_{\lambda} \cap U_{\mu}\right) \times V$ onto itself whenever $U_{\lambda} \cap U_{\mu} \neq \emptyset$,
(iv) $\psi_{\lambda, x} \circ \psi_{\mu, x}^{-1} \in \operatorname{Aut}(V)$ whenever $x \in U_{\lambda} \cap U_{\mu}$.

Then there exists a unique differentiable structure on $E$ that makes each $\psi_{\lambda}$ a diffeomorphism and $\pi$ a submersion. Moreover, endowing $E_{x}$ with the vector space structure that makes each $\psi_{\lambda, x}$, with $x \in U_{\lambda}$, an isomorphism (this is well defined by condition (iv)), then $\pi: E \rightarrow M$ becomes a vector bundle of rank $k$ over $M$.

Note that the second condition implies that each $\psi_{\lambda}$ is a bijection, so that the third one makes sense. To obtain local trivializations of $\pi: E \rightarrow M$, it is enough to choose an isomorphism $S: V \rightarrow \mathbb{R}^{k}$ and then define $\varphi_{\lambda}=(i d \times S) \circ \psi_{\lambda}, \lambda \in L$.

Examples A.2. (i) The dual vector bundle: Given a vector bundle $\pi: E \rightarrow M$ we define a projection $\theta: E^{*} \rightarrow M$ by setting

$$
\theta^{-1}(x)=E^{*}(x)=\operatorname{Hom}\left(E_{x} ; \mathbb{R}\right)
$$

Thus, $E^{*}$ is the disjoint union of the dual vector spaces of the fibers of $\pi$. To make $\theta: E^{*} \rightarrow M$ into a vector bundle, take an atlas $\left(U_{\lambda}, \varphi_{\lambda}\right)_{\lambda \in L}$ of local trivializations of $E$ with transition functions $\left(g_{\lambda \mu}\right)$. Given $\lambda \in L$ and $x \in U_{\lambda}$, the isomorphism $\varphi_{\lambda, x}$ has a transpose $\varphi_{\lambda, x}^{t}:\left(\mathbb{R}^{k}\right)^{*} \rightarrow E_{x}^{*}$, which is also an isomorphism. Let $\psi_{\lambda, x}$ be its inverse. Then,

$$
\psi_{\lambda}(f)=\left(\theta(f), \psi_{\lambda, x}(f)\right)
$$

defines a bijection between $\theta^{-1}\left(U_{\lambda}\right)$ and $U_{\lambda} \times\left(\mathbb{R}^{k}\right)^{*}$. Moreover, if $U_{\lambda} \cap U_{\mu} \neq \emptyset$, then

$$
\psi_{\lambda, x} \circ \psi_{\mu, x}^{-1}=\left(\varphi_{\mu, x} \circ \varphi_{\lambda, x}^{-1}\right)^{t}=\left(g_{\mu \lambda}\right)(x)^{t},
$$

which implies that $\psi_{\lambda} \circ \psi_{\mu}^{-1}$ is a diffeomorphism. Thus, we can apply the above Gluing Principle to obtain the dual vector bundle.
(ii) The homomorphism bundle: Let $\pi: E \rightarrow M$ and $\rho: F \rightarrow M$ be vector bundles with ranks $k$ and $m$, respectively. Define a projection $\sigma: \operatorname{Hom}(E ; F) \rightarrow M$ by making

$$
\sigma^{-1}(x)=\operatorname{Hom}\left(E_{x} ; F_{x}\right),
$$

so that the set $\operatorname{Hom}(E ; F)$ is the disjoint union of the spaces of linear maps from $E_{x}$ to $F_{x}$ for $x$ ranging over $M$.

Let $\left(U_{\lambda}\right)_{\lambda \in L}$ be an open cover of $M$ for which one can define atlases $\left(U_{\lambda}, \varphi_{\lambda}\right)$ for $E$ and $\left(U_{\lambda}, \psi_{\lambda}\right)$ for $F$, with respective transition functions $\left(g_{\lambda \mu}\right)$ and $\left(h_{\lambda \mu}\right)$. Given $\lambda \in L$ and $x \in U_{\lambda}$, we have a homomorphism

$$
\operatorname{Hom}\left(\varphi_{\lambda, x}^{-1}, \psi_{\lambda, x}\right): \operatorname{Hom}\left(E_{x} ; F_{x}\right) \rightarrow \operatorname{Hom}\left(\mathbb{R}^{k} ; \mathbb{R}^{m}\right)
$$

that takes $L \in \operatorname{Hom}\left(E_{x} ; F_{x}\right)$ to $\psi_{\lambda, x} \circ L \circ \varphi_{\lambda, x}^{-1} \in \operatorname{Hom}\left(\mathbb{R}^{k} ; \mathbb{R}^{m}\right)$. Arguing in a similar way as in the construction of the dual vector bundle, we arrive at a vector bundle atlas for $\operatorname{Hom}(E ; F)$ whose transition functions are given by

$$
f_{\lambda \mu}(x)=\operatorname{Hom}\left(g_{\lambda \mu}(x), h_{\lambda \mu}(x)\right)
$$

for $x \in U_{\lambda} \cap U_{\mu}$, hence take values in $\operatorname{Aut}\left(\operatorname{Hom}\left(\mathbb{R}^{k} ; \mathbb{R}^{m}\right)\right)$. When $\pi: E \rightarrow M$ and $\rho: F \rightarrow M$ coincide, we write $\operatorname{End}(E)$ instead of $\operatorname{Hom}(E ; E)$.

More generally, if $E_{1}, \ldots, E_{k}$ and $F$ are vector bundles over $M$, one may define a vector bundle $\sigma: \operatorname{Hom}^{k}\left(E_{1}, \ldots, E_{k} ; F\right) \rightarrow M$ such that

$$
\sigma^{-1}(x)=\operatorname{Hom}^{k}\left(\left(E_{1}\right)_{x}, \ldots,\left(E_{k}\right)_{x} ; F_{x}\right),
$$

the vector space of $k$-linear maps of $\left(E_{1}\right)_{x} \times \cdots \times\left(E_{k}\right)_{x}$ into $F_{x}$.
(iii) The Whitney sum: With notations as in the previous example, we define $E \oplus F \mapsto M$ by setting $(E \oplus F)_{x}=E_{x} \oplus F_{x}$. To obtain an atlas ( $U_{\lambda}, \nu_{\lambda}$ ), start from

$$
\nu_{\lambda, x}=\varphi_{\lambda, x} \oplus \psi_{\lambda, x}: E_{x} \oplus F_{x} \rightarrow \mathbb{R}^{k} \oplus \mathbb{R}^{m}=\mathbb{R}^{k+m},
$$

and the transition function from $\nu_{\mu}$ to $\nu_{\lambda}$ is given by

$$
x \mapsto g_{\lambda \mu}(x) \oplus h_{\lambda \mu}(x) .
$$

(iv) Tensor bundles: The vector bundle $E \otimes F \rightarrow M$ is constructed by setting $(E \otimes F)_{x}=E_{x} \otimes F_{x}$ and starting from

$$
\varphi_{\lambda, x} \otimes \psi_{\lambda, x}: E_{x} \otimes F_{x} \rightarrow \mathbb{R}^{k} \otimes F_{x} \rightarrow \mathbb{R}^{k} \otimes \mathbb{R}^{m}=\mathbb{R}^{k m}
$$

where $\left(U_{\lambda}, \varphi_{\lambda}\right)$ and $\left(U_{\lambda}, \psi_{\lambda}\right)$ are atlases of local trivializations for $E$ and $F$, respectively. Theorem A. 1 can again be applied since the transition functions have the form

$$
x \mapsto g_{\lambda \mu}(x) \otimes h_{\lambda \mu}(x) .
$$

The standard identification

$$
V_{1}^{*} \otimes \cdots \otimes V_{k}^{*} \otimes W \cong \operatorname{Hom}^{k}\left(V_{1}, \ldots, V_{k} ; W\right)
$$

for any vector spaces $V_{1}, \ldots, V_{k}, W$ allows us also to identify the vector bundles

$$
E_{1}^{*} \otimes \cdots \otimes E_{k}^{*} \otimes F \cong \operatorname{Hom}^{k}\left(E_{1}, \ldots, E_{k} ; F\right)
$$

for any vector bundles $E_{1}, \ldots, E_{k}, F$ over $M$.
Vector bundle morphisms: Given vector bundles $\pi_{i}: E_{i} \rightarrow M, 1 \leq i \leq 2$, a differentiable map $\alpha: E_{1} \rightarrow E_{2}$ is called a vector bundle morphism over $M$ if it maps $\pi_{1}^{-1}(x)$ linearly into $\pi_{2}^{-1}(x)$ for every $x \in M$. If $\alpha$ is a bijection (in which case $\alpha^{-1}$ is also a morphism), then $\alpha$ is said to be a vector bundle isomorphism.

More generally, a morphism between vector bundles $\theta: D \rightarrow N$ and $\pi: E \rightarrow M$ over possibly distinct differentiable manifolds $M$ and $N$ is a differentiable map $\hat{f}: D \rightarrow$ $E$ that takes fibers linearly into fibers. The morphism $\hat{f}$ induces a differentiable map $f: N \rightarrow M$ such that $\pi \circ \hat{f}=f \circ \theta$. The map $\hat{f}$ is also said to be a morphism over $f$.
Vector subbundles: Let $\pi: E \rightarrow M$ is a vector bundle of rank $k$. If $F \subset E$ is a subset such that the restriction $\pi_{F}: F \rightarrow M$ has also the structure of a vector bundle of rank $j$ such that the inclusion $i: F \rightarrow E$ is a vector bundle morphism, then $F$ is called a vector subbundle of $E$. In this case, the inclusion is always an embedding and $F \cap E_{x}$ is a vector subspace of $E_{x}$, and this is precisely the linear structure of the fiber $F_{x}$. In particular, a subset of $E$ admits at most one vector bundle structure with respect to which it becomes a vector subbundle of $E$.
The induced vector bundle: Let $\pi: E \rightarrow M$ be a vector bundle and let $f: N \rightarrow M$ be a differentiable map. Define

$$
\hat{E}=\{(x, e) \in N \times E: f(x)=\pi(e)\}
$$

and denote $\hat{\pi}(x, e)=x$ and $\hat{f}(x, e)=e$. For each $x \in N$, set $\hat{E}_{x}=\hat{\pi}^{-1}(x)=\{x\} \times E_{f(x)}$ and $\hat{f}_{x}=\left.\hat{f}\right|_{\hat{E}_{x}}$. Then $\hat{E}_{x}$ has a natural structure of vector space that makes $\hat{f}_{x}$ an isomorphism. Let $\left(U_{\lambda}, \varphi_{\lambda}\right)_{\lambda \in L}$ be an atlas of local trivializations for $E$, and let $\left(g_{\lambda \mu}\right)$ be the corresponding transition functions. Defining

$$
\hat{\varphi}_{\lambda, x}=\varphi_{\lambda, f(x)} \circ \hat{f}_{x}, \quad x \in \hat{U}_{\lambda}=f^{-1}\left(U_{\lambda}\right),
$$

we obtain, for each $\lambda \in L$, a bijection $\hat{\varphi}_{\lambda}: \hat{\pi}^{-1}\left(\hat{U}_{\lambda}\right) \rightarrow \hat{U}_{\lambda} \times \mathbb{R}^{k}$, where $k$ is the rank of $E$. It is immediate to verify that

$$
\hat{\varphi}_{\lambda, x} \circ \hat{\varphi}_{\mu, x}^{-1}=g_{\lambda \mu}(f(x)), \quad x \in \hat{U}_{\lambda} \cap \hat{U}_{\mu} .
$$

Hence the Gluing Principle can be applied, making $\hat{\pi}: \hat{E} \rightarrow N$ into a vector bundle. The map $\hat{f}$ is automatically a morphism over $f$. The total space $\hat{E}$ is usually denoted by $f^{*} E$.
Sections: A local section of the vector bundle $\pi: E \rightarrow M$ over an open set $U \subset M$ is a differentiable mapping $\xi: U \rightarrow E$ such that $\pi \circ \xi=i d_{U}$, that is, $\xi(x) \in E_{x}$ for every $x \in U$; if $U=M$, then $\xi$ is said to be a global section, or simply a section of $\pi$.

The set of sections over $U$ is denoted by $\Gamma(U, E)$, and $\Gamma(M, E)$ is written $\Gamma(E)$ for short. The set $\Gamma(U, E)$ is a module over $C^{\infty}(U)$.

A section $X: M \rightarrow T M$ of the tangent bundle $\pi: T M \rightarrow M$ of a differentiable manifold is a vector field on $M$. We write $\mathfrak{X}(M)=\Gamma(T M)$. If $f: N \rightarrow M$ is a differentiable map and $\pi: E \rightarrow M$ is a vector bundle, then a section $\xi \in \Gamma\left(f^{*} E\right)$ of the induced bundle $f^{*} E$ is also called a section of $E$ along $f$. In particular, a vector field along $f$ is a section of $f^{*} T M$.

Given a vector bundle $\pi: E \rightarrow M$ of rank $k$, a local trivialization $(U, \varphi)$ and a section $\xi \in \Gamma(E)$, there exists a differentiable map $\xi^{\varphi}: U \rightarrow \mathbb{R}^{k}$, called the principal part of $\xi$ with respect to $\varphi$, such that

$$
\varphi(\xi(x))=\left(x, \xi^{\varphi}(x)\right)
$$

for any $x \in U$. Differentiability of $\xi$ is equivalent to differentiability of $\xi^{\varphi}$ for any local trivialization $(U, \varphi)$. In particular, the zero section of $E$, taking each $x \in M$ to the origin of $E_{x}$, is clearly differentiable, for its principal part with respect to any local trivialization is a constant map.

If $A \subset M$ is not necessarily an open subset, then a section of the vector bundle $\pi: E \rightarrow M$ over $A$ is a map $\xi: A \rightarrow E$ such that, for each $x \in A$, there exist an open neighborhood $U_{x}$ of $x$ and a local section $\xi_{x} \in \Gamma\left(U_{x}, E\right)$ such that $\xi_{x}$ and $\xi$ coincide in $U_{x} \cap A$.

A partition of unity argument shows that any section over a closed subset $A \subset M$ can be extended to a global section on $M$. In particular, for every $e \in E$ there is a section $\xi$ such that $\xi(\pi(e))=e$.

Given vector bundles $\pi_{1}: E_{1} \rightarrow M$ and $\pi_{2}: E_{2} \rightarrow M$, it is a basic fact that $\Gamma\left(\operatorname{Hom}\left(E_{1} ; E_{2}\right)\right)$ and $\operatorname{Hom}\left(\Gamma\left(E_{1}\right) ; \Gamma\left(E_{2}\right)\right)$ are naturally isomorphic as $C^{\infty}(M)$-modules. Namely, to each section $s \in \Gamma\left(\operatorname{Hom}\left(E_{1} ; E_{2}\right)\right)$ let $\phi_{s}: \Gamma\left(E_{1}\right) \rightarrow \Gamma\left(E_{2}\right)$ be given by

$$
\phi_{s}(\sigma)(p)=s(p) \sigma(p)
$$

for any $\sigma \in \Gamma\left(E_{1}\right)$. It is easily seen that $\phi_{s} \in \operatorname{Hom}\left(\Gamma\left(E_{1}\right) ; \Gamma\left(E_{2}\right)\right)$ and that the map

$$
\Phi: \Gamma\left(\operatorname{Hom}\left(E_{1} ; E_{2}\right)\right) \rightarrow \operatorname{Hom}\left(\Gamma\left(E_{1}\right) ; \Gamma\left(E_{2}\right)\right)
$$

defined by $\Phi(s)=\phi_{s}$ is a module homomorphism. To construct its inverse, given a module homomorphism $\phi: \Gamma\left(E_{1}\right) \rightarrow \Gamma\left(E_{2}\right)$ define $s \in \Gamma\left(\operatorname{Hom}\left(E_{1} ; E_{2}\right)\right)$ by

$$
s(x)(e)=\phi(\sigma)(x)
$$

for $e \in E_{1}(x)$, where $\sigma$ is any section in $\Gamma\left(E_{1}\right)$ such that $\sigma(x)=e$. Using that $\phi$ is linear over $C^{\infty}(M)$ one can show that this is well defined, that is, the right-hand side
of the preceding expression does not depend on the choice of $\sigma$. The constructed map is easily checked to be the inverse of $\Phi$.

More generally, if $E_{1}, \ldots, E_{k}$ and $F$ are vector bundles over $M$, then there is a natural isomorphism between $\Gamma\left(\operatorname{Hom}^{k}\left(E_{1}, \ldots, E_{k} ; F\right)\right)$ and $\operatorname{Hom}^{k}\left(\Gamma\left(E_{1}\right), \ldots, \Gamma\left(E_{k}\right) ; \Gamma(F)\right)$ as $C^{\infty}(M)$-modules.
Moving frames: Let $\pi: E \rightarrow M$ be a vector bundle of rank $k$. A moving frame on an open subset $U \subset M$ is a set of $k$ sections $\xi_{1}, \ldots, \xi_{k} \subset \Gamma(U, E)$ such that $\left\{\xi_{1}(x), \ldots, \xi_{k}(x)\right\}$ is a basis of $E_{x}$ for every $x \in U$. Each local trivialization $(U, \varphi)$ of $E$ determines a moving frame $\eta_{1}, \ldots, \eta_{k}$ on $U$ by setting

$$
\eta_{i}(x)=\varphi^{-1}\left(x, e_{i}\right), \quad 1 \leq i \leq k
$$

where $\left\{e_{1}, \ldots, e_{k}\right\}$ is the canonical basis of $\mathbb{R}^{k}$. Conversely, a moving frame $\xi_{1}, \ldots, \xi_{k}$ on $U$ determines a local trivialization $\varphi: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^{k}$ given by

$$
\varphi(e)=\left(\pi(e), \varphi_{\pi(e)} e\right),
$$

where, for each $x \in U, \varphi_{x}$ is the isomorphism between $E_{x}$ and $\mathbb{R}^{k}$ determined by the basis $\left\{\xi_{1}(x), \ldots, \xi_{k}(x)\right\}$. In other words,

$$
\varphi^{-1}\left(x, c^{1}, \ldots, c^{k}\right)=\sum_{i=1}^{k} c^{i} \xi_{i}(x)
$$

It follows that a vector bundle of rank $k$ is trivial if and only if it admits a global moving frame.
Semi-Riemannian metrics on vector bundles: Let $\pi: E \rightarrow M$ be a vector bundle and let $g: \Gamma(E) \times \Gamma(E) \rightarrow C^{\infty}(M)$ be a symmetric $C^{\infty}(M)$-bilinear map, or equivalently, a symmetric section of $E^{*} \otimes E^{*}$. Then $g$ is said to be a semi-Riemannian metric on $E$ if for every $e \in E, e \neq 0$, there exists $f \in E$ such that $\pi(e)=\pi(f)$ and $g(e, f) \neq 0$. The index of the vector bundle is the index of $g$. If $g(e, e)>0$ for every $e \in E, e \neq 0$, then $g$ is a Riemannian metric on $E$. Using partitions of unity it is easy to show that any vector bundle admits a semi-Riemannian metric.
Linear connections: Let $\pi: E \rightarrow M$ be a vector bundle. A linear connection on $E$ is an $\mathbb{R}$-bilinear map

$$
\begin{gathered}
\mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E) \\
(X, \xi) \mapsto \nabla_{X} \xi
\end{gathered}
$$

satisfying the properties
(i) $\nabla_{f X} \xi=f \nabla_{X} \xi$,
(ii) $\nabla_{X} f \xi=X(f) \xi+f \nabla_{X} \xi$
for all $f \in C^{\infty}(M), X \in \mathfrak{X}(M)$ and $\xi \in \Gamma(E)$.
From (i) it follows that the map $X \mapsto \nabla_{X} \xi$ is $C^{\infty}(M)$-linear and, consequently, the value of $\nabla_{X} \xi$ at $x \in M$ depends only on the value of $X$ at $x$. It is also easy to see that the operator $\nabla_{X}: \Gamma(E) \rightarrow \Gamma(E)$ is a local operator in the sense that the value of $\nabla_{X} \xi$ at $x \in M$ depends only on the values of $\xi$ in a neighborhood of $x$.

Any section of the trivial bundle $\pi: M \times \mathbb{R}^{k} \rightarrow M$ has the form

$$
\xi(x)=(x, f(x))
$$

for some smooth function $f: M \rightarrow \mathbb{R}^{k}$, called the principal part of $\xi$. The canonical connection on that bundle is defined by requiring that the principal part of $\nabla_{X} \xi$ be the function $X(f)$.

A section $\xi \in \Gamma(U ; E)$ is said to be parallel on $U$ if $\nabla_{X} \xi=0$ for every $X \in \mathfrak{X}(U)$. A vector subbundle $F$ of $E$ is parallel if $\nabla_{X} \xi$ is a section of $F$ for every $X \in \mathfrak{X}(M)$ whenever $\xi$ is a section of $F$.

If $\pi: E \rightarrow M$ is a semi-Riemannian vector bundle, a linear connection $\nabla$ on $E$ is said to be compatible with the metric $g$ on $E$ if

$$
X g(\xi, \eta)=g\left(\nabla_{X} \xi, \eta\right)+g\left(\xi, \nabla_{X} \eta\right)
$$

for all $X \in \mathfrak{X}(M)$ and $\xi, \eta \in \Gamma(E)$.
Induced connection: Let $\pi: E \rightarrow M$ be a vector bundle with connection $\nabla$ and let $f: N \rightarrow M$ be a differentiable map. Then there exists a unique connection $f^{*} \nabla$ on $f^{*} E$, called the induced connection, such that

$$
f^{*} \nabla_{X}(\xi \circ f)=\nabla_{f_{*} X} \xi
$$

for all $X \in \mathfrak{X}(N)$ and $\xi \in \Gamma(E)$. We often use the same symbol $\nabla$ for the induced connection, when there is no risk of confusion. For instance, if $\sigma$ is a curve in $M$ and $\xi \in \Gamma(E)$, we denote $\sigma^{*} \nabla_{d / d t}(\xi \circ \sigma)$ simply by $\nabla_{d / d t} \xi$.

If $\pi: T M \rightarrow M$ is the tangent bundle of a Riemannian manifold $M, \nabla$ is its LeviCivita connection and $f: N \rightarrow M$ is a differentiable map, then the induced connection on $f^{*} T M$ satisfies

$$
\nabla_{X} f_{*} Y-\nabla_{Y} f_{*} X=f_{*}[X, Y]
$$

for all $X, Y \in \mathfrak{X}(N)$.
Parallel transport: Let $\pi: E \rightarrow M$ be a vector bundle with connection $\nabla$. A section $\xi$ of $E$ along a curve $\sigma$ in $M$ (that is, a section $\xi$ of $\sigma^{*} E$ ) is said to be parallel if $\nabla_{d / d t} \xi=0$. If $\xi \in \Gamma(E)$ and $\nabla_{d / d t}(\xi \circ \sigma)=0$ (that is, $\sigma^{*} \nabla_{d / d t}(\xi \circ \sigma)=0$ ), then $\xi$ is said to be parallel along $\sigma$.

Given a curve $\sigma: J \rightarrow M$, then for any $a \in J$ and $e \in E_{\sigma(a)}$ there exists a unique section $\eta$ along $\sigma$ such that $\eta$ is parallel and $\eta(a)=e$. It is called the parallel extension of $e$ along $\sigma$, and its value $\eta(b)$ at any $b \in J$ is the parallel transport of $e$ along $\sigma$ from $\sigma(a)$ to $\sigma(b)$.

The map $\sigma_{b}^{a}: E_{\sigma(a)} \rightarrow E_{\sigma(b)}$ taking each element $e \in E_{\sigma(a)}$ to its parallel transport to $E_{\sigma(b)}$ is an isomorphism which is an isometry if $E$ is a semi-Riemannian vector bundle
and $\nabla$ is compatible with the metric. The connection $\nabla$ can be recovered from the parallel transport by the formula

$$
\nabla_{X} \xi=\left.\frac{d}{d t}\right|_{t=a} \sigma_{a}^{t} \xi(\sigma(t)), \quad X=\sigma^{\prime}(a)
$$

Connections on tensor bundles: If $\pi: E \rightarrow M$ is a vector bundle, there exists a unique connection $\nabla^{*}$ on the dual vector bundle $\theta: E^{*} \rightarrow M$ that preserves duality, that is, that satisfies

$$
X(\omega(\eta))=\left(\nabla_{X}^{*} \omega\right)(\eta)+\omega\left(\nabla_{X} \eta\right)
$$

for all $\omega \in \Gamma\left(E^{*}\right), \eta \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$. Namely, just define the first term on the right-hand side so that the preceding equation is satisfied, and check that this indeed defines a connection on $E^{*}$. For instance,

$$
\begin{aligned}
\left(\nabla_{X}^{*} f \omega\right)(\eta) & =X(f \omega(\eta))-(f \omega)\left(\nabla_{X} \eta\right) \\
& =X(f) \omega(\eta)+f X(\omega(\eta))-f \omega\left(\nabla_{X} \eta\right) \\
& =\left(X(f) \omega+f \nabla_{X}^{*} \omega\right)(\eta)
\end{aligned}
$$

for all $f \in C^{\infty}(M), \omega \in \Gamma\left(E^{*}\right)$ and $\eta \in \Gamma(E)$. One often uses the same symbol $\nabla$ for the dual connection without risk of confusion.

If the vector bundle $\pi: E \rightarrow M$ is equipped with a semi-Riemannian metric $g: \Gamma(E) \times \Gamma(E) \rightarrow C^{\infty}(M)$, then one has the musical isomorphism

$$
\eta \in \Gamma(E) \mapsto \eta^{b} \in \Gamma\left(E^{*}\right) \cong \operatorname{Hom}\left(\Gamma(E) ; C^{\infty}(M)\right)
$$

defined by

$$
\eta^{b}(\xi)=\langle\eta, \xi\rangle
$$

for all $\eta, \xi \in \Gamma(E)$. In this case, the reader may easily check that the dual connection $\nabla^{*}$ was defined so that

$$
\nabla_{X}^{*} \eta^{b}=\left(\nabla_{X} \eta\right)^{b}
$$

for all $X \in \mathfrak{X}(M)$ and $\eta \in \Gamma(E)$.
Now let $E_{1}, \ldots, E_{k}$ be vector bundles over $M$, and let $\nabla^{i}$ be a connection on $E_{i}$ for all $1 \leq i \leq k$. Then there exists a unique way of defining a canonical connection

$$
\nabla: \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)
$$

on the tensor bundle $E=E_{1} \otimes \cdots \otimes E_{k}$ in such a way that

$$
\begin{equation*}
\nabla_{X}\left(\eta_{1} \otimes \cdots \otimes \eta_{k}\right)=\sum_{i=1}^{k} \eta_{1} \otimes \cdots \otimes \nabla_{X}^{i} \eta_{i} \otimes \cdots \otimes \eta_{k} \tag{A.1}
\end{equation*}
$$

for all $X \in \mathfrak{X}(M)$ and $\eta_{i} \in \Gamma\left(E_{i}\right), 1 \leq i \leq k$. Namely, using the identification

$$
\Gamma(E) \cong \operatorname{Hom}^{k}\left(E_{1}^{*}, \ldots, E_{k}^{*} ; C^{\infty}(M)\right)
$$

one may define

$$
\left(\nabla_{X} \eta\right)\left(\omega_{1}, \ldots, \omega_{k}\right)=X\left(\eta\left(\omega_{1}, \ldots, \omega_{k}\right)\right)-\sum_{i=1}^{k} \eta\left(\omega_{1}, \ldots, \nabla_{X}^{i} \omega_{i}, \ldots, \omega_{k}\right)
$$

for all $X \in \mathfrak{X}(M), \eta \in \Gamma(E)$ and $\omega_{i} \in \Gamma\left(E_{i}^{*}\right), 1 \leq i \leq k$, where we have used the symbol $\nabla^{i}$ also for the dual connection on $E_{i}^{*}$. With this definition, one can easily check that (A.1) is satisfied for any $\eta=\eta_{1} \otimes \cdots \otimes \eta_{k} \in \Gamma(E)$. In fact, one may use (A.1) as the definition of the connection $\nabla$ on $E$ once one takes into account that $\nabla_{X}$ is a local operator, that $\Gamma(E)$ is locally generated by sections of the type $\eta_{1} \otimes \cdots \otimes \eta_{k}$ and that the right-hand side of equation is $k$-linear over $\mathbb{R}$ on $\left(\eta_{1}, \ldots, \eta_{k}\right)$, which makes the left-hand side well defined.

Given vector bundles $E$ and $F$, since $\operatorname{Hom}(E ; F)$ is naturally identified with $E^{*} \otimes F$, the preceding definition yields a canonical connection on $\operatorname{Hom}(E ; F)$, which satisfies

$$
\left(\nabla_{X} B\right) \eta=\nabla_{X} B \eta-B\left(\nabla_{X} \eta\right)
$$

for all $X \in \mathfrak{X}(M), B \in \Gamma(\operatorname{Hom}(E ; F)) \cong \operatorname{Hom}(\Gamma(E) ; \Gamma(F))$ and $\eta \in \Gamma(E)$, where we have denoted with the same symbol $\nabla$ the connections on $\operatorname{Hom}(E ; F), E$ and $F$.

One says that $B \in \Gamma(\operatorname{Hom}(E ; F))$ is parallel if $\nabla_{X} B$ is the zero section of $\operatorname{Hom}(E ; F)$ for any $X \in \mathfrak{X}(M)$.

As another case of special interest for us in this book, let $E_{1}, E_{2}$ and $F$ be vector bundles over $M$. Then the canonical connection on $\operatorname{Hom}\left(E_{1}, E_{2} ; F\right) \cong E_{1}^{*} \otimes E_{2}^{*} \otimes F$ is such that

$$
\left(\nabla_{X} B\right)\left(\eta_{1}, \eta_{2}\right)=\nabla_{X} B\left(\eta_{1}, \eta_{2}\right)-B\left(\nabla_{X} \eta_{1}, \eta_{2}\right)-B\left(\eta_{1}, \nabla_{X} \eta_{2}\right)
$$

for all $X \in \mathfrak{X}(M), B \in \Gamma\left(\operatorname{Hom}\left(E_{1}, E_{2} ; F\right)\right) \cong \operatorname{Hom}^{2}\left(\Gamma\left(E_{1}\right), \Gamma\left(E_{2}\right) ; \Gamma(F)\right)$ and $\eta \in \Gamma(E)$, where $\nabla$ stands for the connections on all the vector bundles $\operatorname{Hom}\left(E_{1}, E_{2} ; F\right), E_{1}, E_{2}$ and $F$. Also in this case, that $B \in \Gamma\left(\operatorname{Hom}\left(E_{1}, E_{2} ; F\right)\right)$ is parallel means that $\nabla_{X} B=0$ for any $X \in \mathfrak{X}(M)$.

The horizontal distribution: Let $\pi: E \rightarrow M$ be a vector bundle. Since $\pi$ is a submersion, the kernel of $\pi_{*}$ is a vector subbundle $\mathcal{V}$ of $T E$ with the same rank as that of $E$, called the vertical subbundle of TE. A linear connection $\nabla$ on $\pi: E \rightarrow M$ determines a horizontal map $\beta: \pi^{*} T M \rightarrow T E$, that is, a vector bundle morphism from $\pi^{*} T M$ into $T E$ such that $\pi_{*} \circ \beta=i d_{T M}$.

Namely, given $(e, v) \in \pi^{*} T M$, let $\sigma$ be a curve in $M$ with initial velocity $v$; the parallel extension of $e$ along $\sigma$ is a curve in $E$ whose initial velocity is, by definition, $\beta(e, v)$. It is easy to check that $\beta$ is well defined and satisfies $\pi_{*} \circ \beta=i d_{T M}$. The image subbundle $\mathcal{H}$ of $\beta$ is thus a vector subbundle of $T E$ such that $T E$ is the Whitney sum $T E=\mathcal{V} \oplus \mathcal{H}$. It is called the horizontal subbundle of $T E$.

A curve $\tau$ in $E$ is said to be horizontal if it is everywhere tangent to $\mathcal{H}$. The construction of the parallel transport shows that, given $e \in E$, any curve in $M$ with initial point $\pi(e)$ has a unique horizontal lift with initial point $e$. Note also that a curve $\tau$ in $E$ is horizontal if, and only if, $\tau$ is a parallel section of $E$ along $\pi \circ \tau$.

Let $\pi: E \rightarrow M$ be a vector bundle with a linear connection $\nabla$. If the horizontal distribution $\mathcal{H}$ is integrable, then each $e \in E$ admits a parallel local extension, that is, there exists a local section $\xi$ on a neighborhood of $\pi(e)$ such that $\xi(\pi(e))=e$ and $\xi$ is parallel along any curve in its domain. For if $S$ is an integral manifold of $\mathcal{H}$ and $e \in S$, then the restriction $\left.\pi\right|_{S}$ is a local diffeomorphism, hence there exists a neighborhood $U$ of $\pi(e)$ in $M$ and a local inverse $\xi: U \rightarrow S$. Clearly, $\xi \in \Gamma(U ; E)$ and $\xi(\pi(e))=e$. If $\sigma$ is a curve in $U$, then $\xi \circ \sigma$ is a horizontal curve, that is, $\xi$ is parallel along $\sigma$. For the converse it suffices to observe that the image of any parallel local section is an integral manifold of $\mathcal{H}$.
The curvature tensor: The curvature tensor of a vector bundle $\pi: E \rightarrow M$ with linear connection $\nabla$ is the $\mathbb{R}$-trilinear map $R: \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ defined by

$$
R(X, Y) \xi=\nabla_{X} \nabla_{Y} \xi-\nabla_{Y} \nabla_{X} \xi-\nabla_{[X, Y]} \xi
$$

It is easily seen that $R$ is trilinear over $C^{\infty}(M)$, hence the value of $R(X, Y) \xi$ at $x \in M$ depends only on the values of $X, Y$ and $\xi$ at $x$. In other words, we can also regard $R$ as a section of $\operatorname{Hom}(T M, T M, E ; E)$.

If $\pi: M \times \mathbb{R}^{k} \rightarrow M$ is a product vector bundle and $\nabla$ is its canonical connection, then it follows from the definition of the bracket of vector fields that its curvature tensor is identically zero.

Given a vector bundle $\pi: E \rightarrow M$ with connection $\nabla$ and a differentiable map $f: N \rightarrow M$, the curvature tensor $\bar{R}$ of the induced connection on $f^{*} E$ is given in terms of the curvature tensor $R$ of $E$ at any point $x \in N$ by

$$
\bar{R}(X, Y) e=R\left(f_{*} X, f_{*} Y\right) e
$$

for all $X, Y \in T_{x} N$ and for all $e \in E_{f(x)}$.
Flat connections: A linear connection $\nabla$ on a vector bundle $\pi: E \rightarrow M$ is said to be flat if its curvature tensor $R$ vanishes identically. A fundamental fact is that flatness of $\nabla$ is a necessary and sufficient condition for integrability of the horizontal distribution $\mathcal{H}$.

Theorem A.3. Let $\pi: E \rightarrow M$ be a vector bundle with a linear connection $\nabla$. Then each $e \in E$ has a local parallel extension if and only if $\nabla$ is flat.

A global version of the preceding theorem is as follows.
Theorem A.4. Let $\pi: E \rightarrow M$ be a vector bundle of rank $k$ with a linear connection $\nabla$ over a simply connected manifold. Then the following assertions are equivalent:
(i) The connection $\nabla$ is flat.
(ii) There exists a global parallel moving frame $\xi_{1}, \ldots, \xi_{k}$.
(iii) There exists a parallel vector bundle isomorphism $\Phi: E \rightarrow M \times \mathbb{R}^{k}$.

The proof of the Fundamental theorem of submanifolds relies on the following consequence of Theorem A. 4 for semi-Riemannian vector bundles.

Corollary A.5. Let $\pi: E \rightarrow M$ be a semi-Riemannian vector bundle of rank $k$ with a compatible linear connection $\nabla$ over a simply connected manifold. Then the following assertions are equivalent:
(i) The connection $\nabla$ is flat.
(ii) There exists a global parallel orthonormal moving frame $\xi_{1}, \ldots, \xi_{k}$.
(iii) There exists a parallel vector bundle isometry $\Phi: E \rightarrow M \times \mathbb{R}^{k}$.

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