LECTURE 9: THE WHITNEY EMBEDDING THEOREM

Historically, the word "manifold" (*Mannigfaltigkeit* in German) first appeared in Riemann's doctoral thesis in 1851. At the early times, manifolds are defined extrinsically: they are the set of all possible values of some variables with certain constraints. Translated into modern language,"smooth manifolds" are objects that are (locally) defined by smooth equations and, according to last lecture, are embedded submanifolds in Euclidean spaces. In 1912 Weyl gave an intrinsic definition for smooth manifolds. A natural question is: what is the difference between the extrinsic definition and the intrinsic definition? Is there any "abstract" manifold that cannot be embedded into any Euclidian space?

In 1930s, Whitney and others settled this foundational problem: the two ways of defining smooth manifolds are in fact the same. In fact, Whitney's result is much more stronger than this. He showed that not only one can embed any smooth manifold into some Euclidian space, but that the dimension of the Euclidian space can be chosen to be (as low as) twice the dimension of the manifold itself!

Theorem 0.1 (The Whitney embedding theorem)**.** *Any smooth manifold M of dimension m can be embedded into $\mathbb{R}^{2m+1}$.*

Remark. In 1944, by using completely different techniques (now known as the "Whitney trick"), Whitney was able to prove

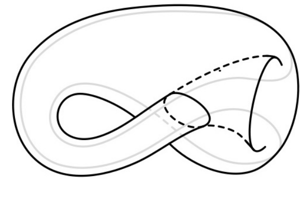
> **Theorem 0.2** (The Strong Whitney Embedding Theorem)**.** *Any smooth manifold M of dimension $m \geq 2$ can be embedded into $\mathbb{R}^{2m}$ (and can be immersed into $\mathbb{R}^{2m-1}$).*

We will not prove this stronger version in this course, but just mention that the Whitney trick was further developed in *h-cobordism theory* by Smale, using which he proved the Poincare conjecture in dimension ≥ 5 in 1961!

Remark. Many shaper results were proved during the second half of 20th century:

- Any smooth compact *orientable* m-manifold can be embedded into $\mathbb{R}^{2m-1}$.
- For $m \neq 2^k$, any smooth m-manifold can be embedded into $\mathbb{R}^{2m-1}$. (But if $m = 2^k$, $\mathbb{RP}^m$ cannot be embedded into $\mathbb{R}^{2m-1}$).
- Any smooth m-manifold can be immersed into $\mathbb{R}^{2m-a(m)}$, where $a(m)$ is the number of 1's that appear in the binary expansion of m.

Here is an example of a 2-manifold which can be immersed into $\mathbb{R}^3$, but cannot be embedded into $\mathbb{R}^3$: the Klein bottle. (Note: "immersed into" $\neq$ "the image is an immersed submanifold", since the immersion need not be injective.)



In what follows we will prove Theorem 0.1. The case when M is compact is easier, and the non-compact case is a bit harder. In both cases the proof can be divided into two steps. In step 1, we can prove that there exists a large K so that M can be injectively immersed into $\mathbb{R}^K$. (When M is compact, the idea of the proof for Step 1 is already explained at the end of last lecture.) Then in step 2, we prove that if $K > 2m+1$, then one can always replace $\mathbb{R}^K$ by some $\mathbb{R}^{K-1}$. For the compact case, this already implies the Whitney embedding theorem. For the non-compact case, we need to do a bit more work to get an embedding from an injective immersion. (The idea is somehow hidden in the proof of Theorem 2.4 in Lecture 8: The condition that N is compact is used to show that f is a closed map, from which one can conclude f is an embedding. In topology we also learned other conditions that guarantee the closedness of a map, e.g. a proper continuous map onto a locally compact Hausdorff space is closed.)

Although the proof of the theorem is a bit technical (which uses Sard's theorem and thus is not quite "constructive"), there is a "naive philosophy" behind the theorem: each small piece of the manifold can be embedded as a small piece of $\mathbb{R}^m$ inside a larger ambient Euclidian space; if you want to embed two "consecutive" pieces, they may "intersect", but it should be possible to avoid the "intersection" if the ambient space that the two pieces lie has dimension greater than the sum of the dimensions of two pieces!

1. The Whitney embedding theorem: Compact Case

We will first prove the Whitney embedding theorem for the simple case where M is *compact*. We start with

Theorem 1.1. *Any compact smooth manifold M admits an injective immersion into $\mathbb{R}^K$ for sufficiently large K.*

Proof. Let $\{\varphi_i, U_i, V_i\}_{1\le i\le k}$ be a finite set of coordinate charts on M that covers M. Let $\{\rho_i \mid 1 \le i \le k\}$ be a P.O.U. subordinate to $\{U_i \mid 1 \le i \le k\}$. Define

$$\Phi : M \to \mathbb{R}^{k(m+1)}, \quad p \mapsto (\rho_1(p)\varphi_1(p), \cdots, \rho_k(p)\varphi_k(p), \rho_1(p), \cdots, \rho_k(p)).$$

We shall prove

- $\boxed{\Phi \text{ is an injective map.}}$

  Suppose $\Phi(p_1) = \Phi(p_2)$. Take an index i so that $\rho_i(p_1) = \rho_i(p_2) \neq 0$. Then $p_1, p_2 \in \operatorname{supp}(\rho_i) \subset U_i$. It follows that $\varphi_i(p_1) = \varphi_i(p_2)$. So we must have $p_1 = p_2$ since φ_i is bijective.
- $\boxed{\Phi \text{ is an immersion, i.e. } d\Phi_p \text{ is injective for any } p \in M.}$

  For any $X_p \in T_pM$, by Leibniz law,

$$\begin{aligned} d\Phi_p(X_p) = (&X_p(\rho_1)\varphi_1(p) + \rho_1(p)(d\varphi_1)_p(X_p), \cdots, \\ &X_p(\rho_k)\varphi_k(p) + \rho_k(p)(d\varphi_k)_p(X_p), X_p(\rho_1), \cdots, X_p(\rho_k)). \end{aligned}$$

It follows that if $d\Phi_p(X_p) = 0$, then $X_p(\rho_i) = 0$ for all i, and thus $\rho_i(p)(d\varphi_i)_p(X_p) = 0$ for all i. Pick an index i so that $\rho_i(p) \neq 0$. We see $(d\varphi_i)_p(X_p) = 0$. Since φ_i is a diffeomorphism, we conclude that $X_p = 0$. So $d\Phi_p$ is injective.

□

Remark. In fact we proved a stronger result:

Theorem 1.2. *If M can be covered by finitely many coordinate charts, then there exists an injective immersion from M into some Euclidian space.*

Next we apply Sard's theorem to prove (note: we don't assume compactness here)

Theorem 1.3. *If a smooth manifold M of dimension m admits an injective immersion into $\mathbb{R}^K$ for some $K > 2m+1$, then it admits an injective immersion into $\mathbb{R}^{K-1}$.*

Proof. Suppose we already have an injective immersion $\Phi : M \to \mathbb{R}^K$ with $K > 2m+1$. We want to produce an injective immersion of M into $\mathbb{R}^{K-1}$. To do so, we study the compositions of Φ with projections from $\mathbb{R}^K$ to all possible $K-1$ dimensional vector subspaces in $\mathbb{R}^K$, and we will show that for "almost all" projections, we still get injective immersions.

Note that any $K-1$ dimensional vector subspace in $\mathbb{R}^K$ is uniquely determined by its normal direction, which is a 1-dimensional line passing the origin, and the set of all 1-dimensional lines passing the origin in $\mathbb{R}^K$ is the real projective space $\mathbb{RP}^{K-1}$, which is a smooth manifold of dimension $K-1$. For any $[v] \in \mathbb{RP}^{K-1}$, we let

$$P_{[v]} = \{u \in \mathbb{R}^N \mid u \cdot v = 0\} \simeq \mathbb{R}^{K-1}$$

be the orthogonal complement space of $[v]$ in $\mathbb{R}^K$. Let

$$\pi_{[v]} : \mathbb{R}^K \to P_{[v]}$$

be the orthogonal projection to this hyperplane. We claim that the set of $[v]$'s for which

$$\Phi_{[v]} = \pi_{[v]} \circ \Phi$$

is not an injective immersion has measure zero in $\mathbb{RP}^{K-1}$. Hence for most $[v] \in \mathbb{RP}^{K-1}$, the map $\Phi_{[v]}$ is an injective immersion from M into some $\mathbb{R}^{K-1}$.

First let's consider $[v]$'s so that $\Phi_{[v]}$ is not injective. Then one can find $p_1 \neq p_2$ so that $\Phi_{[v]}(p_1) = \Phi_{[v]}(p_2)$, i.e. $0 \neq \Phi(p_1) - \Phi(p_2)$ lies in the line $[v]$. In other words,

$$[v] = [\Phi(p_1) - \Phi(p_2)].$$

So $[v]$ must lie in the image of the smooth map

$$\alpha : (M \times M) \setminus \Delta_M \to \mathbb{RP}^{K-1}, \quad (p_1, p_2) \mapsto [\Phi(p_1) - \Phi(p_2)],$$

where $\Delta_M = \{(p,p) \mid p \in M\}$ is the "diagonal" in $M \times M$. Since $(M \times M) \setminus \Delta_M$ is a smooth manifold of dimension $2m < K-1$, Sard's theorem implies that the image of α is of measure zero in $\mathbb{RP}^{K-1}$. So the set of $[v]$'s so that $\Phi_{[v]}$ is not injective is of measure zero.

Next let's consider $[v]$'s so that $\Phi_{[v]}$ is not an immersion. Then there exists some $p \in M$ and some $0 \neq X_p \in T_pM$ so that $(d\Phi_{[v]})_p(X_p) = 0$, i.e.

$$(d\pi_{[v]})_{\Phi(p)}(d\Phi)_p(X_p) = 0.$$

Since $\pi_{[v]}$ is linear, $d\pi_{[v]} = \pi_{[v]}$. A conceptional proof: What is the differential of a smooth map? It's the linear approximation of the smooth map. What if the smooth map is already linear then? It follows that $0 \neq (d\Phi)_p(X_p)$ is in $[v]$, i.e.

$$[v] = [(d\Phi)_p(X_p)].$$

In other words, $[v]$ lies in the image of

$$\beta : TM \setminus \{0\} \to \mathbb{RP}^{K-1}, \quad (p, X_p) \mapsto [(d\Phi)_p(X_p)],$$

where $TM \setminus \{0\} = \{(p, X_p) \mid X_p \neq 0\}$ is an open submanifold of the tangent bundle TM, which is a smooth manifold of dimension $2m$ (c.f. PSet 2). Again since TM has dimension $2m < K - 1$, by Sard's theorem, the image of β is of measure zero. So the set of $[v]$'s so that $\Phi_{[v]}$ is not an immersion is of measure zero. □

In view of the fact

$$(d\Phi_{[v]})_p(X_p) = 0 \Longleftrightarrow (d\Phi_{[v]})_p(\frac{X_p}{|X_p|}) = 0,$$

one can modify the last step and prove

Theorem 1.4. *If a smooth manifold M of dimension m can be embedded into $\mathbb{R}^{2m+1}$, then it can be immersed into $\mathbb{R}^{2m}$.*

Sketch of proof. We first embed M into $\mathbb{R}^{2m+1}$, then repeat the last step in the proof of Theorem 1.3, with the modification that we choose $X_p \in T_pM$ so that $|X_p| = 1$ (here the length of a vector $X_p \in T_pM$ "$\subset T_p\mathbb{R}^{2m+1}$" is the "Euclidian" length). In other words, the map β in the proof above can be replaced by the map

$$\tilde{\beta} : SM \to \mathbb{RP}^{2m}, \quad (p, X_p) \mapsto [(d\Phi)_p(X_p)],$$

where $SM = \{(p, X_p) \mid |X_p| = 1\}$ is the "sphere bundle" of M, which is a smooth manifold of dimension $2m - 1$. (Please write down the details.) □

Remark. Here you already see the advantage of the existence of an embedding into Euclidian spaces: you can now talk about the "length" of tangent vectors (which is an "extrinsic" conception here). (One can also define length of tangent vectors on smooth manifolds intrinsically, which leads to the subject Riemannian Geometry.)

Since any injective immersion from a compact manifold is an embedding, we immediately see

Theorem 1.5 (The Whitney Embedding Theorem, Compact Case)**.** *Any smooth compact manifold M of dimension m can be embedded into $\mathbb{R}^{2m+1}$.*

In combining with Theorem 1.4, we get

Theorem 1.6 (The Whitney Immersion Theorem, Compact Case)**.** *Any smooth compact manifold M of dimension m can be immersed into $\mathbb{R}^{2m}$.*

2. The Whitney embedding theorem: Non-Compact Case

Of course the above arguments fail when the manifold M is non-compact, since we can no longer cover M by finitely many coordinate charts, and an injective immersion from a non-compact manifold could be non-embedding. In general non-compact objects could behave quite differently from compact objects. For example, any compact Lie group must be a linear Lie group. The same conclusion does not hold for non-compact Lie groups. Fortunately for the Whitney embedding theorem, the non-compact version holds as well. We start with a non-compact version of Theorem 1.1:

Theorem 2.1. *Any non-compact smooth manifold M admits an injective immersion into $\mathbb{R}^K$ for sufficiently large K.*

Proof. According to PSet 1 Part 2 Problem 3, one can find a positive smooth exhaustion function f on M. For each $i \in \mathbb{N}$, we define

$$M_i = f^{-1}([i, i+1]).$$

Since M_i is compact, it can be covered by finitely many coordinate charts $U_1, \cdots, U_k$. We let

$$N_i = (U_1 \cup \cdots \cup U_k) \cap f^{-1}((i-0.1, i+1.1)).$$

Then each N_i is an open submanifold of M so that $M_i \subset N_i$. Moreover, $N_i \cap N_j = \emptyset$ if $|i-j| \geq 2$. By construction, each N_i can be covered by finitely many coordinate charts. So according to Theorem 1.2, there is an injective immersion from N_i into some $\mathbb{R}^K$ for K large. Since N_i is a smooth manifold (without boundary) of dimension m, Theorem 1.3 implies that we can find injective immersions φ_i from N_i into $\mathbb{R}^{2m+1}$.

Now pick smooth bump function ρ_i so that $\rho_i = 1$ in an open neighborhood of M_i and $\mathrm{supp}\rho_i \subset N_i$. Define

$$\Phi : M \to \mathbb{R}^{4m+3}, \quad p \mapsto \left(\sum_{i \text{ odd}} \rho_i(p)\varphi_i(p), \sum_{i \text{ even}} \rho_i(p)\varphi_i(p), f(p) \right).$$

Note that near each point $p \in M$, at most one term in each summations above is nonzero. So Φ is a smooth map. It remains to show that Φ is an injective immersion:

- $\boxed{\Phi \text{ is injective}}$ If $\Phi(p_1) = \Phi(p_2)$, then $\exists i \in \mathbb{N}$ so that $f(p_1) = f(p_2) \in [i, i+1]$. So $p_1, p_2 \in M_i \subset N_i$ and $\varphi_i(p_1) = \varphi_i(p_2)$. Since φ_i is injective, we get $p_1 = p_2$.
- $\boxed{\Phi \text{ is an immersion}}$ Suppose $p \in M_i$. Without loss of generality, we assume i is odd. Then for any $0 \neq X_p \in T_pM$,

  $$d\Phi_p(X_p) = ((d\varphi_i)_p(X_p), *, *).$$

  Since φ_i is an immersion on $U_i \ni p$, we get $(d\varphi_i)_p(X_p) \neq 0$. Thus $d\Phi_p(X_p) \neq 0$.

This completes the proof. $\square$

According to Theorem 1.3, any smooth manifold of dimension m admits an injective immersion into $\mathbb{R}^{2m+1}$. But for non-compact manifolds, such injective immersion need not to be an embedding. So one still need some trick.

We recall that a map f between topological spaces is called a *proper map* if the pre-image of any compact set is compact. (So the positive exhaustion function we constructed in PSet 1 Part 2 Problem 3 is proper.) The following proposition is an extension of Theorem 2.4 in Lecture 8: (Note that if N is compact, then continuous map $f : N \to M$ is proper. In some sense, properness is a substitution of compactness. We will use such idea again later.)

Proposition 2.2. *Let $f : N \to M$ be an injective immersion. If f is proper, then f is an embedding.*

Proof. Left as an exercise. ☐

Now we are ready to prove

Theorem 2.3 (The Whitney Embedding Theorem, Non-Compact Case)**.** *Any smooth non-compact manifold M of dimension m can be embedded into $\mathbb{R}^{2m+1}$.*

Proof. By Theorem 2.1 and Theorem 1.3, there exists an injective immersion $\Phi : M \to \mathbb{R}^{2m+1}$. Composing Φ with the diffeomorphism

$$\mathbb{R}^{2m+1} \to B^{2m+1}(1) = \{x \in \mathbb{R}^{2m+1} \mid |x| \le 1\}, \qquad x \mapsto \frac{x}{1+|x|^2}$$

if necessary, we may assume that $|\Phi(p)| \le 1$ for all $p \in M$.

Take any positive smooth exhaustion function f on M, and define

$$\widetilde{\Phi} = (\Phi, f) : M \to \mathbb{R}^{2m+2}, \quad p \mapsto (\Phi(p), f(p)).$$

Repeating the proof of Theorem 1.3, we get another injective immersion

$$\Psi = \pi_{[v]} \circ \widetilde{\Phi} : M \to \mathbb{R}^{2m+1},$$

where $\pi_{[v]}$ is some projection $\pi : \mathbb{R}^{2m+2} \to P_{[v]} \simeq \mathbb{R}^{2m+1}$. Moreover, since almost all $[v] \in \mathbb{RP}^{2m+1}$ work for our purpose, we can always choose $[v]$ so that $[v] \neq [0 : \cdots : 0 : 1]$. We claim that Ψ is proper. So according to Proposition 2.2, Ψ is an embedding.

It remains to prove that Ψ is proper. WLOG, we may assume $v \in S^{2m+1}$. Denote $v = (v', v^{2m+1})$. Then the condition $[v] \neq [0 : \cdots : 0 : 1]$ is equivalent to $|v^{2m+1}| < 1$. Since $|v| = 1$, we have $\pi_{[v]}(x) = x - (x \cdot v)v$. So we get

$$\begin{aligned}\Psi(p) &= (\Phi(p), f(p)) - \left[(\Phi(p), f(p)) \cdot (v', v^{2m+1})\right](v', v^{2m+1}) \\ &= \left(*, f(p)[1 - (v^{2m+1})^2] - (\Phi(p) \cdot v')v^{2m+1}\right).\end{aligned}$$

Now we prove properness. For any compact set $K \subset \mathbb{R}^{2m+1}$, $\exists A > 0$ so that

$$K \subset \{x \mid |x^{2m+1}| < A\}.$$

Since $|\Phi(p)| \le 1, |v^{2m+1}| \le 1$ and $|v'| \le 1$, we have

$$\begin{aligned}&\left|f(p)[1 - (v^{2m+1})^2] - (\Phi(p) \cdot v')v^{2m+1}\right| < A \\ \Longrightarrow &\left|f(p)[1 - (v^{2m+1})^2]\right| \le A + \left|(\Phi(p) \cdot v')v^{2m+1}\right| \le A + 1.\end{aligned}$$

It follows that $\Psi^{-1}(K) \subset f^{-1}([-\frac{A+1}{1-|v^{2m+1}|^2}, \frac{A+1}{1-|v^{2m+1}|^2}])$. But $\Psi^{-1}(K)$ is closed in M since Ψ is continuous, and $f^{-1}([-\frac{A+1}{1-|v^{2m+1}|^2}, \frac{A+1}{1-|v^{2m+1}|^2}])$ is compact in M since f is proper. So $\Psi^{-1}(K)$ is compact. By definition, Ψ is proper. $\square$

In view of Theorem 1.4, we get

Theorem 2.4 (The Whitney Immersion Theorem, Non-Compact Case)**.** *Any smooth non-compact manifold M of dimension m can be immersed into $\mathbb{R}^{2m}$.*

Finally we remark that the Whitney embedding/immersion theorems also holds for smooth manifolds with boundary.