

**POSITIVE COMPLEX SECTIONAL CURVATURE, RICCI FLOW
AND THE DIFFERENTIAL SPHERE THEOREM**

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1. Introduction

A fundamental question in Riemannian geometry is to understand what natural curvature conditions are preserved on a Riemannian manifold M under Ricci flow. The well known conditions are: positive scalar curvature; positive curvature operator; 2-positive curvature operator; positive bisectonal curvature and positive Ricci curvature in three dimensions. It is also known that the nonnegative sectional curvature [N] and nonnegative Ricci curvature [K] are not preserved in general (see also [BW2]). Recently, Brendle and Schoen [BS] in an important paper have added another important condition to this list: positive isotropic curvature. See also [Ng]. This result suggests a program for the classification of compact manifolds with positive isotropic curvature. The fact that these various conditions are preserved yields, after further study, fundamental results in Riemannian geometry. For example, Böhm and Wilking [BW1], in foundational work, exploited the preservation of the positivity and the 2-positivity of the curvature operator to prove that compact manifolds satisfying either one of these conditions are spherical space forms. Brendle and Schoen used the preservation of positive isotropic curvature and an associated curvature condition on $M \times \mathbb{R}^2$ to prove that compact manifolds with (pointwise) 1/4-pinched positive sectional curvature admit metrics of constant positive curvature and are therefore diffeomorphic to spherical space forms. This resolves the long standing open conjecture, the differential sphere conjecture. For a detailed discussion of the history of this problem see the introduction of [BS].

Another natural curvature condition is positive complex sectional curvature, which lies between the positivity of the isotropic curvature and the positivity of the curvature operator. To define this condition on a Riemannian manifold (M, g) consider the complexified tangent bundle $TM \otimes \mathbb{C}$. Extend the metric to be symmetric and linear over \mathbb{C} on $TM \otimes \mathbb{C}$ (not hermitian) and extend the curvature R linearly over \mathbb{C} . Then (M, g) has positive (non-negative) complex sectional curvature if for every $p \in M$ and every linearly independent pair of vectors $Z, W \in T_p M \otimes \mathbb{C}$:

$$\langle R(Z, W)\bar{Z}, \bar{W} \rangle > 0, (\langle R(Z, W)\bar{Z}, \bar{W} \rangle \geq 0).$$

In this short note, we shall show first that the positivity and the nonnegativity of the complex sectional curvature is preserved under Ricci flow. Using this, together with techniques of Böhm and Wilking, one can conclude that if (M, g) is a compact Riemannian manifold with positive complex sectional curvature then the normalized Ricci flow deforms the metric to a metric of constant positive curvature. We then use earlier work of Yau and Zheng [YZ] to show that a metric with strictly (pointwise) 1/4-pinched sectional curvature has positive complex sectional curvature. This gives a direct proof of Brendle-Schoen's recent differential sphere theorem, bypassing any

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discussion of positive isotropic curvature. A further application of our approach is a characterization of space forms using a weaker point-wise $1/4$ -pinched condition.

Theorem 1.1. *Let (M, g_0) be a compact Riemannian manifold. Assume that there exists continuous function $k(p), \delta(p) \geq 0$ such that $\mathcal{P} = \{p | k(p) > 0\}$ is dense and $\delta(p_0) > 0$ for some $p_0 \in \mathcal{P}$, with the property that*

$$\frac{1 + \delta(p)}{4} k(p) \leq R(X, Y, X, Y) / |X \wedge Y|^2 \leq (1 - \delta(p)) k(p).$$

Then the normalized Ricci flow deforms (M, g_0) into a metric of constant curvature. Consequently M is diffeomorphic to a spherical space form.

It turns out that M has nonnegative complex sectional curvature is the same as $M \times \mathbb{R}^2$ has nonnegative isotropic curvature. We discuss this in the last section.

2. Ricci flow preserves positive complex sectional curvature

Let (M, g) be a compact Riemannian n -manifold. Recall that the curvature tensor R is said to have nonnegative complex sectional curvature if $\langle R(Z, W)\bar{Z}, \bar{W} \rangle \geq 0$ for any $p \in M$ and $Z, W \in T_p M \otimes \mathbb{C}$, where $\langle -, - \rangle$ denotes the symmetric inner product. We will henceforth use the notation

$$R(Z, W, \bar{Z}, \bar{W}) = \langle R(Z, W)\bar{Z}, \bar{W} \rangle$$

Let $Z = X + \sqrt{-1}Y$ and $W = U + \sqrt{-1}V$ with $X, Y, U, V \in T_p M$.

Choose Z and W so that they satisfy $\nabla Z = 0$, $\nabla W = 0$, $D_t Z = 0$ and $D_t W = 0$, then, using **[H2]** (see also **[H1]**), we have with respect to the moving frame,

$$(D_t - \Delta) R(Z, W, \bar{Z}, \bar{W}) = R_{ZWpq} R_{\bar{Z}\bar{W}pq} + 2R_{Zp\bar{Z}q} R_{Wp\bar{W}q} - 2R_{Zp\bar{W}q} R_{Wp\bar{Z}q}$$

Here $R_{ZWpq} = R(Z, W, e_p, e_q)$, $p, q = 1, \dots, n$, where $\{e_p\}$, is a orthonormal frame (also a unitary frame of $T_p M \otimes \mathbb{C}$). Following the notation of **[H1]** and **[BW1]** we denote the right hand side above by $Q(R)(Z, W, \bar{Z}, \bar{W})$ (abbreviated as $Q(R)$).

We will show that if t_0 is the first time for which there are vectors $Z, W \in T_p M \otimes \mathbb{C}$ such that,

$$(2.1) \quad R(Z, W, \bar{Z}, \bar{W}) = 0.$$

then,

$$Q(R)(Z, W, \bar{Z}, \bar{W}) \geq 0.$$

We remark that from this fact and the Hamilton maximum principle it follows that Ricci flow preserves positive complex sectional curvature.

Following **[Mo]**, for any complex tangent vectors Z_1 and W_1 and any real number s define,

$$f(s) \doteq R(Z + sZ_1, W + sW_1, \bar{Z} + s\bar{Z}_1, \bar{W} + s\bar{W}_1) \geq 0$$

Using that

$$R(Z, W, \bar{Z}, \bar{W}) = 0,$$

it follows that $f(0) = 0$ and $f''(0) \geq 0$. This then implies

$$(2.2) \quad 0 \leq R(Z_1, W_1, \bar{Z}, \bar{W}) + R(Z_1, W, \bar{Z}_1, \bar{W}) + R(Z_1, W, \bar{Z}, \bar{W}_1) \\ + R(Z, W_1, \bar{Z}_1, \bar{W}) + R(Z, W_1, \bar{Z}, \bar{W}_1) + R(Z, W, \bar{Z}_1, \bar{W}_1).$$

Replacing Z_1 by $\sqrt{-1}Z_1$, and W_1 by $\sqrt{-1}W_1$ we have that

$$(2.3) \quad 0 \leq -R(Z_1, W_1, \bar{Z}, \bar{W}) + R(Z_1, W, \bar{Z}_1, \bar{W}) + R(Z_1, W, \bar{Z}, \bar{W}_1) \\ + R(Z, W_1, \bar{Z}_1, \bar{W}) + R(Z, W_1, \bar{Z}, \bar{W}_1) - R(Z, W, \bar{Z}_1, \bar{W}_1).$$

Adding we have,

$$(2.4) \quad 0 \leq R(Z_1, W, \overline{Z}_1, \overline{W}) + R(Z, W_1, \overline{Z}, \overline{W}_1) + 2\mathcal{R}e(R(Z, W_1, \overline{Z}_1, \overline{W})).$$

The result now follows from a result of [Mo]. See for example Lemma 2.86 of [Chow, et al] (see also pages 11-12 of [H3]).

We have proved:

Theorem 2.1. *The Ricci flow on a compact manifold preserves the cones consisting of: (i) the curvature operators with nonnegative complex sectional curvature and (ii) the curvature operators with positive complex sectional curvature.*

In [BW1], the following concept is introduced.

Definition 2.2. A continuous family $C(s)_{s \in [0, \infty)}$ of closed convex $O(n)$ -invariant cones of full dimension (in the space of algebraic curvature operators) is called a pinching family if

- (1) each $R \in C(s) \setminus \{0\}$ has positive scalar curvature,
- (2) $R^2 + R^\#$ is contained in the interior of the tangent cone of $C(s)$ at R for all $R \in C(s) \setminus \{0\}$ and all $s \in (0, \infty)$.
- (3) $C(s)$ converges in the pointed Hausdorff topology to the one-dimensional cone $\mathbb{R}_+ I$ as $s \rightarrow \infty$.

The proof in [BW1] (see also the argument of [BS]) yields the the following theorem.

Theorem 2.3. *If $C(0)$ is a $O(n)$ -invariant cone preserved under Ricci flow. Assume further that the cone of positive curvature operators is contained in the interior of $C(0)$ and every $R \in C(0)$ has nonnegative Ricci curvature. Then there exists a continuous pinching family $C(s)$ with $C(s) = C(0)$.*

An immediate corollary is:

Corollary 2.4. *If (M, g) is a compact Riemannian manifold with positive complex sectional curvature, then the normalized Ricci flow deforms (M, g) to a Riemannian manifold of constant positive curvature.*

Proof. By Theorem 2.1 we have that the cone consisting of the curvature operators with nonnegative complex sectional curvature is invariant under the Ricci flow. Note that the cone of positive curvature operators is contained in the interior of the above cone. Also it is clear that if the curvature operator R has nonnegative complex sectional curvature it must have nonnegative Ricci curvature. The result now follows from the pinching family construction of [BW1] (see also [BS]) and Theorem 5.1 of [BW1]. q.e.d.

3. 1/4-pinched implies positive complex sectional curvature

Yau and Zheng [YZ] prove that if the sectional curvatures are negative and 1/4 pinched, that is,

$$-1 \leq R(X, Y, X, Y)/|X \wedge Y|^2 \leq -\frac{1}{4}$$

for any linearly independent vectors $X, Y \in TM$ then the complex sectional curvature is non-positive, that is, $R(Z, W, \overline{Z}, \overline{W}) \leq 0$, for any linearly independent vectors $Z, W \in TM \otimes \mathbb{C}$. A slight modification of this argument can be used to show that if at $p \in M$, for any $X, Y \in T_p M$,

$$\frac{1+\delta}{4}k(p) \leq R(X, Y, X, Y)/|X \wedge Y|^2 \leq (1-\delta)k(p),$$

for some $k(p) > 0$ then for any any linearly independent vectors $Z, W \in TM \otimes \mathbb{C}$, $R(Z, W, \overline{Z}, \overline{W}) > 0$. For the sake of the completion we include the argument here.

We start with the lemma of Berger.

Proposition 3.1 (Berger). *Suppose that for any $X, Y \in T_p M$ and $k(p) > 0$*

$$\frac{1+\delta}{4}k(p) \leq R(X, Y, X, Y)/|X \wedge Y|^2 \leq (1-\delta)k(p),$$

If $\{X, Y, U, V\} \in T_p M$ are linearly independent and

$$\Delta = \langle X, U \rangle \langle Y, V \rangle - \langle X, V \rangle \langle Y, U \rangle = 0.$$

Then

$$(3.1) \quad 6|R(X, Y, U, V)| \leq \frac{3-5\delta}{5-3\delta}(2R(X, Y) + 2R(U, V) + R(X, V) + R(V, Y) + R(X, U) + R(U, Y)).$$

Here $R(X, Y) = R(X, Y, X, Y)$.

Proof. See [YZ], proof of Lemma 1. One can let $a = c = 1$ to make the argument more transparent. q.e.d.

Proposition 3.2. *If the sectional curvature is pointwise $1/4$ pinched, in the sense that*

$$\frac{k(p)}{4} \leq R(X, Y, X, Y)/|X \wedge Y|^2 \leq k(p),$$

for some function $k(p) > 0$, then $R(Z, W, \overline{Z}, \overline{W}) \geq 0$. Moreover if

$$\frac{1+\delta}{4}k(p) \leq R(X, Y, X, Y)/|X \wedge Y|^2 \leq (1-\delta)k(p),$$

for some $\delta > 0$, then there exists $\epsilon > 0$ such that $(R - \epsilon I)(Z, W, \overline{Z}, \overline{W}) \geq 0$, where I is the identity (complex extension) of $S^2(\wedge^2(\mathbb{R}^n))$.

Proof. We follow the proof of [YZ]. Define the function

$$f(Z, W) = \frac{R(Z, W, \overline{Z}, \overline{W})}{|Z|^2|W|^2}$$

on $(\mathbb{C}^n)^* \times (\mathbb{C}^n)^*$ where $|Z|^2 = \langle Z, \overline{Z} \rangle$. Clearly f is defined on $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$. To prove the first statement of the Proposition it suffices to show that $f(Z, W) \geq 0$ for all $Z, W \in T_p M \otimes \mathbb{C}$. We shall prove the result by contradiction. Assume that there exist a pair of vector (Z, W) such that $f(Z, W) < 0$. Notice that $R(aZ + bW, cZ + dW, \overline{aZ + bW}, \overline{cZ + dW}) = |ad - bc|^2 R(Z, W, \overline{Z}, \overline{W})$. Hence by replacing (Z, W) by $(Z, W - \frac{\langle W, Z \rangle}{\langle Z, Z \rangle} Z)$ in the case $\langle Z, Z \rangle \neq 0$, or by $(Z - W, Z + W)$ in the case that both $\langle Z, Z \rangle = \langle W, W \rangle = 0$ we can assume $\langle Z, W \rangle = 0$ without changing the sign of $f(Z, W)$ (though its absolute value is changed). Thus we can assume that the minimum of $f(Z, W)$ under the constraint

$$\langle Z, W \rangle = 0.$$

is achieved and is negative. Suppose the minimum is achieved at (Z, W) . Clearly Z, W are linearly independent. Introduce the Lagrange multiplier,

$$F(Z, W) = f(Z, W) + \lambda|\langle Z, W \rangle|^2,$$

Then the minimum point (Z, W) is a critical point of F and hence,

$$R(Z, W)\overline{Z} + f(Z, W)|Z|^2W = 0.$$

Therefore,

$$\langle \bar{Z}, W \rangle = \frac{R(Z, W, \bar{Z}, \bar{Z})}{|Z|^2 f(Z, \bar{W})} = 0.$$

Thus $\langle Z, W \rangle = 0$ and $\langle \bar{Z}, W \rangle = 0$. Writing $Z = X + \sqrt{-1}Y$ and $W = U + \sqrt{-1}V$ we conclude that $\{X, Y\} \perp \{U, V\}$. Observing that $f(\lambda Z, \mu W) = f(Z, W)$, for any complex scalars λ and μ we see that we can adjust Z and W so that $X \perp Y$ and $U \perp V$. Without loss of the generality we may assume that $1 = |X| \geq |Y|$ and $1 = |U| \geq |V|$. Therefore we have,

$$\begin{aligned} R(Z, W, \bar{Z}, \bar{W}) &= R(X, U, X, U) + R(X, V, X, V) + R(Y, U, Y, U) + R(Y, V, Y, V) \\ &\quad - 2R(X, Y, U, V) \\ &\geq R(X, U) + R(X, V) + R(Y, U) + R(Y, V) \\ &\quad - \frac{1}{5}(2R(X, Y) + 2R(U, V) + R(X, V) + R(V, Y) \\ &\quad + R(X, U) + R(U, Y)) \\ &= \frac{4}{5}(R(X, U) + R(X, V) + R(Y, U) + R(Y, V)) \\ &\quad - \frac{2}{5}(R(X, Y) + R(U, V)) \\ &\geq \frac{k}{5}(1 - |V|^2)(1 - |Y|^2) \geq 0. \end{aligned}$$

This contradicts $R(Z, W, \bar{Z}, \bar{W}) < 0$ and therefore proves the first statement of the Proposition. For the second statement, observe that for sufficiently small ϵ , say $\epsilon \leq \frac{\delta}{4}k(p)$, $\tilde{R} = R - \epsilon I$ satisfies the weaker pinching condition. q.e.d.

The consequence is the recent important result of Brendle and Schoen.

Corollary 3.3 (Brendle-Schoen). *Assume that (M, g_0) is a compact Riemannian manifold. Assume that the sectional curvature of g_0 satisfies that*

$$\frac{1 + \delta}{4}k(p) \leq R(X, Y, X, Y)/|X \wedge Y|^2 \leq (1 - \delta)k(p)$$

for some continuous function $k(p) > 0$ and constant $\delta > 0$. Then the normalized Ricci flow deforms it into metric of constant curvature.

In the next section we shall prove a generalized version of this result.

4. Generalization

In this section we shall generalize Corollary 3.3. We first start with the following proposition.

Proposition 4.1. *Let $(M, g(t))$ be a solution the Ricci flow. Assume that at $t = 0$, $R - f_0(x)I$ has nonnegative complex sectional curvature for some continuous $f \geq 0$. Let $f(x, t)$ be the solution to $(D_t - \Delta)f(x, t) = 0$ with the initial data $f(x, 0) = f_0(x)$. Then $\tilde{R} = R - f(x, t)I$ has nonnegative complex sectional curvature for $t > 0$.*

Proof. By Lemma 2.1 of [BW1] is easy to check that

$$(4.1) \quad Q(\tilde{R}) = Q(R) - 2f \operatorname{Ric}(R) \wedge \operatorname{id} + (n-1)f^2 I$$

where id is the identity of $\mathbb{R}^n = T_p M$. Since $f(x, t)$ satisfies $(\frac{\partial}{\partial t} - \Delta)f(x, t) = 0$,

$$(4.2) \quad (D_t - \Delta)\tilde{R} = Q(R)$$

By assumption \tilde{R} has nonnegative complex sectional curvature at $t = 0$. Therefore there is a first time t_0 (possibly at $t = 0$) at which for some $Z, W \in T_p M \otimes \mathbb{C}$ we have $\tilde{R}(Z, W, \bar{Z}, \bar{W}) = 0$. By the maximum principle applied to (4.2) the theorem follows if we can show $Q(R)(Z, W, \bar{Z}, \bar{W}) \geq 0$. Since \tilde{R} is an algebraic curvature operator we can apply the results of Section 2 to conclude that $Q(\tilde{R})(Z, W, \bar{Z}, \bar{W}) \geq 0$. The result then follows using (4.1) if we can show that

$$(4.3) \quad (2f \operatorname{Ric}(R) \wedge \operatorname{id} - (n-1)f^2 \mathbf{I})(Z, W, \bar{Z}, \bar{W}) \geq 0,$$

To verify (4.3) first notice that at (p, t_0) , \tilde{R} has nonnegative Ricci curvature. Hence at (p, t_0)

$$A \doteq \operatorname{Ric}(\tilde{R}) = \operatorname{Ric}(R) - (n-1)f \operatorname{id} \geq 0$$

as element of $S^2(\mathbb{R}^n)$. From this, at (p, t_0) ,

$$\operatorname{Ric}(R) \geq (n-1)f \operatorname{id} \geq 0$$

Thus, at (p, t_0) ,

$$(2f \operatorname{Ric}(R) \wedge \operatorname{id} - (n-1)f^2 \mathbf{I}) = f \operatorname{Ric}(R) \wedge \operatorname{id} + fA \wedge \operatorname{id}.$$

On the other hand

$$\begin{aligned} A \wedge \operatorname{id}(Z, W, \bar{Z}, \bar{W}) &= \frac{1}{2} \langle A(Z) \wedge W + Z \wedge A(W), \bar{Z} \wedge \bar{W} \rangle \\ &= \frac{1}{2} (\langle A(Z), \bar{Z} \rangle |W|^2 + \langle A(W), \bar{W} \rangle |Z|^2) \\ &\quad - \frac{1}{2} (\langle A(W), \bar{Z} \rangle \langle Z, \bar{W} \rangle + \langle A(Z), \bar{W} \rangle \langle W, \bar{Z} \rangle) \end{aligned}$$

which is nonnegative by the Cauchy-Schwartz inequality and $A \geq 0$. Similarly,

$$\operatorname{Ric}(R) \wedge \operatorname{id}(Z, W, \bar{Z}, \bar{W}) \geq 0$$

The result follows.

q.e.d.

Corollary 4.2. *If $f_0(x) \geq 0$ and $f_0(x_0) > 0$ for some x_0 , then R has positive complex sectional curvature for $t > 0$.*

This together with Proposition 3.2 implies the following result.

Corollary 4.3. *Let (M, g_0) be a compact Riemannian manifold. Assume that there exists continuous function $k(p), \delta(p) \geq 0$ such that $\mathcal{P} = \{p \mid k(p) > 0\}$ is dense and $\delta(p_0) > 0$ for some $p_0 \in \mathcal{P}$, with the property that*

$$\frac{1 + \delta(p)}{4} k(p) \leq R(X, Y, X, Y) / |X \wedge Y|^2 \leq (1 - \delta(p)) k(p).$$

Then the normalized Ricci flow deforms (M, g_0) into a metric of constant curvature.

5. Characterization of various invariant curvature cones

In [BS], the authors introduced two invariant curvature cones \tilde{C} and \hat{C} motivated from their result that the nonnegativity of the isotropic curvature is preserved under Ricci flow. Let $\pi : \mathbb{R}^n \times \mathbb{R}^2 \rightarrow \mathbb{R}^n$ be the projection and define $\hat{R}(x, y, z, w) = R(\pi(x), \pi(y), \pi(z), \pi(w))$ where $x, y, z, w \in T(\mathbb{R}^n \times \mathbb{R}^2)$. Recall from [BS] that

$$\hat{C} = \{R \mid \hat{R} \text{ has nonnegative isotropic curvature}\}$$

The \tilde{C} cone is defined similarly using $\mathbb{R}^n \times \mathbb{R}$.

We shall show that in fact the cone of nonnegative complex sectional curvature as used in this paper is the same as \tilde{C} .¹ We are indebted to Nolan Wallach for the following result.

Proposition 5.1. *The following are equivalent:*

- (1) $R \in \tilde{C}$;
- (2) R has non-negative complex sectional curvature.

Proof. It suffices to show that given $Z, W \in \mathbb{C}^n$ linearly independent, there exist extensions $\tilde{Z} = Z + ue_1 + ve_2$ and $\tilde{W} = W + xe_1 + ye_2$ of Z, W to vectors \tilde{Z}, \tilde{W} in $\mathbb{C}^n \times \mathbb{C}^2$ such that $\text{Span}\{\tilde{Z}, \tilde{W}\}$ is an isotropic plane. Here $\{e_1, e_2\}$ is an orthonormal basis of the factor \mathbb{R}^2 in the definition of \tilde{C} .

The existence of such an extension is equivalent to the solution of the matrix equation $XX^t = A$ with

$$X = \begin{pmatrix} u & v \\ x & y \end{pmatrix}, \quad A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

where $a = -\langle Z, Z \rangle$, $b = -\langle W, W \rangle$, $c = -\langle Z, W \rangle$. That the matrix equation can be solved follows from the fact that quadratic form $as^2 + 2cst + bt^2$ can be diagonalized by transformations of $\text{GL}(2, \mathbb{C})$. q.e.d.

The second result characterizes the \tilde{C} cone. Let $Z, W \in \mathbb{C}^n$ be linearly independent. We say the 2-vector $Z \wedge W$ is isotropic if:

$$0 = \langle Z \wedge W, Z \wedge W \rangle \doteq \langle Z, Z \rangle \langle W, W \rangle - \langle Z, W \rangle^2.$$

Proposition 5.2. *The following are equivalent:*

- (1) $R \in \tilde{C}$;
- (2) R is non-negative on any isotropic 2-vector $Z \wedge W$.

Proof. To show (1) implies (2) we suppose that $Z, W \in \mathbb{C}^n$ are linearly independent and satisfy:

$$\langle Z, Z \rangle \langle W, W \rangle - \langle Z, W \rangle^2 = 0.$$

Then there exist complex scalars a, b such that $a^2 = -\langle Z, Z \rangle$, $b^2 = -\langle W, W \rangle$, $ab = -\langle Z, W \rangle$. Hence $\tilde{Z} = Z + ae_1$ and $\tilde{W} = W + be_1$ span an isotropic 2-plane, where e_1 is a unit vector in \mathbb{R} . Since R has non-negative isotropic curvature on $\mathbb{R}^n \times \mathbb{R}$,

$$R(\tilde{Z}, \tilde{W}, \tilde{Z}, \tilde{W}) \geq 0.$$

Hence,

$$R(Z, W, \tilde{Z}, \tilde{W}) \geq 0.$$

The converse is similar and left to the reader. q.e.d.

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¹After the circulation of an earlier version of this paper, the authors were informed by Brendle and Schoen of the following statement: M has nonnegative complex sectional curvature is equivalent to $M \times \mathbb{R}^4$ has nonnegative isotropic curvature. This motivated the current section.

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