214 - de Rham's Theorem

This note presents a proof of de Rham's Theorem. It assumes some familiarity with singular cohomology, presumably much less than what is covered in 215. The argument is taken from Bredon's *Topology and Geometry*, with exposition altered to suit what we have and have not done already.

In order to help distinguish between the cohomology theories in the discussion, we will use $H^k_\Omega(M)$ in place of $H^k(M)$ as used in class when discussing de Rham cohomology. $H^k(M;\mathbb{R})$ will denote smooth singular cohomology.

Theorem 1. *Let M be a smooth manifold. The homomorphism*

$$\Psi : \Omega^k(M) \to \Delta^k(M)$$

where $\Psi(\omega) : \Delta_k(M) \to \mathbb{R}$ is given by

$$\Psi(\omega)(\sigma) = \int_\sigma \omega$$

induces an isomorphism

$$\Psi^* : H^k_\Omega(M) \to H^k(M;\mathbb{R}).$$

The outline of the argument is provided by the following lemma.

Lemma 2. *Let M^n be a smooth manifold and $P(U)$ a statement about open subsets $U \subset M$. If the following three conditions hold:*

1. *$P(U)$ is true for U diffeomorphic to a convex open subset of $\mathbb{R}^n$,*
2. *$P(U), P(V), P(U \cap V) \implies P(U \cup V)$,*
3. *$\{U_\alpha : \alpha \in A\}$ disjoint and $P(U_\alpha)$ for all $\alpha \in A \implies P(\bigcup_{\alpha \in A} U_\alpha)$,*

then $P(M)$ is true.

Proof of Theorem 1 assuming Lemma 2. Let $P(U)$ denote the statement that Theorem 1 is true for U.

If U is diffeomorphic to a convex open subset of $\mathbb{R}^n$, then $P(U)$ is true by the Poincaré Lemma.

Now suppose that $P(U), P(V), P(U \cap V)$ are true. We obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \Omega^k(M) & \to & \Omega^k(U) \oplus \Omega^k(V) & \to & \Omega^k(U \cap V) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \Delta^k(M) & \to & \Delta^k(U) \oplus \Delta^k(V) & \to & \Delta^k(U \cap V) & \to 0 \end{array}$$

where all vertical maps are Ψ and each row is exact. This lifts to a ladder commutative diagram with vertical maps Ψ^* and rows given by the Mayer-Vietoris sequences for de Rham and smooth singular cohomology respectively. The two Ψ^*'s on the left of $\Psi^* : H^k_\Omega(U \cup V) \to H^k(U \cup V; \mathbb{R})$ and the two Ψ^*'s on the right are isomorphisms by assumption, so $\Psi^* : H^k_\omega(U \cup V) \to H^k(U \cup V; \mathbb{R})$ is an isomorphism by the Five Lemma.

Finally, let $\{U_\alpha : \alpha \in A\}$ disjoint and $P(U_\alpha)$ for all $\alpha \in A$. The cohomology of $\bigcup_{\alpha \in A} U_\alpha$ in either theory is the product of the $H^k(U_\alpha)$, so $P(\bigcup_{\alpha \in A} U_\alpha)$ is true. □

Note: A similar proof shows the equivalence of smooth singular cohomology and singular cohomology replacing Ψ by the inclusion map of smooth simplices into simplices. Singular cohomology and de Rham cohomology are then equivalent by transitivity.

Proof of Lemma 2. As in our proof of the Whitney Embedding Theorem, we may cover M by compact sets $C_1, C_2, C_3, \ldots$ such that

$$C_i \cap C_j = \emptyset \quad \text{if} \quad |i - j| > 1.$$

Give M an arbitrary Riemannian metric. Each C_i can be covered by a finite number of geodesically convex[1] sets $U_{1i}, \ldots U_{j_i i}$. We may use small enough open sets so that for

$$\hat{C}_i = \bigcup_{l=1}^{j_i} U_{li},$$

we maintain the disjointness

$$\hat{C}_i \cap \hat{C}_j = \emptyset \quad \text{if} \quad |i - j| > 1.$$

We can also do this in such a way that each geodesically convex subset of U_{li}, including U_{li}, is diffeomorphic to a convex open subsets of $\mathbb{R}^n$, so $P(U_{li})$ is true. Furthermore, $U_{1i} \cap U_{2i}$ is geodesically convex, so $P(U_{1i} \cap U_{2i})$ is true. By Property 2, $P(U_{1i} \cup U_{2i})$ is true. Next,

$$(U_{1i} \cup U_{2i}) \cap U_{3i} = (U_{1i} \cap U_{3i}) \cup (U_{2i} \cap U_{3i}),$$

[1] Recall that a subset C of a Riemannian manifold is geodesically convex if for all $p, q \in C$, there exists a geodesic from p to q in C. This gives us a coordinate free notion of convexity, allowing avoidance of coordinate neighborhood contortions.

So $P(U_{1i} \cap U_{3i})$ and $P(U_{2i} \cap U_{3i})$ imply $P((U_{1i} \cup U_{2i}) \cap U_{3i})$. Since $P(U_{1i} \cup U_{2i})$ and $P(U_{3i})$ are true, this gives $P(U_{1i} \cup U_{2i} \cup U_{3i})$. $P(\hat{C}_i)$ now follows by the obvious induction argument.

Let $\hat{C}_{even} = \bigcup_{n \in \mathbb{N}} \hat{C}_{2n}$ and $\hat{C}_{odd} = \bigcup_{n \in \mathbb{N}} \hat{C}_{2n-1}$. These are both disjoint unions, so Property 3 implies $P(\hat{C}_{even})$ and $P(\hat{C}_{odd})$. $M = \hat{C}_{even} \cup \hat{C}_{odd}$, so it suffices by Property 2 to prove $P(\hat{C}_{even} \cap \hat{C}_{odd})$.

$$\hat{C}_{even} \cap \hat{C}_{odd} = \bigcup_{n \in \mathbb{N}} (\hat{C}_n \cup \hat{C}_{n+2}) \cap \hat{C}_{n+1},$$

with the union over $\mathbb{N}$ disjoint, so we may reduce by Property 3 to showing $P((\hat{C}_n \cup \hat{C}_{n+2}) \cap \hat{C}_{n+1})$. $(\hat{C}_n \cup \hat{C}_{n+2}) \cap \hat{C}_{n+1}$ is a union of geodesically convex sets as before (take intersections of the U_{li}), so we may simply repeat our previous argument for $\hat{C}_i$. $\square$