WHITNEY EMBEDDING THEOREM

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ABSTRACT. The Whitney embedding theorem states that any smooth compact manifold of dimension n can be embedded as a closed submanifold of $\mathbb{R}^{2n}$. In this article, I prove a weaker version of the thoerem, which states that any smooth compact manifold of dimension n can be embedded as a closed submanifold of $\mathbb{R}^{2n+1}$. I do this in two steps: first, embed it in $\mathbb{R}^N$ for some finite N, and then, through iterated projections, push the dimension to $2n+1$.

1. The weak form

1.1. Statement.

Theorem 1 (Whitney embedding: weak form)**.** A smooth compact real manifold of real dimension n can be embedded as a closed submanifold of $\mathbb{R}^N$ for some $N \geq n$.

Some remarks:

- The *closed* part arises automatically because any compact subset of a Hausdorff space is closed.
- It suffices to restrict our attention to connected components. This is because every compact space has only finitely many connected components, and if we obtain embeddings of each, then we can obtain embeddings of their union by simply placing the images sufficiently far away.

1.2. Definition of manifold.

Definition. A **manifold**(defined) of dimension n is a topological space S along with an open cover U_i, i ranging over an index set I, and homeomorphisms $\phi_i : U_i \to V_i$ with V_i open in $\mathbb{R}^n$.

- Each open set U_i along with the mapping ϕ_i is termed a **coordinate chart**(defined).
- The collection of coordinate charts corresopnding to an open cover is termed an **atlas**(defined).

1.3. Definition of differential manifold.

Definition. A **differential manifold**(defined) of dimension n is a topological space S along with an open cover U_I, i ranging over an index set I, and homeomorphisms $\phi_i : U_i \to V_i$, where V_i is open in $\mathbb{R}^n$, such that the following holds:

Consider the set $U_i \cap U_j$. Let W denote $\phi_i(V_i \cap V_j)$. Then the composite mapping $\phi_j.\phi_i^{-1}$ is a diffeomorphism between open sets in $\mathbb{R}^n$.

1.4. Concept of differentiable embedding.

Definition. Let M_1 and M_2 be two differential manifolds. A **differentiable embedding**(defined) of M_1 in M_2 is a mapping $f : M_1 \to M_2$ such that:

- f is injective, viz $f(p) = f(q) \implies p = q$.
- f is differentiable at all points.
- At any point $p \in M_1$, the mapping $df_p : T_p(M_1) \to T_{f(p)}M_2$ is injective. That is, if v and w are two tangent vectors at p, then $df_p(v) = df_p(w) \implies v = w$.

With this definition, we are in a position to understand the statement of the Whitney embedding theorem: given any differential manifold that is *compact, connected* of dimension n as a topological space, there is a differentiable embedding into $\mathbb{R}^N$.

1.5. **Compactness translates to a finite atlas.**

Observation 1. A compact manifold has a finite atlas, that is, there is a finite collection of coordinate charts U_i with mappings ϕ_i to open sets V_i in $\mathbb{R}^n$.

Proof. Take any atlas of the manifold. Choose a finite subcover of the open cover for this atlas, and we have a finite atlas. □

1.6. **Partition of unity.**

Definition. Let X be a topological space, and U_i be a point-finite open cover of X (that is, an open cover such that every point of X is contained in only finitely many members). A **partition of unity**(defined) corresponding to U_i is a collection of continuous functions $f_i : X \to [0, 1]$ such that the support of f_i is contained in U_i for each i, and such that at each point p, the sum of the values $f_i(p)$ is 1.

Theorem 2. Given a compact space and a finite open cover of it, there exists a partition of unity for that finite open cover.

1.7. **Cartesian product of all the functions.**

Definition. Let $f_i : A \to B$ be a collection of n functions. Then the Cartesian product of f_i is defined as a function $f : A \to B^n$ such that the i^{th} coordinate projection of f is f_i.

We now have all the ingredients for the proof.

1.8. **The embedding.** Let M be a compact n-dimensional differential manifold, and (U_i, ϕ_i) be a finite atlas (with m elements). let f_i be a partition of unity corresponding to the open cover U_i. Consider a mapping $\mu_{ij} : p \mapsto x_i(\phi_j(p))f_j(p)$ where x_i denotes the i^{th} coordinate. Clearly, μ_{ij} are zero outside U_j, because the support lies there, and they are also continuous inside U_j, because there they are the products of two continuous functions. Take the direct product of all the μ_{ij}s and all the f_js. Call this direct product Φ. Φ is a mapping from M to $\mathbb{R}^{m(n+1)}$.

Claim. The mapping Φ described above is injective.

Proof. Suppose p and q are two distinct points in the manifold M such that $\Phi(p) = \Phi(q)$. Then clearly, there is some f_j such that $f_j(p) = f_j(q) \neq 0$ (the not equal to zero comes because the f_js form a partition of unity). Thus, both p and q lie inside the ball U_j.

Note that $f_j(p) = f_j(q) \neq 0$, and the local coordinate chart ϕ_j must have at least one differing coordinate for p and q. If this is the i^{th} coordinate, then $x_i(f_j(p)) \neq x_i(f_j(q))$, and hence $\Phi(p) \neq \Phi(q)$, leading to a contradiction. □

Claim. The differential of Φ is injective.

Proof. Because writing every coordinate of the differential is cumbersome, I abbreviate the expression by clubbing together the coordinates for the differentials in the directions μ_{ij} with the same i and varying j. That is, I club these together as vectors.

Then, applying the product rule, the differential of Φ at p sends an element $v \in T_pM$ to:

$$((df_1)_p(v)\phi_1(p) + f_1(p)(d\phi_1)_p(v), \ldots, (df_k)_p(v)\phi_k(p) + f_k(p)(d\phi_k)_p(v), (df_1)_p(v), \ldots, (df_k)_p(v))$$

where $(df_i)_p(v)\phi_i(p) + f_i(p)(d\phi_i)_p(v)$ is itself a n-tuple.

Suppose, at a point p, there are two vectors v and w such that $d\Phi_p(v) = d\Phi_p(w)$. Then, $(df_k)_p(v) = (df_k)_p(w)$ for all functions f_k. because the f_is form a partition of unity, there is some f_i such that $f_i(p) \neq 0$. Putting $k = i$, we obtain that $(df_i)_p(v) = (df_i)_p(w)$.

Also, the vectors $(df_i)_p(v)\phi_i(p) + f_i(p)(d\phi_i)_p(v)$ and $(df_i)_p(w)\phi_i(p) + f_i(p)(d\phi_i)_p(w)$ are equal. This gives us:

$$\begin{aligned}
(df_i)_p(v)\phi_i(p) + f_i(p)(d\phi_i)_p(v) &= (df_i)_p(w)\phi_i(p) + f_i(p)(d\phi_i)_p(w) \\
\implies f_i(p)(d\phi_i)_p(v) &= f_i(p)(d\phi_i)_p(w) \text{ because } (df_i)_p(v) = (df_i)_p(w) \\
\implies (d\phi_i)_p(v) &= (d\phi_i)_p(w) \text{ because } f_i(p) \neq 0
\end{aligned}$$

$\implies v = w$ because each ϕ_i is a differentiable embedding

Hence, starting that $d\Phi_p(v) = d\Phi_p(w)$ we obtain $v = w$. Thus, the mapping $d\Phi$ is injective on every tangent space. □

1.9. **The "put-together proof".**

2. The form we want

Theorem 3 (Whitney embedding: medium form)**.** Any smooth compact real manifold of real dimension n can be embedded as a closed submanifold of $\mathbb{R}^{2n+1}$.

The proof idea is to show that if there is an embedding in $\mathbb{R}^N$, then there is also an embedding in $\mathbb{R}^{N-1}$ provided $N > 2n + 1$.

Roughly speaking, we start with an embedding in $\mathbb{R}^N$, and then try to locate a hyperplane such that projecting the embedding on that hyperplane gives a differentiable embedding. In order to do this, we need a result that uses the fact that the *actual dimension of the manifold is low* to argue that *there is a direction* such that all lines in that direction intersect the manifold exactly once, and hence projecting to the perpendicular hyperplane maintains injectivity. We also need to show that since *the actual dimension of the manifold is low*, the differential mapping is also injective.

The result we use for this purpose is Sard's theorem, or rather a corollary of Sard's theorem.

2.1. **Sard's theorem: a recall for application.** First, some terminology:

Definition. Let $f : M_1 \to M_2$ be a differentiable mapping between differential manifolds M_1 and M_2.

1. A **regular point**(defined) for f is a point $p \in M_1$ such that the induced mapping $df : T_pM_1 \to T_{f(p)}M_2$ is surjective.
2. A **critical point**(defined) for f is a point that is not a critical point.
3. A **critical value**(defined) for f is a point in M_2 that occurs as the image of a critical point via f.
4. A **regular value**(defined) for f is a point in M_2 whose inverse image is nonempty and does not contain any critical points.

Theorem 4 (Sard's theorem)**.** Let $f : M_1 \to M_2$ be a differentiable map between differential manifolds M_1 and M_2. Then, the set of critical values of f has Lebesgue measure zero in M_2.

An immediate consequence of this is what we'll be using:

Corollary 1 (Mapping lower to higher dimensions)**.** Suppose M_1 and M_2 are differential manifolds, such that the dimension of M_1 is strictly smaller than the dimension of M_2. Then, the image of M_1 in M_2 is a set of measure zero.

Proof. Because the dimension of M_1 is strictly smaller than the dimension of M_2, th map df can nevr be surjective. Hence, every point is a critical point, and thus, every point in the image is a critical value. Thus, applying Sard's theorem (theorem 4) we obtain that the image has measure zero. □

2.2. **Assuring injectivity.**

Lemma 1 (Full-measureset of hyperplanes)**.** Suppose $N > 2n+1$. Suppose there is an embedding Φ of an n-dimensional differential manifold M in $\mathbb{R}^n$. Then, there exists a full-measure set of hyperplanes Π in $\mathbb{R}^n$ such that the composite mapping $p_\Pi.\Phi$ is injective from M to Π.

Proof. Note that $p_\Pi.\Phi$ is injective if and only if every line orthogonal to Π intersects the manifoldat exactly one point. In other words, we want that *most* directions are *not* available as directions of lines joining pairs of points in the manifold. This reduces to demanding that the mapping from pairs of distinct points in the manifold, to *directions* in $\mathbb{R}^N$, have an image of measure zero.

Let's put this formally.

Let Δ be the diagonal in $M \times M$ Define a mapping:

$$a : M \times M \setminus \Delta \to \mathbb{RP}^{N-1}$$

where $a(x, y)$ is the line through the origin parallel to the line joining $\Phi(x)$ and $\Phi(y)$ in $\mathbb{R}^N$.

Note that the left hand side has dimension $2n$, while the right hand side has dimension $N-1$, which is at least $2n+1$. We can apply corollary 1 to the map a and conclude that the image of a is a measure zero set. Hence, the complement is a full-measure set, and thus, the collection of hyperplanes Π for which $p_\Pi.\Phi$ is injective is a full-measure set. □

2.3. **Assuring a differentiable embedding.**

Lemma 2 (Differential injectivity)**.** Suppose $N > 2n+1$. Suppose there is an embedding Φ of an n-dimensional differential manifold M in $\mathbb{R}^n$. Then, there exists a full-measure set of hyperplanes Π in $\mathbb{R}^n$ such that the composite mapping $p_\Pi.\Phi$ has injective differential everywhere.

Proof. Injectivity of the differential translates to the condition that the line orthogonal to the hyperplane does *not* occur as the image of any tangent space via the differential mapping. Thus, we need to make sure that the differential mapping misses out a full-measure set of vectors.

The differential mapping is a mapping from *tangent directions* to the manifold to *directions* in $\mathbb{R}^N$. Let's formalize this.

Consider the projectivization of the tangent bundle of M, denote as $\mathbb{P}(TM)$. This is a fiber bundle over M with fiber $\mathbb{RP}^{n-1}$. The total space is thus a smooth manifold of dimension $2n-1$. Define a mapping:

$$b : \mathbb{P}(TM) \to \mathbb{RP}^{N-1}$$

given as $l \in T_x(M) \mapsto D_x(\Phi(l)) \in \mathbb{R}^N$.

The dimension of the left side is $2n-1$, while the dimension of the right side is at least $2n+1$. Hence, we can apply corollary 1 and conclude that the image has measure zero. Hence, the collection of hyperplanes Π for which the differential is injective, is of full measure. □

2.4. **Putting the pieces together.** We have obtained that:

- For a full-measure set of hyperplanes, projecting the embedding to the hyperplane gives an injective mapping.

- For a full-measure set of hyperplanes, projecting the embedding to the hyperplane gives a mapping with differential injective.

Intersecting these two full-measure sets, we obtain another full-measure set of hyperplanes where both $p_\Pi.\Phi$ and the differential are injective.

Hence, starting with a differentiable embedding of M in $\mathbb{R}^N$, we have obtained a differentiable embedding of M in $\mathbb{R}^{N-1}$.