DIFFERENTIABLE MANIFOLDS

Course **C3.2b** 2010

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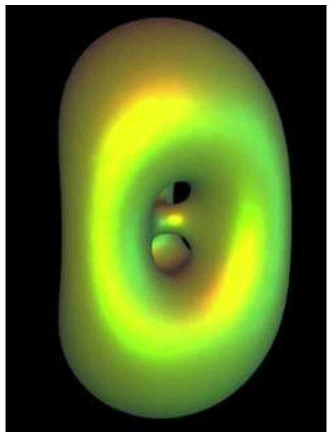
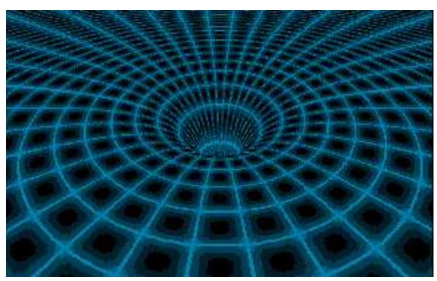
Contents

1 Introduction

This is an introductory course on differentiable manifolds. These are higher dimensional analogues of surfaces like this:



This is the image to have, but we shouldn't think of a manifold as always sitting inside a fixed Euclidean space like this one, but rather as an abstract object. One of the historical driving forces of the theory was General Relativity, where the manifold is four-dimensional spacetime, wormholes and all:



Spacetime is not part of a bigger Euclidean space, it just exists, but we need to learn how to do analysis on it, which is what this course is about.

Another input to the subject is from mechanics – the dynamics of complicated mechanical systems involve spaces with many degrees of freedom. Just think of the different configurations that an Anglepoise lamp can be put into:

How many degrees of freedom are there? How do we describe the dynamics of this if we hit it?

The first idea we shall meet is really the defining property of a manifold – to be able to describe points locally by n real numbers, *local coordinates*. Then we shall need to define analytical objects (*vector fields, differential forms* for example) which are independent of the choice of coordinates. This has a double advantage: on the one hand it enables us to discuss these objects on topologically non-trivial manifolds like spheres, and on the other it also provides the language for expressing the equations of mathematical physics in a coordinate-free form, one of the fundamental principles of relativity.

The most basic example of analytical techniques on a manifold is the theory of differential forms and the exterior derivative. This generalizes the grad, div and curl of ordinary three-dimensional calculus. A large part of the course will be occupied with this. It provides a very natural generalization of the theorems of Green and Stokes in three dimensions and also gives rise to *de Rham cohomology* which is an analytical way of approaching the algebraic topology of the manifold. This has been important in an enormous range of areas from algebraic geometry to theoretical physics.

More refined use of analysis requires extra data on the manifold and we shall simply define and describe some basic features of Riemannian metrics. These generalize the first fundamental form of a surface and, in their Lorentzian guise, provide the substance of general relativity. A more complete story demands a much longer course, but here we shall consider just two aspects which draw on the theory of differential forms: the study of geodesics via a vector field, the geodesic flow, on the cotangent bundle, and some basic properties of harmonic forms.

Certain standard technical results which we shall require are proved in the Appendix so as not to interrupt the development of the theory.

A good book to accompany the course is: *An Introduction to Differential Manifolds* by Dennis Barden and Charles Thomas (Imperial College Press £22 (paperback)).

2 Manifolds

2.1 Coordinate charts

The concept of a manifold is a bit complicated, but it starts with defining the notion of a *coordinate chart.*

Definition 1 *A coordinate chart on a set X is a subset $U \subseteq X$ together with a bijection*

$$\varphi : U \to \varphi(U) \subseteq \mathbf{R}^n$$

onto an open set $\varphi(U)$ in $\mathbf{R}^n$.

Thus we can parametrize points x of U by n coordinates $\varphi(x) = (x_1, \ldots, x_n)$.

We now want to consider the situation where X is covered by such charts and satisfies some consistency conditions. We have

Definition 2 *An n-dimensional atlas on X is a collection of coordinate charts $\{U_\alpha, \varphi_\alpha\}_{\alpha \in I}$ such that*

- *X is covered by the $\{U_\alpha\}_{\alpha \in I}$*
- *for each $\alpha, \beta \in I$, $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbf{R}^n$*
- *the map*

  $$\varphi_\beta \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

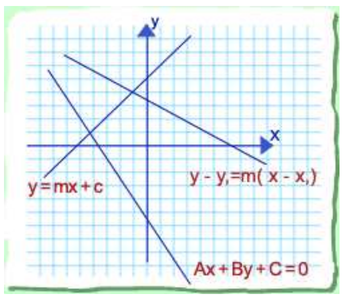
  is C^∞ with C^∞ inverse.

Recall that $F(x_1, \ldots, x_n) \in \mathbf{R}^n$ is C^∞ if it has derivatives of all orders. We shall also say that F is *smooth* in this case. It is perfectly possible to develop the theory of manifolds with less differentiability than this, but this is the normal procedure.

Examples:

1. Let $X = \mathbf{R}^n$ and take $U = X$ with $\varphi = id$. We could also take X to be any open set in $\mathbf{R}^n$.

2. Let X be the set of straight lines in the plane:



Each such line has an equation $Ax + By + C = 0$ where if we multiply A, B, C by a non-zero real number we get the same line. Let U_0 be the set of non-vertical lines. For each line $\ell \in U_0$ we have the equation

$$y = mx + c$$

where m, c are uniquely determined. So $\varphi_0(\ell) = (m, c)$ defines a coordinate chart $\varphi_0 : U_0 \to \mathbf{R}^2$. Similarly if U_1 consists of the non-horizontal lines with equation

$$x = \tilde{m}y + \tilde{c}$$

we have another chart $\varphi_1 : U_1 \to \mathbf{R}^2$.

Now $U_0 \cap U_1$ is the set of lines $y = mx + c$ which are not horizontal, so $m \neq 0$. Thus

$$\varphi_0(U_0 \cap U_1) = \{(m, c) \in \mathbf{R}^2 : m \neq 0\}$$

which is open. Moreover, $y = mx + c$ implies $x = m^{-1}y - cm^{-1}$ and so

$$\varphi_1 \varphi_0^{-1}(m, c) = (m^{-1}, -cm^{-1})$$

which is smooth with smooth inverse. Thus we have an atlas on the space of lines.

3. Consider $\mathbf{R}$ as an additive group, and the subgroup of integers $\mathbf{Z} \subset \mathbf{R}$. Let X be the quotient group $\mathbf{R}/\mathbf{Z}$ and $p : \mathbf{R} \to \mathbf{R}/\mathbf{Z}$ the quotient homomorphism.

Set $U_0 = p(0, 1)$ and $U_1 = p(-1/2, 1/2)$. Since any two elements in the subset $p^{-1}(a)$ differ by an integer, p restricted to $(0, 1)$ or $(-1/2, 1/2)$ is injective and so we have coordinate charts

$$\varphi_0 = p^{-1} : U_0 \to (0, 1), \quad \varphi_1 = p^{-1} : U_1 \to (-1/2, 1/2).$$

Clearly U_0 and U_1 cover $\mathbf{R}/\mathbf{Z}$ since the integer $0 \in U_1$.

We check:

$$\varphi_0(U_0 \cap U_1) = (0, 1/2) \cup (1/2, 1), \quad \varphi_1(U_0 \cap U_1) = (-1/2, 0) \cup (0, 1/2)$$

which are open sets. Finally, if $x \in (0, 1/2)$, $\varphi_1\varphi_0^{-1}(x) = x$ and if $x \in (1/2, 1)$, $\varphi_1\varphi_0^{-1}(x) = x - 1$. These maps are certainly smooth with smooth inverse so we have an atlas on $X = \mathbf{R}/\mathbf{Z}$.

4. Let X be the extended complex plane $X = \mathbf{C} \cup \{\infty\}$. Let $U_0 = \mathbf{C}$ with $\varphi_0(z) = z \in \mathbf{C} \cong \mathbf{R}^2$. Now take

$$U_1 = \mathbf{C} \backslash \{0\} \cup \{\infty\}$$

and define $\varphi_1(\tilde{z}) = \tilde{z}^{-1} \in \mathbf{C}$ if $\tilde{z} \neq \infty$ and $\varphi_1(\infty) = 0$. Then

$$\varphi_0(U_0 \cap U_1) = \mathbf{C} \backslash \{0\}$$

which is open, and

$$\varphi_1\varphi_0^{-1}(z) = z^{-1} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}.$$

This is a smooth and invertible function of (x, y). We now have a 2-dimensional atlas for X, the extended complex plane.

5. Let X be n-dimensional *real projective space*, the set of 1-dimensional vector subspaces of $\mathbf{R}^{n+1}$. Each subspace is spanned by a non-zero vector v, and we define $U_i \subset \mathbf{R}\mathrm{P}^n$ to be the subset for which the i-th component of $v \in \mathbf{R}^{n+1}$ is non-zero. Clearly X is covered by $U_1, \ldots, U_{n+1}$. In U_i we can uniquely choose v such that the ith component is 1, and then U_i is in one-to-one correspondence with the hyperplane $x_i = 1$ in $\mathbf{R}^{n+1}$, which is a copy of $\mathbf{R}^n$. This is therefore a coordinate chart

$$\varphi_i : U_i \to \mathbf{R}^n.$$

The set $\varphi_i(U_i \cap U_j)$ is the subset for which $x_j \neq 0$ and is therefore open. Furthermore

$$\varphi_i\varphi_j^{-1} : \{x \in \mathbf{R}^{n+1} : x_j = 1, x_i \neq 0\} \to \{x \in \mathbf{R}^{n+1} : x_i = 1, x_j \neq 0\}$$

is

$$v \mapsto \frac{1}{x_i} v$$

which is smooth with smooth inverse. We therefore have an atlas for $\mathbf{R}\mathrm{P}^n$.

2.2 The definition of a manifold

All the examples above are actually manifolds, and the existence of an atlas is sufficient to establish that, but there is a minor subtlety in the actual definition of a manifold due to the fact that there are lots of choices of atlases. If we had used a different basis for $\mathbf{R}^2$, our charts on the space X of straight lines would be different, but we would like to think of X as an object independent of the choice of atlas. That's why we make the following definitions:

Definition 3 *Two atlases $\{(U_\alpha, \varphi_\alpha)\}$, $\{(V_i, \psi_i)\}$ are compatible if their union is an atlas.*

What this definition means is that all the extra maps $\psi_i \varphi_\alpha^{-1}$ must be smooth. Compatibility is clearly an equivalence relation, and we then say that:

Definition 4 *A differentiable structure on X is an equivalence class of atlases.*

Finally we come to the definition of a manifold:

Definition 5 *An n-dimensional differentiable manifold is a space X with a differentiable structure.*

The upshot is this: to prove something is a manifold, all you need is to find one atlas. The definition of a manifold takes into account the existence of many more atlases.

Many books give a slightly different definition – they start with a topological space, and insist that the coordinate charts are homeomorphisms. This is fine if you see the world as a hierarchy of more and more sophisticated structures but it suggests that in order to prove something is a manifold you first have to define a topology. As we'll see now, the atlas does that for us.

First recall what a topological space is: a set X with a distinguished collection of subsets V called *open sets* such that

1. $\emptyset$ and X are open
2. an arbitrary union of open sets is open
3. a finite intersection of open sets is open

Now suppose M is a manifold. We shall say that a subset $V \subseteq M$ is open if, for each α, $\varphi_\alpha(V \cap U_\alpha)$ is an open set in $\mathbf{R}^n$. One thing which is immediate is that $V = U_\beta$ is open, from Definition 2.

We need to check that this gives a topology. Condition 1 holds because $\varphi_\alpha(\emptyset) = \emptyset$ and $\varphi_\alpha(M \cap U_\alpha) = \varphi_\alpha(U_\alpha)$ which is open by Definition 1. For the other two, if V_i is a collection of open sets then because φ_α is bijective

$$\varphi_\alpha((\cup V_i) \cap U_\alpha) = \cup \varphi_\alpha(V_i \cap U_\alpha)$$
$$\varphi_\alpha((\cap V_i) \cap U_\alpha) = \cap \varphi_\alpha(V_i \cap U_\alpha)$$

and then the right hand side is a union or intersection of open sets. Slightly less obvious is the following:

Proposition 2.1 *With the topology above $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a homeomorphism.*

Proof: If $V \subseteq U_\alpha$ is open then $\varphi_\alpha(V) = \varphi_\alpha(V \cap U_\alpha)$ is open by the definition of the topology, so φ_α^{-1} is certainly continuous.

Now let $W \subset \varphi_\alpha(U_\alpha)$ be open, then $\varphi_\alpha^{-1}(W) \subseteq U_\alpha$ and U_α is open in M so we need to prove that $\varphi_\alpha^{-1}(W)$ is open in M. But

$$\varphi_\beta(\varphi_\alpha^{-1}(W) \cap U_\beta) = \varphi_\beta \varphi_\alpha^{-1}(W \cap \varphi_\alpha(U_\alpha \cap U_\beta)) \tag{1}$$

From Definition 2 the set $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open and hence its intersection with the open set W is open. Now $\varphi_\beta \varphi_\alpha^{-1}$ is C^∞ with C^∞ inverse and so certainly a homeomorphism, and it follows that the right hand side of (1) is open. Thus the left hand side $\varphi_\beta(\varphi_\alpha^{-1} W \cap U_\beta)$ is open and by the definition of the topology this means that $\varphi_\alpha^{-1}(W)$ is open, i.e. φ_α is continuous. □

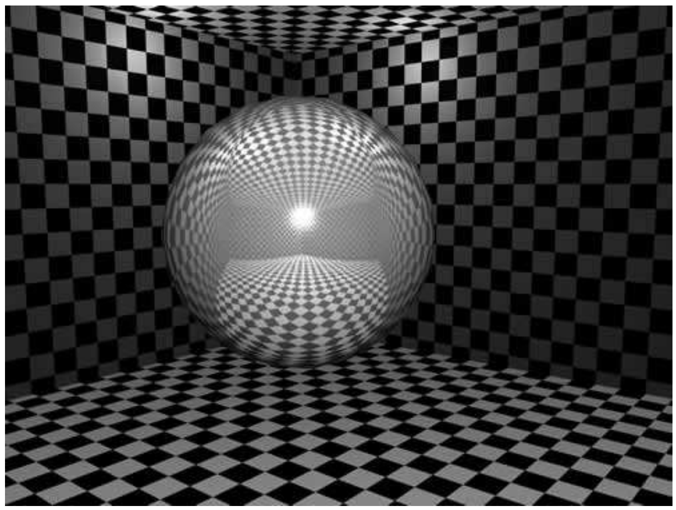
To make any reasonable further progress, we have to make two assumptions about this topology which will hold for the rest of these notes:

- **the manifold topology is Hausdorff**
- **in this topology we have a countable basis of open sets**

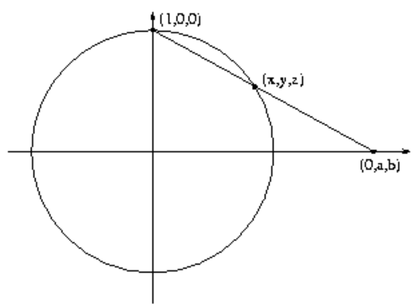
Without these assumptions, manifolds are not even metric spaces, and there is not much analysis that can reasonably be done on them.

2.3 Further examples of manifolds

We need better ways of recognizing manifolds than struggling to find explicit coordinate charts. For example, the sphere is a manifold



and although we can use stereographic projection to get an atlas:



there are other ways. Here is one.

Theorem 2.2 *Let $F : U \to \mathbf{R}^m$ be a C^∞ function on an open set $U \subseteq \mathbf{R}^{n+m}$ and take $c \in \mathbf{R}^m$. Assume that for each $a \in F^{-1}(c)$, the derivative*

$$DF_a : \mathbf{R}^{n+m} \to \mathbf{R}^m$$

is surjective. Then $F^{-1}(c)$ has the structure of an n-dimensional manifold which is Hausdorff and has a countable basis of open sets.

Proof: Recall that the derivative of F at a is the linear map $DF_a : \mathbf{R}^{n+m} \to \mathbf{R}^m$ such that

$$F(a + h) = F(a) + DF_a(h) + R(a, h)$$

where $R(a,h)/\|h\| \to 0$ as $h \to 0$.

If we write $F(x_1,\ldots,x_{n+m}) = (F_1,\ldots,F_m)$ the derivative is the Jacobian matrix

$$\frac{\partial F_i}{\partial x_j}(a) \qquad 1 \le i \le m, 1 \le j \le n+m$$

Now we are given that this is surjective, so the matrix has rank m. Therefore by reordering the coordinates $x_1,\ldots,x_{n+m}$ we may assume that the square matrix

$$\frac{\partial F_i}{\partial x_j}(a) \qquad 1 \le i \le m, 1 \le j \le m$$

is invertible.

Now define

$$G : U \to \mathbf{R}^{n+m}$$

by

$$G(x_1,\ldots,x_{n+m}) = (F_1,\ldots,F_m,x_{m+1},\ldots,x_{n+m}). \tag{2}$$

Then DG_a is invertible.

We now apply the *inverse function theorem* to G, a proof of which is given in the Appendix. It tells us that there is a neighbourhood V of a, and W of $G(a)$ such that $G : V \to W$ is invertible with smooth inverse. Moreover, the formula (2) shows that G maps $V \cap F^{-1}(c)$ to the intersection of W with the copy of $\mathbf{R}^n$ given by $\{x \in \mathbf{R}^{n+m} : x_i = c_i, 1 \le i \le m\}$. This is therefore a coordinate chart φ.

If we take two such charts $\varphi_\alpha, \varphi_\beta$, then $\varphi_\alpha\varphi_\beta^{-1}$ is a map from an open set in $\{x \in \mathbf{R}^{n+m} : x_i = c_1, 1 \le i \le m\}$ to another one which is the restriction of the map $G_\alpha G_\beta^{-1}$ of (an open set in) $\mathbf{R}^{n+m}$ to itself. But this is an invertible C^∞ map and so we have the requisite conditions for an atlas.

Finally, in the induced topology from $\mathbf{R}^{n+m}$, G_α is a homeomorphism, so open sets in the manifold topology are the same as open sets in the induced topology. Since $\mathbf{R}^{n+m}$ is Hausdorff with a countable basis of open sets, so is $F^{-1}(c)$. $\square$

We can now give further examples of manifolds:

Examples: 1. Let

$$S^n = \{x \in \mathbf{R}^{n+1} : \sum_1^{n+1} x_i^2 = 1\}$$

be the unit n-sphere. Define $F : \mathbf{R}^{n+1} \to \mathbf{R}$ by

$$F(x) = \sum_1^{n+1} x_i^2.$$

This is a C^∞ map and

$$DF_a(h) = 2\sum_i a_i h_i$$

is non-zero (and hence surjective in the 1-dimensional case) so long as a is not identically zero. If $F(a) = 1$, then

$$\sum_1^{n+1} a_i^2 = 1 \neq 0$$

so $a \neq 0$ and we can apply Theorem 2.2 and deduce that the sphere is a manifold.

2. Let $O(n)$ be the space of $n \times n$ orthogonal matrices: $AA^T = 1$. Take the vector space M_n of dimension n^2 of all real $n \times n$ matrices and define the function

$$F(A) = AA^T$$

to the vector space of *symmetric* $n \times n$ matrices. This has dimension $n(n+1)/2$. Then $O(n) = F^{-1}(I)$.

Differentiating F we have

$$DF_A(H) = HA^T + AH^T$$

and putting $H = KA$ this is

$$KAA^T + AA^T K^T = K + K^T$$

if $AA^T = I$, i.e. if $A \in F^{-1}(I)$. But given any symmetric matrix S, taking $K = S/2$ shows that DF_I is surjective and so, applying Theorem 2.2 we find that $O(n)$ is a manifold. Its dimension is

$$n^2 - n(n+1)/2 = n(n-1)/2.$$

2.4 Maps between manifolds

We need to know what a smooth map between manifolds is. Here is the definition:

Definition 6 *A map $F : M \to N$ of manifolds is a smooth map if for each point $x \in M$ and chart $(U_\alpha, \varphi_\alpha)$ in M with $x \in U_\alpha$ and chart (V_i, ψ_i) of N with $F(x) \in V_i$, the set $F^{-1}(V_i)$ is open and the composite function*

$$\psi_i F \varphi_\alpha^{-1}$$

on $\varphi_\alpha(F^{-1}(V_i) \cap U_\alpha)$ is a C^∞ function.

Note that it is enough to check that the above holds for one atlas – it will follow from the fact that $\varphi_\alpha \varphi_\beta^{-1}$ is C^∞ that it then holds for all compatible atlases.

Exercise 2.3 *Show that a smooth map is continuous in the manifold topology.*

The natural notion of equivalence between manifolds is the following:

Definition 7 *A diffeomorphism $F : M \to N$ is a smooth map with smooth inverse.*

Example: Take two of our examples above – the quotient group $\mathbf{R}/\mathbf{Z}$ and the 1-sphere, the circle, S^1. We shall show that these are diffeomorphic. First we define a map

$$G : \mathbf{R}/\mathbf{Z} \to S^1$$

by

$$G(x) = (\cos 2\pi x, \sin 2\pi x).$$

This is clearly a bijection. Take $x \in U_0 \subset \mathbf{R}/\mathbf{Z}$ then we can represent the point by $x \in (0, 1)$. Within the range $(0, 1/2)$, $\sin 2\pi x \neq 0$, so with $F = x_1^2 + x_2^2$, we have $\partial F/\partial x_2 \neq 0$. The use of the inverse function theorem in Theorem 2.2 then says that x_1 is a local coordinate for S^1, and in fact on the whole of $(0, 1/2)$ $\cos 2\pi x$ is smooth and invertible. We proceed by taking the other similar open sets to check fully that G is a smooth, bijective map. To prove that its inverse is smooth, we can rely on the inverse function theorem, since $\sin 2\pi x \neq 0$ in the interval.

3 Tangent vectors and cotangent vectors

3.1 Existence of smooth functions

The most fundamental type of map between manifolds is a smooth map

$$f : M \to \mathbf{R}.$$

We can add these and multiply by constants so they form a vector space $C^\infty(M)$, the space of C^∞ functions on M. In fact, under multiplication it is also a commutative ring. So far, all we can assert is that the constant functions lie in this space, so let's see why there are lots and lots of global C^∞ functions. We shall use bump functions and the Hausdorff property.

First note that the following function of one variable is C^∞:

$$\begin{aligned} f(t) &= e^{-1/t} \quad t > 0 \\ &= 0 \qquad t \leq 0 \end{aligned}$$

Now form

$$g(t) = \frac{f(t)}{f(t) + f(1-t)}$$

so that g is identically 1 when $t \geq 1$ and vanishes if $t \leq 0$. Next write

$$h(t) = g(t+2)g(2-t).$$

This function vanishes if $|t| \geq 2$ and is 1 where $|t| \leq 1$: it is completely flat on top.

Finally make an n-dimensional version

$$k(x_1, \ldots, x_n) = h(x_1)h(x_2)\ldots h(x_n).$$

In the sup norm, this is 1 if $|x| \leq 1$, so $k(r^{-1}x)$ is identically 1 in a ball of radius r and 0 outside a ball of radius $2r$.

We shall use this construction several times later on. For the moment, let M be any manifold and (U, φ_U) a coordinate chart. Choose a function k of the type above whose support (remember $\operatorname{supp} f = \overline{\{x : f(x) \neq 0\}}$) lies in $\varphi_U(U)$ and define

$$f : M \to \mathbf{R}$$

by

$$\begin{aligned} f(x) &= k \circ \varphi_U(x) \quad x \in U \\ &= 0 \qquad x \in M \backslash U. \end{aligned}$$

Is this a smooth function? The answer is yes: by definition f is smooth for points in the coordinate neighbourhood U. But $\operatorname{supp} k$ is closed and bounded in $\mathbf{R}^n$ and so compact and since φ_U is a homeomorphism, f is zero on the complement of a compact set in M. But a compact set in a Hausdorff space is closed, so its complement is open. If $y \neq U$ then there is a neighbourhood of y on which f is identically zero, in which case clearly f is smooth at y.

3.2 The derivative of a function

Smooth functions exist in abundance. The question now is: we know what a differentiable function is – so what is its derivative? We need to give some coordinate-independent definition of derivative and this will involve some new concepts. The derivative at a point $a \in M$ will lie in a vector space T_a^* called the cotangent space.

First let's address a simpler question – what does it mean for the derivative to vanish? This is more obviously a coordinate-invariant notion because on a compact manifold any function has a maximum, and in any coordinate system in a neighbourhood of that point, its derivative must vanish. We can check that: if $f : M \to \mathbf{R}$ is smooth then the composition

$$g = f\varphi_\alpha^{-1}$$

is a C^∞ function of $x_1, \dots x_n$. Suppose its derivative vanishes at $\varphi_U(a)$ and now take a different chart φ_β with $h = f\varphi_\beta^{-1}$. Then

$$g = f\varphi_\alpha^{-1} = f\varphi_\beta^{-1}\varphi_\beta\varphi_\alpha^{-1} = h\varphi_\beta\varphi_\alpha^{-1}.$$

But from the definition of an atlas, $\varphi_\beta\varphi_\alpha^{-1}$ is smooth with smooth inverse, so

$$g(x_1, \dots, x_n) = h(y_1(x), \dots, y_n(x))$$

and by the chain rule

$$\frac{\partial g}{\partial x_i} = \sum_j \frac{\partial h}{\partial y_j}(y(a))\frac{\partial y_j}{\partial x_i}(a).$$

Since $y(x)$ is invertible, its Jacobian matrix is invertible, so that $Dg_a = 0$ if and only if $Dh_{y(a)} = 0$. We have checked then that the vanishing of the derivative at a point a is independent of the coordinate chart. We let $Z_a \subset C^\infty(M)$ be the subset of functions whose derivative vanishes at a. Since Df_a is linear in f the subset Z_a is a vector subspace.

Definition 8 *The cotangent space T_a^* at $a \in M$ is the quotient space*

$$T_a^* = C^\infty(M)/Z_a.$$

The derivative of a function f at a is its image in this space and is denoted $(df)_a$.

Here we have simply defined the derivative as all functions modulo those whose derivative vanishes. It's almost a tautology, so to get anywhere we have to prove something about T_a^*. First note that if ψ is a smooth function on a neighbourhood of x, we can multiply it by a bump function to extend it to M and then look at its image in $T_a^* = C^\infty(M)/Z_a$. But its derivative in a coordinate chart around a is independent of the bump function, because all such functions are identically 1 in a neighbourhood of a. Hence we can actually define the derivative at a of smooth functions which are only defined in a neighbourhood of a. In particular we could take the coordinate functions $x_1, \ldots, x_n$. We then have

Proposition 3.1 *Let M be an n-dimensional manifold, then*

- *the cotangent space T_a^* at $a \in M$ is an n-dimensional vector space*
- *if (U, φ) is a coordinate chart around x with coordinates $x_1, \ldots, x_n$, then the elements $(dx_1)_a, \ldots (dx_n)_a$ form a basis for T_a^**
- *if $f \in C^\infty(M)$ and in the coordinate chart, $f\varphi^{-1} = \phi(x_1, \ldots, x_n)$ then*

$$(df)_a = \sum_i \frac{\partial \phi}{\partial x_i}(\varphi(a))(dx_i)_a \tag{3}$$

Proof: If $f \in C^\infty(M)$, with $f\varphi^{-1} = \phi(x_1, \ldots, x_n)$ then

$$f - \sum \frac{\partial \phi}{\partial x_i}(\varphi(a))x_i$$

is a (locally defined) smooth function whose derivative vanishes at a, so

$$(df)_a = \sum \frac{\partial f}{\partial x_i}(\varphi(a))(dx_i)_a$$

and $(dx_1)_a, \ldots (dx_n)_a$ span T_a^*.

If $\sum_i \lambda_i (dx_i)_a = 0$ then $\sum_i \lambda_i x_i$ has vanishing derivative at a and so $\lambda_i = 0$ for all i.
$\square$

Remark: It is rather heavy handed to give two symbols f, ϕ for a function and its representation in a given coordinate system, so often in what follows we shall use just f. Then we can write (3) as

$$df = \sum \frac{\partial f}{\partial x_i} dx_i.$$

With a change of coordinates $(x_1, \ldots, x_n) \to (y_1(x), \ldots, y_n(x))$ the formalism gives

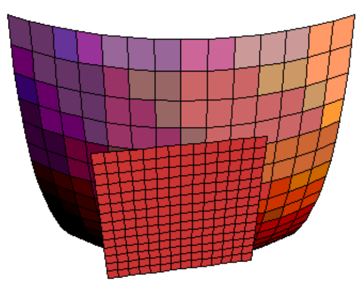
$$df = \sum_j \frac{\partial f}{\partial y_j} dy_j = \sum_{i,j} \frac{\partial f}{\partial y_j} \frac{\partial y_j}{\partial x_i} dx_i.$$

Definition 9 *The tangent space T_a at $a \in M$ is the dual space of the cotangent space T_a^*.*

This is admittedly a roundabout way of defining T_a, but since the double dual $(V^*)^*$ of a finite dimensional vector space is naturally isomorphic to V the notation is consistent. If $x_1, \ldots, x_n$ is a local coordinate system at a and $(dx_1)_a, \ldots, (dx_n)_a$ the basis of T_a^* defined in (3.1) then the dual basis for the tangent space T_a is denoted

$$\left(\frac{\partial}{\partial x_1}\right)_a, \ldots, \left(\frac{\partial}{\partial x_n}\right)_a.$$

This definition at first sight seems far away from our intuition about the tangent space to a surface in $\mathbf{R}^3$:



The problem arises because our manifold M does not necessarily sit in Euclidean space and we have to define a tangent space intrinsically. There are two ways around this: one would be to consider functions $f : \mathbf{R} \to M$ and equivalence classes of these, instead of functions the other way $f : M \to \mathbf{R}$. Another, perhaps more useful, one is provided by the notion of *directional derivative*. If f is a function on a surface in $\mathbf{R}^3$, then for every tangent direction $\mathbf{u}$ at a we can define the derivative of f at a in the direction $\mathbf{u}$, which is a real number: $\mathbf{u} \cdot \nabla f(a)$ or $Df_a(u)$. Imitating this gives the following:

Definition 10 *A tangent vector at a point $a \in M$ is a linear map $X_a : C^\infty(M) \to \mathbf{R}$ such that*

$$X_a(fg) = f(a)X_a g + g(a)X_a f.$$

This is the formal version of the Leibniz rule for differentiating a product.

Now if $\xi \in T_a$, it lies in the dual space of $T_a^* = C^\infty(M)/Z_a$ and so

$$f \mapsto \xi((df)_a)$$

is a linear map from $C^\infty(M)$ to $\mathbf{R}$. Moreover from (3),

$$d(fg)_a = f(a)(dg)_a + g(a)(df)_a$$

and so

$$X_a(f) = \xi((df)_a)$$

is a tangent vector at a. In fact, any tangent vector is of this form, but the price paid for the nice algebraic definition in (10) which is the usual one in textbooks is that we need a lemma to prove it.

Lemma 3.2 *Let X_a be a tangent vector at a and f a smooth function whose derivative at a vanishes. Then $X_a f = 0$.*

Proof: Use a coordinate system near a. By the fundamental theorem of calculus,

$$\begin{aligned} f(x) - f(a) &= \int_0^1 \frac{\partial}{\partial t} f(a + t(x-a))dt \\ &= \sum_i (x_i - a_i) \int_0^1 \frac{\partial f}{\partial x_i}(a + t(x-a))dt. \end{aligned}$$

If $(df)_a = 0$ then

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(a + t(x-a))dt$$

vanishes at $x = a$, as does $x_i - a_i$. Now although these functions are defined locally, using a bump function we can extend them to M, so that

$$f = f(a) + \sum_i g_i h_i \tag{4}$$

where $g_i(a) = h_i(a) = 0$.

By the Leibnitz rule

$$X_a(1) = X_a(1.1) = 2X_a(1)$$

which shows that X_a annihilates constant functions. Applying the rule to (4)

$$X_a(f) = X_a(\sum_i g_i h_i) = \sum_i (g_i(a) X_a h_i + h_i(a) X_a g_i) = 0.$$

This means that $X_a : C^\infty(M) \to \mathbf{R}$ annihilates Z_a and is well defined on $T_a^* = C^\infty(M)/Z_a$ and so $X_a \in T_a$. □

The vectors in the tangent space are therefore the tangent vectors as defined by (10). Locally, in coordinates, we can write

$$X_a = \sum_i^n c_i \left(\frac{\partial}{\partial x_i}\right)_a$$

and then

$$X_a(f) = \sum_i c_i \frac{\partial f}{\partial x_i}(a) \tag{5}$$

3.3 Derivatives of smooth maps

Suppose $F : M \to N$ is a smooth map and $f \in C^\infty(N)$. Then $f \circ F$ is a smooth function on M.

Definition 11 *The derivative at $a \in M$ of the smooth map $F : M \to N$ is the homomorphism of tangent spaces*

$$DF_a : T_a M \to T_{F(a)} N$$

defined by

$$DF_a(X_a)(f) = X_a(f \circ F).$$

This is an abstract, coordinate-free definition. Concretely, we can use (5) to see that

$$\begin{aligned} DF_a \left(\frac{\partial}{\partial x_i}\right)_a (f) &= \frac{\partial}{\partial x_i}(f \circ F)(a) \\ &= \sum_j \frac{\partial F_j}{\partial x_i}(a) \frac{\partial f}{\partial y_j}(F(a)) = \sum_j \frac{\partial F_j}{\partial x_i}(a) \left(\frac{\partial}{\partial y_j}\right)_{F(a)} f \end{aligned}$$

Thus the derivative of F is an invariant way of defining the Jacobian matrix.

With this definition we can give a generalization of Theorem 2.2 – the proof is virtually the same and is omitted.

Theorem 3.3 *Let $F : M \to N$ be a smooth map and $c \in N$ be such that at each point $a \in F^{-1}(c)$ the derivative DF_a is surjective. Then $F^{-1}(c)$ is a smooth manifold of dimension* $\dim M - \dim N$.

In the course of the proof, it is easy to see that the manifold structure on $F^{-1}(c)$ makes the inclusion

$$\iota : F^{-1}(c) \subset M$$

a smooth map, whose derivative is injective and maps isomorphically to the kernel of DF. So when we construct a manifold like this, its tangent space at a is

$$T_a \cong \operatorname{Ker} DF_a.$$

This helps to understand tangent spaces for the case where F is defined on $\mathbf{R}^n$:

Examples:

1. The sphere S^n is $F^{-1}(1)$ where $F : \mathbf{R}^{n+1} \to \mathbf{R}$ is given by

$$F(x) = \sum_i x_i^2.$$

So here

$$DF_a(x) = 2\sum_i x_i a_i$$

and the kernel of DF_a consists of the vectors orthogonal to a, which is our usual vision of the tangent space to a sphere.

2. The orthogonal matrices $O(n)$ are given by $F^{-1}(I)$ where $F(A) = AA^T$. At $A = I$, the derivative is

$$DF_I(H) = H + H^T$$

so the tangent space to $O(n)$ at the identity matrix is $\operatorname{Ker} DF_I$, the space of skew-symmetric matrices $H = -H^T$.

The examples above are of manifolds $F^{-1}(c)$ sitting inside M and are examples of *submanifolds*. Here we shall adopt the following definition of a submanifold, which is often called an *embedded submanifold*:

Definition 12 *A manifold M is a submanifold of N if there is an inclusion map*

$$\iota : M \to N$$

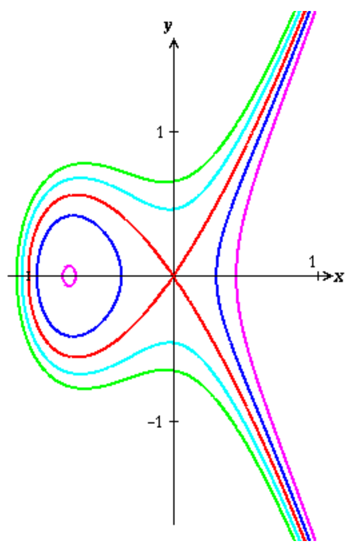
such that

- *ι is smooth*
- *$D\iota_x$ is injective for each $x \in M$*
- *the manifold topology of M is the induced topology from N*

Remark: The topological assumption avoids a situation like this:

$$\iota(t) = (t^2 - 1, t(t^2 - 1)) \in \mathbf{R}^2$$

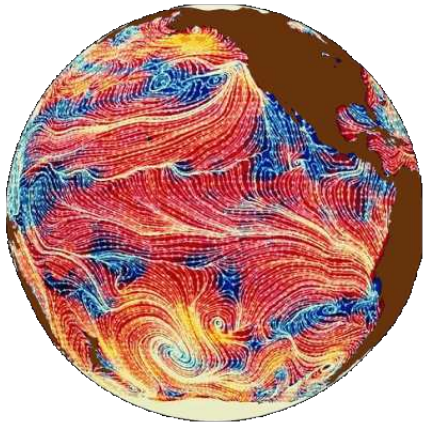
for $t \in (-1, \infty)$. This is smooth and injective with injective derivative: it is the part of the singular cubic $y^2 = x^2(x + 1)$ consisting of the left hand loop and the part in the first quadrant. Any open set in $\mathbf{R}^2$ containing 0 intersects the curve in a t-interval $(-1, -1 + \delta)$ *and* an interval $(1 - \delta', 1 + \delta')$. Thus $(1 - \delta', 1 + \delta')$ on its own is not open in the induced topology.



4 Vector fields

4.1 The tangent bundle

Think of the wind velocity at each point of the earth.



This is an example of a vector field on the 2-sphere S^2. Since the sphere sits inside $\mathbf{R}^3$, this is just a smooth map $X : S^2 \to \mathbf{R}^3$ such that $X(x)$ is tangential to the sphere at x.

Our problem now is to define a vector field intrinsically on a general manifold M, without reference to any ambient space. We know what a tangent vector at $a \in M$ is – a vector in T_a – but we want to describe a smoothly varying family of these. To do this we need to fit together all the tangent spaces as a ranges over M into a single manifold called the tangent bundle. We have n degrees of freedom for $a \in M$ and n for each tangent space T_a so we expect to have a $2n$-dimensional manifold. So the *set* to consider is

$$TM = \bigcup_{x \in M} T_x$$

the disjoint union of all the tangent spaces.

First let (U, φ_U) be a coordinate chart for M. Then for $x \in U$ the tangent vectors

$$\left(\frac{\partial}{\partial x_1}\right)_x, \ldots, \left(\frac{\partial}{\partial x_n}\right)_x$$

provide a basis for each T_x. So we have a bijection

$$\psi_U : U \times \mathbf{R}^n \to \bigcup_{x \in U} T_x$$

defined by

$$\psi_U(x, y_1, \ldots, y_n) = \sum_1^n y_i \left(\frac{\partial}{\partial x_i}\right)_x .$$

Thus

$$\Phi_U = (\varphi_U, id) \circ \psi_U^{-1} : \bigcup_{x \in U} T_x \to \varphi_U(U) \times \mathbf{R}^n$$

is a coordinate chart for

$$V = \bigcup_{x \in U} T_x.$$

Given U_α, U_β coordinate charts on M, clearly

$$\Phi_\alpha(V_\alpha \cap V_\beta) = \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbf{R}^n$$

which is open in $\mathbf{R}^{2n}$. Also, if $(x_1, \ldots, x_n)$ are coordinates on U_α and $(\tilde{x}_1, \ldots, \tilde{x}_n)$ on U_β then

$$\left(\frac{\partial}{\partial x_i}\right)_x = \sum_j \frac{\partial \tilde{x}_j}{\partial x_i} \left(\frac{\partial}{\partial \tilde{x}_j}\right)_x$$

the dual of (3). It follows that

$$\Phi_\beta \Phi_\alpha^{-1}(x_1, \ldots, x_n, y_1 \ldots, y_n) = (\tilde{x}_1, \ldots, \tilde{x}_n, \sum_j \frac{\partial \tilde{x}_1}{\partial x_i} y_i, \ldots, \sum_i \frac{\partial \tilde{x}_n}{\partial x_i} y_i).$$

and since the Jacobian matrix is smooth in x, linear in y and invertible, $\Phi_\beta \Phi_\alpha^{-1}$ is smooth with smooth inverse and so (V_α, Φ_α) defines an atlas on TM.

Definition 13 *The tangent bundle of a manifold M is the $2n$-dimensional differentiable structure on TM defined by the above atlas.*

The construction brings out a number of properties. First of all the projection map

$$p : TM \to M$$

which assigns to $X_a \in T_a M$ the point a is smooth with surjective derivative, because in our local coordinates it is defined by

$$p(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1, \ldots, x_n).$$

The inverse image $p^{-1}(a)$ is the vector space T_a and is called a *fibre* of the projection. Finally, TM is Hausdorff because if X_a, X_b lie in different fibres, since M is Hausdorff we can separate $a, b \in M$ by open sets U, U' and then the open sets $p^{-1}(U), p^{-1}(U')$ separate X_a, X_b in TM. If X_a, Y_a are in the same tangent space then they lie in a coordinate neighbourhood which is homeomorphic to an open set of $\mathbf{R}^{2n}$ and so can be separated there. Since M has a countable basis of open sets and $\mathbf{R}^n$ does, it is easy to see that TM also has a countable basis.

We can now define a vector field:

Definition 14 *A vector field on a manifold is a smooth map*

$$X : M \to TM$$

such that

$$p \circ X = id_M.$$

This is a clear global definition. What does it mean? We just have to spell things out in local coordinates. Since $p \circ X = id_M$,

$$X(x_1, \ldots, x_n) = (x_1, \ldots, x_n, y_1(x), \ldots, y_n(x))$$

where $y_i(x)$ are smooth functions. Thus the tangent vector $X(x)$ is given by

$$X(x) = \sum_i y_i(x) \left(\frac{\partial}{\partial x_i}\right)_x$$

which is a smoothly varying field of tangent vectors.

Remark: We shall meet other manifolds Q with projections $p : Q \to M$ and the general terminology is that a smooth map $s : M \to Q$ for which $p \circ s = id_M$ is called a *section*. When $Q = TM$ is the tangent bundle we always have the *zero section* given by the vector field $X = 0$. Using a bump function ψ we can easily construct other vector fields by taking a coordinate system, some locally defined smooth functions $y_i(x)$ and writing

$$X(x) = \sum_i y_i(x) \left(\frac{\partial}{\partial x_i}\right)_x.$$

Multiplying by ψ and extending gives a global vector field.

Remark: Clearly we can do a similar construction using the cotangent spaces T_a^* instead of the tangent spaces T_a, and using the basis

$$(dx_1)_x, \ldots, (dx_n)_x$$

instead of the dual basis

$$\left(\frac{\partial}{\partial x_1}\right)_x, \ldots, \left(\frac{\partial}{\partial x_1}\right)_x.$$

This way we form the *cotangent bundle* T^*M. The derivative of a function f is then a map $df : M \to T^*M$ satisfying $p \circ df = id_M$, though not every such map of this form is a derivative.

Perhaps we should say here that the tangent bundle and cotangent bundle are examples of *vector bundles*. Here is the general definition:

Definition 15 *A real vector bundle of rank m on a manifold M is a manifold E with a smooth projection $p : E \to M$ such that*

- *each fibre $p^{-1}(x)$ has the structure of an m-dimensional real vector space*
- *each point $x \in M$ has a neighbourhood U and a diffeomorphism*

$$\psi_U : p^{-1}(U) \cong U \times \mathbf{R}^m$$

  such that ψ_U is a linear isomorphism from the vector space $p^{-1}(x)$ to the vector space $\{x\} \times \mathbf{R}^m$

- *on the intersection $U \cap V$*

$$\psi_U \psi_V^{-1} : U \cap V \times \mathbf{R}^m \to U \cap V \times \mathbf{R}^m$$

  is of the form

$$(x, v) \mapsto (x, g_{UV}(x)v)$$

  where $g_{UV}(x)$ is a smooth function on $U \cap V$ with values in the space of invertible $m \times m$ matrices.

For the tangent bundle g_{UV} is the Jacobian matrix of a change of coordinates and for the cotangent bundle, its inverse transpose.

4.2 Vector fields as derivations

The algebraic definition of tangent vector in Definition 10 shows that a vector field X maps a C^∞ function to a function on M:

$$X(f)(x) = X_x(f)$$

and the local expression for X means that

$$X(f)(x) = \sum_i y_i(x) \left(\frac{\partial}{\partial x_i}\right)_x (f) = \sum_i y_i(x) \frac{\partial f}{\partial x_i}(x).$$

Since the $y_i(x)$ are smooth, $X(f)$ is again smooth and satisfies the Leibnitz property

$$X(fg) = f(Xg) + g(Xf).$$

In fact, any linear transformation with this property (called a derivation of the algebra $C^\infty(M)$) is a vector field:

Proposition 4.1 *Let $X : C^\infty(M) \to C^\infty(M)$ be a linear map which satisfies*

$$X(fg) = f(Xg) + g(Xf).$$

Then X is a vector field.

Proof: For each $a \in M$, $X_a(f) = X(f)(a)$ satisfies the conditions for a tangent vector at a, so X defines a map $X : M \to TM$ with $p \circ X = id_M$, and so locally can be written as

$$X_x = \sum_i y_i(x) \left(\frac{\partial}{\partial x_i}\right)_x.$$

We just need to check that the $y_i(x)$ are smooth, and for this it suffices to apply X to a coordinate function x_i extended by using a bump function in a coordinate neighbourhood. We get

$$Xx_i = y_i(x)$$

and since by assumption X maps smooth functions to smooth functions, this is smooth. □

The characterization of vector fields given by Proposition 4.1 immediately leads to a way of combining two vector fields X, Y to get another. Consider both X and Y as linear maps from $C^\infty(M)$ to itself and compose them. Then

$$\begin{aligned} XY(fg) &= X(f(Yg) + g(Yf)) = (Xf)(Yg) + f(XYg) + (Xg)(Yf) + g(XYf) \\ YX(fg) &= Y(f(Xg) + g(Xf)) = (Yf)(Xg) + f(YXg) + (Yg)(Xf) + g(YXf) \end{aligned}$$

and subtracting and writing $[X, Y] = XY - YX$ we have

$$[X, Y](fg) = f([X, Y]g) + g([X, Y]f)$$

which from Proposition 4.1 means that $[X, Y]$ is a vector field.

Definition 16 *The Lie bracket of two vector fields X, Y is the vector field $[X, Y]$.*

Example: If $M = \mathbf{R}$ then $X = fd/dx, Y = gd/dx$ and so

$$[X, Y] = (fg' - gf')\frac{d}{dx}.$$

We shall later see that there is a geometrical origin for the Lie bracket.

4.3 One-parameter groups of diffeomorphisms

Think of wind velocity (assuming it is constant in time) on the surface of the earth as a vector field on the sphere S^2. There is another interpretation we can make. A particle at position $x \in S^2$ moves after time t seconds to a position $\varphi_t(x) \in S^2$. After a further s seconds it is at

$$\varphi_{t+s}(x) = \varphi_s(\varphi_t(x)).$$

What we get this way is a homomorphism of groups: from the additive group $\mathbf{R}$ to the group of diffeomorphisms of S^2 under the operation of composition. The technical definition is the following:

Definition 17 *A one-parameter group of diffeomorphisms of a manifold M is a smooth map*

$$\varphi : M \times \mathbf{R} \to M$$

such that (writing $\varphi_t(x) = \varphi(x, t)$)

- $\varphi_t : M \to M$ *is a diffeomorphism*
- $\varphi_0 = id$
- $\varphi_{s+t} = \varphi_s \circ \varphi_t$.

We shall show that vector fields generate one-parameter groups of diffeomorphisms, but only under certain hypotheses. If instead of the whole surface of the earth our manifold is just the interior of the UK and the wind is blowing East-West, clearly after however short a time, some particles will be blown offshore, so we cannot hope for $\varphi_t(x)$ that works for all x and t. The fact that the earth is compact is one reason why it works there, and this is one of the results below. The idea, nevertheless, works locally and is a useful way of understanding vector fields as "infinitesimal diffeomorphisms" rather than as abstract derivations of functions.

To make the link with vector fields, suppose φ_t is a one-parameter group of diffeomorphisms and f a smooth function. Then

$$f(\varphi_t(a))$$

is a smooth function of t and we write

$$\frac{\partial}{\partial t} f(\varphi_t(a))|_{t=0} = X_a(f).$$

It is straightforward to see that, since $\varphi_0(a) = a$ the Leibnitz rule holds and this is a tangent vector at a, and so as $a = x$ varies we have a vector field. In local coordinates we have

$$\varphi_t(x_1, \ldots, x_n) = (y_1(x,t), \ldots, y_n(x,t))$$

and

$$\begin{aligned} \frac{\partial}{\partial t} f(y_1, \ldots, y_n) &= \sum_i \frac{\partial f}{\partial y_i}(y) \frac{\partial y_i}{\partial t}(x)|_{t=0} \\ &= \sum_i c_i(x) \frac{\partial f}{\partial x_i}(x) \end{aligned}$$

which yields the vector field

$$X = \sum_i c_i(x) \frac{\partial}{\partial x_i}.$$

We now want to reverse this: go from the vector field to the diffeomorphism. The first point is to track that "trajectory" of a single particle.

Definition 18 *An integral curve of a vector field X is a smooth map $\varphi : (\alpha, \beta) \subset \mathbf{R} \to M$ such that*

$$D\varphi_t \left(\frac{d}{dt} \right) = X_{\varphi(t)}.$$

Example: Suppose $M = \mathbf{R}^2$ with coordinates (x, y) and $X = \partial/\partial x$. The derivative $D\varphi$ of the smooth function $\varphi(t) = (x(t), y(t))$ is

$$D\varphi \left(\frac{d}{dt} \right) = \frac{dx}{dt} \frac{\partial}{\partial x} + \frac{dy}{dt} \frac{\partial}{\partial y}$$

so the equation for an integral curve of X is

$$\begin{aligned} \frac{dx}{dt} &= 1 \\ \frac{dy}{dt} &= 0 \end{aligned}$$

which gives

$$\varphi(t) = (t + a_1, a_2).$$

In our wind analogy, the particle at (a_1, a_2) is transported to $(t + a_1, a_2)$.

In general we have:

Theorem 4.2 *Given a vector field X on a manifold M and $a \in M$ there exists a maximal integral curve of X through a.*

By "maximal" we mean that the interval (α, β) is maximal – as we saw above it may not be the whole of the real numbers.

Proof: First consider a coordinate chart (U_γ, ψ_γ) around a then if

$$X = \sum_i c_i(x) \frac{\partial}{\partial x_i}$$

the equation

$$D\varphi_t \left(\frac{d}{dt} \right) = X_{\varphi(t)}$$

can be written as the system of ordinary differential equations

$$\frac{dx_i}{dt} = c_i(x_1, \dots, x_n).$$

The existence and uniqueness theorem for ODE's (see Appendix) asserts that there is some interval on which there is a unique solution with initial condition

$$(x_1(0), \dots, x_n(0)) = \psi_\gamma(a).$$

Suppose $\varphi : (\alpha, \beta) \to M$ is *any* integral curve with $\varphi(0) = a$. For each $x \in (\alpha, \beta)$ the subset $\varphi([0, x]) \subset M$ is compact, so it can be covered by a finite number of coordinate charts, in each of which we can apply the existence and uniqueness theorem to intervals $[0, \alpha_1], [\alpha_1, \alpha_2], \dots, [\alpha_n, x]$. Uniqueness implies that these local solutions agree with φ on any subinterval containing 0.

We then take the maximal open interval on which we can define φ. □

To find the one-parameter group of diffeomorphisms we now let $a \in M$ vary. In the example above, the integral curve through (a_1, a_2) was $t \mapsto (t + a_1, a_2)$ and this defines the group of diffeomorphisms

$$\varphi_t(x_1, x_2) = (t + x_1, x_2).$$

Theorem 4.3 *Let X be a vector field on a manifold M and for $(t, x) \in \mathbf{R} \times M$, let $\varphi(t, x) = \varphi_t(x)$ be the maximal integral curve of X through x. Then*

- *the map $(t, x) \mapsto \varphi_t(x)$ is smooth*
- *$\varphi_t \circ \varphi_s = \varphi_{t+s}$ wherever the maps are defined*
- *if M is compact, then $\varphi_t(x)$ is defined on $\mathbf{R} \times M$ and gives a one-parameter group of diffeomorphisms.*

Proof: The previous theorem tells us that for each $a \in M$ we have an open interval $(\alpha(a), \beta(a))$ on which the maximal integral curve is defined. The local existence theorem also gives us that there is a solution for initial conditions in a neighbourhood of a so the set

$$\{(t, x) \in \mathbf{R} \times M : t \in (\alpha(x), \beta(x))\}$$

is open. This is the set on which $\varphi_t(x)$ is maximally defined.

The theorem (see Appendix) on smooth dependence on initial conditions tells us that $(t, x) \mapsto \varphi_t(x)$ is smooth.

Consider $\varphi_t \circ \varphi_s(x)$. If we fix s and vary t, then this is the unique integral curve of X through $\varphi_s(x)$. But $\varphi_{t+s}(x)$ is an integral curve which at $t = 0$ passes through $\varphi_s(x)$. By uniqueness they must agree so that $\varphi_t \circ \varphi_s = \varphi_{t+s}$. (Note that $\varphi_t \circ \varphi_{-t} = id$ shows that we have a diffeomorphism wherever it is defined).

Now consider the case where M is compact. For each $x \in M$, we have an open interval $(\alpha(x), \beta(x))$ containing 0 and an open set $U_x \subseteq M$ on which $\varphi_t(x)$ is defined. Cover M by $\{U_x\}_{x \in M}$ and take a finite subcovering $U_{x_1}, \ldots, U_{x_N}$, and set

$$I = \bigcap_1^N (\alpha(x_i), \beta(x_i))$$

which is an open interval containing 0. By construction, for $t \in I$ we get

$$\varphi_t : I \times M \to M$$

which defines an integral curve (though not necessarily maximal) through each point $x \in M$ and with $\varphi_0(x) = x$. We need to extend to all real values of t.

If $s, t \in \mathbf{R}$, choose n such that $(|s| + |t|)/n \in I$ and define (where multiplication is composition)

$$\varphi_t = (\varphi_{t/n})^n, \qquad \varphi_s = (\varphi_{s/n})^n.$$

Now because $t/n, s/n$ and $(s + t)/n$ lie in I we have

$$\varphi_{t/n}\varphi_{s/n} = \varphi_{(s+t)/n} = \varphi_{s/n}\varphi_{t/n}$$

and so because $\varphi_{t/n}$ and $\varphi_{s/n}$ commute, we also have

$$\begin{aligned} \varphi_t \varphi_s &= (\varphi_{t/n})^n (\varphi_{s/n})^n \\ &= (\varphi_{(s+t)/n})^n \\ &= \varphi_{s+t} \end{aligned}$$

which completes the proof. □

4.4 The Lie bracket revisited

All the objects we shall consider will have the property that they can be transformed naturally by a diffeomorphism, and the link between vector fields and diffeomorphisms we have just observed provides an "infinitesimal' version of this.

Given a diffeomorphism $F : M \to M$ and a smooth function f we get the transformed function $f \circ F$. When $F = \varphi_t$, generated according to the theorems above by a vector field X, we then saw that

$$\frac{\partial}{\partial t} f(\varphi_t)|_{t=0} = X(f).$$

So: *the natural action of diffeomorphisms on functions specializes through one-parameter groups to the derivation of a function by a vector field.*

Now suppose Y is a vector field, considered as a map $Y : M \to TM$. With a diffeomorphism $F : M \to M$, its derivative $DF_x : T_x \to T_{F(x)}$ gives

$$DF_x(Y_x) \in T_{F(x)}.$$

This defines a new vector field $\tilde{Y}$ by

$$\tilde{Y}_{F(x)} = DF_x(Y_x) \tag{6}$$

Thus for a function f,

$$(\tilde{Y})(f \circ F) = (Yf) \circ F \tag{7}$$

Now if $F = \varphi_t$ for a one-parameter group, we have $\tilde{Y}_t$ and we can differentiate to get

$$\dot{Y} = \left.\frac{\partial}{\partial t}\tilde{Y}_t\right|_{t=0}$$

From (7) this gives

$$\dot{Y}f + Y(Xf) = XYf$$

so that $\dot{Y} = XY - YX$ is the natural derivative defined above. Thus *the natural action of diffeomorphisms on vector fields specializes through one-parameter groups to the Lie bracket* $[X, Y]$.

5 Tensor products

We have so far encountered vector fields and the derivatives of smooth functions as analytical objects on manifolds. These are examples of a general class of objects called *tensors* which we shall encounter in more generality. The starting point is pure linear algebra.

Let V, W be two finite-dimensional vector spaces over $\mathbf{R}$. We are going to define a new vector space $V \otimes W$ with two properties:

- if $v \in V$ and $w \in W$ then there is a product $v \otimes w \in V \otimes W$
- the product is bilinear:

$$\begin{aligned}(\lambda v_1 + \mu v_2) \otimes w &= \lambda v_1 \otimes w + \mu v_2 \otimes w \\ v \otimes (\lambda w_1 + \mu w_2) &= \lambda v \otimes w_1 + \mu v \otimes w_2\end{aligned}$$

In fact, it is the properties of the vector space $V \otimes W$ which are more important than what it is (and after all what is a real number? Do we always think of it as an equivalence class of Cauchy sequences of rationals?).

Proposition 5.1 *The tensor product $V \otimes W$ has the universal property that if $B : V \times W \to U$ is a bilinear map to a vector space U then there is a unique linear map*

$$\beta : V \otimes W \to U$$

such that $B(v, w) = \beta(v \otimes w)$.

There are various ways to define $V \otimes W$. In the finite-dimensional case we can say that $V \otimes W$ is *the dual space of the space of bilinear forms on $V \times W$*: i.e. maps $B : V \times W \to \mathbf{R}$ such that

$$\begin{aligned}B(\lambda v_1 + \mu v_2, w) &= \lambda B(v_1, w) + \mu B(v_2, w) \\ B(v, \lambda w_1 + \mu w_2) &= \lambda B(v, w_1) + \mu B(v, w_2)\end{aligned}$$

Given $v, w \in V, W$ we then define $v \otimes w \in V \otimes W$ as the map

$$(v \otimes w)(B) = B(v, w).$$

This satisfies the universal property because given $B : V \times W \to U$ and $\xi \in U^*$, $\xi \circ B$ is a bilinear form on $V \times W$ and defines a linear map from U^* to the space of bilinear forms. The dual map is the required homomorphism β from $V \otimes W$ to $(U^*)^* = U$.

A bilinear form B is uniquely determined by its values $B(v_i, w_j)$ on basis vectors $v_1, \ldots, v_m$ for V and $w_1, \ldots w_n$ for W which means the dimension of the vector space of bilinear forms is mn, as is its dual space $V \otimes W$. In fact, we can easily see that the mn vectors

$$v_i \otimes w_j$$

form a basis for $V \otimes W$. It is important to remember though that a typical element of $V \otimes W$ can only be written as a *sum*

$$\sum_{i,j} a_{ij} v_i \otimes w_j$$

and not as a pure product $v \otimes w$.

Taking $W = V$ we can form multiple tensor products

$$V \otimes V, \quad V \otimes V \otimes V = \otimes^3 V, \quad \ldots$$

We can think of $\otimes^p V$ as the dual space of the space of p-fold *multilinear forms* on V.

Mixing degrees we can even form the *tensor algebra*:

$$T(V) = \oplus_{k=0}^{\infty} (\otimes^k V).$$

An element of $T(V)$ is a finite sum

$$\lambda 1 + v_0 + \sum v_i \otimes v_j + \ldots + \sum v_{i_1} \otimes v_{i_2} \ldots \otimes v_{i_p}$$

of products of vectors $v_i \in V$. The obvious multiplication process is based on extending by linearity the product

$$(v_1 \otimes \ldots \otimes v_p)(u_1 \otimes \ldots \otimes u_q) = v_1 \otimes \ldots \otimes v_p \otimes u_1 \otimes \ldots \otimes u_q$$

It is associative, but noncommutative.

For the most part we shall be interested in only a quotient of this algebra, called the *exterior algebra*.

5.1 The exterior algebra

Let $T(V)$ be the tensor algebra of a real vector space V and let $I(V)$ be the *ideal* generated by elements of the form

$$v \otimes v$$

where $v \in V$. So $I(V)$ consists of all sums of multiples by $T(V)$ on the left and right of these generators.

Definition 19 *The exterior algebra of V is the quotient*

$$\Lambda^* V = T(V)/I(V).$$

If $\pi : T(V) \to \Lambda^* V$ is the quotient projection then we set

$$\Lambda^p V = \pi(\otimes^p V)$$

and call this the p-fold *exterior power* of V. We can think of this as the dual space of the space of multilinear forms $M(v_1, \ldots, v_p)$ on V which vanish if any two arguments coincide – the so-called alternating multilinear forms. If $a \in \otimes^p V, b \in \otimes^q V$ then $a \otimes b \in \otimes^{p+q} V$ and taking the quotient we get a product called the exterior product:

Definition 20 *The exterior product of $\alpha = \pi(a) \in \Lambda^p V$ and $\beta = \pi(b) \in \Lambda^q V$ is*

$$\alpha \wedge \beta = \pi(a \otimes b).$$

Remark: If $v_1, \ldots, v_p \in V$ then we define an element of the dual space of the space of alternating multilinear forms by

$$v_1 \wedge v_2 \wedge \ldots \wedge v_p(M) = M(v_1, \ldots, v_p).$$

The key properties of the exterior algebra follow:

Proposition 5.2 *If $\alpha \in \Lambda^p V, \beta \in \Lambda^q V$ then*

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha.$$

Proof: Because for $v \in V$, $v \otimes v \in I(V)$, it follows that $v \wedge v = 0$ and hence

$$0 = (v_1 + v_2) \wedge (v_1 + v_2) = 0 + v_1 \wedge v_2 + v_2 \wedge v_1 + 0.$$

So interchanging any two entries from V in an expression like

$$v_1 \wedge \ldots \wedge v_k$$

changes the sign.

Write α as a linear combination of terms $v_1 \wedge \ldots \wedge v_p$ and β as a linear combination of $w_1 \wedge \ldots \wedge w_q$ and then, applying this rule to bring w_1 to the front we see that

$$(v_1 \wedge \ldots \wedge v_p) \wedge (w_1 \wedge \ldots \wedge w_q) = (-1)^p w_1 \wedge v_1 \wedge \ldots v_p \wedge w_2 \wedge \ldots \wedge w_q.$$

For each of the q w_i's we get another factor $(-1)^p$ so that in the end

$$(w_1 \wedge \ldots \wedge w_q)(v_1 \wedge \ldots \wedge v_p) = (-1)^{pq} (v_1 \wedge \ldots \wedge v_p)(w_1 \wedge \ldots \wedge w_q).$$

□

Proposition 5.3 *If* $\dim V = n$ *then* $\dim \Lambda^n V = 1$.

Proof: Let $w_1, \ldots, w_n$ be n vectors on V and relative to some basis let M be the square matrix whose columns are $w_1, \ldots, w_n$. then

$$B(w_1, \ldots, w_n) = \det M$$

is a non-zero n-fold multilinear form on V. Moreover, if any two of the w_i coincide, the determinant is zero, so this is a non-zero alternating n-linear form – an element in the dual space of $\Lambda^n V$.

On the other hand, choose a basis $v_1, \ldots, v_n$ for V, then anything in $\otimes^n V$ is a linear combination of terms like $v_{i_1} \otimes \ldots \otimes v_{i_n}$ and so anything in $\Lambda^n V$ is, after using Proposition 5.2, a linear combination of $v_1 \wedge \ldots \wedge v_n$.

Thus $\Lambda^n V$ is non-zero and at most one-dimensional hence is one-dimensional. □

Proposition 5.4 *let* $v_1, \ldots, v_n$ *be a basis for* V*, then the* $\binom{n}{p}$ *elements* $v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_p}$ *for* $i_1 < i_2 < \ldots < i_p$ *form a basis for* $\Lambda^p V$*.*

Proof: By reordering and changing the sign we can get any exterior product of the v_i's so these elements clearly span $\Lambda^p V$. Suppose then that

$$\sum a_{i_1 \ldots i_p} v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_p} = 0.$$

Because $i_1 < i_2 < \ldots < i_p$, each term is uniquely indexed by the subset $\{i_1, i_2, \ldots, i_p\} = I \subseteq \{1, 2, \ldots, n\}$, and we can write

$$\sum_I a_I v_I = 0 \tag{8}$$

If I and J have a number in common, then $v_I \wedge v_J = 0$, so if J has $n - p$ elements, $v_I \wedge v_J = 0$ unless J is the complementary subset I' in which case the product is a multiple of $v_1 \wedge v_2 \ldots \wedge v_n$ and by Proposition 5.3 this is non-zero. Thus, multiplying (8) by each term $v_{I'}$ we deduce that each coefficient $a_I = 0$ and so we have linear independence. □

Proposition 5.5 *The vector* v *is linearly dependent on the linearly independent vectors* $v_1, \ldots, v_p$ *if and only if* $v_1 \wedge v_2 \wedge \ldots \wedge v_p \wedge v = 0$.

Proof: If v is linearly dependent on $v_1, \ldots, v_p$ then $v = \sum a_i v_i$ and expanding

$$v_1 \wedge v_2 \wedge \ldots \wedge v_p \wedge v = v_1 \wedge v_2 \wedge \ldots \wedge v_p \wedge (\sum_1^p a_i v_i)$$

gives terms with repeated v_i, which therefore vanish. If not, then $v_1, v_2 \ldots, v_p, v$ can be extended to a basis and Proposition 5.4 tells us that the product is non-zero. □

Proposition 5.6 *If $A : V \to W$ is a linear transformation, then there is an induced linear transformation*

$$\Lambda^p A : \Lambda^p V \to \Lambda^p W$$

such that

$$\Lambda^p A(v_1 \wedge \ldots \wedge v_p) = Av_1 \wedge Av_2 \wedge \ldots \wedge Av_p.$$

Proof: From Proposition 5.4 the formula

$$\Lambda^p A(v_1 \wedge \ldots \wedge v_p) = Av_1 \wedge Av_2 \wedge \ldots \wedge Av_p$$

actually defines what $\Lambda^p A$ is on basis vectors but doesn't prove it is independent of the choice of basis. But the universal property of tensor products gives us

$$\otimes^p A : \otimes^p V \to \otimes^p W$$

and $\otimes^p A$ maps the ideal $I(V)$ to $I(W)$ so defines $\Lambda^p A$ invariantly. □

Proposition 5.7 *If* $\dim V = n$*, then the linear transformation* $\Lambda^n A : \Lambda^n V \to \Lambda^n V$ *is given by* $\det A$.

Proof: From Proposition 5.3, $\Lambda^n V$ is one-dimensional and so $\Lambda^n A$ is multiplication by a real number $\lambda(A)$. So with a basis $v_1, \ldots, v_n$,

$$\Lambda^n A(v_1 \wedge \ldots \wedge v_n) = Av_1 \wedge Av_2 \wedge \ldots Av_n = \lambda(A) v_1 \wedge \ldots \wedge v_n.$$

But

$$Av_i = \sum_j A_{ji} v_j$$

and so

$$\begin{aligned} Av_1 \wedge Av_2 \wedge \ldots \wedge Av_n &= \sum A_{j_1,1} v_{j_1} \wedge A_{j_2,2} v_{j_2} \wedge \ldots \wedge A_{j_n,n} v_{j_n} \\ &= \sum_{\sigma \in S_n} A_{\sigma 1,1} v_{\sigma 1} \wedge A_{\sigma 2,2} v_{\sigma 2} \wedge \ldots \wedge A_{\sigma n,n} v_{\sigma n} \end{aligned}$$

where the sum runs over all permutations σ. But if σ is a transposition then the term $v_{\sigma 1} \wedge v_{\sigma 2} \ldots \wedge v_{\sigma n}$ changes sign, so

$$Av_1 \wedge Av_2 \wedge \ldots \wedge Av_n = \sum_{\sigma \in S_n} \operatorname{sgn} \sigma A_{\sigma 1,1} A_{\sigma 2,2} \ldots A_{\sigma n,n} v_1 \wedge \ldots \wedge v_n$$

which is the definition of $(\det A) v_1 \wedge \ldots \wedge v_n$. □

6 Differential forms

6.1 The bundle of p-forms

Now let M be an n-dimensional manifold and T_x^* the cotangent space at x. We form the p-fold exterior power

$$\Lambda^p T_x^*$$

and, just as we did for the tangent bundle and cotangent bundle, we shall make

$$\Lambda^p T^* M = \bigcup_{x \in M} \Lambda^p T_x^*$$

into a vector bundle and hence a manifold.

If $x_1, \ldots, x_n$ are coordinates for a chart (U, φ_U) then for $x \in U$, the elements

$$dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p}$$

for $i_1 < i_2 < \ldots < i_p$ form a basis for $\Lambda^p T_x^*$. The $\binom{n}{p}$ coefficients of $\alpha \in \Lambda^p T_x^*$ then give a coordinate chart Ψ_U mapping to the open set

$$\varphi_U(U) \times \Lambda^p \mathbf{R}^n \subseteq \mathbf{R}^n \times \mathbf{R}^{\binom{n}{p}}.$$

When $p = 1$ this is just the coordinate chart we used for the cotangent bundle:

$$\Phi_U(x, \sum y_i dx_i) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$$

and on two overlapping coordinate charts we there had

$$\Phi_\beta \Phi_\alpha^{-1}(x_1, \ldots, x_n, y_1 \ldots, y_n) = (\tilde{x}_1, \ldots, \tilde{x}_n, \sum_j \frac{\partial \tilde{x}_i}{\partial x_1} y_i, \ldots, \sum_i \frac{\partial \tilde{x}_i}{\partial x_n} y_i).$$

For the p-th exterior power we need to replace the Jacobian matrix

$$J = \frac{\partial \tilde{x}_i}{\partial x_j}$$

by its induced linear map

$$\Lambda^p J : \Lambda^p \mathbf{R}^n \to \Lambda^p \mathbf{R}^n.$$

It's a long and complicated expression if we write it down in a basis but it is invertible and each entry is a polynomial in C^∞ functions and hence gives a smooth map with smooth inverse. In other words,

$$\Psi_\beta \Psi_\alpha^{-1}$$

satisfies the conditions for a manifold of dimension $n + \binom{n}{p}$.

Definition 21 *The bundle of p-forms of a manifold M is the differentiable structure on $\Lambda^p T^* M$ defined by the above atlas. There is a natural projection $p : \Lambda^p T^* M \to M$ and a section is called a differential p-form*

Examples:

1. A zero-form is a section of $\Lambda^0 T^*$ which by convention is just a smooth function f.

2. A 1-form is a section of the cotangent bundle T^*. From our definition of the derivative of a function, it is clear that df is an example of a 1-form. We can write in a coordinate system

$$df = \sum_j \frac{\partial f}{\partial x_j} dx_j.$$

By using a bump function we can extend a locally-defined p-form like $dx_1 \wedge dx_2 \wedge \ldots \wedge dx_p$ to the whole of M, so sections always exist. In fact, it will be convenient at various points to show that any function, form, or vector field can be written as a sum of these local ones. This involves the concept of *partition of unity.*

6.2 Partitions of unity

Definition 22 *A partition of unity on M is a collection $\{\varphi_i\}_{i \in I}$ of smooth functions such that*

- $\varphi_i \geq 0$

- $\{\operatorname{supp} \varphi_i : i \in I\}$ *is locally finite*
- $\sum_i \varphi_i = 1$

Here *locally finite* means that for each $x \in M$ there is a neighbourhood U which intersects only finitely many supports $\operatorname{supp} \varphi_i$.

In the appendix, the following general theorem is proved:

Theorem 6.1 *Given any open covering $\{V_\alpha\}$ of a manifold M there exists a partition of unity $\{\varphi_i\}$ on M such that* $\operatorname{supp} \varphi_i \subset V_{\alpha(i)}$ *for some $\alpha(i)$.*

We say that such a partition of unity is *subordinate* to the given covering.

Here let us just note that in the case when M is compact, life is much easier: for each point $x \in \{V_\alpha\}$ we take a coordinate neighbourhood $U_x \subset \{V_\alpha\}$ and a bump function which is 1 on a neighbourhood V_x of x and whose support lies in U_x. Compactness says we can extract a finite subcovering of the $\{V_x\}_{x \in X}$ and so we get smooth functions $\psi_i \geq 0$ for $i = 1, \ldots, N$ and equal to 1 on V_{x_i}. In particular the sum is positive, and defining

$$\varphi_i = \frac{\psi_i}{\sum_1^N \psi_i}$$

gives the partition of unity.

Now, not only can we create global p-forms by taking local ones, multiplying by φ_i and extending by zero, but conversely if α is *any* p-form, we can write it as

$$\alpha = (\sum_i \varphi_i)\alpha = \sum_i (\varphi_i \alpha)$$

which is a sum of extensions of locally defined ones.

At this point, it may not be clear why we insist on introducing these complicated exterior algebra objects, but there are two motivations. One is that the algebraic theory of determinants is, as we have seen, part of exterior algebra, and multiple integrals involve determinants. We shall later be able to integrate p-forms over p-dimensional manifolds.

The other is the appearance of the skew-symmetric cross product in ordinary three-dimensional calculus, giving rise to the curl differential operator taking vector fields to vector fields. As we shall see, to do this in a coordinate-free way, and in all dimensions, we have to dispense with vector fields and work with differential forms instead.

6.3 Working with differential forms

We defined a differential form in Definition 21 as a section of a vector bundle. In a local coordinate system it looks like this:

$$\alpha = \sum_{i_1 < i_2 < \ldots < i_p} a_{i_1 i_2 \ldots i_p}(x) dx_{i_1} \wedge dx_{i_2} \ldots \wedge dx_{i_p} \tag{9}$$

where the coefficients are smooth functions. If $x(y)$ is a different coordinate system, then we write the derivatives

$$dx_{i_k} = \sum_j \frac{\partial x_{i_k}}{\partial y_j} dy_j$$

and substitute in (9) to get

$$\alpha = \sum_{j_1 < j_2 < \ldots < j_p} \tilde{a}_{j_1 j_2 \ldots j_p}(y) dy_{j_1} \wedge dy_{j_2} \ldots \wedge dy_{j_p}.$$

Example: Let $M = \mathbf{R}^2$ and consider the 2-form $\omega = dx_1 \wedge dx_2$. Now change to polar coordinates on the open set $(x_1, x_2) \neq (0, 0)$:

$$x_1 = r \cos\theta, \quad x_2 = r \sin\theta.$$

We have

$$\begin{aligned} dx_1 &= \cos\theta dr - r \sin\theta d\theta \\ dx_2 &= \sin\theta dr + r \cos\theta d\theta \end{aligned}$$

so that

$$\omega = (\cos\theta dr - r \sin\theta d\theta) \wedge (\sin\theta dr + r \cos\theta d\theta) = r dr \wedge d\theta.$$

We shall often write

$$\Omega^p(M)$$

as the infinite-dimensional vector space of all p-forms on M.

Although we first introduced vector fields as a means of starting to do analysis on manifolds, in many ways differential forms are better behaved. For example, suppose we have a smooth map

$$F : M \to N.$$

The derivative of this gives at each point $x \in M$ a linear map

$$DF_x : T_x M \to T_{F(x)} N$$

but if we have a *section* of the tangent bundle TM – a vector field X – then $DF_x(X_x)$ doesn't in general define a vector field on N – it doesn't tell us what to choose in $T_a N$ if $a \in N$ is not in the image of F.

On the other hand suppose α is a section of $\Lambda^p T^* N$ – a p-form on N. Then the dual map

$$DF'_x : T^*_{F(x)} N \to T^*_x M$$

defines

$$\Lambda^p(DF'_x) : \Lambda^p T^*_{F(x)} N \to \Lambda^p T^*_x M$$

and then

$$\Lambda^p(DF'_x)(\alpha_{F(x)})$$

is defined for all x and is a section of $\Lambda^p T^* M$ – a p-form on M.

Definition 23 *The pull-back of a p-form $\alpha \in \Omega^p(N)$ by a smooth map $F : M \to N$ is the p-form $F^*\alpha \in \Omega^p(M)$ defined by*

$$(F^*\alpha)_x = \Lambda^p(DF'_x)(\alpha_{F(x)}).$$

Examples:

1. The pull-back of a 0-form $f \in C^\infty(N)$ is just the composition $f \circ F$.
2. Let $F : \mathbf{R}^3 \to \mathbf{R}^2$ be given by

$$F(x_1, x_2, x_3) = (x_1 x_2, x_2 + x_3) = (x, y)$$

and take

$$\alpha = x dx \wedge dy.$$

Then

$$\begin{aligned} F^*\alpha &= (x \circ F) d(x \circ F) \wedge d(y \circ F) \\ &= x_1 x_2 d(x_1 x_2) \wedge d(x_2 + x_3) \\ &= x_1 x_2 (x_1 dx_2 + x_2 dx_1) \wedge d(x_2 + x_3) \\ &= x_1^2 x_2 dx_2 \wedge dx_3 + x_1 x_2^2 dx_1 \wedge dx_2 + x_1 x_2^2 dx_1 \wedge dx_3 \end{aligned}$$

From the algebraic properties of the maps

$$\Lambda^p A : \Lambda^p V \to \Lambda^p V$$

we have the following straightforward properties of the pull-back:

- $(F \circ G)^* \alpha = G^*(F^* \alpha)$
- $F^*(\alpha + \beta) = F^*\alpha + F^*\beta$
- $F^*(\alpha \wedge \beta) = F^*\alpha \wedge F^*\beta$

6.4 The exterior derivative

We now come to the construction of the basic differential operator on forms – the exterior derivative which generalizes the grads, divs and curls of three-dimensional calculus. The key feature it has is that it is defined naturally by the manifold structure without any further assumptions.

Theorem 6.2 *On any manifold M there is a natural linear map*

$$d : \Omega^p(M) \to \Omega^{p+1}(M)$$

called the exterior derivative such that

1. *if $f \in \Omega^0(M)$, then $df \in \Omega^1(M)$ is the derivative of f*
2. *$d^2 = 0$*
3. *$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ if $\alpha \in \Omega^p(M)$*

Examples: Before proving the theorem, let's look at $M = \mathbf{R}^3$, following the rules of the theorem, to see d in all cases $p = 0, 1, 2$.

$\mathbf{p} = 0$: by definition

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3$$

which we normally would write as grad f.

$\mathbf{p} = 1$: take a 1-form

$$\alpha = a_1 dx_1 + a_2 dx_2 + a_3 dx_3$$

then applying the rules we have

$$\begin{aligned} d(a_1 dx_1 + a_2 dx_2 + a_3 dx_3) &= da_1 \wedge dx_1 + da_2 \wedge dx_2 + da_3 \wedge dx_3 \\ &= \left(\frac{\partial a_1}{\partial x_1} dx_1 + \frac{\partial a_1}{\partial x_2} dx_2 + \frac{\partial a_1}{\partial x_3} dx_3 \right) \wedge dx_1 + \ldots \end{aligned}$$

$$= \left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1} \right) dx_3 \wedge dx_1 + \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \right) dx_1 \wedge dx_2 + \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3} \right) dx_2 \wedge dx_3.$$

The coefficients of this define what we would call the curl of the vector field **a** but **a** has now become a 1-form α and not a vector field and $d\alpha$ is a 2-form, not a vector field. The geometrical interpretation has changed. Note nevertheless that the invariant statement $d^2 = 0$ is equivalent to $\operatorname{curl} \operatorname{grad} f = 0$.

$\mathbf{p} = 2$: now we have a 2-form

$$\beta = b_1 dx_2 \wedge dx_3 + b_2 dx_3 \wedge dx_1 + b_3 dx_1 \wedge dx_2$$

and

$$\begin{aligned} d\beta &= \frac{\partial b_1}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3 + \frac{\partial b_2}{\partial x_2} dx_1 \wedge dx_2 \wedge dx_3 + \frac{\partial b_3}{\partial x_3} dx_1 \wedge dx_2 \wedge dx_3 \\ &= \left(\frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2} + \frac{\partial b_3}{\partial x_3} \right) dx_1 \wedge dx_2 \wedge dx_3 \end{aligned}$$

which would be the divergence of a vector field **b** but in our case is applied to a 2-form β. Again $d^2 = 0$ is equivalent to $\operatorname{div} \operatorname{curl} \mathbf{b} = 0$.

Here we see familiar formulas, but acting on unfamiliar objects. The fact that we can pull differential forms around by smooth maps will give us a lot more power, even in three dimensions, than if we always considered these things as vector fields.

Let us return to the Theorem 6.2 now and give its proof.

Proof: We shall define $d\alpha$ by first breaking up α as a sum of terms with support in a local coordinate system (using a partition of unity), define a local d operator using a coordinate system, and then show that the result is independent of the choice.

So, to begin with, write a p-form locally as

$$\alpha = \sum_{i_1 < i_2 < \ldots < i_p} a_{i_1 i_2 \ldots i_p}(x) dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p}$$

and define

$$d\alpha = \sum_{i_1<i_2<\ldots<i_p} da_{i_1 i_2 \ldots i_p} \wedge dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p}.$$

When $p = 0$, this is just the derivative, so the first property of the theorem holds.

For the second part, we expand

$$d\alpha = \sum_{j, i_1<i_2<\ldots<i_p} \frac{\partial a_{i_1 i_2 \ldots i_p}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p}$$

and then calculate

$$d^2\alpha = \sum_{j,k,i_1<i_2<\ldots<i_p} \frac{\partial^2 a_{i_1 i_2 \ldots i_p}}{\partial x_j \partial x_k} dx_k \wedge dx_j \wedge dx_{i_1} \wedge dx_{i_2} \ldots \wedge dx_{i_p}.$$

The term

$$\frac{\partial^2 a_{i_1 i_2 \ldots i_p}}{\partial x_j \partial x_k}$$

is symmetric in j, k but it multiplies $dx_k \wedge dx_j$ in the formula which is skew-symmetric in j and k, so the expression vanishes identically and $d^2\alpha = 0$ as required.

For the third part, we check on decomposable forms

$$\begin{aligned} \alpha &= f dx_{i_1} \wedge \ldots \wedge dx_{i_p} = f dx_I \\ \beta &= g dx_{j_1} \wedge \ldots \wedge dx_{j_q} = g dx_J \end{aligned}$$

and extend by linearity. So

$$\begin{aligned} d(\alpha \wedge \beta) &= d(fg dx_I \wedge dx_J) \\ &= d(fg) \wedge dx_I \wedge dx_J \\ &= (f dg + g df) \wedge dx_I \wedge dx_J \\ &= (-1)^p f dx_I \wedge dg \wedge dx_J + df \wedge dx_I \wedge g dx_J \\ &= (-1)^p \alpha \wedge d\beta + d\alpha \wedge \beta \end{aligned}$$

So, using one coordinate system we have defined an operation d which satisfies the three conditions of the theorem. Now represent α in coordinates $y_1, \ldots, y_n$:

$$\alpha = \sum_{i_1<i_2<\ldots<i_p} b_{i_1 i_2 \ldots i_p} dy_{i_1} \wedge dy_{i_2} \wedge \ldots \wedge dy_{i_p}$$

and define in the same way

$$d'\alpha = \sum_{i_1<i_2<\ldots<i_p} db_{i_1 i_2 \ldots i_p} \wedge dy_{i_1} \wedge dy_{i_2} \wedge \ldots \wedge dy_{i_p}.$$

We shall show that $d = d'$ by using the three conditions.

From (1) and (3),

$$d\alpha = d(\sum b_{i_1 i_2 \ldots i_p} dy_{i_1} \wedge dy_{i_2} \ldots \wedge dy_{i_p}) =$$

$$\sum db_{i_1 i_2 \ldots i_p} \wedge dy_{i_1} \wedge dy_{i_2} \wedge \ldots \wedge dy_{i_p} + \sum b_{i_1 i_2 \ldots i_p} d(dy_{i_1} \wedge dy_{i_2} \wedge \ldots \wedge dy_{i_p})$$

and from (3)

$$d(dy_{i_1} \wedge dy_{i_2} \wedge \ldots \wedge dy_{i_p}) = d(dy_{i_1}) \wedge dy_{i_2} \wedge \ldots \wedge dy_{i_p} - dy_{i_1} \wedge d(dy_{i_2} \wedge \ldots \wedge dy_{i_p}).$$

From (1) and (2) $d^2 y_{i_1} = 0$ and continuing similarly with the right hand term, we get zero in all terms.

Thus on each coordinate neighbourhood U $d\alpha = \sum_{i_1<i_2<\ldots<i_p} db_{i_1 i_2 \ldots i_p} \wedge dy_{i_1} \wedge dy_{i_2} \wedge \ldots \wedge dy_{i_p} = d'\alpha$ and $d\alpha$ is thus globally well-defined. □

One important property of the exterior derivative is the following:

Proposition 6.3 *Let $F : M \to N$ be a smooth map and $\alpha \in \Omega^p(N)$. then*

$$d(F^*\alpha) = F^*(d\alpha).$$

Proof: Recall that the derivative $DF_x : T_x M \to T_{F(x)} N$ was defined in (11) by

$$DF_x(X_x)(f) = X_x(f \circ F)$$

so that the dual map $DF_x' : T^*_{F(x)} N \to T^*_x M$ satisfies

$$DF_x'(df)_{F(x)} = d(f \circ F)_x.$$

From the definition of pull-back this means that

$$F^*(df) = d(f \circ F) = d(F^* f) \tag{10}$$

Now if

$$\alpha = \sum_{i_1<i_2<\ldots<i_p} a_{i_1i_2\ldots i_p}(x)dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p},$$

$$F^*\alpha = \sum_{i_1<i_2<\ldots<i_p} a_{i_1i_2\ldots i_p}(F(x))F^*dx_{i_1} \wedge F^*dx_{i_2} \wedge \ldots \wedge F^*dx_{i_p}$$

by the multiplicative property of pull-back and then using the properties of d and (10)

$$\begin{aligned} d(F^*\alpha) &= \sum_{i_1<i_2<\ldots<i_p} d(a_{i_1i_2\ldots i_p}(F(x))) \wedge F^*dx_{i_1} \wedge F^*dx_{i_2} \wedge \ldots \wedge F^*dx_{i_p} \\ &= \sum_{i_1<i_2<\ldots<i_p} F^*da_{i_1i_2\ldots i_p} \wedge F^*dx_{i_1} \wedge F^*dx_{i_2} \wedge \ldots \wedge F^*dx_{i_p} \\ &= F^*(d\alpha). \end{aligned}$$

□

6.5 The Lie derivative of a differential form

Suppose φ_t is the one-parameter (locally defined) group of diffeomorphisms defined by a vector field X. Then there is a naturally defined *Lie derivative*

$$\mathcal{L}_X\alpha = \frac{\partial}{\partial t}\varphi_t^*\alpha\Big|_{t=0}$$

of a p-form α by X. It is again a p-form. We shall give a useful formula for this involving the exterior derivative.

Proposition 6.4 *Given a vector field X on a manifold M, there is a linear map*

$$i_X : \Omega^p(M) \to \Omega^{p-1}(M)$$

(called the interior product) such that

- $i_X df = X(f)$
- $i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^p\alpha \wedge i_X\beta$ *if* $\alpha \in \Omega^p(M)$

The proposition tells us exactly how to work out an interior product: if

$$X = \sum_i a_i \frac{\partial}{\partial x_i},$$

and $\alpha = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_p$ is a basic p-form then

$$i_X \alpha = a_1 dx_2 \wedge \ldots \wedge dx_p - a_2 dx_1 \wedge dx_3 \wedge \ldots \wedge dx_p + \ldots \qquad (11)$$

In particular

$$i_X(i_X \alpha) = a_1 a_2 dx_3 \wedge \ldots \wedge dx_p - a_2 a_1 dx_3 \wedge \ldots \wedge dx_p + \ldots = 0.$$

Example: Suppose

$$\alpha = dx \wedge dy, \quad X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

then

$$i_X \alpha = xdy - ydx.$$

The interior product is just a linear algebra construction. Above we have seen how to work it out when we write down a form as a sum of basis vectors. We just need to prove that it is well-defined and independent of the way we do that, which motivates the following more abstract proof:

Proof: In Remark 5.1 we defined $\Lambda^p V$ as the dual space of the space of alternating p-multilinear forms on V. If M is an alternating $(p-1)$-multilinear form on V and ξ a linear form on V then

$$(\xi M)(v_1, \ldots, v_p) = \xi(v_1)M(v_2, \ldots, v_p) - \xi(v_2)M(v_1, v_3, \ldots, v_p) + \ldots \qquad (12)$$

is an alternating p-multilinear form. So if $\alpha \in \Lambda^p V$ we can define $i_\xi \alpha \in \Lambda^{p-1} V$ by

$$(i_\xi \alpha)(M) = \alpha(\xi M).$$

Taking $V = T^*$ and $\xi = X \in V^* = (T^*)^* = T$ gives the interior product. Equation (12) gives us the rule (11) for working out interior products. $\square$

Here then is the formula for the Lie derivative:

Proposition 6.5 *The Lie derivative $\mathcal{L}_X \alpha$ of a p-form α is given by*

$$\mathcal{L}_X \alpha = d(i_X \alpha) + i_X d\alpha.$$

Proof: Consider the right hand side

$$R_X(\alpha) = d(i_X\alpha) + i_X d\alpha.$$

Now i_X reduces the degree p by 1 but d increases it by 1, so R_X maps p-forms to p-forms. Also,

$$d(d(i_X\alpha) + i_X d\alpha) = di_X d\alpha = (di_X + i_X d)d\alpha$$

because $d^2 = 0$, so R_X commutes with d. Finally, because

$$\begin{aligned} i_X(\alpha \wedge \beta) &= i_X\alpha \wedge \beta + (-1)^p \alpha \wedge i_X\beta \\ d(\alpha \wedge \beta) &= d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \end{aligned}$$

we have

$$R_X(\alpha \wedge \beta) = (R_X\alpha) \wedge \beta + \alpha \wedge R_X(\beta).$$

On the other hand

$$\varphi_t^*(d\alpha) = d(\varphi_t^*\alpha)$$

so differentiating at $t = 0$, we get

$$\mathcal{L}_X d\alpha = d(\mathcal{L}_X\alpha)$$

and

$$\varphi_t^*(\alpha \wedge \beta) = \varphi_t^*\alpha \wedge \varphi_t^*\beta$$

and differentiating this, we have

$$\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X\alpha \wedge \beta + \alpha \wedge \mathcal{L}_X\beta.$$

Thus both $\mathcal{L}_X$ and R_X preserve degree, commute with d and satisfy the same Leibnitz identity. Hence, if we write a p-form as

$$\alpha = \sum_{i_1 < i_2 < \ldots < i_p} a_{i_1 i_2 \ldots i_p}(x) dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p}$$

$\mathcal{L}_X$ and R_X will agree so long as they agree on functions. But

$$R_X f = i_X df = X(f) = \frac{\partial}{\partial t} f(\varphi_t)\bigg|_{t=0} = \mathcal{L}_X f$$

so they do agree. □

6.6 de Rham cohomology

In textbooks on vector calculus, you may read not only that curl grad $f = 0$, but also that if a vector field $\mathbf{a}$ satisfies curl $\mathbf{a} = 0$, then it can be written as $\mathbf{a} = \operatorname{grad} f$ for some function f. Sometimes the statement is given with the proviso that the open set of $\mathbf{R}^3$ on which $\mathbf{a}$ is defined satisfies the topological condition that it is simply connected (any closed path can be contracted to a point).

In the language of differential forms on a manifold, the analogue of the above statement would say that if a 1-form α satisfies $d\alpha = 0$, and M is simply-connected, there is a function f such that $df = \alpha$.

While this is true, the criterion of simply connectedness is far too strong. We want to know when the kernel of

$$d : \Omega^1(M) \to \Omega^2(M)$$

is equal to the image of

$$d : \Omega^0(M) \to \Omega^1(M).$$

Since $d^2 f = 0$, the second vector space is contained in the first and what we shall do is simply to study the quotient, which becomes a topological object in its own right, with an algebraic structure which can be used to say many things about the global topology of a manifold.

Definition 24 *The p-th de Rham cohomology group of a manifold M is the quotient vector space:*

$$H^p(M) = \frac{\operatorname{Ker} d : \Omega^p(M) \to \Omega^{p+1}(M)}{\operatorname{Im} d : \Omega^{p-1}(M) \to \Omega^p(M)}$$

Remark:

1. Although we call it the cohomology *group*, it is simply a real vector space. There are analogous structures in algebraic topology where the additive group structure is more interesting.

2. Since there are no forms of degree -1, the group $H^0(M)$ is the space of functions f such that $df = 0$. Now each connected component M_i of M is an open set of M and hence a manifold. The mean value theorem tells us that on any open ball in a coordinate neighbourhood of M_i, $df = 0$ implies that f is equal to a constant c, and the subset of M_i on which $f = c$ is open and closed and hence equal to M_i.

Thus if M is connected, the de Rham cohomology group $H^0(M)$ is naturally isomorphic to $\mathbf{R}$: the constant value c of the function f. In general $H^0(M)$ is the vector space of real valued functions on the set of components. Our assumption that M

has a countable basis of open sets means that there are at most countably many components. When M is compact, there are only finitely many, since components provide an open covering. In fact, the cohomology groups of a compact manifold are finite-dimensional vector spaces for all p, though we shall not prove that here.

It is convenient in discussing the exterior derivative to introduce the following terminology:

Definition 25 *A form $\alpha \in \Omega^p(M)$ is closed if $d\alpha = 0$.*

Definition 26 *A form $\alpha \in \Omega^p(M)$ is exact if $\alpha = d\beta$ for some $\beta \in \Omega^{p-1}(M)$.*

The de Rham cohomology group $H^p(M)$ is by definition the quotient of the space of closed p-forms by the subspace of exact p-forms. Under the quotient map, a closed p-form α defines a cohomology class $[\alpha] \in H^p(M)$, and $[\alpha'] = [\alpha]$ if and only if $\alpha' - \alpha = d\beta$ for some β.

Here are some basic features of the de Rham cohomology groups:

Proposition 6.6 *The de Rham cohomology groups of a manifold M of dimension n have the following properties:*

- *$H^p(M) = 0$ if $p > n$*
- *for $a \in H^p(M), b \in H^q(M)$ there is a bilinear product $ab \in H^{p+q}(M)$ which satisfies*
$$ab = (-1)^{pq} ba$$
- *if $F : M \to N$ is a smooth map, it defines a natural linear map*
$$F^* : H^p(N) \to H^p(M)$$
which commutes with the product.

Proof: The first part is clear since $\Lambda^p T^* = 0$ for $p > n$.

For the product, this comes directly from the exterior product of forms. If $a = [\alpha], b = [\beta]$ we define
$$ab = [\alpha \wedge \beta]$$

but we need to check that this really does define a cohomology class. Firstly, since α, β are closed,

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta = 0$$

so there is a class defined by $\alpha \wedge \beta$. Suppose we now choose a different representative $\alpha' = \alpha + d\gamma$ for a. Then

$$\alpha' \wedge \beta = (\alpha + d\gamma) \wedge \beta = \alpha \wedge \beta + d(\gamma \wedge \beta)$$

using $d\beta = 0$, so $d(\gamma \wedge \beta) = d\gamma \wedge \beta$. Thus $\alpha' \wedge \beta$ and $\alpha \wedge \beta$ differ by an exact form and define the same cohomology class. Changing β gives the same result.

The last part is just the pull-back operation on forms. Since

$$dF^*\alpha = F^* d\alpha$$

F^* defines a map of cohomology groups. And since

$$F^*(\alpha \wedge \beta) = F^*\alpha \wedge F^*\beta$$

it respects the product. □

Perhaps the most important property of the de Rham cohomology, certainly the one that links it to algebraic topology, is the deformation invariance of the induced maps F. We show that if F_t is a smooth family of smooth maps, then the effect on cohomology is independent of t. As a matter of terminology (because we have only defined smooth maps of manifolds) we shall say that a map

$$F : M \times [a, b] \to N$$

is smooth if it is the restriction of a smooth map on the product with some slightly bigger open interval $M \times (a - \epsilon, b + \epsilon)$.

Theorem 6.7 *Let $F : M \times [0, 1] \to N$ be a smooth map. Set $F_t(x) = F(x, t)$ and consider the induced map on de Rham cohomology $F_t^* : H^p(N) \to H^p(M)$. Then*

$$F_1^* = F_0^*.$$

Proof: Represent $a \in H^p(N)$ by a closed p-form α and consider the pull-back form $F^*\alpha$ on $M \times [0, 1]$. We can decompose this uniquely in the form

$$F^*\alpha = \beta + dt \wedge \gamma \tag{13}$$

where β is a p-form on M (also depending on t) and γ is a $(p-1)$-form on M, depending on t. In a coordinate system it is clear how to do this, but more invariantly, the form β is just $F_t^*\alpha$. To get γ in an invariant manner, we can think of

$$(x, s) \mapsto (x, s + t)$$

as a local one-parameter group of diffeomorphisms of $M \times (a, b)$ which generates a vector field $X = \partial/\partial t$. Then

$$\gamma = i_X F^*\alpha.$$

Now α is closed, so from (13),

$$0 = d_M\beta + dt \wedge \frac{\partial\beta}{\partial t} - dt \wedge d_M\gamma$$

where d_M is the exterior derivative in the variables of M. It follows that

$$\frac{\partial\beta}{\partial t} = d_M\gamma.$$

Now integrating with respect to the parameter t, and using

$$\frac{\partial}{\partial t}F_t^*\alpha = \frac{\partial\beta}{\partial t}$$

we obtain

$$F_1^*\alpha - F_0^*\alpha = \int_0^1 \frac{\partial}{\partial t}F_t^*\alpha\ dt = d\int_0^1 \gamma\ dt.$$

So the closed forms $F_1^*\alpha$ and $F_0^*\alpha$ differ by an exact form and

$$F_1^*(a) = F_0^*(a).$$

□

Here is an immediate corollary:

Proposition 6.8 *The de Rham cohomology groups of $M = \mathbf{R}^n$ are zero for $p > 0$.*

Proof: Define $F : \mathbf{R}^n \times [0, 1] \to \mathbf{R}^n$ by

$$F(x, t) = tx.$$

Then $F_1(x) = x$ which is the identity map, and so

$$F_1^* : H^p(\mathbf{R}^n) \to H^p(\mathbf{R}^n)$$

is the identity.

But $F_0(x) = 0$ which is a constant map. In particular the derivative vanishes, so the pull-back of any p-form of degree greater than zero is the zero map. So for $p > 0$

$$F_0^* : H^p(\mathbf{R}^n) \to H^p(\mathbf{R}^n)$$

vanishes.

From Theorem 6.7 $F_0^* = F_1^*$ and we deduce that $H^p(\mathbf{R}^n)$ vanishes for $p > 0$. Of course $\mathbf{R}^n$ is connected so $H^0(\mathbf{R}^n) \cong \mathbf{R}$. □

Exercise 6.9 *Show that the previous proposition holds for a star shaped region in* $\mathbf{R}^n$*: an open set* U *with a point* $a \in U$ *such that for each* $x \in U$ *the straight-line segment* $\overline{ax} \subset U$*. This is usually called the Poincaré lemma.*

The same argument above can be used for the map $F_t : M \times \mathbf{R}^n \to M \times \mathbf{R}^n$ *given by* $F_t(a, x) = (a, tx)$ *to show that* $H^p(M \times \mathbf{R}^n) \cong H^p(M)$*.*

We are in no position yet to calculate many other de Rham cohomology groups, but here is a first non-trivial example. Consider the case of $\mathbf{R}/\mathbf{Z}$, diffeomorphic to the circle. In the atlas given earlier, we had $\varphi_1\varphi_0^{-1}(x) = x$ or $\varphi_1\varphi_0^{-1}(x) = x - 1$ so the 1-form $dx = d(x-1)$ is well-defined, and nowhere zero. It is not the derivative of a function, however, since $\mathbf{R}/Z$ is compact and any function must have a minimum where $df = 0$. We deduce that

$$H^1(\mathbf{R}/\mathbf{Z}) \neq 0.$$

On the other hand, suppose that $\alpha = g(x)dx$ is any 1-form (necessarily closed because it is the top degree). Then g is a periodic function: $g(x+1) = g(x)$. To solve $df = \alpha$ means solving $f'(x) = g(x)$ which is easily done on $\mathbf{R}$ by:

$$f(x) = \int_0^x g(s)ds.$$

But we want $f(x+1) = f(x)$ which will only be true if

$$\int_0^1 g(x)dx = 0.$$

Thus in general

$$\alpha = g(x)dx = \left(\int_0^1 g(s)ds\right) dx + df$$

and any 1-form is of the form $cdx + df$. Thus $H^1(\mathbf{R}/\mathbf{Z}) \cong \mathbf{R}$.

We can use this in fact to start an inductive calculation of the de Rham cohomology of the n-sphere.

Theorem 6.10 *For $n > 0$, $H^p(S^n) \cong \mathbf{R}$ if $p = 0$ or $p = n$ and is zero otherwise.*

Proof: We have already calculated the case of $n = 1$ so suppose that $n > 1$.

Clearly the group vanishes when $p > n$, the dimension of S^n, and for $n > 0$, S^n is connected and so $H^0(S^n) \cong \mathbf{R}$.

Decompose S^n into open sets U, V, the complement of closed balls around the North and South poles respectively. By stereographic projection these are diffeomorphic to open balls in $\mathbf{R}^n$. If α is a closed p-form for $1 < p < n$, then by the Poincaré lemma $\alpha = du$ on U and $\alpha = dv$ on V for some $(p-1)$ forms u, v. On the intersection $U \cap V$,

$$d(u - v) = \alpha - \alpha = 0$$

so $(u - v)$ is closed. But

$$U \cap V \cong S^{n-1} \times \mathbf{R}$$

so

$$H^{p-1}(U \cap V) \cong H^{p-1}(S^{n-1})$$

and by induction this vanishes, so on $U \cap V$, $u - v = dw$.

Now look at $U \cap V$ as a product with a finite open interval: $S^{n-1} \times (-2, 2)$. We can find a bump function $\varphi(s)$ which is 1 for $s \in (-1, 1)$ and has support in $(-2, 2)$. Take slightly smaller sets $U' \subset U, V' \subset V$ such that $U' \cap V' = S^{n-1} \times (-1, 1)$. Then φw extends by zero to define a form on S^n and we have u on U' and $v + d(\varphi w)$ on V' with $u = v + dw = v + d(\varphi w)$ on $U' \cap V'$. Thus we have defined a $(p-1)$ form β on S^n such that $\beta = u$ on U' and $v + d(\varphi w)$ on V' and $\alpha = d\beta$ on U' and V' and so globally $\alpha = d\beta$. Thus the cohomology class of α is zero.

This shows that we have vanishing of $H^p(S^n)$ for $1 < p < n$.

When $p = 1$, in the argument above $u - v$ is a function on $U \cap V$ and since $d(u - v) = 0$ it is a constant c if $U \cap V$ is connected, which it is for $n > 1$. Then $d(v + c) = \alpha$ and the pair of functions u on U and $v + c$ on V agree on the overlap and define a function f such that $df = \alpha$.

When $p = n$ the form $u - v$ defines a class in $H^{n-1}(U \cap V) \cong H^{n-1}(S^{n-1}) \cong \mathbf{R}$. So let ω be an $(n-1)$ form on S^{n-1} whose cohomology class is non-trivial and pull it back to $S^{n-1} \times (-2, 2)$ by the projection onto the first factor. Then $H^{n-1}(S^{n-1} \times (-2, 2))$ is generated by $[\omega]$ and we have

$$u - v = \lambda\omega + dw$$

for some $\lambda \in \mathbf{R}$. If $\lambda = 0$ we repeat the process above, so $H^n(S^n)$ is at most one-dimensional. Note that λ is linear in α and is independent of the choice of u and v – if we change u by a closed form then it is exact since $H^{p-1}(U) = 0$ and we can incorporate it into w.

All we need now is to find a class in $H^n(S^n)$ for which $\lambda \neq 0$. To do this consider

$$\varphi dt \wedge \omega$$

extended by zero outside $U \cap V$. Then

$$\left(\int_{-2}^{t} \varphi(s) ds\right) \omega$$

vanishes for $t < -2$ and so extends by zero to define a form u on U such that $du = \alpha$. When $t > 2$ this is non-zero but we can change this to

$$v = \left(\int_{-2}^{t} \varphi(s) ds\right) \omega - \left(\int_{-2}^{2} \varphi(s) ds\right) \omega$$

which does extend by zero to V and still satisfies $dv = \alpha$. Thus taking the difference, λ above is the positive number

$$\lambda = \int_{-2}^{2} \varphi(s) ds.$$

□

To get more information on de Rham cohomology we need to study the other aspect of differential forms: integration.

7 Integration of forms

7.1 Orientation

Recall the change of variables formula in a multiple integral:

$$\int f(y_1,\ldots,y_n)dy_1dy_2\ldots dy_n = \int f(y_1(x),\ldots,y_n(x))|\det \partial y_i/\partial x_j|dx_1dx_2\ldots dx_n$$

and compare to the change of coordinates for an n-form on an n-dimensional manifold:

$$\begin{aligned}
\theta &= f(y_1,\ldots,y_n)dy_1\wedge dy_2\wedge\ldots\wedge dy_n \\
&= f(y_1(x),\ldots,y_n(x))\sum_i \frac{\partial y_1}{\partial x_i}dx_i\wedge\ldots\wedge\sum_p \frac{\partial y_n}{\partial x_p}dx_p \\
&= f(y_1(x),\ldots,y_n(x))(\det \partial y_i/\partial x_j)dx_1\wedge dx_2\ldots\wedge dx_n
\end{aligned}$$

The only difference is the absolute value, so that if we can sort out a consistent sign, then we should be able to assign a coordinate-independent value to the integral of an n-form over an n-dimensional manifold. The sign question is one of *orientation*.

Definition 27 *An n-dimensional manifold is said to be orientable if it has an everywhere non-vanishing n-form ω.*

Definition 28 *Let M be an n-dimensional orientable manifold. An orientation on M is an equivalence class of non-vanishing n-forms ω where $\omega\sim\omega'$ if $\omega' = f\omega$ with $f>0$.*

Clearly a connected orientable manifold has two orientations: the equivalence classes of $\pm\omega$.

Example:

1. Let $M\subset \mathbf{R}^{n+1}$ be defined by $f(x)=c$, with $df(a)\neq 0$ if $f(a)=c$. By Theorem 2.2, M is a manifold and moreover, if $\partial f/\partial x_i\neq 0$, $x_1,\ldots,x_{i-1},x_{i+1},x_{n+1}$ are local coordinates. Consider, on such a coordinate patch,

$$\omega = (-1)^i\frac{1}{\partial f/\partial x_i}dx_1\wedge\ldots\wedge dx_{i-1}\wedge dx_{i+1}\ldots\wedge dx_{n+1} \qquad (14)$$

This is non-vanishing.

Now M is defined by $f(x) = c$ so that on M

$$\sum_j \frac{\partial f}{\partial x_j} dx_j = 0$$

and if $\partial f/\partial x_j \neq 0$

$$dx_j = -\frac{1}{\partial f/\partial x_j}(\partial f/\partial x_i dx_i + \ldots).$$

Substituting in (14) we get

$$\omega = (-1)^j \frac{1}{\partial f/\partial x_j} dx_1 \wedge \ldots \wedge dx_{j-1} \wedge dx_{j+1} \ldots \wedge dx_{n+1}.$$

The formula (14) therefore defines for all coordinate charts a non-vanishing n-form, so M is orientable.

The obvious example is the sphere S^n with

$$\omega = (-1)^i \frac{1}{x_i} dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \ldots \wedge dx_{n+1}.$$

2. Consider real projective space $\mathbf{R}\mathrm{P}^n$ and the smooth map

$$p : S^n \to \mathbf{R}\mathrm{P}^n$$

which maps a unit vector in $\mathbf{R}^{n+1}$ to the one-dimensional subspace it spans. Concretely, if $x_1 \neq 0$, we use $x = (x_2, \ldots, x_{n+1})$ as coordinates on S^n and the usual coordinates $(x_2/x_1, \ldots, x_{n+1}/x_1)$ on $\mathbf{R}\mathrm{P}^n$, then

$$p(x) = \frac{1}{\sqrt{1 - \|x\|^2}} x. \tag{15}$$

This is smooth with smooth inverse

$$q(y) = \frac{1}{\sqrt{1 + \|y\|^2}} y$$

so we can use $(x_2, \ldots, x_{n+1})$ as local coordinates on $\mathbf{R}\mathrm{P}^n$.

Let $\sigma : S^n \to S^n$ be the diffeomorphism $\sigma(x) = -x$. Then

$$\sigma^*\omega = (-1)^i \frac{1}{-x_i} d(-x_1) \wedge \ldots \wedge d(-x_{i-1}) \wedge d(-x_{i+1}) \ldots \wedge d(-x_{n+1}) = (-1)^{n-1}\omega.$$

Suppose $\mathbf{R}\mathrm{P}^n$ is orientable, then it has a non-vanishing n-form θ. Since the map (15) has a local smooth inverse, the derivative of p is invertible, so that $p^*\theta$ is a non-vanishing n-form on S^n and so

$$p^*\theta = f\omega$$

for some non-vanishing smooth function f. But $p \circ \sigma = p$ so that

$$f\omega = p^*\theta = \sigma^* p^*\theta = (f \circ \sigma)(-1)^{n-1}\omega.$$

Thus, if n is even,

$$f \circ \sigma = -f$$

and if $f(a) > 0$, $f(-a) < 0$. But $\mathbf{R}\mathrm{P}^n = p(S^n)$ and S^n is connected so $\mathbf{R}\mathrm{P}^n$ is connected. This means that f must vanish somewhere, which is a contradiction.

Hence $\mathbf{R}\mathrm{P}^{2m}$ is not orientable.

There is a more sophisticated way of seeing the non-vanishing form on S^n which gives many more examples. First note that a non-vanishing n-form on an n-dimensional manifold is a non-vanishing section of the rank 1 vector bundle $\Lambda^n T^* M$. The top exterior power has a special property: suppose $U \subset V$ is an m-dimensional vector subspace of an n-dimensional space V, then V/U has dimension $n-m$. There is then a *natural* isomorphism

$$\Lambda^m U \otimes \Lambda^{n-m}(V/U) \cong \Lambda^n V. \tag{16}$$

To see this let $u_1, \ldots, u_m$ be a basis of U and $v_1, \ldots, v_{n-m}$ vectors in V/U. By definition there exist vectors $\tilde{v}_1, \ldots, \tilde{v}_{n-m}$ such that $v_i = \tilde{v}_i + U$. Consider

$$u_1 \wedge u_2 \ldots \wedge u_m \wedge \tilde{v}_1 \wedge \ldots \wedge \tilde{v}_{n-m}.$$

This is independent of the choice of $\tilde{v}_i$ since any two choices differ by a linear combination of u_i, which is annihilated by $u_1 \wedge \ldots u_m$. This map defines the isomorphism. Because it is natural it extends to the case of vector bundles.

Suppose now that M of dimension n is defined as the subset $f^{-1}(c)$ of $\mathbf{R}^n$ where $f : \mathbf{R}^n \to \mathbf{R}^m$ has surjective derivative on M. This means that the 1-forms $df_1, \ldots df_m$ are linearly independent at the points of $M \subset \mathbf{R}^n$. We saw that in this situation, the tangent space $T_a M$ of M at a is the subspace of $T_a \mathbf{R}^n$ annihilated by the derivative of f, or equivalently the 1-forms df_i. Another way of saying this is that the cotangent space $T_a^* M$ is the quotient of $T_a^* R^n$ by the subspace U spanned by $df_1, \ldots, df_m$. From (16) we have an isomorphism

$$\Lambda^m U \otimes \Lambda^{n-m}(T^* M) \cong \Lambda^n T^* \mathbf{R}^n.$$

Now $df_1 \wedge df_2 \wedge \ldots \wedge df_m$ is a non-vanishing section of $\Lambda^m U$, and $dx_1 \wedge \ldots \wedge dx_n$ is a non-vanishing section of $\Lambda^n T^*\mathbf{R}^n$ so the isomorphism defines a non-vanishing section ω of $\Lambda^{n-m}T^*M$.

All such manifolds, and not just the sphere, are therefore orientable. In the case $m = 1$, where M is defined by a single real-valued function f, we have

$$df \wedge \omega = dx_1 \wedge dx_2 \ldots \wedge dx_n.$$

If $\partial f/\partial x_n \neq 0$, then $x_1, \ldots, x_{n-1}$ are local coordinates and so from this formula we see that

$$\omega = (-1)^{n-1}\frac{1}{\partial f/\partial x_n}dx_1 \wedge \ldots \wedge dx_{n-1}$$

as above.

Remark: Any compact manifold M^m can be embedded in $\mathbf{R}^N$ for some N, but the argument above shows that M is not always cut out by $N - m$ globally defined functions with linearly independent derivatives, because it would then have to be orientable.

Orientability helps in integration through the following:

Proposition 7.1 *A manifold is orientable if and only if it has a covering by coordinate charts such that*

$$\det\left(\frac{\partial y_i}{\partial x_j}\right) > 0$$

on the intersection.

Proof: Assume M is orientable, and let ω be a non-vanishing n-form. In a coordinate chart

$$\omega = f(x_1, \ldots, x_n)dx_1 \wedge \ldots dx_n.$$

After possibly making a coordinate change $x_1 \mapsto c - x_1$, we have coordinates such that $f > 0$.

Look at two such overlapping sets of coordinates. Then

$$\begin{aligned}
\omega &= g(y_1, \ldots, y_n)dy_1 \wedge \ldots \wedge dy_n \\
&= g(y_1(x), \ldots, y_n(x))(\det \partial y_i/\partial x_j)dx_1 \wedge dx_2 \ldots \wedge dx_n \\
&= f(x_1, \ldots, x_n)dx_1 \wedge \ldots dx_n
\end{aligned}$$

Since $f > 0$ and $g > 0$, the determinant $\det \partial y_i/\partial x_j$ is also positive.

Conversely, suppose we have such coordinates. Take a partition of unity $\{\varphi_\alpha\}$ subordinate to the coordinate covering and put

$$\omega = \sum \varphi_\alpha dy_1^\alpha \wedge dy_2^\alpha \wedge \ldots \wedge dy_n^\alpha.$$

Then on a coordinate neighbourhood U_β with coordinates $x_1, \ldots, x_n$ we have

$$\omega|_{U_\beta} = \sum \varphi_\alpha \det(\partial y_i^\alpha/\partial x_j) dx_1 \wedge \ldots dx_n.$$

Since $\varphi_\alpha \geq 0$ and $\det(\partial y_i^\alpha/\partial x_j)$ is positive, this is non-vanishing. □

Now suppose M is orientable and we have chosen an orientation. We shall define the integral

$$\int_M \theta$$

of any n-form θ of compact support on M.

We first choose a coordinate covering as in Proposition 7.1. On each coordinate neighbourhood U_α we have

$$\theta|_{U_\alpha} = f_\alpha(x_1, \ldots, x_n) dx_1 \wedge \ldots \wedge dx_n.$$

Take a partition of unity φ_i subordinate to this covering. Then

$$\varphi_i \theta|_{U_\alpha} = g_i(x_1, \ldots, x_n) dx_1 \wedge \ldots \wedge dx_n$$

where g_i is a smooth function of compact support on the whole of $\mathbf{R}^n$. We then *define*

$$\int_M \theta = \sum_i \int_M \varphi_i \theta = \sum_i \int_{\mathbf{R}^n} g_i(x_1, \ldots, x_n) dx_1 dx_2 \ldots dx_n.$$

Note that since θ has compact support, its support is covered by finitely many open sets on which $\varphi_i \neq 0$, so the above is a finite sum.

The integral is well-defined precisely because of the change of variables formula in integration, and the consistent choice of sign from the orientation.

7.2 Stokes' theorem

The theorems of Stokes and Green in vector calculus are special cases of a single result in the theory of differential forms, which by convention is called Stokes' theorem. We begin with a simple version of it:

Theorem 7.2 *Let M be an oriented n-dimensional manifold and $\omega \in \Omega^{n-1}(M)$ be of compact support. Then*

$$\int_M d\omega = 0.$$

Proof: Use a partition of unity subordinate to a coordinate covering to write

$$\omega = \sum \varphi_i \omega.$$

Then on a coordinate neighbourhood

$$\varphi_i \omega = a_1 dx_2 \wedge \ldots \wedge dx_n - a_2 dx_1 \wedge dx_3 \wedge \ldots \wedge dx_n + \ldots$$

and

$$d(\varphi_i \omega) = \left(\frac{\partial a_1}{\partial x_1} + \ldots + \frac{\partial a_n}{\partial x_n} \right) dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n.$$

From the definition of the integral, we need to sum each

$$\int_{\mathbf{R}^n} \left(\frac{\partial a_1}{\partial x_1} + \ldots + \frac{\partial a_n}{\partial x_n} \right) dx_1 dx_2 \ldots dx_n.$$

Consider

$$\int_{\mathbf{R}^n} \frac{\partial a_1}{\partial x_1} dx_1 dx_2 \ldots dx_n.$$

By Fubini's theorem we evaluate this as a repeated integral

$$\int_{\mathbf{R}} \int_{\mathbf{R}} \ldots \left(\int \frac{\partial a_1}{\partial x_1} dx_1 \right) dx_2 dx_3 \ldots dx_n.$$

But a_1 has compact support, so vanishes if $|x_1| \geq N$ and thus

$$\int_{\mathbf{R}} \frac{\partial a_1}{\partial x_1} dx_1 = [a_1]_{-N}^{N} = 0.$$

The other terms vanish in a similar way. □

Theorem 7.2 has an immediate payoff for de Rham cohomology:

Proposition 7.3 *Let M be a compact orientable n-dimensional manifold. Then the de Rham cohomology group $H^n(M)$ is non-zero.*

Proof: Since M is orientable, it has a non-vanishing n-form θ. Because there are no $n+1$-forms, it is closed, and defines a cohomology class $[\theta] \in H^n(M)$.

Choose the orientation defined by θ and integrate: we get

$$\int_M \theta = \sum \int f_i dx_1 dx_2 \dots dx_n$$

which is positive since each $f_i \geq 0$ and is positive somewhere.

Now if the cohomology class $[\theta] = 0$, $\theta = d\omega$, but then Theorem 7.2 gives

$$\int_M \theta = \int_M d\omega = 0$$

a contradiction. □

Here is a topological result which follows directly from the proof of the above fact:

Theorem 7.4 *Every vector field on an even-dimensional sphere S^{2m} vanishes somewhere.*

Proof: Suppose for a contradiction that there is a non-vanishing vector field. For the sphere, sitting inside $\mathbf{R}^{2m+1}$, we can think of a vector field as a smooth map

$$v : S^{2m} \to \mathbf{R}^{2m+1}$$

such that $(x, v(x)) = 0$ and if v is non-vanishing we can normalize it to be a unit vector. So assume $(v(x), v(x)) = 1$.

Now define $F_t : S^{2m} \to \mathbf{R}^{2m+1}$ by

$$F_t(x) = \cos t\, x + \sin t\, v(x).$$

Since $(x, v(x)) = 0$, we have

$$(\cos t\, x + \sin t\, v(x), \cos t\, x + \sin t\, v(x)) = 1$$

so that F_t maps the unit sphere to itself. Moreover,

$$F_0(x) = x, \qquad F_\pi(x) = -x.$$

Now let ω be the standard orientation form on S^{2m}:

$$\omega = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{2m}/x_{2m+1}.$$

We see that

$$F_0^*\omega = \omega, \quad F_\pi^*\omega = -\omega.$$

But by Theorem 6.7, the maps F_0^*, F_π^* on $H^{2m}(S^{2m})$ are equal. We deduce that the de Rham cohomology class of ω is equal to its negative and so must be zero, but this contradicts that fact that its integral is positive. Thus the vector field must have a zero. □

Green's theorem relates a surface integral to a volume integral, and the full version of Stokes' theorem does something similar for manifolds. The manifolds we have defined are analogues of a surface – the sphere for example. We now need to find analogues of the solid ball that the sphere bounds. These are still called manifolds, but with a *boundary.*

Definition 29 *An n-dimensional manifold with boundary is a set M with a collection of subsets U_α and maps*

$$\varphi_\alpha : U_\alpha \to (\mathbf{R}^n)^+ = \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_n \geq 0\}$$

such that

- $M = \cup_\alpha U_\alpha$
- $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ *is a bijection onto an open set of* $(\mathbf{R}^n)^+$ *and* $\varphi_\alpha(U_\alpha \cap U_\beta)$ *is open for all* α, β,
- $\varphi_\beta\varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ *is the restriction of a* C^∞ *map from a neighbourhood of* $\varphi_\alpha(U_\alpha \cap U_\beta) \subseteq (\mathbf{R}^n)^+ \subset \mathbf{R}^n$ *to* $\mathbf{R}^n$.

The boundary ∂M of M is defined as

$$\partial M = \{x \in M : \varphi_\alpha(x) \in \{(x_1, \ldots, x_{n-1}, 0) \in \mathbf{R}^n\}$$

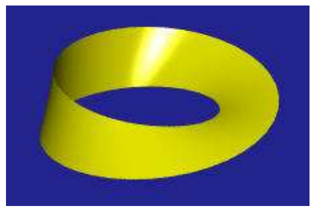
and these charts define the structure of an $(n-1)$-manifold on ∂M.

Example:

1. The model space $(\mathbf{R}^n)^+$ is a manifold with boundary $x_n = 0$.

2. The unit ball $\{x \in \mathbf{R}^n : \|x\| \leq 1\}$ is a manifold with boundary S^{n-1}.

3. The Möbius band is a 2-dimensional manifold with boundary the circle:



4. The cylinder $I \times S^1$ is a 2-dimensional manifold with boundary the union of two circles – a manifold with two components.

We can define differential forms etc. on manifolds with boundary in a straightforward way. Locally, they are just the restrictions of smooth forms on some open set in $\mathbf{R}^n$ to $(\mathbf{R}^n)^+$. A form on M restricts to a form on its boundary.

Proposition 7.5 *If a manifold M with boundary is oriented, there is an induced orientation on its boundary.*

Proof: We choose local coordinate systems such that ∂M is defined by $x_n = 0$ and $\det(\partial y_i/\partial x_j) > 0$. So, on overlapping neighbourhoods,

$$y_i = y_i(x_1, \ldots, x_n), \quad y_n(x_1, \ldots, x_{n-1}, 0) = 0.$$

Then the Jacobian matrix has the form

$$\begin{pmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 & & \partial y_1/\partial x_n \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \partial y_n/\partial x_n \end{pmatrix} \tag{17}$$

From the definition of manifold with boundary, $\varphi_\beta \varphi_\alpha^{-1}$ maps $x_n > 0$ to $y_n > 0$, so y_n has the property that if $x_n = 0, y_n = 0$ and if $x_n > 0, y_n > 0$. It follows that

$$\left.\frac{\partial y_n}{\partial x_n}\right|_{x_n=0} > 0.$$

From (17) the determinant of the Jacobian for ∂M is given by

$$\det(J_{\partial M}) \left.\frac{\partial y_n}{\partial x_n}\right|_{x_n=0} = \det(J_M)$$

so if $\det(J_M) > 0$ so is $\det(J_{\partial M})$.

□

Remark: The boundary of an oriented manifold has an induced orientation, but there is a convention about which one to choose: for a surface in $\mathbf{R}^3$ this is the choice of an "inward" or "outward" normal. Our choice will be that if $dx_1 \wedge \ldots \wedge dx_n$ defines the orientation on M with $x_n \geq 0$ defining M locally, then $(-1)^n dx_1 \wedge \ldots \wedge dx_{n-1}$ (the "outward" normal) is the induced orientation on ∂M. The boundary of the cylinder gives opposite orientations on the two circles. The Möbius band is not orientable, though its boundary the circle of course is.

We can now state the full version of Stokes' theorem:

Theorem 7.6 *(Stokes' theorem) Let M be an n-dimensional oriented manifold with boundary ∂M and let $\omega \in \Omega^{n-1}(M)$ be a form of compact support. Then, using the induced orientation*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof: We write again

$$\omega = \sum \varphi_i \omega$$

and then

$$\int_M d\omega = \sum \int_M d(\varphi_i \omega).$$

We work as in the previous version of the theorem, with

$$\varphi_i \omega = a_1 dx_2 \wedge \ldots \wedge dx_n - a_2 dx_1 \wedge dx_3 \wedge \ldots \wedge dx_n + \ldots + (-1)^{n-1} a_n dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{n-1}$$

(7.2), but now there are two types of open sets. For those which do not intersect ∂M the integral is zero by Theorem 7.2. For those which do, we have

$$\begin{aligned}
\int_M d(\varphi_i\omega) &= \int_{x_n\geq 0}\left(\frac{\partial a_1}{\partial x_1}+\ldots+\frac{\partial a_n}{\partial x_n}\right)dx_1dx_2\ldots dx_n \\
&= \int_{\mathbf{R}^{n-1}}[a_n]_0^\infty dx_1\ldots dx_{n-1} \\
&= -\int_{\mathbf{R}^{n-1}} a_n(x_1,x_2,\ldots,x_{n-1},0)dx_1\ldots dx_{n-1} \\
&= \int_{\partial M}\varphi_i\omega
\end{aligned}$$

□

where the last line follows since

$$\varphi_i\omega|_{\partial M} = (-1)^{n-1}a_n dx_1\wedge dx_2\wedge\ldots\wedge dx_{n-1}$$

and we use the induced orientation $(-1)^n dx_1\wedge\ldots\wedge dx_{n-1}$.

An immediate corollary is the following classical result, called the *Brouwer fixed point theorem.*

Theorem 7.7 *Let B be the unit ball $\{x\in\mathbf{R}^n : \|x\|\leq 1\}$ and let $F : B\to B$ be a smooth map from B to itself. Then F has a fixed point.*

Proof: Suppose there is no fixed point, so that $F(x)\neq x$ for all $x\in B$. For each $x\in B$, extend the straight line segment $\overline{F(x)x}$ until it meets the boundary sphere of B in the point $f(x)$. Then we have a smooth function

$$f : B\to\partial B$$

such that if $x\in\partial B$, $f(x)=x$.

Let ω be the standard non-vanishing $(n-1)$-form on $S^{n-1}=\partial B$, with

$$\int_{\partial B}\omega = 1.$$

Then

$$1=\int_{\partial B}\omega=\int_{\partial B}f^*\omega$$

since f is the identity on S^{n-1}. But by Stokes' theorem,

$$\int_{\partial B} f^*\omega = \int_B d(f^*\omega) = \int_B f^*(d\omega) = 0$$

since $d\omega = 0$ as ω is in the top dimension on S^{n-1}.

The contradiction $1 = 0$ means that there must be a fixed point. □

8 The degree of a smooth map

By using integration of forms we have seen that for a compact orientable manifold of dimension n the de Rham cohomology group $H^n(M)$ is non-zero, and that this fact enabled us to prove some global topological results about such manifolds. We shall now refine this result, and show that the group is (for a compact, connected, orientable manifold) just one-dimensional. This gives us a concrete method of determining the cohomology class of an n-form: it is exact if and only if its integral is zero.

8.1 de Rham cohomology in the top dimension

First a lemma:

Lemma 8.1 *Let $U^n = \{x \in \mathbf{R}^n : |x_i| < 1\}$ and let $\omega \in \Omega^n(\mathbf{R}^n)$ be a form with support in U^n such that*

$$\int_{U^n} \omega = 0.$$

Then there exists $\beta \in \Omega^{n-1}(\mathbf{R}^n)$ with support in U^n such that $\omega = d\beta$.

Proof: We prove the result by induction on the dimension n, but we make the inductive assumption that ω and β depend smoothly on a parameter $\lambda \in \mathbf{R}^m$, and also that if ω vanishes identically for some λ, so does β.

Consider the case $n = 1$, so $\omega = f(x, \lambda)dx$. Clearly taking

$$\beta(x, \lambda) = \int_{-1}^{x} f(u, \lambda)du \tag{18}$$

gives us a function with $d\beta = \omega$. But also, since f has support in U, there is a $\delta > 0$ such that f vanishes for $x > 1 - \delta$ or $x < -1 + \delta$. Thus

$$\int_{-1}^{x} f(u, \lambda)du = \int_{-1}^{1} f(u, \lambda)du = 0$$

for $x > 1-\delta$ and similarly for $x < -1+\delta$ which means that β itself has support in U. If $f(x,\lambda) = 0$ for all x, then from the integration (18) so does $\beta(x,\lambda)$.

Now assume the result for dimensions less than n and let

$$\omega = f(x_1, \ldots, x_n, \lambda) dx_1 \wedge \ldots \wedge dx_n$$

be the given form. Fix $x_n = t$ and consider

$$f(x_1, \ldots, x_{n-1}, t, \lambda) dx_1 \wedge \ldots \wedge dx_{n-1}$$

as a form on $\mathbf{R}^{n-1}$, depending smoothly on t and λ. Its integral is no longer zero, but if σ is a bump function on U^{n-1} such that the integral of $\sigma dx_1 \wedge \ldots \wedge dx_{n-1}$ is 1, then putting

$$g(t,\lambda) = \int_{U^{n-1}} f(x_1, \ldots, x_{n-1}, t, \lambda) dx_1 \wedge \ldots \wedge dx_{n-1}$$

we have a form

$$f(x_1, \ldots, x_{n-1}, t, \lambda) dx_1 \wedge \ldots \wedge dx_{n-1} - g(t,\lambda) \sigma dx_1 \wedge \ldots \wedge dx_{n-1}$$

with support in U^{n-1} and zero integral. Apply induction to this and we can write it as $d\gamma$ where γ has support in U^{n-1}.

Now put $t = x_n$, and consider $d(\gamma \wedge dx_n)$. The x_n-derivative of γ doesn't contribute because of the dx_n factor, and σ is independent of x_n, so we get

$$d(\gamma \wedge dx_n) = f(x_1, \ldots, x_{n-1}, x_n, \lambda) dx_1 \wedge \ldots \wedge dx_n - g(x_n, \lambda) \sigma dx_1 \wedge \ldots \wedge dx_n.$$

Putting

$$\xi(x_1, \ldots, x_n, \lambda) = (-1)^{n-1} \left(\int_{-1}^{x_n} g(t,\lambda) dt \right) \sigma dx_1 \wedge \ldots \wedge dx_{n-1}$$

also gives

$$d\xi = g(x_n, \lambda) \sigma dx_1 \wedge \ldots \wedge dx_n.$$

We can therefore write

$$f(x_1, \ldots, x_{n-1}, x_n, \lambda) dx_1 \wedge \ldots \wedge dx_n = d(\gamma \wedge dx_n + \xi) = d\beta.$$

Now by construction β has support in $|x_i| < 1$ for $1 \leq i \leq n-1$, but what about the x_n direction? Since $f(x_1, \ldots, x_{n-1}, t, \lambda)$ vanishes for $t > 1-\delta$ or $t < -1+\delta$, the inductive assumption tells us that γ does also for $x_n > 1-\delta$. As for ξ, if $t > 1-\delta$,

$$\begin{aligned}
\int_{-1}^{t} g(s,\lambda)ds &= \int_{-1}^{t}\left(\int_{U^{n-1}} f(x_1,\ldots,x_{n-1},t,\lambda)dx_1\wedge\ldots\wedge dx_{n-1}\right)dt \\
&= \int_{-1}^{1}\left(\int_{U^{n-1}} f(x_1,\ldots,x_{n-1},t,\lambda)dx_1\wedge\ldots\wedge dx_{n-1}\right)dt \\
&= \int_{U^n} f\left(x_1,\ldots,x_n,\lambda\right)dx_1\wedge\ldots\wedge dx_n \\
&= 0
\end{aligned}$$

by assumption. Thus the support of ξ is in U^n. Again, examining the integrals, if $f(x,\lambda)$ is identically zero for some λ, so is β.

□

Using the lemma, we prove:

Theorem 8.2 *If M is a compact, connected orientable n-dimensional manifold, then $H^n(M) \cong \mathbf{R}$.*

Proof: Take a covering by coordinate neighbourhoods which map to $U^n = \{x \in \mathbf{R}^n : |x_i| < 1\}$ and a corresponding partition of unity $\{\varphi_i\}$. By compactness, we can assume we have a finite number $U_1,\ldots,U_N$ of open sets. Using a bump function, fix an n-form α_0 with support in U_1 and

$$\int_M \alpha_0 = 1.$$

Thus, by Theorem 7.3 the cohomology class $[\alpha_0]$ is non-zero. To prove the theorem we want to show that for any n-form α,

$$[\alpha] = c[\alpha_0]$$

i.e. that $\alpha = c\alpha_0 + d\gamma$.

Given α use the partition of unity to write

$$\alpha = \sum \varphi_i\alpha$$

then by linearity it is sufficient to prove the result for each $\varphi_i\alpha$, so we may assume that the support of α lies in a coordinate neighbourhood U_m. Because M is connected we can connect $p \in U_1$ and $q \in U_m$ by a path and by the connectedeness of open intervals we can assume that the path is covered by a sequence of U_i's, each of which intersects the next: i.e. renumbering, we have

$$p \in U_1, \quad U_i \cap U_{i+1} \neq \emptyset, \quad q \in U_m.$$

Now for $1 \leq i \leq m-1$ choose an n-form α_i with support in $U_i \cap U_{i+1}$ and integral 1. On U_1 we have

$$\int (\alpha_0 - \alpha_1) = 0$$

and so applying Lemma 8.1, there is a form β_0 with support in U_1 such that

$$\alpha_0 - \alpha_1 = d\beta_1.$$

Continuing, we get altogether

$$\begin{array}{rcl} \alpha_0 - \alpha_1 & = & d\beta_1 \\ \alpha_1 - \alpha_2 & = & d\beta_2 \\ \dots & = & \dots \\ \alpha_{m-2} - \alpha_{m-1} & = & d\beta_{m-1} \end{array}$$

and adding, we find

$$\alpha_0 - \alpha_{m-1} = d(\sum_i \beta_i) \tag{19}$$

On U_m, we have

$$\int \alpha = c = \int c\alpha_{m-1}$$

and applying the Lemma again, we get $\alpha - c\alpha_{m-1} = d\beta$ and so from (19)

$$\alpha = c\alpha_{m-1} + d\beta = c\alpha_0 + d(\beta - c\sum_i \beta_i)$$

as required.

□

Theorem 8.2 tells us that for a compact connected oriented n-dimensional manifold, $H^n(M)$ is one-dimensional. Take a form ω_M whose integral over M is 1, then $[\omega_M]$ is a natural basis element for $H^n(M)$. Suppose

$$F : M \to N$$

is a smooth map of compact connected oriented manifolds of the same dimension n. Then we have the induced map

$$F^* : H^n(N) \to H^n(M)$$

and relative to our bases

$$F^*[\omega_N] = k[\omega_M] \tag{20}$$

for some real number k. We now show that k is an *integer*.

Theorem 8.3 *Let M, N be oriented, compact, connected manifolds of the same dimension n, and $F : M \to N$ a smooth map. There exists an integer, called the degree of F such that*

- *if $\omega \in \Omega^n(N)$ then*

$$\int_M F^*\omega = \deg F \int_N \omega$$

- *if a is a regular value of F then*

$$\deg F = \sum_{x \in F^{-1}(a)} \operatorname{sgn}(\det DF_x)$$

Remark:

1. A *regular value* for a smooth map $F : M \to N$ is a point $a \in N$ such that for each $x \in F^{-1}(a)$, the derivative DF_x is surjective. When $\dim M = \dim N$ this means that DF_x is invertible. Sard's theorem (a proof of which is in the Appendix) shows that for any smooth map most points in N are regular values.

2. The expression $\operatorname{sgn}(\det DF_x)$ in the theorem can be interpreted in two ways, but depends crucially on the notion of orientation – consistently associating the right sign for all the points $x \in F^{-1}(a)$. The straightforward approach uses Proposition 7.1 to associate to an orientation a class of coordinates whose Jacobians have positive determinant. If $\det DF_x$ is written as a Jacobian matrix in such a set of coordinates for M and N, then $\operatorname{sgn}(\det DF_x)$ is just the sign of the determinant. More invariantly, $DF_x : T_xM \mapsto T_aN$ defines a linear map

$$\Lambda^n(DF_x') : \Lambda T^*N_a \to \Lambda T_x^*M.$$

Orientations on M and N are defined by non-vanishing forms ω_M, ω_N and

$$\Lambda^n(DF_x')(\omega_N) = \lambda\omega_M.$$

Then $\operatorname{sgn}(\det DF_x)$ is the sign of λ.

3. Note the immediate corollary of the theorem: if F is not surjective, then $\deg F = 0$.

Proof: For the first part of the theorem, the cohomology class of ω is $[\omega] = c[\omega_N]$ and so integrating (and using Proposition 7.2),

$$\int_N \omega = c \int_N \omega_N = c.$$

Using the number k in (20),

$$F^*[\omega] = cF^*[\omega_N] = ck[\omega_M]$$

and integrating,

$$\int_M F^*\omega = ck \int_M \omega_M = ck = k \int_N \omega.$$

For the second part, since DF_x is an isomorphism at all points in $F^{-1}(a)$, from Theorem 3.3, $F^{-1}(a)$ is a zero-dimensional manifold. Since it is compact (closed inside a compact space M) it is a finite set of points. The inverse function theorem applied to these m points shows that there is a coordinate neighbourhood U of $a \in N$ such that $F^{-1}(U)$ is a disjoint union of m open sets U_i such that

$$F : U_i \to U$$

is a diffeomorphism.

Let σ be an n-form supported in U with $\int_N \sigma = 1$ and consider the diffeomorphism $F : U_i \to U$. Then by the coordinate invariance of integration of forms, and using the orientations on M and N,

$$\int_{U_i} F^*\sigma = \operatorname{sgn} DF_{x_i} \int_U \sigma = \operatorname{sgn} DF_{x_i}.$$

Hence, summing

$$\int_M F^*\sigma = \sum_i \operatorname{sgn} DF_{x_i}$$

and this is from the first part

$$k = k \int_N \sigma = \int_M F^*\sigma$$

which gives

$$k = \sum_i \operatorname{sgn} DF_{x_i}.$$

□

Example: Let M be the extended complex plane: $M = \mathbf{C} \cup \{\infty\}$. This is a compact, connected, orientable 2-manifold. In fact it is the 2-sphere. Consider the map $F : M \to M$ defined by

$$\begin{aligned} F(z) &= z^k + a_1 z^{k-1} + \ldots + a_k, \quad z \neq \infty \\ F(\infty) &= \infty \end{aligned}$$

This is smooth because in coordinates near $z = \infty$, F is defined (for $w = 1/z$) by

$$w \mapsto \frac{w^k}{1 + a_1 w + \ldots a_k w^k}.$$

To find the degree of F, consider

$$F_t(z) = z^k + t(a_1 z^{k-1} + \ldots + a_k).$$

This is a smooth map for all t and by Theorem 6.7 the action on cohomology is independent of t, so

$$\deg F = \deg F_0$$

where $F_0(z) = z^k$.

We can calculate this degree by taking a 2-form, with $|z| = r$ and $z = x + iy$

$$f(r) dx \wedge dy = f(r) r dr \wedge d\theta$$

with $f(r)$ of compact support. Then the degree is given by

$$\deg F_0 \int_{\mathbf{R}^2} f(r) r dr \wedge d\theta = \int_{\mathbf{R}^2} f(r^k) r^k d(r^k) k d\theta = k \int_{\mathbf{R}^2} f(r) r dr \wedge d\theta.$$

Thus $\deg F = k$. If $k > 0$ this means in particular that F is surjective and therefore takes the value 0 somewhere, so that

$$z^k + a_1 z^{k-1} + \ldots + a_k = 0$$

has a solution. This is the *fundamental theorem of algebra.*

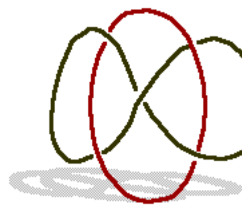
Example: Take two smooth maps $f_1, f_2 : S^1 \to \mathbf{R}^3$. These give two circles in $\mathbf{R}^3$ – suppose they are disjoint. Define

$$F : S^1 \times S^1 \to S^2$$

by

$$F(s,t) = \frac{f_1(s) - f_2(t)}{\|f_1(s) - f_2(t)\|}.$$

The degree of this map is called the *linking number*.



Example: Let $M \subset \mathbf{R}^3$ be a compact surface and $\mathbf{n}$ its unit normal. The *Gauss map* is the map

$$F : M \to S^2$$

defined by $F(x) = \mathbf{n}(x)$. If we work out the degree by integration, we take the standard 2-form ω on S^2. Then one finds that

$$\int_M F^*\omega = \int_M K\sqrt{EG - F^2}dudv$$

where K is the Gaussian curvature. The degree tells us this integral is 2π times an integer, which by the Gauss-Bonnet theorem is the Euler characteristic of M.

9 Riemannian metrics

Differential forms and the exterior derivative provide one piece of analysis on manifolds which, as we have seen, links in with global topological questions. There is much more one can do when one introduces a Riemannian metric. Since the whole subject of Riemannian geometry is a huge one, we shall here look at only two aspects which relate to the use of differential forms: the study of harmonic forms and of geodesics. In particular, we ignore completely here questions related to curvature.

9.1 The metric tensor

In informal terms, a Riemannian metric on a manifold M is a smoothly varying positive definite inner product on the tangent spaces T_x. To make global sense of this, note that an inner product is a bilinear form, so at each point x we want a vector in the tensor product

$$T_x^* \otimes T_x^*.$$

We can put, just as we did for the exterior forms, a vector bundle structure on

$$T^*M \otimes T^*M = \bigcup_{x \in M} T_x^* \otimes T_x^*.$$

The conditions we need to satisfy for a vector bundle are provided by two facts we used for the bundle of p-forms:

- each coordinate system $x_1, \ldots, x_n$ defines a basis $dx_1, \ldots, dx_n$ for each T_x^* in the coordinate neighbourhood and the n^2 elements

  $$dx_i \otimes dx_j, \quad 1 \leq i, j \leq n$$

  give a corresponding basis for $T_x^* \otimes T_x^*$

- the Jacobian of a change of coordinates defines an invertible linear transformation $J : T_x^* \to T_x^*$ and we have a corresponding invertible linear transformation $J \otimes J : T_x^* \otimes T_x^* \to T_x^* \otimes T_x^*$.

Given this, we define:

Definition 30 *A Riemannian metric on a manifold M is a section g of $T^* \otimes T^*$ which at each point is symmetric and positive definite.*

In a local coordinate system we can write

$$g = \sum_{i,j} g_{ij}(x) dx_i \otimes dx_j$$

where $g_{ij}(x) = g_{ji}(x)$ and is a smooth function, with $g_{ij}(x)$ positive definite. Often the tensor product symbol is omitted and one simply writes

$$g = \sum_{i,j} g_{ij}(x) dx_i dx_j.$$

Example:

1. The Euclidean metric on $\mathbf{R}^n$ is defined by

$$g = \sum dx_i \otimes dx_i.$$

So

$$g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \delta_{ij}.$$

2. A submanifold of $\mathbf{R}^n$ has an induced Riemannian metric: the tangent space at x can be thought of as a subspace of $\mathbf{R}^n$ and we take the Euclidean inner product on $\mathbf{R}^n$.

Given a smooth map $F : M \to N$ and a metric g on N, we can pull back g to a section F^*g of $T^*M \otimes T^*M$:

$$(F^*g)_x(X, Y) = g_F(x)(DF_x(X), DF_x(Y)).$$

If DF_x is invertible, this will again be positive definite, so in particular if F is a diffeomorphism.

Definition 31 *A diffeomorphism $F : M \to N$ between two Riemannian manifolds is an isometry if $F^*g_N = g_M$.*

Example: Let $M = \{(x, y) \in \mathbf{R}^2 : y > 0\}$ and

$$g = \frac{dx^2 + dy^2}{y^2}.$$

If $z = x + iy$ and

$$F(z) = \frac{az + b}{cz + d}$$

with a, b, c, d real and $ad - bc > 0$, then

$$F^*dz = (ad - bc)\frac{dz}{(cz + d)^2}$$

and

$$F^*y = y \circ F = \frac{1}{i}\left(\frac{az + b}{cz + d} - \frac{a\bar{z} + b}{c\bar{z} + d}\right) = \frac{ad - bc}{|cz + d|^2}y.$$

Then

$$F^*g = (ad - bc)^2\frac{dx^2 + dy^2}{|(cz + d)^2|^2}\frac{|cz + d|^4}{(ad - bc)^2y^2} = \frac{dx^2 + dy^2}{y^2} = g.$$

So these Möbius transformations are isometries of a Riemannian metric on the upper half-plane.

With a Riemannian metric one can define the length of a curve:

Definition 32 *Let M be a Riemannian manifold and $\gamma : [0, 1] \to M$ a smooth map (i.e. a smooth curve in M). The length of the curve is*

$$\ell(\gamma) = \int_0^1 \sqrt{g(\gamma', \gamma')}dt$$

where $\gamma'(t) = D\gamma_t(d/dt)$.

With this definition, any Riemannian manifold is a metric space: define

$$d(x, y) = \inf\{\ell(\gamma) \in \mathbf{R} : \gamma(0) = x, \gamma(1) = y\}.$$

Are Riemannian manifolds special? No, because:

Proposition 9.1 *Any manifold admits a Riemannian metric.*

Proof: Take a covering by coordinate neighbourhoods and a partition of unity subordinate to the covering. On each open set U_α we have a metric

$$g_\alpha = \sum_i dx_i^2$$

in the local coordinates. Define

$$g = \sum \varphi_i g_{\alpha(i)}.$$

This sum is well-defined because the supports of φ_i are locally finite. Since $\varphi_i \geq 0$ at each point every term in the sum is positive definite or zero, but at least one is positive definite so the sum is positive definite. □

9.2 The geodesic flow

Consider any manifold M and its cotangent bundle T^*M, with projection to the base $p : T^*M \to M$. Let X be tangent vector to T^*M at the point $\xi_a \in T_a^*$. Then

$$Dp_{\xi_a}(X) \in T_aM$$

so

$$\theta(X) = \xi_a(Dp_{\xi_a}(X))$$

defines a canonical 1-form θ on T^*M. In coordinates $(x, y) \mapsto \sum_i y_i dx_i$, the projection p is

$$p(x, y) = x$$

so if

$$X = \sum a_i \frac{\partial}{\partial x_i} + \sum b_i \frac{\partial}{\partial y_i}$$

then

$$\theta(X) = \sum_i y_i dx_i(Dp_{\xi_a}X) = \sum_i y_i a_i$$

which gives

$$\theta = \sum_i y_i dx_i.$$

We now take the exterior derivative

$$\omega = -d\theta = \sum dx_i \wedge dy_i$$

which is the *canonical* 2-*form* on the cotangent bundle. It is non-degenerate, so that the map

$$X \mapsto i_X\omega$$

from the tangent bundle of T^*M to its cotangent bundle is an isomorphism.

Now suppose f is a smooth function on T^*M. Its derivative is a 1-form df. Because of the isomorphism above, there is a unique vector field X on T^*M such that

$$i_X\omega = df.$$

If g is another function with vector field Y, then

$$Y(f) = df(Y) = i_Y i_X\omega = -i_X i_Y\omega = -X(g) \tag{21}$$

On a Riemannian manifold we shall see next that there is a natural function on T^*M. In fact a metric defines an inner product on T^* as well as on T, for the map

$$X \mapsto g(X, -)$$

defines an isomorphism from T to T^*. In concrete terms, if g^* is the inner product on T^*, then

$$g^*(\sum_j g_{ij} dx_j, \sum_k g_{kl} dx_l) = g_{ik}$$

which means that

$$g^*(dx_j, dx_k) = g^{jk}$$

where g^{jk} denotes the inverse matrix to g_{jk}.

We consider the function on T^*M defined by

$$H(\xi_a) = g^*(\xi_a, \xi_a).$$

In local coordinates this is

$$H(x, y) = \sum_{ij} g^{ij}(x) y_i y_j.$$

Definition 33 *The vector field X on T^*M given by $i_X\omega = dH$ is called the geodesic flow of the metric g.*

Definition 34 *If $\gamma : (a, b) \to T^*M$ is an integral curve of the geodesic flow, then the curve $p(\gamma)$ in M is called a geodesic.*

In local coordinates, if the geodesic flow is

$$X = \sum a_i \frac{\partial}{\partial x_i} + \sum b_i \frac{\partial}{\partial y_i}$$

then

$$i_X\omega = \sum_k (a_k dy_k - b_k dx_k) = dH = \sum_{ij} \frac{\partial g^{ij}}{\partial x_k} dx_k y_i y_j + 2 \sum_{ij} g^{ij} y_i dy_j.$$

Thus the integral curves are solutions of

$$\frac{dx_k}{dt} = 2 \sum_j g^{kj} y_j \quad (22)$$

$$\frac{dy_k}{dt} = -\sum_{ij} \frac{\partial g^{ij}}{\partial x_k} y_i y_j \quad (23)$$

Before we explain why this is a geodesic, just note the qualitative behaviour of these curves. For each point $a \in M$, choose a point $\xi_a \in T^*_a$ and consider the unique integral curve starting at ξ_a. Equation (22) tells us that the projection of the integral curve is parallel at a to the tangent vector X_a such that $g(X_a, -) = \xi_a$. Thus these curves have the property that through each point and in each direction there passes one geodesic.

Geodesics are normally thought of as curves of shortest length, so next we shall link up this idea with the definition above. Consider the variational problem of looking for critical points of the length functional

$$\ell(\gamma) = \int_0^1 \sqrt{g(\gamma', \gamma')}dt$$

for curves with fixed end-points $\gamma(0) = a, \gamma(1) = b$. For simplicity assume a, b are in the same coordinate neighbourhood. If

$$F(x, z) = \sum_{ij} g_{ij}(x) z_i z_j$$

then the first variation of the length is

$$\begin{aligned} \delta\ell &= \int_0^1 \frac{1}{2}F^{-1/2}\left(\frac{\partial F}{\partial x_i}\dot{x}_i + \frac{\partial F}{\partial z_i}\frac{d\dot{x}_i}{dt}\right)dt \\ &= \int_0^1 \frac{1}{2}F^{-1/2}\frac{\partial F}{\partial x_i}\dot{x}_i - \frac{d}{dt}\left(\frac{1}{2}F^{-1/2}\frac{\partial F}{\partial z_i}\right)\dot{x}_i dt. \end{aligned}$$

on integrating by parts with $\dot{x}_i(0) = \dot{x}_i(1) = 0$. Thus a critical point of the functional is given by

$$\frac{1}{2}F^{-1/2}\frac{\partial F}{\partial x_i} - \frac{d}{dt}\left(\frac{1}{2}F^{-1/2}\frac{\partial F}{\partial z_i}\right) = 0$$

If we parametrize this critical curve by arc length:

$$s = \int_0^t \sqrt{g(\gamma', \gamma')}dt$$

then $F = 1$, and the equation simplifies to

$$\frac{\partial F}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial F}{\partial z_i}\right) = 0.$$

But this is

$$\sum \frac{\partial g_{jk}}{\partial x_i}\frac{dx_j}{dt}\frac{dx_k}{dt} - \frac{d}{dt}\left(2g_{ik}\frac{dx_k}{dt}\right) = 0 \tag{24}$$

But now define y_i by

$$\frac{dx_k}{dt} = 2\sum_j g^{kj} y_j$$

as in the first equation for the geodesic flow (22) and substitute in (24) and we get

$$4\sum \frac{\partial g_{jk}}{\partial x_i} g^{ja} y_a g^{kb} y_b - \frac{d}{dt}\left(4 g_{ik} g^{ka} y_a\right) = 0$$

and using

$$\sum_j g^{ij} g_{jk} = \delta^i_k$$

this yields

$$-\frac{\partial g^{jk}}{\partial x_i} y_j y_k = \frac{dy_i}{dt}$$

which is the second equation for the geodesic flow. (Here we have used the formula for the derivative of the inverse of a matrix G: $D(G^{-1}) = -G^{-1}DGG^{-1}$).

The formalism above helps to solve the geodesic equations when there are isometries of the metric. If $F : M \to M$ is a diffeomorphism of M then its natural action on 1-forms induces a diffeomorphism of T^*M. Similarly with a one-parameter group φ_t. Differentiating at $t = 0$ this means that a vector field X on M induces a vector field $\tilde{X}$ on T^*M. Moreover, the 1-form θ on T^*M is canonically defined and hence invariant under the induced action of any diffeomorphism. This means that

$$\mathcal{L}_{\tilde{X}}\theta = 0$$

and therefore, using (6.5) that

$$i_{\tilde{X}} d\theta + d(i_{\tilde{X}}\theta) = 0$$

so since $\omega = -d\theta$

$$i_{\tilde{X}}\omega = df$$

where $f = i_{\tilde{X}}\theta$.

Proposition 9.2 *The function f above is $f(\xi_x) = \xi_x(X_x)$.*

Proof: Write in coordinates

$$\tilde{X} = \sum a_i \frac{\partial}{\partial x_i} + \sum b_i \frac{\partial}{\partial y_i}$$

where $\theta = \sum_i y_i dx_i$. Since $\tilde{X}$ projects to the vector field X on M, then

$$X = \sum a_i \frac{\partial}{\partial x_i}$$

and

$$i_{\tilde{X}}\theta = \sum_i a_i y_i = \xi_x(X_x)$$

by the definition of θ. □

Now let M be a Riemannian manifold and H the function on T^*M defined by the metric as above. If φ_t is a one-parameter group of *isometries*, then the induced diffeomorphisms of T^*M will preserve the function H and so the vector field $\tilde{Y}$ will satisfy

$$\tilde{Y}(H) = 0.$$

But from (21) this means that $X(f) = 0$ where X is the geodesic flow and f the function $i_{\tilde{Y}}\theta$. This function is constant along the geodesic flow, and is therefore a constant of integration of the geodesic equations.

Example: Consider the metric

$$g = \frac{dx_1^2 + dx_2^2}{x_2^2}$$

on the upper half plane and its geodesic flow X.

The map $(x_1, x_2) \mapsto (x_1 + t, x_2)$ is clearly a one-parameter group of isometries (the Möbius transformations $z \mapsto z + t$) and defines the vector field

$$Y = \frac{\partial}{\partial x_1}.$$

On the cotangent bundle this gives the function

$$f(x, y) = y_1$$

which is constant on the integral curve.

The map $z \mapsto e^t z$ is also an isometry with vector field

$$Z = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$$

so that

$$g(x,y) = x_1y_1 + x_2y_2$$

is constant.

We also have automatically that $H = x_2^2(y_1^2 + y_2^2)$ is constant since

$$X(H) = i_X i_X \omega = 0.$$

We therefore have three equations for the integral curves of the geodesic flow:

$$\begin{aligned} y_1 &= c_1 \\ x_1y_1 + x_2y_2 &= c_2 \\ x_2^2(y_1^2 + y_2^2) &= c_3 \end{aligned}$$

Eliminating y_1, y_2 gives the geodesics:

$$(c_1x_1 - c_2)^2 + c_1^2x_2^2 = c_3.$$

If $c_1 = 0$ this is a half-line $x_2 = const.$. Otherwise it is a semicircle with centre on the x_1 axis. These are the straight lines of non-Euclidean geometry.

9.3 Harmonic forms

We mentioned above that a metric g defines an inner product not just on T_a but also an inner product g^* on T_a^*. With this we can define an inner product on the pth exterior power $\Lambda^p T_a^*$:

$$(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_p, \beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_p) = \det g^*(\alpha_i, \beta_j) \quad (25)$$

In particular, on an n-manifold there is an inner product on each fibre of the bundle $\Lambda^n T^*$. Since each fibre is one-dimensional there are only two unit vectors $\pm u$.

Definition 35 *Let M be an oriented Riemannian manifold, then the volume form is the unique n-form ω of unit length in the equivalence class defined by the orientation.*

In local coordinates,the definition of the inner product (25) gives

$$(dx_1 \wedge \ldots \wedge dx_n, dx_1 \wedge \ldots \wedge dx_n) = \det g_{ij}^* = (\det g_{ij})^{-1}$$

Thus if $dx_1 \wedge \ldots \wedge dx_n$ defines the orientation,

$$\omega = \sqrt{\det g_{ij}} dx_1 \wedge \ldots \wedge dx_n.$$

On a compact manifold we can integrate this to obtain the total volume – so a metric defines not only lengths but also volumes.

Now take $\alpha \in \Lambda^p T_a^*, \beta \in \Lambda^{n-p} T_a^*$ and define $f_\beta : \Lambda^p T_a^* \to \mathbf{R}$ by

$$f_\beta(\alpha)\omega = \beta \wedge \alpha.$$

But we have an inner product, so any linear map on $\Lambda^p T_a^*$ is of the form

$$\alpha \mapsto (\alpha, \gamma)$$

for some $\gamma \in \Lambda^p T_a^*$, so we have a well-defined linear map $\beta \mapsto \gamma_\beta$ from $\Lambda^{n-p}T^*$ to $\Lambda^p T^*$, satisfying

$$(\gamma_\beta, \alpha)\omega = \beta \wedge \alpha.$$

We use a different symbol for this:

Definition 36 *The Hodge star operator is the linear map* $* : \Omega^p(M) \to \Omega^{n-p}(M)$ *with the property that at each point*

$$(\alpha, \beta)\omega = \alpha \wedge *\beta.$$

Example: If $e_1, \ldots, e_n$ is an orthonormal basis of the space of one-forms at a point, then

$$*(e_1 \wedge \ldots \wedge e_p) = e_{p+1} \wedge \ldots \wedge e_n.$$

Exercise 9.3 *Show that on p-forms,* $*^2 = (-1)^{p(n-p)}$.

On a Riemannian manifold we can use the star operator to define new differential operators on forms. In particular, consider the operator

$$d^* : \Omega^p(M) \to \Omega^{p-1}(M)$$

defined by

$$d^* = (-1)^{np+n+1} * d * .$$

The notation is suggestive, in fact:

Proposition 9.4 *Let M be an oriented Riemannian manifold with volume form ω and let $\alpha \in \Omega^p(M), \beta \in \Omega^{p-1}(M)$ be forms of compact support. Then*

$$\int_M (d^*\alpha, \beta)\omega = \int_M (\alpha, d\beta)\omega.$$

Proof: We have

$$\int_M (d^*\alpha, \beta)\omega = (-1)^{np+n+1}\int_M (*d*\alpha, \beta)\omega = (-1)^{np+n+1}\int_M (\beta, *d*\alpha)\omega = (-1)^{np+n+1}\int_M \beta\wedge **d*\alpha$$

from the definition of d^* and $*$. But on the $n-p+1$-form $d*\alpha$, $*^2 = (-1)^{(n-p+1)(p-1)}$ so this is

$$(-1)^{np+n+1+(n-p+1)(p-1)}\int_M \beta\wedge d*\alpha = (-1)^p\int_M \beta\wedge d*\alpha.$$

Now

$$d(\beta\wedge *\alpha) = d\beta\wedge *\alpha + (-1)^{p-1}\beta\wedge d*\alpha.$$

Integrating $d(\beta\wedge *\alpha)$ gives zero from the first version of Stokes' theorem (7.2), so we get

$$(-1)^p\int_M \beta\wedge d*\alpha = \int_M d\beta\wedge *\alpha = \int_M (\alpha, d\beta)\omega.$$

□

Definition 37 *Let M be an oriented Riemannian manifold, then the Laplacian on p-forms is the differential operator $\Delta : \Omega^p(M)\to\Omega^p(M)$ defined by*

$$\Delta = dd^* + d^*d.$$

Example: Suppose $M = \mathbf{R}^3$ with the Euclidean metric and $\alpha = a_1 dx_1$, then

$$\Delta(a_1dx_1) = (dd^* + d^*d)(a_1dx_1)$$

so

$$dd^*(a_1dx_1) = -d*d*(a_1dx_1) = -d*d(a_1dx_2\wedge dx_3) = -d*\frac{\partial a_1}{\partial x_1}dx_1\wedge dx_2\wedge dx_3$$

$$= -d\frac{\partial a_1}{\partial x_1} = -\frac{\partial^2 a_1}{\partial x_1^2}dx_1 - \frac{\partial^2 a_1}{\partial x_2\partial x_1}dx_2 - \frac{\partial^2 a_1}{\partial x_3\partial x_1}dx_3$$

and

$$d^*d(a_1dx_1) = d^*(\frac{\partial a_1}{\partial x_2}dx_2\wedge dx_1 + \frac{\partial a_1}{\partial x_3}dx_3\wedge dx_1) = *d(\frac{\partial a_1}{\partial x_2}dx_3 - \frac{\partial a_1}{\partial x_3}dx_2)$$

$$= *(\frac{\partial^2 a_1}{\partial x_1\partial x_2}dx_1\wedge dx_3 + \frac{\partial^2 a_1}{\partial x_2^2}dx_2\wedge dx_3 - \frac{\partial^2 a_1}{\partial x_1\partial x_3}dx_1\wedge dx_2 - \frac{\partial^2 a_1}{\partial x_3^2}dx_3\wedge dx_2)$$

$$= \frac{\partial^2 a_1}{\partial x_1 \partial x_2} dx_2 - \frac{\partial^2 a_1}{\partial x_2^2} dx_1 + \frac{\partial^2 a_1}{\partial x_1 \partial x_3} dx_3 - \frac{\partial^2 a_1}{\partial x_3^2} dx_1.$$

Adding, we get

$$\Delta(a_1 dx_1) = -\left(\frac{\partial^2 a_1}{\partial x_1^2} + \frac{\partial^2 a_1}{\partial x_2^2} + \frac{\partial^2 a_1}{\partial x_3^2} \right) dx_1$$

which is the negative of the usual Laplacian on the coefficient a_1. By linearity the same is true for a general 1-form $a_1 dx_1 + a_2 dx_2 + a_3 dx_3$.

When $p = 0$ we have

$$\Delta f = d^* df = (-1)^{n+n+1} * d * df = - * d * df$$

and this is sometimes called the *Laplace-Beltrami operator*, though there are differing conventions about sign:

Example:

1. Take $M = \mathbf{R}^n$ with the Euclidean metric.

$$\begin{aligned} df &= \sum_i \frac{\partial f}{\partial x_i} dx_i \\ *df &= \frac{\partial f}{\partial x_1} dx_2 \wedge \ldots \wedge dx_n + \ldots \\ d * df &= \frac{\partial^2 f}{\partial x_1^2} dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n + \ldots \\ \Delta f = - * d * df &= -\sum_i \frac{\partial^2 f}{\partial x_i^2} \end{aligned}$$

2. Take M to be the upper half-plane with metric

$$g = \frac{1}{y^2}(dx^2 + dy^2).$$

Then

$$\omega = \frac{1}{y^2} dx \wedge dy \quad * dx = dy, \quad *dy = -dx.$$

So

$$\begin{aligned} \Delta f &= - * d(\frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx) \\ &= -y^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) \end{aligned}$$

It follows that the real and imaginary part of a holomorphic function of $z = x + iy$ satisfy $\Delta f = 0$, just as in the case of the Euclidean metric.

Definition 38 *A differential form $\alpha \in \Omega^p(M)$ is a harmonic form if $\Delta\alpha = 0$.*

On a compact manifold harmonic forms play a very important role, which there is no time to explore in this course. Here is the starting point:

Proposition 9.5 *Let M be a compact oriented Riemannian manifold. Then*

- *a p-form is harmonic if and only if $d\alpha = 0$ and $d^*\alpha = 0$*
- *in each de Rham cohomology class there is at most one harmonic form.*

Proof: Clearly if $d\alpha = d^*\alpha = 0$, then $(dd^* + d^*d)\alpha = 0$. Suppose conversely $\Delta\alpha = 0$, then

$$0 = \int_M (\Delta\alpha, \alpha)\omega = \int_M (dd^* + d^*d)\alpha, \alpha)\omega = \int_M (d^*\alpha, d^*\alpha)\omega + \int_M (d\alpha, d\alpha)\omega.$$

But these last two terms are non-negative and vanish if and only if $d\alpha = d^*\alpha = 0$.

Suppose α, α' are harmonic forms in the same cohomology class, then

$$\alpha - \alpha' = d\beta.$$

But then

$$0 = d^*\alpha - d^*\alpha' = d^*d\beta$$

and

$$0 = \int_M (d^*d\beta, \beta)\omega = \int_M (d\beta, d\beta)\omega$$

which gives $d\beta = 0$ and $\alpha = \alpha'$. □

The theorem of W.V.D. Hodge says that there exists in each cohomology class a harmonic form which, as we have seen, is unique. This result was a profound influence on geometry in the last half of the 20th century. The proof is beyond the scope of this course, but the interested reader with a week or two to spare can find a proof in: *Foundations of Differentiable manifolds and Lie Groups* by F. Warner, Graduate Texts in Mathematics **94**, Springer 1983. There is a natural interpretation of the result: the harmonic form α in a cohomology class is the one of smallest $\mathcal{L}^2$ norm, because any other is of the form $\alpha + d\beta$ and

$$\int_M (\alpha + d\beta, \alpha + d\beta)\omega = \int_M (\alpha, \alpha)\omega + \int_M (d\beta, d\beta)\omega \geq \int_M (\alpha, \alpha)\omega$$

since

$$\int_M (\alpha, d\beta)\omega = \int_M (d^*\alpha, \beta)\omega = 0.$$

There are some immediate consequences of the Hodge theorem. First note that:

Proposition 9.6 *The Laplacian Δ commutes with $*$.*

Proof:

$$\begin{aligned}
(dd^* + d^*d) * \alpha &= (-1)^{n(n-p)+n+1} d * d * *\alpha + (-1)^{n(n-p+1)+n+1} * d * d * \alpha \\
&= (-1)^{n(n-p)+n+1+p(n-p)} d * d\alpha + (-1)^{n+pn+1} * d * d * \alpha \\
&= (-1)^{p+1} d * d\alpha + (-1)^{n+pn+1} * d * d * \alpha
\end{aligned}$$

and

$$\begin{aligned}
*(dd^* + d^*d)\alpha &= (-1)^{np+n+1} * d * d * \alpha + (-1)^{n(p+1)+n+1} d * d * *\alpha \\
&= (-1)^{np+n+1} * d * d * \alpha + (-1)^{n(p+1)+n+p(n-p)+1} d * d\alpha \\
&= (-1)^{np+n+1} * d * d * \alpha + (-1)^{p+1} d * d\alpha
\end{aligned}$$

□

It follows from the proposition that $*$ maps harmonic forms to harmonic forms. since $*^2 = (-1)^{p(n-p)}$ it is invertible and so it maps the space of harmonic p-forms isomorphically to the space of harmonic $n-p$ forms. One consequence of the Hodge theorem is that

$$\dim H^p(M) = \dim H^{n-p}(M).$$

This we saw for $p = 0$ rather differently in Theorem 8.2.